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Moment Problems in Hereditary Function Spaces

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Abstract: We introduce a concept of hereditary set of multi-indices, and consider vector spaces of functions generated by families associated to such sets of multi-indices, called hereditary function spaces. Existence and uniquenes of representing measures for some abstract truncated moment problems are investigated in this framework, by adapting the concept of idempotent and that of dimensional stability, and using some techniques involving $C^\star$-algebras and commuting self-adjoint multiplication operators.

Keywords: hereditary set of multi-indices; relative idempotent; relative multiplicativity; dimensional stability

MSC: Primary: 44A60 Secondary: 46C05; 47B15.

1 Introduction

The use of finite families of multi-indices with appropriate properties to investigate the associated truncated moment problems appears in several works, as for instance in [8, 9], and more recently in [17, 18]. In this work, we deal with the concept of a hereditary family of multi-indices, which happens to be compatible with our techniques, developped in some older works for ordinary truncated moment problems (see [14–16] etc.).

As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. For a fixed integer $n \in \mathbb{N}$, the Cartesian product $\mathbb{Z}_+^n$ is said to be the set of multi-indices of lenght n.

Let $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, and $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$ be arbitrary. Then $\mathbf{t}^\mathbf{k}$ means the monomial $t_1^{k_1} \dots t_n^{k_n}$, and $|\mathbf{k}| = k_1 + \dots + k_n$. By $\mathfrak{B}(\mathbb{R}^n)$ we denote the set of all Borel subsets of $\mathbb{R}^n$. In the set $\mathbb{Z}_+^n$ we consider the order relation "≤" given by $\mathbf{k} \le \mathbf{p}$ whenever $k_j \le p_j,\ j = 1, \dots, n$, where $\mathbf{k} = (k_1, \dots, k_n),\ \mathbf{p} = (p_1, \dots, p_n)$.

Let $\mathbb{K}$ be an arbitrary subset of $\mathbb{Z}_+^n$, and let $\mathcal{P}_\mathbb{K}^n$ be the complex vector space spanned by the set of monomials $\{\mathbf{t}^\mathbf{k} : \mathbf{k} \in \mathbb{K}\}$. Let $\Gamma = (\gamma_\mathbf{k})_{\mathbf{k}\in\mathbb{K}}$ be an arbitrary set of real numbers. *The* $\mathbb{K}$*-truncated moment problem* consists of finding necessary and sufficient condition in terms of Γ, insuring the existence of a non-negative measure μ on $\mathfrak{B}(\mathbb{R}^n)$ such that each monomial $\mathbf{t}^\mathbf{k}$ is μ-integrable, and

$$\gamma_\mathbf{k} = \int \mathbf{t}^\mathbf{k} d\mu(\mathbf{t}), \quad \mathbf{k} \in \mathbb{K}. \tag{1}$$

Equivalently, for the assignment $\mathbf{t}^\mathbf{k} \mapsto \gamma_\mathbf{k}$, which induces a linear functional on $\mathcal{P}_\mathbb{K}^n$, say $\Lambda_\mathbb{K}$, one looks for a non-negative measure μ on $\mathfrak{B}(\mathbb{R}^n)$ such that $\Lambda_\mathbb{K}(p) = \int p(\mathbf{t})d\mu(\mathbf{t})$ for all $p \in \mathcal{P}_\mathbb{K}^n$.

The moment problems with respect to a given set of multi-indices $\mathbb{K}$ can be stated in a more abstract context, for functions more general that polynomials. Let us introduce such a convenient framework.

Let $(\Omega, \mathfrak{S})$ be a *measurable space*, that is, Ω is an arbitrary (nonempty) set and $\mathfrak{S}$ is a σ-algabra of subsets of Ω. Let also $\mathcal{F}$ be a vector space consisting of $\mathfrak{S}$-measurable complex-valued functions on Ω, invariant under complex conjugation. Assume that we have a linear map $\Lambda : \mathcal{F} \mapsto \mathbb{C}$. A natural question is to investigate the

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existence of a positive measure μ on Ω such that

$$\Lambda(f) = \int_{\Omega} f(\omega) d\mu(\omega), \quad f \in \mathcal{F}.$$

Thanks to an argument originating in [12], in many situations of interest we may restrict ourselves to the case when the space $\mathcal{F}$ is finite dimensional. The finite dimensionality of the space $\mathcal{F}$ leads to the possibility to replace an existing measure μ by another one consisting of a finite number of atoms, via an argument going back to [13] (see also [1]).

An extremal condition, firstly stated for polynomial moment problem (see [5]) is still very useful in our more general framework. Specifically, it insures the uniqueness of the representing measure when such a measure exists (see Proposition 1). Concerning the existence of a representing measure, we present two main results. For the first one, we use the concept of idempotent element, introduced in [15], which plays a central role in our development. In hereditary function spaces having a finite number of generators, the existence of representing measures is characterized by the existence of orthogonal bases consisting of idempotents, satisfying a certain "multiplicativity condition" (see Theorem 1).

The concept of dimensional stability (discussed in [14]) goes back to the concept of *flatness*, introduced in [3] in the context of spaces of polynomials. This property is used to prove another main results of this paper (see Theorem 2).

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2 Hereditary Function Spaces

2.1 Hereditary Sets of Indices

Let us first define the maps $S_j : \mathbb{Z}_+^n \mapsto \mathbb{Z}_+^n$ via the formulas

$$S_j(k_1, \dots, k_j, \dots, k_n) = (k_1, \dots, k_j + 1, \dots, k_n), \; (k_1, \dots, k_j, \dots, k_n) \in \mathbb{Z}_+^n, \tag{2}$$

for all $j = 1, \dots, n$, which are, in fact, mutually commuting shifts.

The following type of sets of indices also appears in other works, sometimes under different names, associated with different techniques (see [8, 9, 18] etc.).

Definition 1. A subset $\mathbb{K} \subset \mathbb{Z}_+^n$ is said to be *hereditary* if for every $\mathbf{k} \in \mathbb{K}$ and $\mathbf{r} \in \mathbb{Z}_+^n$ such that $\mathbf{r} \le \mathbf{k}$, we have $\mathbf{r} \in \mathbb{K}$.

Example 1. (1) *Let* $\mathbb{K} = \mathbb{K}_m = \{\mathbf{k} \in \mathbb{Z}_+^n : |\mathbf{k}| \le m\}$, *for some fixed* $m \in \mathbb{N}$. *Then* $\mathbb{K}$ *is hereditary.*

(2) *Let* $\mathbb{K} = \mathbb{K}_{\mathbf{d}} = \{\mathbf{k} \in \mathbb{Z}_+^n : \mathbf{k} \le \mathbf{d}\}$, *where* $\mathbf{d} \in \mathbb{Z}_+^n$ *is fixed. Then* $\mathbb{K}$ *is hereditary.*

(3) *Let* $\mathbf{k}_1, \dots, \mathbf{k}_r$ *be fixed elements of* $\mathbb{Z}_+^n$. *Then the set*

$$\bigcup_{j=1}^{r} \{\mathbf{k} \in \mathbb{Z}_+^n : \mathbf{k} \le \mathbf{k}_j\}$$

is hereditary.

Lemma 1. *Let* $\mathbb{K}_j \subset \mathbb{Z}_+^n$ $(j = 1, 2)$ *be hereditary. Then* $\mathbb{K} = \mathbb{K}_1 + \mathbb{K}_2 \subset \mathbb{Z}_+^n$ *is also hereditary.*

The easy proof is left to the reader. This lemma shows that the family of hereditary sets contains also the sets obtained by adding and combining those from Example 1.

Remark 1. Let $\mathbb{K} \subset \mathbb{Z}_+^n$ be a hereditary finite set. We define, by recurrence, the sets of indices $\mathbb{K}_r = \{\mathbf{S}^{\mathbf{p}}\mathbf{k} : |\mathbf{p}| \le r, \; \mathbf{k} \in \mathbb{K}, r \ge 0\}$, so $\mathbb{K}_0 = \mathbb{K}$, and $\mathbf{S} = (S_1, \dots, S_n)$ is given by formula (2). Note that we have $\mathbb{K}_0 \subset \mathbb{K}_1 \subset \mathbb{K}_2 \subset \cdots$. In fact, $\mathbb{K}_r = \{\mathbf{S}^{\mathbf{p}}\mathbf{S}^{\mathbf{k}}\mathbf{0}, |\mathbf{p}| \le r, \; \mathbf{k} \in \mathbb{K}\}$ for all $r \ge 0$. Moreover, the set $\mathbb{K}_\infty = \cup_{r \ge 0}\mathbb{K}_r$ is also hereditary.

