

MOMENTS OF RANDOMLY STOPPED SUMS

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1. Introduction. Let $(\Omega, \mathfrak{F}, P)$ be a probability space, let x_1, x_2, \dots be a sequence of random variables on Ω , and let \mathfrak{F}_n be the σ -algebra generated by x_1, \dots, x_n , with $\mathfrak{F}_0 = (\phi, \Omega)$. A *stopping variable* (of the sequence x_1, x_2, \dots) is a random variable t on Ω with positive integer values such that the event $[t = n] \in \mathfrak{F}_n$ for every $n \geq 1$. Let $S_n = \sum_{i=1}^n x_i$; then $S_t = S_{t(\omega)}(\omega) = \sum_{i=1}^t x_i$ is a randomly stopped sum. We shall always assume that

$$(1) \quad E|x_n| < \infty, \quad E(x_{n+1} | \mathfrak{F}_n) = 0, \quad (n \geq 1).$$

The moments of S_t have been investigated since the advent of Sequential Analysis, beginning with Wald [9], whose theorem states that for *independent, identically distributed* (iid) x_i with $Ex_i = 0$, $Et < \infty$ implies that $ES_t = 0$. For higher moments of S_t , the known results [1, 3, 4, 5, 10] are not entirely satisfactory. We shall obtain theorems for ES_t^r ($r = 2, 3, 4$); the case $r = 2$ is of special interest in applications. For iid x_i with $Ex_i = 0$ and $Ex_i^2 = \sigma^2 < \infty$, we shall show that $Et < \infty$ implies $ES_t^2 = \sigma^2 Et$.

2. The second moment. It follows from assumption (1) that $(S_n, \mathfrak{F}_n; n \geq 1)$ is a *martingale*; i.e., that

$$(2) \quad E|S_n| < \infty, \quad E(S_{n+1} | \mathfrak{F}_n) = S_n \quad (n \geq 1).$$

The following well-known fact ([3], p. 302) will be stated as

LEMMA 1. *Let $(S_n, \mathfrak{F}_n; n \geq 1)$ be a martingale and let t be any stopping variable such that*

$$(3) \quad E|S_t| < \infty, \quad \liminf \int_{[t > n]} |S_n| = 0;$$

then

$$(4) \quad E(S_t | \mathfrak{F}_n) = S_n \quad \text{if } t \geq n \quad (n \geq 1),$$

and hence $ES_t = ES_1$.

LEMMA 2. *If $E \sum_{i=1}^t |x_i| < \infty$, then (3) holds.*

PROOF. $|S_t| \leq \sum_{i=1}^t |x_i|$, so that $E|S_t| < \infty$, and

$$\lim \int_{[t > n]} |S_n| \leq \lim \int_{[t > n]} \sum_{i=1}^t |x_i| = 0.$$

In the remainder of this section we shall suppose, in addition to (1) that

$$(5) \quad Ex_n^2 < \infty \quad (n \geq 1)$$

and we define for $n \geq 1$

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$$(6) \quad Z_n = S_n^2 - \sum_1^n x_i^2.$$

The sequence $(Z_n, \mathfrak{F}_n; n \geq 1)$ is also a martingale, with $EZ_1 = 0$.

For any stopping variable t , let $t(n) = \min(n, t)$; then Lemma 1 applies to Z_n and $t(n)$, so that $EZ_{t(n)} = 0$, and hence

$$(7) \quad ES_{t(n)}^2 = E \sum_1^{t(n)} x_i^2.$$

Letting $n \rightarrow \infty$ we have $S_{t(n)}^2 \rightarrow S_t^2$ and $\sum_1^{t(n)} x_i^2 \uparrow \sum_1^t x_i^2$. Hence, by Fatou's lemma and (7),

$$(8) \quad ES_t^2 \leq \lim ES_{t(n)}^2 = \lim E \sum_1^{t(n)} x_i^2 = E \sum_1^t x_i^2.$$

The question now arises under what circumstances equality holds in (8). (By Lemma 1 this will be the case if (3) holds with S replaced by Z , but, as we shall see, this requirement is unnecessarily stringent.) According to (8), *we need only consider the case in which $ES_t^2 < \infty$, and it will suffice to prove that*

$$(9) \quad ES_t^2 \geq ES_{t(n)}^2 \quad (n \geq 1).$$

LEMMA 3. *If*

$$(10) \quad \liminf \int_{[t>n]} |S_n| = 0,$$

then $ES_t^2 = E \sum_1^t x_i^2$.

