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Moment Tensor Representation of Surface Wave Sources*

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Summary

The equations representing Love and Rayleigh waves due to point sources are derived as linear combinations of the moment tensor elements. Using an analogy in free oscillation excitation, exact expressions for the wave-number transform of the expansion coefficients are obtained in the case of a step function source time history.

The solution to the well-known problem of Love and Rayleigh wave propagation from a point source in a layered elastic half-space can be expressed as a linear combination of the moment tensor elements. The advantages to such an approach are: (1) that the expansion is linear and therefore amenable to the stochastic least-squares estimation procedure (Foster 1961) and (2) that the expansion is complete consisting of monopole, dipole and quadrupole terms (Knopoff & Randall 1970) which ensures reliability in the estimated source parameters. An easy way to demonstrate this expansion is to merge the moment tensor formalism of Gilbert (1970) with the vector harmonic expressions given by Saito (1967).

The Love and Rayleigh wave problems separate out onto vector components when written in terms of the vector cylindrical harmonics. These are:

$$\left.\begin{aligned}
\vec{R}_m{}^1(kr,\phi) &\equiv Y_m(kr,\phi)\hat{z} = (0,0,Y_m)\\
\vec{R}_m{}^2(kr,\phi) &\equiv \frac{1}{k}\nabla Y_m(kr,\phi) = \left(\frac{\partial Y_m}{\partial(kr)},\frac{1}{kr}\frac{\partial Y_m}{\partial\phi},0\right)\\
\vec{L}_m(kr,\phi) &\equiv \frac{1}{k}\nabla x\vec{R}_m{}^1(kr,\phi) = \left(\frac{1}{kr}\frac{\partial Y_m}{\partial\phi},-\frac{\partial Y_m}{\partial(kr)},0\right)\\
\vec{R}^2 &= \hat{z}x\vec{L} \qquad \vec{L} = \vec{R}^2x\hat{z}\\
(\nabla^2+k^2)&Y_m(kr,\phi) = 0\\
Y_m(kr,\phi) &\doteq H_m{}^+(kr)e^{im\phi}
\end{aligned}\right\} \tag{1}$$

where $H_m{}^+$ is the outward travelling Hankel wave function.

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The expansion for the Fourier transform of the elastic displacement field is

$$\vec{S}(r,\phi,z,\omega) = \sum_{m=-\infty}^{+\infty} \int_0^\infty kdk \left\{ \sum_j a_j(\omega,k,m) l(\omega_j,k,z)\, \vec{L}_m(kr,\phi) \right.$$
$$\left. + \sum_j b_j(\omega,k,m)[r_1(\omega_j,k,z)\vec{R}_m^{\,1}(kr,\phi) + r_2(\omega_j,k,z)\vec{R}_m^{\,2}(kr,\phi)] \right\} \tag{2}$$

which satisfies the homogeneous equations of motion:

$$D\vec{S}_H(r,\phi,z,\omega_j) = -\rho\omega_j^2\, \vec{S}_H(r,\phi,z,\omega_j) \tag{3}$$

where D stands for the differential operator corresponding to the divergence of the stress tensor. For (3) to be true, the depth dependences: l, r_1 and r_2 must be solutions to the Love and Rayleigh wave equations

$$\left.\begin{aligned}
&\frac{d}{dz}\begin{pmatrix} l \\ T_l \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{\mu} \\ \mu k^2 - \rho\omega^2 & 0 \end{pmatrix} \begin{pmatrix} l \\ T_l \end{pmatrix} \qquad \begin{aligned} &T_l \equiv \mu(dl/dz) \\ &T_l = 0 \quad \text{at} \quad z = 0 \\ &|l| \to 0 \quad \text{as} \quad z \to \infty \end{aligned} \\
&\frac{d}{dz}\begin{pmatrix} r_2 \\ r_1 \\ T_r^2 \\ T_r^1 \end{pmatrix} = \begin{pmatrix} 0 & -k & \dfrac{1}{\mu} & 0 \\ \dfrac{k\lambda}{\lambda+2\mu} & 0 & 0 & \dfrac{1}{\lambda+2\mu} \\ -\rho\omega^2 + 4k^2\mu\left(\dfrac{\lambda+\mu}{\lambda+2\mu}\right) & 0 & 0 & \dfrac{-k\lambda}{\lambda+2\mu} \\ 0 & -\rho\omega^2 & k & 0 \end{pmatrix} \begin{pmatrix} r_2 \\ r_1 \\ T_r^2 \\ T_r^1 \end{pmatrix} \\
&T_r^1 \equiv (\lambda+2\mu)\frac{dr_1}{dz} - k\lambda r_2 \qquad T_r^2 \equiv \mu\left(kr_1 + \frac{dr_2}{dz}\right) \\
&T_r^1,\ T_r^2 = 0 \quad @ \quad z = 0 \\
&|r_1|,\ |r_2| \to 0 \quad \text{as} \quad z \to \infty
\end{aligned}\right\}, \tag{4}$$

These two sets of equations represent self-adjoint boundary problems (e.g. Atkinson 1964). Consequently, the eigenvalues ($-\omega_j^2$ in both cases) are real and the vector eigenfunctions can be orthogonalized. The orthogonality relations are

$$\left.\begin{aligned}
&\int_0^\infty dz\rho(z) l^*(\omega_i,k,z) l(\omega_j,k,z) = 0 \qquad i \neq j \\
&\int_0^\infty dz\rho(z)[r_1^*(\omega_i,k,z) r_1(\omega_j,k,z) + r_2^*(\omega_i,k,z) r_2(\omega_j,k,z)] = 0 \qquad i \neq j
\end{aligned}\right\} \tag{5}$$

where * indicates complex conjugate.