2.2 Function Spaces

Let $(\Omega, \mathfrak{S})$ be a measurable space, and let also $\mathcal{M}_{\mathfrak{S}}(\Omega)$ be the algebra of all complex-valued $\mathfrak{S}$-measurable functions on Ω (that is, $f^{-1}(B) \in \mathfrak{S}$ for each $f \in \mathcal{M}_{\mathfrak{S}}(\Omega)$ and all Borel sets $B \in \mathfrak{B}(\mathbb{C})$). In fact, in most of the cases we may restrict ouselves to the case when Ω is a Hausdorff space and $\mathfrak{S}$ is the σ-algabra of all Borel subsets of Ω and, when working with finite atomic measures, even to the case when Ω is an arbitrary (nonempty) set and $\mathfrak{S}$ is a family of all subsets of Ω.

For convenience, and following [14–16], a vector subspace $\mathcal{F} \subset \mathcal{M}_{\mathfrak{S}}(\Omega)$ such that $1 \in \mathcal{F}$ and if $f \in \mathcal{F}$, then $\bar{f} \in \mathcal{F}$, is said to be a *function space*.

Fixing a function space $\mathcal{F}$, let $\mathcal{F}^{(2)}$ be the vector space spanned by all products of the form fg with $f, g \in \mathcal{F}$, which is itself a function space. We have $\mathcal{F} \subset \mathcal{F}^{(2)}$, and $\mathcal{F} = \mathcal{F}^{(2)}$ when $\mathcal{F}$ is an algebra.

For any vector subspace $\mathcal{T} \subset \mathcal{F}$ invariant under complex conjugation, the symbol $\mathcal{R}\mathcal{T}$ will designate the "real part" of $\mathcal{T}$, that is $\{f \in \mathcal{T}; f = \bar{f}\}$.

Important examples of function spaces are associated with the space $\mathcal{P}^n$ of all polynomials in $n \geq 1$ real variables, denoted as above by $t_1, \ldots, t_n$, with complex coefficients. For every integer $m \geq 0$, let $\mathcal{P}^n_m$ be the subspace of $\mathcal{P}^n$ consisting of all polynomials p with $\deg(p) \leq m$, where $\deg(p)$ is the total degree of p. Both $\mathcal{P}^n_m$ and $\mathcal{P}^n$ are function spaces (of continuous functions) on $\mathbb{R}^n$. In fact, $\mathcal{P}^n_m = \mathcal{P}^n_{\mathbb{K}_m}$, with $\mathbb{K}_m$ as in Example 1(1). Similarly, $\mathcal{P}^n_{\mathbf{d}} = \mathcal{P}^n_{\mathbb{K}_{\mathbf{d}}}$, with $\mathcal{P}^n_{\mathbb{K}_{\mathbf{d}}}$ as in Example 1(2) is also a function space.

Let $\mathcal{F}$ be a function space and let $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ be a linear map with the following properties:

(1) $\Lambda(\bar{f}) = \overline{\Lambda(f)}$ for all $f \in \mathcal{F}^{(2)}$;
(2) $\Lambda(|f|^2) \geq 0$ for all $f \in \mathcal{F}$;
(3) $\Lambda(1) = 1$.

Adapting some terminology from [10] to our context (see also [14–16]), a linear map Λ with the properties (1)-(3) is said to be a *unital square positive functional*, briefly a *uspf*.

Every uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ satisfies the *Cauchy-Schwarz inequality*

$$|\Lambda(fg)|^2 \leq \Lambda(|f|^2)\Lambda(|g|^2), \; f, g \in \mathcal{F}.$$

A simple example of a uspf is given by a probability measure μ and a function space $\mathcal{F}$ on $(\Omega, \mathfrak{S})$, consisting of square μ-integrable functions. Then the map $\mathcal{F}^{(2)} \ni f \mapsto \int_\Omega f d\mu \in \mathbb{C}$ is a uspf, as one can easily see.

Fixing a function space $\mathcal{F}$ and a uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$, we have a semi-inner product given by the equality

$$\langle f, g \rangle_0 = \Lambda(f\bar{g}), \; f, g \in \mathcal{F}.$$

Then we set

$$\mathcal{I}_{\mathcal{F}} = \{f \in \mathcal{F}; \langle f, f \rangle_0 = 0\} = \{f \in \mathcal{F}; \Lambda(|f|^2) = 0\},$$

which is a vector subspace of $\mathcal{F}$. Moreover, the quotient $\mathcal{H}_{\mathcal{F}} := \mathcal{F}/\mathcal{I}_{\mathcal{F}}$ is an inner product space, with the inner product given by

$$\langle \hat{f}, \hat{g} \rangle = \Lambda(f\bar{g}), \tag{3}$$

where $\hat{f} = f + \mathcal{I}_{\mathcal{F}}$ is the equivalence class of $f \in \mathcal{F}$ modulo $\mathcal{I}_{\mathcal{F}}$.

When the quotient $\mathcal{H}_{\mathcal{F}}$ is finite dimensional, it is actually a Hilbert space, which will be said to be the Hilbert space *associated* to $(\mathcal{F}, \Lambda)$. This will be the case in most of the results of this work.

When the function space $\mathcal{F}$ and a uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ are given, we shall use the notation $\mathcal{I}_{\mathcal{F}}$, $\mathcal{H}_{\mathcal{F}}$, $\hat{f}$, with the meaning from above, if not otherwise specified.

Problem 1. The *(abstract) moment problem* for a given uspf $\varLambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$, where $\mathcal{F}$ is a fixed function space on $(\Omega, \mathfrak{S})$, means to find necessary and sufficient conditions insuring the existence of a probability measure μ, defined on $\mathfrak{S}$, such that $\mathcal{F}$ consists of square μ-integrable functions and $\varLambda(f) = \int_\Omega f d\mu$, $f \in \mathcal{F}^{(2)}$. When such a measure μ exists, it is said to be a *representing measure of $\varLambda$ (with support) in Ω*.

In the classical moment problem on spaces of polynomials, the role of the uspf $\varLambda$ is played by the so-called Riesz functional (see for instance [7]).

In some special cases, a uspf $\varLambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ may have an *atomic representing measure in Ω*, which (in this text) means that there exists a finite subset $\Omega_\varLambda = \{\omega_1, \dots, \omega_d\} \subset \Omega$ consisting of distinct points, and positive numbers $\lambda_1, \dots, \lambda_d$, with $\lambda_1 + \cdots + \lambda_d = 1$, such that $\varLambda(f) = \sum_{j=1}^d \lambda_j f(\omega_j)$ for all $f \in \mathcal{F}^{(2)}$.

When we want to specify the number of points $\{\omega_1, \dots, \omega_d\}$, the corresponding atomic measure will be called a *d-atomic representing measure* (of $\varLambda$ in Ω). Of course, in this case we can write $\varLambda(f) = \int_\Omega f(\omega) d\mu(\omega)$, where μ is a probability measure defined on a σ-algebra $\mathfrak{S}$ containing $\Omega_\varLambda$ and its subsets, such that $\mu(\{\omega_j\}) = \lambda_j\, j = 1, \dots, d$, supported by $\Omega_\varLambda$. In particular, we may take as $\mathfrak{S}$ the family of all subsets of Ω.

When $\mathcal{F}$ is finite dimensional, more generally if $\mathcal{H}_\mathcal{F}$ is finite dimensional, and the uspf $\varLambda$ on $\mathcal{F}^{(2)}$ has an arbitrary representing measure, then one expects that this measure may be replaced by an atomic one. Such a property goes back to Tchakaloff (see Corollary 2 in [13]).

An extremality condition insures the uniqueness of an atomic representing measure, when such a representing measure does exist. The following result is in the spirit of [5] (see also Proposition 2 from [16]).