PROOF. We may suppose that $ES_t^2 < \infty$ whence, by (10) and Lemma 1, (4) holds. Hence

$$\begin{aligned} ES_t^2 &= \int_{[t \leq n]} S_t^2 + \int_{[t > n]} (S_n + (S_t - S_n))^2 \\ &\geq \int_{[t \leq n]} S_t^2 + \int_{[t > n]} S_n^2 + 2 \int_{[t > n]} S_n E(S_t - S_n | \mathfrak{F}_n) = ES_{t(n)}^2. \end{aligned}$$

LEMMA 4. *If*

$$(11) \quad \liminf \int_{[t>n]} S_n^2 < \infty,$$

then (10) holds.

PROOF. Suppose (10) does not hold; then $\liminf \int_{[t>n]} |S_n| = \epsilon > 0$. Hence for any constant $0 < a < \infty$,

$$\liminf \int_{[t>n]} S_n^2 \geq a \liminf \int_{[t>n, |S_n|>a]} |S_n| = a\epsilon,$$

which contradicts (11), since a may be arbitrarily large.

LEMMA 5. *If $E \sum_1^t x_i^2 < \infty$, then (11) holds.*

PROOF. Setting $S_0 = 0$ we have

$$\begin{aligned} \int_{[t>n]} S_n^2 &= \sum_{i=1}^n (\int_{[t>i]} S_i^2 - \int_{[t>i-1]} S_{i-1}^2) \\ &\leq \sum_{i=1}^n \int_{[t \geq i]} (S_i^2 - S_{i-1}^2) \leq \sum_1^\infty \int_{[t \geq i]} x_i^2 = E \sum_1^t x_i^2 < \infty. \end{aligned}$$

From Lemmas 1-5 we have

THEOREM 1. *Let $(S_n, \mathfrak{F}_n; n \geq 1)$ be a martingale with $ES_n^2 < \infty$ and let t be any stopping variable. Set $x_1 = S_1, x_{n+1} = S_{n+1} - S_n$. Then*

$$(12) \quad ES_t^2 \leq E \sum_1^t x_i^2.$$

If any one of the four conditions

$$(13) \quad \liminf \int_{[t>n]} |S_n| = 0, \quad \liminf \int_{[t>n]} S_n^2 < \infty, \\ E \sum_1^t |x_i| < \infty, \quad E \sum_1^t x_i^2 < \infty$$

holds, then

$$(14) \quad ES_t^2 = E \sum_1^t x_i^2.$$

If either $E \sum_1^t |x_i| < \infty$ or $E \sum_1^t x_i^2 < \infty$, then (3) and (4) hold.

Theorem 1 generalizes (a) and (b) of Theorem II of [1]. In order to apply it, we first verify

LEMMA 6. For any stopping variable t and any $r > 0$,

$$E \sum_1^t |x_i|^r = E \sum_1^t E(|x_i|^r | \mathcal{F}_{i-1}).$$

PROOF.

$$E \sum_1^t |x_i|^r = \sum_{j=1}^{\infty} \int_{[t=j]} \sum_{i=1}^j |x_i|^r = \sum_{i=1}^{\infty} \int_{[t \geq i]} |x_i|^r \\ = \sum_{i=1}^{\infty} \int_{[t \geq i]} E(|x_i|^r | \mathcal{F}_{i-1}) = E \sum_1^t E(|x_i|^r | \mathcal{F}_{i-1}).$$

For independent x_n , we have from Theorem 1 and Lemmas 1 and 6

THEOREM 2. Let x_1, x_2, \dots be independent with $Ex_n = 0, E|x_n| = a_n, Ex_n^2 = \sigma_n^2 < \infty (n \geq 1)$ and let $S_n = \sum_1^n x_i$. Then if t is a stopping variable, either of the two relations

$$(15) \quad E \sum_1^t a_i < \infty, \quad E \sum_1^t \sigma_i^2 < \infty$$

implies that $ES_t = 0$ and

$$(16) \quad ES_t^2 = E \sum_1^t x_i^2 = E \sum_1^t \sigma_i^2.$$

If $\sigma_n^2 = \sigma^2 < \infty$, then $Et < \infty$ implies

$$(17) \quad ES_t^2 = E \sum_1^t x_i^2 = \sigma^2 Et.$$

Some stronger sufficient conditions for (17) have been given in ([10], [1], [5], [3] (p. 351), [4]).