Since the Love and Rayleigh wave modes are a complete set of propagating surface waves, any particular surface wave solution can be expressed as a linear combination of them

$$(D+\rho\omega^2)\vec{S}_p(r,\phi,z,\omega) = -\vec{f}(r,\phi,z,\omega) \tag{6}$$

where $\vec{f}$ is the force density of the source. Expressions for the expansion coefficients a_j and b_j can be found by inverting (6) using the Fourier–Bessel transform and the orthogonality of the surface wave modes. These are

$$\left.\begin{aligned} a_j(\omega,k,m) &= -\frac{1}{2\pi I_L(\omega_j,k)}\,\frac{1}{\omega^2-\omega_j^2}\int dV\, l^*(\omega_j,k,z)\,\vec{L}_m{}^\dagger(kr,\phi)\vec{f}(r,\phi,z,\omega) \\ b_j(\omega,k,m) &= -\frac{1}{2\pi I_R(\omega_j,k)}\,\frac{1}{\omega^2-\omega_j^2}\int dV\,[r_1{}^*(\omega_j,k,z)\vec{R}_m{}^{1\dagger}(kr,\phi) \\ &\qquad + r_2{}^*(\omega_j,k,z)\vec{R}_m{}^{2\dagger}(kr,\phi)]\vec{f}(r,\phi,z,\omega) \end{aligned}\right\} \tag{7}$$

where † indicates Hermitian adjoint. The normalization integrals are

$$\left.\begin{aligned} I_L(\omega_j,k) &= \int_0^\infty dz\rho(z)|l(\omega_j,k,z)|^2 \\ I_R(\omega_j,k) &= \int_0^\infty dz\rho(z)[|r_1(\omega_j,k,z)|^2+|r_2(\omega_j,k,z)|^2]. \end{aligned}\right\} \tag{8}$$

For point sources, the force density is derived from a stress tensor which can be expressed in terms of a moment tensor (Gilbert 1970):

$$\vec{f} = -\nabla.T = -\nabla.[M\delta(\vec{r}-\vec{r}_s)]. \tag{9}$$

With no loss of generality, the position of the point source can be taken to be the origin

$$\vec{r}_s = (0,0,z_s). \tag{10}$$

In this case, the expressions (7) can be integrated for a step function source time dependence as Gilbert (1970) did for the free oscillation expansion to give exact expressions for a_j and b_j. These are

$$\begin{aligned} a_j(t,k,m) = -\Bigg\{&\frac{1-\cos\omega_j t}{(2\pi)^2 I_L(\omega_j,k)\omega_j^2} - \frac{i\pi k}{2} l^*(\omega_j,k,z_s)(\delta_2{}^m-\delta^m_{-2})M_{xx} \\ &+\frac{i\pi k}{2} l^*(\omega_j,k,z_s)(\delta_2{}^m-\delta^m_{-2})M_{yy} - \pi k l^*(\omega_j,k,z_s)(\delta_2{}^m+\delta^m_{-2})M_{xy} \\ &-i\pi\frac{dl^*(\omega_j,k,z_s)}{dz}(\delta_1{}^m-\delta^m_{-1})M_{xz} - \pi\frac{dl^*(\omega_j,k,z_s)}{dz}(\delta_1{}^m+\delta^m_{-1})M_{yz}\Bigg\} \end{aligned}$$

$$b_j(t,k,m) = -\frac{1-\cos\omega_j t}{(2\pi)^2 I_R(\omega_j,k)\omega_j^2}\Bigg\{\left[-\pi k\delta_0^m+\frac{\pi k}{2}(\delta_2^m+\delta_{-2}^m)\right]r_2^*(\omega_j,k,z_s)M_{xx}$$

$$+\left[-\pi k\delta_0^m-\frac{\pi k}{2}(\delta_2^m+\delta_{-2}^m)\right]r_2^*(\omega_j,k,z_s)M_{yy}-i\pi k r_2^*(\omega_j,k,z_s)(\delta_2^m-\delta_{-2}^m)M_{xy}$$

$$+2\pi\frac{dr_1^*(\omega_j,k,z_s)}{dz}\delta_0^m M_{zz}+\pi(\delta_1^m+\delta_{-1}^m)\bigg[kr_1^*(\omega_j,k,z_s)$$

$$+\frac{dr_2^*(\omega_j,k,z_s)}{dz}\bigg]M_{xz}-i\pi(\delta_1^m-\delta_{-1}^m)\bigg[kr_1^*(\omega_j,k,z_s)$$

$$+\frac{dr_2^*(\omega_j,k,z_s)}{dz}\bigg]M_{yz}\Bigg\} \tag{11}$$

In both relations δ is the Kronecker delta and z_s refers to the source depth. These expressions are to be evaluated at the retarded time: $t-kr/\omega_j$.

The analogy with the free oscillation expansion is complete except for the replacement of one modal index sum (over l) with an integral over k in (2). This remaining integral must be done numerically when there are no analytic expressions for the dispersion relations and eigenfunctions. The procedure corresponds to summing over a very dense set of wavenumber modes.

To recapitulate, the derivation of equation (11) requires the following assumptions:

(1) the source and receiver are in a plane layered half space,

(2) the source is stationary and acts at only one point in the half space,

(3) the source has a step function time dependence,

(4) all body wave effects are ignored, and

(5) the material is unstressed before the source acts.

With these assumptions, the six elements of the moment tensor provide a complete description of the point source. It can be uniquely decomposed into three source mechanisms: (1) symmetric or explosive source, (2) compensated linear vector dipole (CLVD) source, and (3) double-couple source, provided that the principal stress directions for the latter two are coincident (Knopoff & Randall 1970). As such, the expansion is a good candidate for the application of linear estimation procedures as Randall (1972) has done for the spherical case and should provide discriminant information between explosive and earthquake sources.

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