Proposition 1. *Let $\mathcal{F}$ be a function space on Ω, and let $\varLambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ be a uspf. Assume that the associated Hilbert space $\mathcal{H}_\mathcal{F}$ is finite dimensional. Then there exists at most one d-atomic representing measure of the uspf $\varLambda$, with support in Ω, having $d := \dim \mathcal{H}_\mathcal{F}$ atoms.*

Proof. If the uspf $\varLambda$ has a d-atomic representing measure in Ω with $d := \dim \mathcal{H}_\mathcal{F}$ atoms, say μ, with finite support $\mathfrak{Z} \subset \Omega$, it follows that the map $\mathcal{H}_\mathcal{F} \ni \hat{f} \mapsto f|\mathfrak{Z} \in L^2(\mathfrak{Z}, \mu)$ is a unitary operator. Indeed, we have only to note that the map $\hat{f} \mapsto f|\mathfrak{Z}$ from $\mathcal{H}_\mathcal{F}$ into $L^2(\mathfrak{Z}, \mu)$ is correctly defined, linear and injective. It is also surjective because the dimension of $\mathcal{H}_\mathcal{F}$ equals the dimension of $L^2(\mathfrak{Z}, \mu)$. As we have $\|\hat{f}\|^2 = \int_\mathfrak{Z} |f|^2 d\mu$ for all $\hat{f}$, this map is actually a unitary transformation.

Now assume that there exists another d-atomic representing measure of $\varLambda$ in Ω, with support $\varXi := \{\xi_1, \dots, \xi_d\}$. As in the previous case, the map $\hat{f} \mapsto f|\varXi$ induces a unitary operator from $\mathcal{H}_\mathcal{F}$ onto $L^2(\varXi, \nu)$.

We now extend μ (resp. ν) to Ω by setting $\mu(\Omega \setminus \mathfrak{Z}) = 0$ (resp. $\nu(\Omega \setminus \varXi) = 0$). If $\zeta \in \mathfrak{Z} \setminus \varXi$, for the characteristic function χ of the set $\{\zeta\}$ (defined on Ω) we must have

$$0 \neq \int_\Omega \chi d\mu = \langle \chi, 1 \rangle_0 = \int_\Omega \chi d\nu = 0,$$

which is impossible, so $\mathfrak{Z} \subset \varXi$. A similar argument shows that $\varXi \subset \mathfrak{Z}$. Therefore, $\mathfrak{Z} = \varXi$. In fact, this argument shows that the weights of both measures at a given point in the support must be the same. □

2.3 Generators of Function Spaces

Let $\mathcal{F}$ be a a function space on Ω. Let also $\mathbb{K} \subset \mathbb{Z}_+^n$ be a subset containing $\mathbf{0} = (0, \dots, 0)$, and let $\theta = (\theta_1, \dots, \theta_n)$ be an n-tuple of elements of $\mathcal{RF}$.

If the family $\{\theta^\alpha : \alpha \in \mathbb{K}\}$ spans the space $\mathcal{F}$, we say that the function space $\mathcal{F}$ is $\mathbb{K}$-*generated by* θ. In addition, if the set $\mathbb{K}$ is hereditary, we say that the function space $\mathcal{F}$ is *hereditary*.

Obviously, when $\mathbb{K}$ is finite, the space $\mathcal{F}$ is of finite dimension, and if $\mathbb{K} = \{\mathbf{k} : |\mathbf{k}| \leq 1\}$, the space $\mathcal{F}$ is $\mathbb{K}$-generated by θ if and only if $\mathcal{F}$ is the span of $\{\theta_1, \dots, \theta_n\}$. Moreover, if $\mathcal{F}$ is $\mathbb{K}$-generated by θ, then $\mathcal{F}^{(2)}$ is $\mathbb{K}_2$-generated by θ, where $\mathbb{K}_2 = \mathbb{K} + \mathbb{K}$.

As a matter of fact, if $\mathcal{F}$ is a function space on Ω that is $\mathbb{K}$-generated by an n-tuple $\theta = (\theta_1, \dots, \theta_n)$ of elements of $\mathcal{RF}$, we must have the equality, $\mathcal{F} = \{p \circ \theta; p \in \mathcal{P}^n_{\mathbb{K}}\}$, where θ is regarded as a function from Ω into $\mathbb{R}^n$, where $\mathcal{P}^n_{\mathbb{K}}$ is the complex space of polynomials $\mathbb{K}$-generated by $\mathbf{t} = (t_1, \dots, t_n)$.

2.4 Idempotents

In the following we recall the concept of an idempotent (see [15, 16]), and present some of its properties.

We fix a function space $\mathcal{F}$ and a uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$, having a finite dimensional associated Hilbert space $\mathcal{H}_{\mathcal{F}}$, whose norm is denoted by $\| \star \|$. We denote by $\mathcal{RH}_{\mathcal{F}}$ the real Hilbert space given by the quotient $\mathcal{RF}/\mathcal{RI}_{\mathcal{F}}$.

Definition 2. *An element $\iota \in \mathcal{RH}_{\mathcal{F}}$ is said to be an idempotent (associated to $\mathcal{F}$) if*

$$\|\iota\|^2 = \langle \iota, \hat{1} \rangle. \tag{4}$$

We set $\mathcal{ID}(\mathcal{H}_{\mathcal{F}}) := \{\iota \in \mathcal{RH}_{\mathcal{F}}; \langle \iota, \hat{1} \rangle = \|\iota\|^2 \neq 0\}$, that is, the family of all nonnull idempotents of $\mathcal{H}_{\mathcal{F}}$.

The following result is a version of Lemma 4 from [15].

Lemma 2. *If $\{\eta_1, \dots, \eta_d\} \subset \mathcal{RH}_{\mathcal{F}}$ is an orthonormal basis with $\langle \eta_j, \hat{1} \rangle \neq 0$, $j = 1, \dots, d$, the set $\{\langle \eta_1, \hat{1} \rangle \eta_1, \dots \langle \eta_d, \hat{1} \rangle \eta_d\}$ is an orthogonal basis of $\mathcal{H}_{\mathcal{F}}$ consisting of idempotents. Moreover,*

$$\langle \eta_1, \hat{1} \rangle \eta_1 + \cdots + \langle \eta_d, \hat{1} \rangle \eta_d = \hat{1},$$

where $d = \dim \mathcal{H}_{\mathcal{F}}$.

As noticed in [15], in the space $\mathcal{H}_{\mathcal{F}}$ there are infinitely many orthogonal bases consisting of idempotents.

Corollary 1. *There are functions $b_1, \dots, b_d \in \mathcal{RF}$ such that $\|b_j\|_0^2 = \langle b_j, 1 \rangle_0 > 0$, $\langle b_j, b_k \rangle_0 = 0$ for all $j, k = 1, \dots, d$, $j \neq k$, and every $f \in \mathcal{F}$ can be uniquely represented as*

$$f = \sum_{j=1}^{d} \langle b_j, 1 \rangle_0^{-1} \langle f, b_j \rangle_0 b_j + f_0,$$

with $f_0 \in \mathcal{I}_{\mathcal{F}}$ and $d = \dim \mathcal{H}_{\mathcal{F}}$.

3 Relative Multiplicativity

As done in [15, 16], we can also characterize the existence of a representing measure for unital square positive functionals in this context. The following is a basic concept, which generalizes a corresponding one from [15], Definition 3.

Definition 3. Let $\mathcal{F}$ be a hereditary function space $\mathbb{K}$-generated by $\theta = (\theta_1, \dots, \theta_n) \subset \mathcal{RF}$, endowed with a uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$. Assume that the space $\mathcal{H}_{\mathcal{F}}$ is finite dimensional, and let $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ be an orthogonal basis in $\mathcal{H}_{\mathcal{F}}$ consisting of idempotent elements. We say that the tuple θ is $\mathcal{B}$-*multiplicative* if

$$\Lambda(\theta^{\mathbf{p}} b_j)\Lambda(\theta^{\mathbf{q}} b_j) = \Lambda(b_j)\Lambda(\theta^{\mathbf{p}+\mathbf{q}} b_j), \tag{5}$$

whenever $\mathbf{p} + \mathbf{q} \in \mathbb{K}$, $j = 1, \dots, d$.

The next result is an extension of Theorem 2 from [15] (and of Theorem 8 from [16] as well).

Theorem 1. *Let $\mathcal{F}$ be a hereditary function space $\mathbb{K}$-generated by $\theta = (\theta_1, \dots, \theta_n) \subset \mathcal{RF}$, and endowed with a uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$. Assume that the space $\mathcal{H}_{\mathcal{F}}$ is finite dimensional.*

The uspf Λ has a uniquely determined representing measure on Ω consisting of $d := \dim \mathcal{H}_{\mathcal{F}}$ atoms if and only if there exists an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ of $\mathcal{H}_{\mathcal{F}}$, consisting of idempotent elements, such that θ is $\mathcal{B}$-multiplicative, and $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$.