COROLLARY 1. Let x_1, x_2, \dots be independent with $Ex_n = 0, Ex_n^2 = 1$, and define t^* (resp. t_*) = 1st $n \geq 1$ such that $|S_n| > n^{\frac{1}{2}}$ (resp. $<$) ($= \infty$ otherwise). Then $Et^* = Et_* = \infty$.

PROOF. If $Et^* < \infty$, then t^* is a genuine stopping variable, i.e., $P(t^* < \infty) = 1$, and by the definition of t^* and (17),

$$Et^* = ES_{t^*}^2 > Et^*,$$

a contradiction; similarly for t_* .

We note that both t^* and t_* are genuine stopping variables if the x_n are, in addition, identically distributed.

The example $P[x_n = 1] = P[x_n = -1] = \frac{1}{2}$ shows that the $>$ ($<$) cannot be

replaced by \geq (\leq), since $Ex_n = 0$, $Ex_n^2 = 1$, and $t^* = t_* = 1$. On the other hand, if t^* is redefined as the first $n > 1$ for which $|S_n| \geq n^{\frac{1}{2}}$, Et^* is again infinite; similarly for t_* .

Corollary 1 is a generalization of Theorem 1 of [2]. The following corollary generalizes Theorem 2 of [2].

COROLLARY 2. *Let x_1, x_2, \dots be independent with $Ex_n = 0$, $Ex_n^2 = 1$, $P[|x_n| \leq a < \infty] = 1$. For $0 < c < 1$ and $m = 1, 2, \dots$, define $t = \text{first } n \geq m \text{ such that } |S_n| > cn^{\frac{1}{2}}$. Then $Et < \infty$.*

PROOF. For $k = m, m + 1, \dots$, put $t' = \min(t, k)$ and $A_k = [\omega : m < t \leq k]$. Then t' is a stopping variable and by Theorem 2

$$kP[t > k] + \int_{[t \leq k]} t = Et' = ES_{t'}^2 = \int_{[t > k]} S_k^2 + \int_{[t \leq k]} S_t^2$$

or

$$kP[t > k] + \int_{A_k} t \leq c^2 kP[t > k] + \int_{A_k} (ct^{\frac{1}{2}} + a)^2 + m.$$

Hence

$$(1 - c^2)(kP[t > k] + \int_{A_k} t) \leq 2ac \int_{A_k} t^{\frac{1}{2}} + O(1).$$

Therefore, as $k \rightarrow \infty$, $\int_{A_k} t = O(1)$ and $P[t > k] = O(k^{-1}) = o(1)$, so that t is a genuine stopping variable and $Et < \infty$.

COROLLARY 3. *If x_1, x_2, \dots are iid with $Ex_n = 0$, $Ex_n^2 = \sigma^2$, $P[|x_n| \leq a < \infty] = 1$, and if $ES_t^2 < \infty$ for a stopping variable t , then $Et < \infty$ if and only if*

$$(18) \quad \liminf nP[t > n] = 0.$$

PROOF. The ‘‘only if’’ part is obvious. Now suppose (18) holds. Then since $\int_{[t > n]} |S_n| \leq nP[t > n]$, the first condition of (13) holds and hence $\sigma^2 Et = ES_t^2 < \infty$, so that $Et < \infty$ if $\sigma^2 > 0$. (If $\sigma^2 = 0$, then $P[x_n = 0] = 1$ and hence t is equal a.e. to a fixed positive integer, so $Et < \infty$ in this case too.)

Applied to the case $P[x_i = 1] = P[x_i = -1] = \frac{1}{2}$, with $t = \text{first } n \geq 1 \text{ such that } S_t = 1$, we have by Wald’s theorem $Et = \infty$, but by Corollary 3 the stronger result $\liminf nP[t > n] > 0$.