Proof. Let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\mathcal{F}$ consisting of idempotent elements. Hence, every element $\hat{f} \in \mathcal{H}_\mathcal{F}$ has a unique representation of the form $\hat{f} = \sum_{j=1}^d \langle \hat{b}_j, \hat{1}\rangle^{-1}\langle \hat{f}, \hat{b}_j\rangle \hat{b}_j$, via Corollary 1.

We consider on $\mathcal{H}_\mathcal{F}$ the linear functionals $\delta_j(\hat{f}) = \langle \hat{b}_j, \hat{1}\rangle^{-1}\langle \hat{f}, \hat{b}_j\rangle$, $j = 1, \ldots, d$, so $\hat{f} = \sum_{j=1}^d \delta_j(\hat{f})\hat{b}_j$ for all $\hat{f} \in \mathcal{H}$. In particular, $\delta_j(\hat{b}_j) = 1$ and $\delta_j(\hat{b}_k) = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$. In other words, the set $\Delta := \{\delta_1, \ldots, \delta_d\}$ is the dual basis of $\mathcal{B}$.

Next, we define the functions $\hat{f}_\Delta : \Delta \mapsto \mathbb{C}$ by $\hat{f}_\Delta(\delta) = \delta(\hat{f})$ for all $\hat{f} \in \mathcal{H}_\mathcal{F}$ and $\delta \in \Delta$. Setting $\mathcal{H}_\Delta := \{\hat{f}_\Delta; \hat{f} \in \mathcal{H}_\mathcal{F}\}$, we have a linear map $\mathcal{H}_\mathcal{F} \ni \hat{f} \mapsto \hat{f}_\Delta \in \mathcal{H}_\Delta$, which is surjective by definition, and injective because $\hat{f}_\Delta = 0$ implies $\hat{f} = 0$. In other words, the map $\mathcal{H}_\mathcal{F} \ni \hat{f} \mapsto \hat{f}_\Delta \in \mathcal{H}_\Delta$ is a linear isomorphism. In addition, $\hat{f}_\Delta = \sum_{k=1}^d \hat{f}_\Delta(\delta_j)\widehat{b_k}_\Delta$ for all $\hat{f} \in \mathcal{H}_\mathcal{F}$. In fact, the function $\widehat{b_k}_\Delta$ is the characteristic function of the set $\{\delta_k\}$, $k = 1, \ldots, d$.

Setting $C(\Delta) := \{\phi : \Delta \mapsto \mathbb{C}\}$ regarded as a finite dimensional C^*-algebra, we must have $\mathcal{H}_\Delta = C(\Delta)$ as linear spaces, because both have the same dimension, which is equal to the dimension of $\mathcal{H}_\mathcal{F}$. In addition, $\mathcal{H}_\Delta$ also has a multiplicative structure, so that $\mathcal{H}_\Delta$ and $C(\Delta)$ are isomorphic as C^*-algebras. Indeed, the product of two functions from $\mathcal{H}_\Delta$, say $\hat{f}_\Delta = \sum_{j=1}^d \delta_j(\hat{f})\widehat{b}_{j\Delta}$, $\hat{g}_\Delta = \sum_{j=1}^d \delta_j(\hat{g})\widehat{b}_{j\Delta}$, is given by

$$\hat{f}_\Delta \hat{g}_\Delta = \sum_{j=1}^d \delta_j(\hat{f})\delta_j(\hat{g})\widehat{b}_{j\Delta},$$

which coincides with the product of $C(\Delta)$. In particular, $\hat{f}_\Delta\hat{g}_\Delta$ is an element of $\mathcal{H}_\Delta$, and the C^*-algebra structure of $C(\Delta)$ is inherited by $\mathcal{H}_\Delta$.

We now assume that θ is $\mathcal{B}$-multiplicative, and that $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$.

We note that the space $\mathcal{H}_\mathcal{F}$ is spanned by the family $\{\widehat{\theta^{\mathbf{k}}} : \mathbf{k} \in \mathbb{K}\}$, by hypothesis, so the vector space $\mathcal{H}_\Delta$ is spanned by the family $\{\widehat{\theta^{\mathbf{k}}}_\Delta : \mathbf{k} \in \mathbb{K}\}$, while the C^*-algebra $\mathcal{H}_\Delta$ is generated by the family $\{\widehat{\theta_1}_\Delta, \ldots, \widehat{\theta_n}_\Delta\}$. We need a more explicit relation between these families, obtained by using (6), which will be proved in the following. Because we have

$$\Lambda(\theta^{\mathbf{p}} b_j)\Lambda(\theta^{\mathbf{q}} b_j) = \Lambda(b_j)^2\delta_j(\widehat{\theta^{\mathbf{p}}})\delta_j(\widehat{\theta^{\mathbf{q}}}) =$$
$$\Lambda(b_j)\Lambda(\theta^{\mathbf{p}+\mathbf{q}} b_j) = \Lambda(b_j)^2\delta_j(\widehat{\theta^{\mathbf{p}+\mathbf{q}}})$$

whenever $\mathbf{p} + \mathbf{q} \in \mathbb{K}$ and $j = 1, \ldots, d$, via (5), we infer that

$$\widehat{\theta^{\mathbf{p}}}_\Delta \widehat{\theta^{\mathbf{q}}}_\Delta = \widehat{\theta^{\mathbf{p}+\mathbf{q}}}_\Delta,$$

whenever $\mathbf{p} + \mathbf{q} \in \mathbb{K}$. Hence, by recurrence, we deduce that

$$\widehat{\theta^{\mathbf{k}}}\Delta = (\hat{\theta}_\Delta)^{\mathbf{k}}, \ \ \mathbf{k} \in \mathbb{K}. \tag{6}$$

The hypothesis $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, allows us to find a point $\zeta_j \in \Omega$ such that $\theta(\zeta_j) = \delta_j(\hat{\theta})$ for each $j = 1, \ldots, d$.

Let $f \in \mathcal{F}$ be a fixed element. As $\mathcal{F}$ is $\mathbb{K}$-generated by θ, there exists a polynomial p, with complex coefficients of the form $P(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{K}} c_{\mathbf{k}}\mathbf{t}^{\mathbf{k}}$ such that $f = p \circ \theta$. Then we have $\hat{f} = p \circ \hat{\theta}$, and so $\hat{f}_\Delta = p \circ \hat{\theta}_\Delta$, via (6). Hence, we must have

$$\delta_j(\hat{f}) = \hat{f}_\Delta(\delta_j) = p(\hat{\theta}_\Delta(\delta_j)) = (p \circ \theta)(\zeta_j) = f(\zeta_j), \ \ j = 1, \ldots, d.$$

Now, if $f, g \in \mathcal{F}$, because $f = \sum_{j=1}^n \delta_j(\hat{f})b_j + f_0$, $f = \sum_{k=1}^n \delta_k(\hat{g})b_+ g_0$, via Corollary 1, with $f_0, g_0 \in \mathcal{I}_\mathcal{F}$, we obtain

$$\Lambda(fg) = \sum_{j,k=1}^n \delta_j(\hat{f})\delta_k(\hat{g})\Lambda(b_jb_k) = \sum_{j=1}^n \delta_j(\hat{f})\delta_j(\hat{g})\Lambda(b_j).$$

Next, using the computations from above, if $\phi = \sum_{l\in L} f_l g_l$ is an arbitrary element in $\mathcal{F}^{(2)}$, with $f_l, g_j \in \mathcal{F}$ for all $l \in L$, L finite, we have

$$\Lambda(\phi) = \sum_{l\in L}\Lambda(f_lg_l) = \sum_{l\in L}\sum_{j=1}^d \Lambda(b_j)f_l(\zeta_j)g_l(\zeta_j) = \sum_{j=1}^d \Lambda(b_j)\phi(\zeta_j). \tag{7}$$

Moreover, $\Lambda(b_j) > 0$ for all j and $\sum_{j=1}^{d} \Lambda(b_j) = 1$, by Lemma 2. Consequently, the uspf Λ has a representing measure on Ω. In addition, this measure is uniquely determined by Proposition 1.