COROLLARY 4. *Let $(x_n, n \geq 1)$ satisfy $E(x_{n+1} | \mathcal{F}_n) = 0$ and let $E(x_{n+1}^2 | \mathcal{F}_n) = \sigma_{n+1}^2 < \infty$ be constant for $n \geq 0$. Then for $\epsilon > 0$,*

$$P[\max_{n \leq m} |S_n| \geq \epsilon] \leq \epsilon^{-2} \sum_1^m \sigma_n^2.$$

If moreover $\sup_{n \geq 1} |x_n| = z$ with $Ez < \infty$, then

$$(19) \quad P[\max_{n \leq m} |S_n| \geq \epsilon] \geq 1 - [E(\epsilon + z)^2 / \sum_1^m \sigma_n^2].$$

PROOF. Define $t = \text{first } n \geq 1 \text{ such that } |S_n| \geq \epsilon$. Then $t' = \min(t, m)$ is a bounded stopping variable. Hence, by (14) and Lemma 6,

$$\epsilon^2 P[\max_{n \leq m} |S_n| \geq \epsilon] = \epsilon^2 P[t \leq m] \leq ES_{t'}^2 = E \sum_1^{t'} \sigma_n^2 \leq \sum_1^m \sigma_n^2.$$

If $Ez < \infty$, then

$$E(\epsilon + z)^2 \geq ES_{t'}^2 = E \sum_1^{t'} \sigma_n^2 \geq \int_{[t \geq m]} \sum_1^m \sigma_j^2 = (\sum_1^m \sigma_j^2) P[t \geq m]$$

and (19) holds.

The first part of Corollary 4 is a special case of submartingale inequalities ([6], p. 391), and the second part generalizes slightly one of the Kolmogorov inequalities ([6], p. 235) which requires that z be constant.

3. The fourth moment. The analysis in the case of the fourth moment of S_t is somewhat easier than that of the third moment and consequently is presented first. In this section Ex_n^4 will be supposed finite and $Ex_1 = 0$. Define for $r = 1, 2, 3, 4$, and $n = 1, 2, \dots$

$$(20) \quad \begin{aligned} u_{r,n} &= E(x_n^r | \mathcal{F}_{n-1}), & U_{r,n} &= \sum_1^n u_{r,j}, \\ v_{r,n} &= E(|x_n|^r | \mathcal{F}_{n-1}), & V_{r,n} &= \sum_1^n v_{r,j}, \\ T_{r,n} &= \sum_1^n |x_j|^r, & T_{1,n} &= T_n. \end{aligned}$$

In these terms, Lemma 6 asserts that $ET_{r,t} = EV_{r,t}$.

LEMMA 7. *If $ES_t^2 < \infty$ and $\liminf \int_{[t>n]} |S_n| = 0$, then*

$$E(S_t^2 | \mathcal{F}_n) \geq S_n^2 \quad \text{and} \quad E(|S_t| | \mathcal{F}_n) \geq |S_n| \quad \text{for } t > n.$$

PROOF. For any $A \in \mathcal{F}_n$, by Lemma 1

$$\int_{A[t>n]} \dot{S}_t^2 = \int_{A[t>n]} [S_n^2 + 2S_n(S_t - S_n) + (S_t - S_n)^2] \geq \int_{A[t>n]} S_n^2.$$

Hence the first inequality of the lemma holds, and the second inequality follows immediately from Lemma 1 and the fact that $E(|S_t| | \mathcal{F}_n) \geq |E(S_t | \mathcal{F}_n)|$.

THEOREM 3. *If t is a stopping variable such that $E[t \sum_1^t E(x_j^4 | \mathcal{F}_{j-1})] < \infty$, then $ES_t^4 < \infty$ and*

$$(21) \quad ES_t^4 = EU_{4,t} + 4ES_t U_{3,t} + 6ES_t^2 U_{2,t} - 6E \sum_1^t u_{2,j} U_{2,j}.$$