Conversely, assume that there exists a finite family $\{\zeta_1, \dots, \zeta_d\} \subset \Omega$, consisting of distinct points, such that

$$\Lambda(\phi) = \sum_{j=1}^{d} \lambda_j \phi(\zeta_j), \ \forall \phi \in \mathcal{F}^{(2)},$$

where $\lambda_j > 0$ for all j, $\sum_{j=1}^{d} \lambda_j = 1$, and $d = \dim \mathcal{H}_{\mathcal{F}}$.

Set $\mathfrak{Z} = \{\zeta_1, \dots, \zeta_d\}$, and let $C(\mathfrak{Z}) = \{h : \mathfrak{Z} \mapsto \mathbb{C}\}$, regarded as a C^*-algebra. As we must have $\mathcal{I}_{\mathcal{F}} = \{f \in \mathcal{F}; f|\mathfrak{Z} = 0\}$, there exists a map $\rho : \mathcal{H}_{\mathcal{F}} \mapsto C(\mathfrak{Z})$ given by $\hat{f} \mapsto f|\mathfrak{Z}$, which is correctly defined, linear, and injective. In fact, this map is also surjective, because $\dim \mathcal{H}_{\mathcal{F}} = \dim C(\mathfrak{Z})$. We denote by $\chi_k \in C(\mathfrak{Z})$ the characteristic function of the set $\{\zeta_k\}$, and by $\hat{b}_k \in \mathcal{R}\mathcal{H}_{\mathcal{F}}$ the element with $\rho(\hat{b}_k) = \chi_k$, $k = 1, \dots, d$. Then the set $\mathcal{B} := \{\hat{b}_1, \dots, \hat{b}_d\}$ is a family of orthogonal idempotents in $\mathcal{H}_{\mathcal{F}}$, which is actually a basis. Moreover, $b_j(\zeta_j) = 1$ and $b_k(\zeta_j) = 0$ for all $j, k = 1, \dots, d$, $j \neq k$.

Setting $\delta_j(\hat{f}) = f(\zeta_j)$, $f \in \mathcal{F}$, $j = 1, \dots, d$, and $\Delta := \{\delta_1, \dots, \delta_d\}$, we infer that Δ is the dual basis of $\mathcal{B}$, and we have

$$\delta_j(\widehat{\theta^{\mathbf{k}}}) = \theta^{\mathbf{k}}(\zeta_j) = (\theta_1(\zeta_j)^{k_1} \cdots \theta_n(\zeta_j)^{k_n}) = \delta_j(\hat{\theta}^{\mathbf{k}}),$$

whenever $\mathbf{k} \in \mathbb{K}$ and $j = 1, \dots, d$, showing that θ is $\mathcal{B}$-multiplicative. In addition, the obvious equality $\delta_j(\hat{\theta}) = \theta(\zeta_j)$, $j = 1, \dots, d$, concludes the proof of Theorem 1. □

Corollary 2. *Let $\mathcal{F}$ be a function space on Ω, spanned by the n-tuple θ. A uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ has a uniquely determined representing measure on Ω consisting of $d := \dim \mathcal{H}_{\mathcal{F}}$ atoms if either*

(1) *there exists an orthogonal basis $\mathcal{B}$ of $\mathcal{H}$ consisting of idempotent elements such that $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$, or*

(2) $\theta(\Omega) = \mathbb{R}^n$.

Proof. Because $\mathcal{F}$ is spanned by θ, the property (5) is automatically fulfilled. To get the assertion (1) from the statement we need the inclusion $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$, in order to apply the previous theorem, while to get (2), such an inclusion is always true, for an arbitrary orthogonal basis consisting of idempotents. □

4 Dimensional Stability and Consequences

In this section we intend to extend and recapture, in the present context, some results regarding the dimensional stability, developed in [14]. We also recall that the concept of dimensional stability in function spaces of polynomials, as approached in [14], is equivalent to that of *flatness*, due to Curto and Fialkow (see [3, 4]).

Remark 2. Let $\mathcal{F}$ be a function space, and let $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ be a uspf. We assume that the quotient space $\mathcal{H}_{\mathcal{F}} = \mathcal{F}/\mathcal{I}_{\mathcal{F}}$ is finite dimensional, so it is a Hilbert space. Let also $\mathcal{G}$ be a function subspace of $\mathcal{F}$, so $\Lambda|\mathcal{G}^{(2)}$ is a uspf. If $\mathcal{I}_{\mathcal{G}}$ and $\mathcal{H}_{\mathcal{G}}$ are defined by replacing $\mathcal{F}$ by $\mathcal{G}$, we have an isometry

$$\mathcal{H}_{\mathcal{G}} \ni g + \mathcal{I}_{\mathcal{G}} \mapsto g + \mathcal{I}_{\mathcal{F}} \in \mathcal{H}_{\mathcal{F}}. \tag{8}$$

In particular, $\mathcal{H}_{\mathcal{G}}$ is also a Hilbert space, and $\dim \mathcal{H}_{\mathcal{G}} \le \dim \mathcal{H}_{\mathcal{F}}$.

We say that the uspf Λ is *dimensionally stable at* $\mathcal{G}$ if $\dim \mathcal{H}_{\mathcal{G}} = \dim \mathcal{H}_{\mathcal{F}}$. In this case, the isometry (8) is surjective, that is, (8) is a unitary transformation. This is equivalent to the fact that for every $f \in \mathcal{F}$ there exists a $g \in \mathcal{G}$ such that $f - g \in \mathcal{I}_{\mathcal{F}}$. Note that if $f \in \mathcal{R}\mathcal{F}$, we can choose $g \in \mathcal{R}\mathcal{G}$ such that $f - g \in \mathcal{R}\mathcal{I}_{\mathcal{F}}$, because $\mathcal{I}_{\mathcal{F}} = \mathcal{R}\mathcal{I}_{\mathcal{F}} + i\mathcal{R}\mathcal{I}_{\mathcal{F}}$.

The next result is in the spirit of Lemma 5 from [16].

Lemma 3. *Let $\mathcal{F}$ be a function space, let $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ be a uspf, and let $\theta = \{\theta_1, \ldots, \theta_n\}$ be in $\mathcal{RF}$. Let also $\mathcal{G}$ be a function subspace of $\mathcal{F}$ such that $\theta_j\mathcal{G} \subset \mathcal{F}$ for all $j = 1, \ldots, n$, and that Λ is dimensionally stable at $\mathcal{G}$. Then $(\sum_{j=1}^n \theta_j\mathcal{I}_\mathcal{F}) \cap \mathcal{F} \subset \mathcal{I}_\mathcal{F}$. In particular, $\theta_j\mathcal{I}_\mathcal{G} \subset \mathcal{I}_\mathcal{F}$ for all $j = 1, \ldots, n$.*

Proof. Let $f = \sum_{j=1}^n \theta_j f_j \in \mathcal{F}$ with $f_j \in \mathcal{I}_\mathcal{F}$ for all $j = 1, \ldots, n$, and let $g \in \mathcal{G}$. Then

$$|\Lambda(fg)| \leq \sum_{j=1}^n |\Lambda(\theta_j f_j g)| \leq \sum_{j=1}^n (\Lambda(|f_j|^2)^{1/2}(\Lambda(|\theta_j g|^2)^{1/2} = 0.$$

by the Cauchy-Schwarz inequality.

Next, let $h \in \mathcal{G}$ be such that $\bar{f} - h \in \mathcal{I}_\mathcal{F}$, which exists because of the dimensional stability of Λ at $\mathcal{G}$. Then

$$\Lambda(|f|^2) = \Lambda(fh) + \Lambda(f(\bar{f} - h)) = 0$$

by the previous computation and the Cauchy-Schwarz inequality. Therefore $f \in \mathcal{I}_\mathcal{F}$.

The last assertion is obvious. □

Remark 3. We keep the notation and assumptions of Lemma 3. First of all, let $J : \mathcal{H}_\mathcal{G} \mapsto \mathcal{H}_\mathcal{F}$ be the unitary transformation given by (8). Then we define some operators $M_j : \mathcal{H}_\mathcal{G} \mapsto \mathcal{H}_\mathcal{F}$ by the equalities $M_j(g + \mathcal{I}_\mathcal{G}) = \theta_j g + \mathcal{I}_\mathcal{F}$ for all $j = 1, \ldots, m$ and $g \in \mathcal{G}$, which are correctly defined, via Lemma 3. Next, we may consider on the Hilbert space $\mathcal{H}_\mathcal{F}$ the linear operators $T_j = M_j J^{-1}$ for all $j = 1, \ldots, n$. Note that, fixing $f \in \mathcal{F}$ and choosing $g \in \mathcal{G}$ such that $f - g \in \mathcal{I}_\mathcal{F}$, we have $T_j(f + \mathcal{I}_\mathcal{F}) = \theta_j g + \mathcal{I}_\mathcal{F}$ for all j.