PROOF. Set $Y_n = S_n^4 - 6S_n^2 U_{2,n} - 4S_n U_{3,n} - U_{4,n} + 6 \sum_{j=1}^n u_{2,j} U_{2,j}$ and $t' = \min(t, k)$. Since $\{Y_n, \mathcal{F}_n; n \geq 1\}$ is a martingale with $EY_1 = 0$, by Lemma 1,

$$\begin{aligned} ES_{t'}^4 &= 6ES_{t'}^2 U_{2,t'} + 4ES_{t'} U_{3,t'} + EU_{4,t'} - 6E(\sum_{j=1}^{t'} u_{2,j} U_{2,j}) \\ &\leq 6(E^{\frac{1}{2}} S_{t'}^4)(E^{\frac{1}{2}} U_{2,t'}^2) + 4(E^{\frac{1}{2}} S_{t'}^4)(E^{\frac{3}{4}} V_{3,t'}^{4/3}) + EU_{4,t'}, \end{aligned}$$

whence, if $ES_{t'}^4 > 0$,

$$(22) \quad E^{\frac{1}{2}} S_{t'}^4 \leq 6E^{\frac{1}{2}} U_{2,t'}^2 + 4(E^{\frac{3}{4}} V_{3,t'}^{4/3})(ES_{t'}^4)^{-\frac{1}{2}} + (EU_{4,t'})(ES_{t'}^4)^{-\frac{1}{2}}.$$

Now if $p > 1, r > 0$,

$$(23) \quad \begin{aligned} V_{r,n} &= \sum_{j=1}^n E\{|x_j|^r | \mathcal{F}_{j-1}\} \leq n^{(p-1)/p} (\sum_{j=1}^n E^p\{|x_j|^r | \mathcal{F}_{j-1}\})^{1/p} \\ &\leq n^{(p-1)/p} (\sum_{j=1}^n E\{|x_j|^{pr} | \mathcal{F}_{j-1}\})^{1/p} = n^{(p-1)/p} V_{pr,n}^{1/p} \end{aligned}$$

and thus setting $p = 2, r = 2$ and then $p = \frac{4}{3}, r = 3$,

$$(24) \quad EU_{2,t}^2 = EV_{2,t}^2 \leq EtV_{4,t} < \infty, \quad EV_{3,t}^{4/3} \leq Et^{\frac{1}{3}} V_{4,t} < \infty.$$

Moreover, $EU_{4,t} \leq E(tU_{4,t}) < \infty$ and $E(\sum_{j=1}^t u_{2,j} U_{2,j}) \leq EU_{2,t}^2 < \infty$. Thus,

the LHS of (22) is a bounded function of k , implying via Fatou's lemma that $ES_t^4 < \infty$.

Since $|Y_n| \leq S_n^4 + 6S_n^2U_{2,n} + 4|S_n|V_{3,n} + U_{4,n} + 6 \sum_{j=1}^n u_{2,j}U_{2,j} = Y_n'$ (say), it follows from the preceding that

$$E|Y_t| \leq EY_t' \leq ES_t^4 + 6(E^{\frac{1}{2}}S_t^4)(E^{\frac{1}{2}}U_{2,t}^2) + 4(E^{\frac{1}{2}}S_t^4)(E^{\frac{3}{4}}V_{3,t}^{\frac{4}{3}}) + EU_{4,t} + 6EU_{2,t}^2 < \infty.$$

From (24), $ET_{2,t} = EU_{2,t} < \infty$. Thus, (8) of Section 2 and Lemmas 4 and 5 are valid, whence by Lemma 7, $E\{S_t^2 | \mathfrak{F}_k\} \geq S_k^2$ for $t > k$, $k = 1, 2, \dots$. Consequently,

$$\int_{[t>n]} S_t^4 = \int_{[t>n]} [S_n^4 + 2S_n^2(S_t^2 - S_n^2) + (S_t^2 - S_n^2)^2] \geq \int_{[t>n]} S_n^4 + 2 \int_{[t>n]} S_n^2 E\{S_t^2 - S_n^2 | \mathfrak{F}_n\} \geq \int_{[t>n]} S_n^4$$

implying $\int_{[t>n]} S_n^4 = o(1)$ and concomitantly

$$\begin{aligned} \int_{[t>n]} S_n^2 U_{2,n} &\leq (\int_{[t>n]} S_n^4)^{\frac{1}{2}} (\int_{[t>n]} U_{2,t}^2)^{\frac{1}{2}} = o(1), \\ \int_{[t>n]} |S_n| V_{3,n} &\leq (\int_{[t>n]} S_n^4)^{\frac{1}{2}} (\int_{[t>n]} V_{3,t}^{\frac{4}{3}})^{\frac{3}{4}} = o(1), \\ (25) \quad \int_{[t>n]} U_{4,n} &\leq \int_{[t>n]} U_{4,t} = o(1), \\ \int_{[t>n]} \sum_{j=1}^n u_{2,j} U_{2,j} &\leq \int_{[t>n]} U_{2,n}^2 \leq \int_{[t>n]} U_{2,t}^2 = o(1). \end{aligned}$$

Thus, $\int_{[t>n]} |Y_n| \leq \int_{[t>n]} Y_n' = o(1)$ and by Lemma 1, $EY_t = EY_1 = 0$.