As noticed in Remark 2, if $f \in \mathcal{RF}$ we can choose $g \in \mathcal{RG}$ such that $f - g \in \mathcal{RI}_\mathcal{F}$. Therefore, $T_j(\mathcal{RH}_\mathcal{F}) \subset \mathcal{RH}_\mathcal{F}$ for all $j = 1, \ldots, n$.

Proposition 2. *The linear maps T_j, $j = 1, \ldots, n$, are self-adjoint operators, and $T = (T_1, \ldots, T_n)$ is a commuting tuple on $\mathcal{H}_\mathcal{F}$.*

Proof. Let $f_k \in \mathcal{F}$ and $g_k \in \mathcal{G}$ be such that $f_k - g_k \in \mathcal{I}_\mathcal{F}$ $(k = 1, 2)$. Then

$$\begin{aligned}\langle T_j^\star(f_1 + \mathcal{I}_m), f_2 + \mathcal{I}_m\rangle &= \langle f_1 + \mathcal{I}_m, \theta_j g_2 + \mathcal{I}_m\rangle = \langle f_1, \theta_j g_2\rangle_0 \\ &= \langle \theta_j g_1, f_2\rangle_0 = \langle T_j(f_1 + \mathcal{I}_m), f_2 + \mathcal{I}_m\rangle,\end{aligned}$$

via Lemma 3 and Remark 3. Hence $T_1, \ldots, T_n$ are self-adjoint.

We prove now that $T_1, \ldots, T_n$ mutually commute. It suffices to show that $M_j J^{-1} M_k = M_k J^{-1} M_j$ for all $j, k = 1, \ldots, n$. To show this, fix a function $f \in \mathcal{G}$. Thus $M_j(f + \mathcal{I}_\mathcal{G}) = \theta_j f + \mathcal{I}_\mathcal{F}$. We can choose $g_j \in \mathcal{G}$ such that $\theta_j f - g_j \in \mathcal{I}_\mathcal{F}$. Therefore, $J^{-1}(\theta_j f + \mathcal{I}_\mathcal{F}) = g_j + \mathcal{I}_\mathcal{G}$, and $M_k(g_j + \mathcal{I}_\mathcal{G}) = \theta_k g_j + \mathcal{I}_\mathcal{F}$.

Similarly, $M_k(f + \mathcal{I}_\mathcal{G}) = \theta_k f + \mathcal{I}_\mathcal{F}$, and we can choose $g_k \in \mathcal{G}$ such that $\theta_k f - g_k \in \mathcal{I}_\mathcal{F}$, so $M_j(g_k + \mathcal{I}_\mathcal{G}) = \theta_j g_k + \mathcal{I}_\mathcal{F}$. To complete the proof, it suffices to show that $\theta_k g_j - \theta_j g_k \in \mathcal{I}_\mathcal{F}$. Indeed, note that $\theta_j\theta_k f - \theta_j g_k \in \theta_j\mathcal{I}_\mathcal{F}$ and $\theta_k\theta_j f - \theta_k g_j \in \theta_k\mathcal{I}_\mathcal{F}$. Consequently,

$$\theta_k g_j - \theta_j g_k \in (\theta_k\mathcal{I}_\mathcal{F} + \theta_j\mathcal{I}_\mathcal{F}) \cap \mathcal{F} \subset \mathcal{I}_\mathcal{F},$$

via Lemma 3. Consequently, $T_1, \ldots, T_n$ mutually commute. □

Remark 4. Let $\mathcal{G}$ be a hereditary function space $\mathbb{K}$-generated by $\theta = (\theta_1, \ldots, \theta_n) \subset \mathcal{RG}$, where $\mathbb{K} \subset \mathbb{Z}_+^n$ is finite. We set $\mathcal{F} = \sum_{j=0}^n \theta_j\mathcal{G}$, where $\theta_0 = 1$. It is clear that $\mathcal{F}$ is a hereditary function space $\mathbb{K}_+$-generated by θ, with $\mathbb{K}_+ = \{W_j\mathbf{k} : \mathbf{k} \in \mathbb{K},\ j = 0, 1, \ldots, n\}$, where W_0 is the identity and, for $j = 1, \ldots, n$, W_j are the maps defined by relation (2).

Next, let $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ be a uspf, and assume that Λ is dimensionally stable at $\mathcal{G}$. We want to show that $T^\mathbf{k}(1 + \mathcal{I}_\mathcal{F}) = \theta^\mathbf{k} + \mathcal{I}_\mathcal{F}$ for all $\mathbf{k} \in \mathbb{K}_+$.

If $\mathbf{k} \in \mathbb{K}$, so $\theta^\mathbf{k} \in \mathcal{G}$, because of Remark 3 it follows that $T_j(1 + \mathcal{I}_\mathcal{F}) = \theta_j + \mathcal{I}_\mathcal{F}$, whence, inductively, $T^\mathbf{k}(1 + \mathcal{I}_\mathcal{F}) = \theta^\mathbf{k} + \mathcal{I}_\mathcal{F}$.

If $\mathbf{k} \in \mathbb{K}_+$, we may assume, with no loss of generality, that $\mathbf{k} = W_j\mathbf{p}$ for some $\mathbf{p} \in \mathbb{K}$ and $j \in \{1, \ldots, n\}$. Then, using the previous assertion and Remark 3, we deduce that

$$T^\mathbf{k}(1 + \mathcal{I}_\mathcal{F}) = T_j T^\mathbf{p}(1 + \mathcal{I}_\mathcal{F}) = T_j(\theta^\mathbf{p} + \mathcal{I}_\mathcal{F}) = \theta^\mathbf{k} + \mathcal{I}_\mathcal{F}.$$

In particular, if P is a polynomial of the form $P(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{K}_+} c_{\mathbf{k}}\mathbf{t}^{\mathbf{k}}$, we must have $P(T) = \sum_{\mathbf{k}\in\mathbb{K}_+} c_{\mathbf{k}} T^{\mathbf{k}}$. In addition, if $\hat{f} \in \mathcal{H}_{\mathcal{F}}$ is an arbitrary element, because $\mathcal{F}$ is $\mathbb{K}_+$-generated, and so there exists a polynomial $P_f \in \mathcal{P}^n_{\mathbb{K}_+}$ such that $f = P_f(\theta)$, we must have $\hat{f} = P_f(T)(1 + \mathcal{I}_{\mathcal{F}})$.

Note also that, if $P \in \mathcal{P}^n$ is arbitrary, there exists a polynomial $Q_P \in \mathcal{P}^n_{\mathbb{K}_+}$ such that $P(T) = Q_P(T)$, because $P(T)$ belongs to the $C^\star$-algebra generated by the n-tuple T.

The following assertion is now obtained as an application of Theorem 1. See also Theorem 2.11 and Corollary 2.13 from [14] (proved in a different way), as well as Corollary 7.11 from [3] or Theorem 9 from [16].

Theorem 2. *Let $\mathcal{G}$ be a hereditary function space $\mathbb{K}$-generated by $\theta = (\theta_1, \dots, \theta_n) \subset \mathcal{R}\mathcal{G}$, where $\mathbb{K} \subset \mathbb{Z}^n_+$ is finite. Let also $\mathcal{F} = \sum_{j=0}^n \theta_j \mathcal{G}$ $(\theta_0 = 1)$, and let $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ be a uspf such that Λ is dimensionally stable at $\mathcal{G}$. Then we have:*

(1) *there exists an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ of $\mathcal{H}_{\mathcal{F}}$ consisting of idempotent elements such that $\theta = (\theta_1, \dots, \theta_n)$ is $\mathcal{B}$-multiplicative;*

(2) *the uspf Λ has a d-atomic representing measure with support in Ω, where $d := \dim\mathcal{H}_{\mathcal{F}}$, if and only if $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$;*

(3) *if the uspf Λ has an atomic representing measure with support in Ω, this atomic measure is uniquely determined.*

Proof. (1) First of all, note that $\mathcal{H}_{\mathcal{F}} = \{p(T)\hat{1}; p \in \mathcal{P}^n_{\mathbb{K}_+}\}$, via Remark 4.

Next, we want to apply Theorem 1 to show that there exists an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ of $\mathcal{H}_{\mathcal{F}}$ consisting of idempotent elements, such that θ is $\mathcal{B}$-multiplicative.

We first consider the commuting n-tuple $T = (T_1, \dots, T_n)$, consisting of self-adjoint operators, acting in $\mathcal{H}_{\mathcal{F}}$, given by Proposition 2. The spectral theorem for n-tuples of commuting self-adjoint operators (see for instance [2]) implies the existence of commuting self-adjoint projections $E_j = E(\{\xi^{(j)}\})$, $j = 1, \dots, d$, such that $h(T) = \sum_{j=1}^d h(\xi^{(j)})E_j$ for every function $h : \sigma(T) \mapsto \mathbb{C}$, where $\sigma(T) := \{\xi^{(1)}, \dots, \xi^{(d)}\}$ is the joint spectrum of T, which coincides with the support of E. Moreover, if the function h is real-valued, the operator $h(T)$ is self-adjoint. In addition, because the space $\mathcal{R}\mathcal{H}$ is invariant under $T_1, \dots, T_n$ (see Remark 3), it must be also invariant under $h(T)$, whenever h is real-valued. In particular, $\mathcal{R}\mathcal{H}_{\mathcal{F}}$ is invariant under E_j, $j = 1, \dots, d$.

We now construct an orthogonal family $\{\hat{b}_1, \dots, \hat{b}_d\}$ of $\mathcal{H}_{\mathcal{F}}$ consisting of idempotents. Because $\sum_{j=1}^d E_j$ is the identity on $\mathcal{H}_{\mathcal{F}}$, setting $\hat{b}_j = E_j\hat{1} \in \mathcal{R}\mathcal{H}_{\mathcal{F}}, j = 1, \dots, d$, we obtain a decomposition $\hat{1} = \sum_{j=1}^d \hat{b}_j$. As $E_j \neq 0$, we must have $E_j\hat{g} = \hat{g} \neq 0$ for some $\hat{g} = q \circ \theta + \mathcal{I}_m = q(T)(1 + \mathcal{I}_m)$, with $q \in \mathcal{P}^n_{\mathbb{K}_+}$, via Remark 4. Assuming $\hat{b}_j = 0$, we would obtain $E_j\hat{g} = \hat{g} = q(T)\hat{b}_j = 0$, which is not possible. Therefore, $\hat{b}_j \neq 0$ for all $j = 1, \dots, d$. Note also that $\langle \hat{b}_j, \hat{1}\rangle = \langle \hat{b}_j, \hat{b}_j\rangle > 0$, so $\hat{b}_j$ is an idempotent for all $j = 1, \dots, d$. In other words, $\{\hat{b}_1, \dots, \hat{b}_d\}$ is an orthogonal family in $\mathcal{H}_{\mathcal{F}}$ consisting of idempotent elements.

To show that $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ is a basis of $\mathcal{H}_{\mathcal{F}}$ it suffices to show that $\dim(\mathcal{H}_{\mathcal{F}}) = d$. For, we consider the sub-$C^\star$-algebra $\mathcal{C}_T$ generated by T in the $C^\star$-algebra of all linear (automatically bounded) operators acting in $\mathcal{H}_{\mathcal{F}}$. Therefore, we must have $\mathcal{C}_T = \{p(T); p \in \mathcal{P}^n\}$. In fact, choosing an element $p(T)$ with $p \in \mathcal{P}^n$, we may replace p by a polynomial $q \in \mathcal{P}^n_{\mathbb{K}_+}$, such that $p(T) = q(T)$, by Remark 4. Consequently, $\mathcal{C}_T = \{p(T); p \in \mathcal{P}^n_{\mathbb{K}_+}\}$.

As mentioned above, the spectral theorem allows us to write

$$p(T) = \sum_{j=1}^d p(\xi^{(j)})E_j,\ p \in \mathcal{P}^n_{\mathbb{K}_+}.$$

In particular, $\{E_1, \dots, E_d\}$, which is clearly a linearly independent family of operators, is actually an algebraic basis of (the linear space) $\mathcal{C}_T$. Note also that

$$p(T)\hat{1} = \sum_{j=1}^d p(\xi^{(j)})\hat{b}_j,\ p \in \mathcal{P}^n_m.$$

Therefore, using the equality $\mathcal{H}_{\mathcal{F}} = \{p(T)\hat{1}; p \in \mathcal{P}^n_m\}$ mentioned above, we deduce that $\dim(\mathcal{H}_{\mathcal{F}}) = \dim(\mathcal{C}_T) = d$. In particular, $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ is an orthogonal basis of $\mathcal{H}$, consisting of idempotents. In addition, considering the measure $\nu(\star) = \langle E(\star)\hat{1}, \hat{1}\rangle$ on $\sigma(T)$, and putting $\lambda_j = \langle E_j\hat{1}, \hat{1}\rangle = \langle \hat{b}_j, \hat{1}\rangle$, we have

$$\langle \theta^{\mathbf{p}}, b_j\rangle_0 \langle \theta^{\mathbf{q}}, b_j\rangle_0 = \langle T^{\mathbf{p}}\hat{1}, E_j\hat{1}\rangle\langle T^{\mathbf{q}}\hat{1}, E_j\hat{1}\rangle =$$

$$\int\limits_{\{\xi^{(j)}\}} t^{\mathbf{p}} d\nu(t) \int\limits_{\{\xi^{(j)}\}} t^{\mathbf{q}} d\nu(t) = \lambda_j^2 (\xi^{(j)})^{\mathbf{p}} (\xi^{(j)})^{\mathbf{q}} =$$

$$\lambda_j \int\limits_{\{\xi^{(j)}\}} t^{\mathbf{p}+\mathbf{q}} d\nu(t) = \lambda_j \langle \theta^{\mathbf{p}+\mathbf{q}}, b_j\rangle_0$$

whenever $\mathbf{p} + \mathbf{q} \in \mathbb{K}_+$, and $j = 1, \dots, d$. In other words, θ is $\mathcal{B}$-multiplicative, which concludes the assertion (1) from the statement.

To obtain the assertion (2) from the statement, we recall that the dual basis $\Delta := \{\delta_1, \dots, \delta_d\}$ of $\mathcal{B}$ is given by $\delta_j(\hat{f}) = \langle \hat{b}_j, \hat{1}\rangle^{-1}\langle \hat{f}, \hat{b}_j\rangle$, $j = 1, \dots, d$. In particular,

$$\delta_j(\widehat{\theta_k}) = \lambda_j^{-1}\langle E_j T_k \hat{1}, \hat{1}\rangle = \int\limits_{\{\xi^{(j)}\}} t_k d\nu(t) = \xi_k^{(j)},\ j, k = 1, \dots, d.$$

Theorem 1 shows that the uspf Λ of $\mathcal{H}_{\mathcal{F}}$ has a representing measure on Ω consisting of $d := \dim\mathcal{H}_{\mathcal{F}}$ atoms if and only if $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, which concludes the proof of (2).

(3) An explicit form of the integral representation whose existence is given in (2) is obtained as for equation (7). Specifically, choosing $\zeta_j \in \Omega$ such that $\xi^{(j)} = \delta_j(\zeta_j)$, $j = 1, \dots, d$, we deduce the equality

$$\Lambda(\phi) = \sum_{j=1}^{d} \lambda_j \phi(\zeta_j),\ \ \phi \in \mathcal{F}^{(2)},$$

providing a (d-atomic) representing measure for Λ.

Let μ be this representing measure, with support $\mathfrak{Z} := \{\zeta_1, \dots, \zeta_d\}$ and weights $\lambda_j = \mu(\xi^{(j)})$, $j = 1, \dots, d$. Assume that Λ has another atomic representing measure in Ω, say ν, with support $\Sigma := \{\sigma_1, \dots, \sigma_g\} \subset \Omega$. Then necessarily, $g \geq d = \dim(\mathcal{H}_{\mathcal{F}})$, and the map $\mathcal{H}_{\mathcal{F}} \ni \hat{f} \mapsto f|\Sigma \in L^2(\Xi, \nu)$ is an isometry.