Alternative expressions for ES_t^4 are possible as indicated in

THEOREM 4. *If $E(t \sum_{j=1}^t E\{x_j^4 | \mathfrak{F}_{j-1}\}) < \infty$, then setting $S_0 = 0$,*

$$ES_t^4 = 6E \sum_{j=1}^t S_{j-1}^2 u_{2,j} + 4E \sum_{j=1}^t S_{j-1} u_{3,j} + EU_{4,t}.$$

The proof of Theorem 4 is similar to that of Theorem 3 and will be omitted.

COROLLARY. *If $E(tU_{4,t}) < \infty$, then*

$$\begin{aligned} E(6 \sum_{j=1}^t S_{j-1}^2 u_{2,j} + 4 \sum_{j=2}^t S_{j-1} u_{3,j}) &= 6ES_t^2 U_{2,t} + 4ES_t U_{3,t} - 6E(\sum_{j=1}^t u_{2,j} U_{2,j}). \end{aligned}$$

It is intuitively clear that terms with like coefficients are equal, and indeed we have

LEMMA 8. *If $E(tU_{4,t}) < \infty$, then $ES_t U_{3,t} = E(\sum_{j=2}^t S_{j-1} u_{3,j})$ and $E(S_t^2 U_{2,t}) = E(\sum_{j=2}^t S_{j-1}^2 u_{2,j}) + E(\sum_{j=1}^t u_{2,j} U_{2,j})$.*

PROOF. It suffices to verify the first of the two relationships since the second will then follow from the corollary to Theorem 4. Suppose first that

$$(26) \quad E(\sum_{j=1}^t |x_j| V_{r,j}) < \infty.$$

Then

$$\begin{aligned} \sum_{k=1}^{\infty} \int_{[t=k]} \sum_{j=1}^k x_j U_{r,j} &= \sum_{j=1}^{\infty} \int_{[t \geq j]} x_j U_{r,j} \\ &= \sum_{j=1}^{\infty} \int_{[t \geq j]} E(x_j | \mathfrak{F}_{j-1}) U_{r,j} = 0, \end{aligned}$$

whence

$$\begin{aligned}
 E(\sum_{j=1}^t S_{j-1}u_{r,j}) &= \sum_{k=1}^\infty \int_{[t=k]} [\sum_{j=2}^k S_{j-1}u_{r,j} + \sum_{j=1}^k x_j U_{r,j}] \\
 (27) \qquad \qquad \qquad &= \sum_{k=1}^\infty \int_{[t=k]} S_k U_{r,k} \\
 &= ES_t U_{r,t}.
 \end{aligned}$$

Thus, if $t' = \min(t, N)$, (27) holds with t replaced by t' irrespective of (26). However,

$$\begin{aligned}
 (28) \qquad ES_t U_{3,t} &= \sum_{k=1}^N \int_{[t=k]} S_k U_{3,k} + \int_{[t>N]} S_t U_{3,t} \\
 &= ES_{t'} U_{3,t'} - \int_{[t>N]} S_N U_{3,N} + \int_{[t>N]} S_t U_{3,t},
 \end{aligned}$$

and analogously

$$\begin{aligned}
 (29) \quad E(\sum_{j=2}^t S_{j-1}u_{3,j}) &= E(\sum_{j=2}^{t'} S_{j-1}u_{3,j}) - \int_{[t>N]} \sum_{j=2}^N S_{j-1}u_{3,j} \\
 &\qquad \qquad \qquad + \int_{[t>N]} \sum_{j=2}^t S_{j-1}u_{3,j}.
 \end{aligned}$$

Now $E|S_t U_{3,t}| \leq EY_{t'} < \infty$, and employing Lemma 7,

$$\begin{aligned}
 E \sum_1^t |S_{j-1}u_{3,j}| &= \sum_{k=1}^\infty \int_{[t=k]} \sum_1^k |S_{j-1}u_{3,j}| = \sum_{j=1}^\infty \int_{[t \geq j]} |S_{j-1}u_{3,j}| \\
 &\leq \sum_1^\infty \int_{[t \geq j]} |S_t u_{3,j}| \leq E|S_t| V_{3,t} \leq EY_{t'} < \infty.
 \end{aligned}$$

These facts plus (25) imply that all unwanted terms of (28) and (29) are $o(1)$ and the result follows.

Identities and inequalities analogous to (27) abound and several of these will be catalogued as

LEMMA 9. $E(\sum_{n=1}^t S_n^2) \leq EtS_t^2$ under the conditions of Lemma 7.

$$E(\sum_{n=1}^t S_n) = EtS_t \quad \text{if} \quad EtT_t < \infty.$$

$$E(\sum_{n=1}^t T_n) \leq EtT_t \quad \text{if} \quad EtT_t < \infty.$$

PROOF.

$$\begin{aligned}
 E \sum_{n=1}^t S_n^2 &= \sum_{k=1}^\infty \int_{[t=k]} \sum_{n=1}^k S_n^2 \\
 &= \sum_{n=1}^\infty \int_{[t \geq n]} S_n^2 \leq \sum_{n=1}^\infty \int_{[t \geq n]} E(S_t^2 | \mathfrak{F}_n) \\
 &= \sum_{n=1}^\infty \int_{[t \geq n]} S_t^2 = \sum_{n=1}^\infty \sum_{k=n}^\infty \int_{[t=k]} S_t^2 \\
 &= \sum_{k=1}^\infty k \int_{[t=k]} S_t^2 = EtS_t^2
 \end{aligned}$$

employing Lemma 7. Similarly,

$$E(\sum_{n=1}^t T_n) = \sum_{n=1}^\infty \int_{[t \geq n]} T_n \leq \sum_{n=1}^\infty \int_{[t \geq n]} T_t = EtT_t.$$

Finally,

$$\begin{aligned}
 E(\sum_{n=1}^t S_n) &= \sum_{n=1}^\infty \int_{[t \geq n]} S_n = \sum_{n=1}^\infty \int_{[t \geq n]} E(S_t | \mathfrak{F}_n) = \sum_{n=1}^\infty \int_{[t \geq n]} S_t \\
 &= EtS_t
 \end{aligned}$$

in view of Lemmas 1 and 2 and the validity of interchanging the order of summation and integration.

4. The third moment. In this section $E(|x_n|^3)$ will be supposed finite and $Ex_1 = 0$. Define

$$\begin{aligned}
 (30) \quad Y_n &= S_n^3 - 3S_n U_{2,n} - U_{3,n}, \\
 W_n &= S_n^3 - 3 \sum_{j=1}^n S_{j-1} u_{2,j} - U_{3,n}, \\
 Z_n &= S_n^3 - 3 \sum_{j=1}^n S_j u_{2,j} - U_{3,n}.
 \end{aligned}$$

It is readily checked that $(Y_n, \mathfrak{F}_n; n \geq 1)$, $(W_n, \mathfrak{F}_n; n > 1)$, $(Z_n, \mathfrak{F}_n; n > 1)$ are all martingales and that $EY_1 = EW_1 = EZ_1 = 0$.

THEOREM 5. *If $EV_{3,t} < \infty$ and $EV_{1,t}^3 < \infty$, or equivalently if $ET_t^3 < \infty$, then $E|S_t|^3 < \infty$ and $ES_t^3 = 3E(\sum_{j=1}^t S_{j-1} u_{2,j}) + EU_{3,t}$.*