Let B_j be the linear operator on $L^2(\Sigma, \nu)$ given by $B_j h = \theta_j h$ for all $j = 1, \dots, n$ and $h \in L^2(\Sigma, \nu)$. Then $B = (B_1, \dots, B_n)$ is an n-tuple of commuting self-adjoint operators. With $T = (T_1, \dots, T_n)$ as before, fixing $\hat{f} \in \mathcal{H}_{\mathcal{F}}$ with $f \in \mathcal{F}$, and choosing $g \in \mathcal{G}$ with $h := f - g \in \mathcal{I}_{\mathcal{F}}$, so $T_j\hat{f} = \widehat{\theta_j g}$, $(f - g)|\Sigma = 0$, and $(\theta_j g)|\Sigma = (\theta_j f)|\Sigma = B_j(f|\Sigma)$. In other words, identifying the Hilbert space $\mathcal{H}_{\mathcal{F}}$ with the (Hilbert) subspace $\{f|\Sigma; f \in \mathcal{F}\}$, we see that B_j is an extension of the operator T_j for all $j = 1, \dots, n$. In particular, the spectral measure E of T is the restriction of the spectral measure E_B of B to $\mathcal{H}_{\mathcal{F}}$.

We now consider the elements $E_B(\{\sigma_j\})(1|\Sigma)$, which must belong to $\mathcal{H}_{\mathcal{F}}$, because $\mathcal{H}_{\mathcal{F}}$ is invariant under E_B. Therefore, setting $\hat{c}_j = E_B(\{\sigma_j\})(1|\Sigma) = E(\{\sigma_j\})\hat{1}$, $j = 1, \dots, g$, as in the previous part of the proof, $\{\hat{c}_1, \dots, \hat{c}_g\}$ is an orthogonal family of nonnull idempotent elements of $\mathcal{H}_{\mathcal{F}}$. Consequently, we must have $g = d$, and so $\dim(L^2(\Xi, \nu)) = d$. We may now apply Proposition 1, to get the assertion (3). □

Remark 5. Fixing a $\mathbb{K}$-generated space $\mathcal{G}$ by a family $\theta = (\theta_1, \dots, \theta_n) \subset \mathcal{R}\mathcal{G}$, we have a sequence of hereditary function spaces $\{\mathcal{F}_r : r \geq 0\}$ given by

$$\mathcal{F}_r = \sum_{j=0}^{n} \theta_j \mathcal{F}_{r-1}\,(\theta_0 = 1,\ r \geq 1),$$

where $\mathcal{F}_0 = \mathcal{G}$.

The next result is an extension of Theorem 2.6 from [14] (see also Theorem 10 from [16], and their predecessors, namely Theorem 7.8 and Corollary 7.9 from [3]).

Theorem 3. *Let $\mathcal{G}$ be a hereditary function space $\mathbb{K}$-generated by $\theta = (\theta_1, \dots, \theta_n) \subset \mathcal{R}\mathcal{G}$ in Ω, where $\mathbb{K} \subset \mathbb{Z}_+^n$ is finite. Let also $\mathcal{F}_r = \sum_{j=0}^{n} \theta_j \mathcal{F}_{r-1}\,(\theta_0 = 1,\ r \geq 1)$, where $\mathcal{F}_0 = \mathcal{G}$. We fix a uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$, supposed to be* *dimensionally stable at $\mathcal{G}$, where $\mathcal{F} = \mathcal{F}_1$. Also set $\mathcal{F}_\infty$ to be the space $\cup_{r\geq 0}\mathcal{F}_r$. Then $\mathcal{F}_\infty$ is a function space with $\mathcal{F}_\infty^{(2)} = \mathcal{F}_\infty$, and the uspf Λ can be uniquely extended to a uspf $\Lambda_\infty : \mathcal{F}_\infty \mapsto \mathbb{C}$, having a d-atomic measure in Ω, where $d = \dim(\mathcal{H}_\mathcal{G})$.*

Proof. Using the arguments from the proof of Theorem 2, we have

$$\Lambda(\phi) = \sum_{j=1}^{d} \lambda_j \phi(\zeta_j),\ \phi \in \mathcal{F}^{(2)}.$$

Next, it is easily seen that each $\mathcal{F}_r$ $(r \geq 0)$ is a function space, $\mathbb{K}_r$-generated by θ, where $\mathbb{K}_r = \{S_j\mathbf{k} : \mathbf{k} \in \mathbb{K}_{r-1}\}$ $(r \geq 1,\ j = 1, \ldots, n)$, with $\mathbb{K}_0 = \mathbb{K}$ and $\mathbb{K}_1 = \mathbb{K}_+$.

A direct extension of this formula allows us to define

$$\Lambda_\infty(\psi) = \sum_{j=1}^{d} \lambda_j \psi(\zeta_j),\ \psi \in \mathcal{F}_\infty,$$

which is a uspf on $\mathcal{F}_\infty$. We want to show that Λ_∞ is uniquely determined.

Let $\Lambda'_\infty, \Lambda''_\infty$ be two uspf $\mathcal{F}_\infty$, both of them extending Λ. For $r \geq 1$, let $\mathcal{F}_r = \{p \circ \theta; p \in \mathcal{P}^n_{\mathbb{K}_r}\}$, $\mathcal{I}'_r = \{f \in \mathcal{F}_r; \Lambda'(|f|^2) = 0\}$, $\mathcal{I}''_r = \{f \in \mathcal{F}_r; \Lambda''(|f|^2) = 0\}$. Clearly, $\mathcal{I}_\mathcal{F} \subset \mathcal{I}'_r \cap \mathcal{I}''_r$ for all $r \geq 1$.

We shall show by induction with respect to r that for every element $f \in \mathcal{F}_r$ there is an element $f_r \in \mathcal{G}$, such that $f - f_r \in \mathcal{I}'_r \cap \mathcal{I}''_r$. The assertion is obvious for $r = 1$, via the stability at $\mathcal{G}$.

Assume the property true for an $r \geq 1$, and let us prove it for $r + 1$. It suffices to prove it for an element of the form $f = \theta^\mathbf{p}$, with $\mathbf{p} \in \mathbb{K}_{r+1}$. In this case there exists a number $j \in \{1, \ldots, n\}$ and a multi-index $\mathbf{k} \in \mathbb{K}_r$ such that $\theta^\mathbf{p} = \theta_j\theta^\mathbf{k}$. By the induction hypothesis, we can find a function $f_\mathbf{k} \in \mathcal{G}$ such that $\theta^\mathbf{k} - f_\mathbf{k} \in \mathcal{I}'_r \cap \mathcal{I}''_r$. Therefore, $\theta^\mathbf{p} - \theta_j f_\mathbf{k} \in \mathcal{I}'_{r+1} \cap \mathcal{I}''_{r+1}$, by the Cauchy-Schwarz inequality. Further, $\theta_j f_\mathbf{k} \in \mathcal{F}$ and so we can find a function $f_{j,\mathbf{k}} \in \mathcal{G}$ such that $\theta_j f_\mathbf{k} - f_{j,\mathbf{k}} \in \mathcal{I}_1$, via the stability at $\mathcal{G}$. Consequently,

$$\theta^\mathbf{p} - f_\mathbf{p} = \theta^\mathbf{p} - \theta_j f_\mathbf{k} + \theta_j f_\mathbf{k} - f_{j,\mathbf{k}} \in \mathcal{I}'_{r+1} \cap \mathcal{I}''_{r+1} + \mathcal{I}_1 = \mathcal{I}'_{r+1} \cap \mathcal{I}''_{r+1},$$

where $f_\mathbf{p} = f_{j,\mathbf{k}}$.

Finally, noting that for every $r \geq 1$ there exists an $s \geq r$ such that $\mathcal{F}_r^{(2)} \subset \mathcal{F}_s$, and so for every $f \in \mathcal{F}_r^{(2)}$ we can find an element $f_s \in \mathcal{G}$ such that $f - f_s \in \mathcal{I}'_s \cap \mathcal{I}''_s$, we deduce that

$$\Lambda'(f) = \Lambda(f - f_s) = \Lambda''(f),$$

showing the uniqueness of the extensions Λ', Λ'' of the uspf Λ. □

Remark 6. From the proof of the previous theorem, we deduce that $\mathcal{H}_r := \mathcal{F}_r/\mathcal{I}_r (r \geq 1)$ are unitarily equivalent Hilbert spaces. This assertion is true even for $r = \infty$.

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