PROOF. Suppose that $EV_{3,t} < \infty$, $EV_{1,t}^3 < \infty$. (Their equivalence with $ET_t^3 < \infty$ will be deferred to Lemma 10). Then

$$\begin{aligned}
 (31) \quad E|S_t|^3 &= \sum_{k=1}^{\infty} \int_{[t=k]} \sum_{n=1}^k (|S_n|^3 - |S_{n-1}|^3) \leq \sum_{k=1}^{\infty} \sum_{n=1}^k \int_{[t=k]} (|x_n|^3 \\
 &+ 3|S_{n-1}|x_n^2 + 3S_{n-1}^2|x_n|) \\
 &\leq 6 \sum_{k=1}^{\infty} \sum_{n=1}^k \int_{[t=k]} (|x_n|^3 + S_{n-1}^2|x_n|) \\
 &= 6[E(\sum_{n=1}^t |x_n|^3) + E(\sum_{n=1}^t S_{n-1}^2|x_n|)].
 \end{aligned}$$

By Lemma 6,

$$(32) \quad E(\sum_{n=1}^t |x_n|^3) = EV_{3,t} < \infty.$$

On the other hand, $ES_t^2 \leq ET_t^2 \leq 1 + ET_t^3 < \infty$ and

$$\int_{[t \geq k]} |S_k| \leq \int_{[t \geq k]} T_k \leq \int_{[t \geq k]} T_t \leq \int_{[t \geq k]} (1 + T_t^3) = o(1)$$

in view of the asserted equivalence. Thus, Lemma 7 holds, whence

$$\begin{aligned}
 (33) \quad E(\sum_{n=1}^t S_{n-1}^2|x_n|) &= \sum_{k=1}^{\infty} \sum_{n=1}^k \int_{[t=k]} S_{n-1}^2|x_n| = \sum_{n=1}^{\infty} \int_{[t \geq n]} S_{n-1}^2 v_{1,n} \\
 &\leq \sum_{n=1}^{\infty} \int_{[t \geq n]} E(S_t^2 | \mathfrak{F}_{n-1}) v_{1,n} = \sum_{n=1}^{\infty} \int_{[t \geq n]} S_t^2 v_{1,n} \\
 &= \sum_{k=1}^{\infty} \sum_{n=1}^k \int_{[t=k]} S_t^2 v_{1,n} = ES_t^2 V_{1,t} \leq (E^{2/3}|S_t|^3)(E^3 V_{1,t}^3).
 \end{aligned}$$

Replace t by $t' = \min(t, k)$ in (31). Then from (32) and (33),

$$E|S_{t'}|^3 \leq 6EV_{3,t'} + 6(E^{2/3}|S_{t'}|^3)(E^3 V_{1,t'}^3) = O(1) + O(1)E^{2/3}|S_{t'}|^3$$

whence, by Fatou's lemma,

$$(34) \quad E|S_t|^3 < \infty.$$

Next, (34) implies that the expectation in the LHS of (33) is finite whence

$$\begin{aligned}
 (35) \quad E(\sum_{n=1}^t |S_{n-1}| u_{2,n}) &= \sum_{n=1}^{\infty} \int_{[t \geq n]} |S_{n-1}| x_n^2 = E(\sum_{n=1}^t |S_{n-1}| x_n^2) \\
 &\leq E[\sum_{n=1}^t (|x_n|^3 + |S_{n-1}|^2|x_n|)] < \infty.
 \end{aligned}$$

Combining (34) and (35), $E |W_t| < \infty$. Since, paralleling (31),

$$\int_{[t>k]} |S_k|^3 \leq 6 \int_{[t>k]} \sum_{n=1}^k (|x_n|^3 + S_{n-1}^2 |x_n|) = o(1),$$

$\int_{[t>k]} |W_k| = o(1)$ and the theorem follows from Lemma 1.

LEMMA 10. $EV_{3,t} < \infty$ and $EV_{1,t}^3 < \infty$ if and only if $ET_t^3 < \infty$.

PROOF. Suppose $EV_{3,t} < \infty$ and $EV_{1,t}^3 < \infty$. The argument of (31) with T_t replacing S_t yields

$$ET_t^3 \leq 6 \sum_{k=1}^\infty \sum_{n=1}^k \int_{[t=k]} (|x_n|^3 + T_{n-1}^2 |x_n|).$$

The inequality of (33) also obtains with T replacing S in view of the fact that $T_t \geq T_{n-1}$ on the set $[t \geq n]$. Thus, analogously, $ET_{t'}^3 \leq O(1) + O(1)E^{2/3}T_{t'}^3$, implying $ET_t^3 < \infty$.

Conversely, if $ET_t^3 < \infty$, clearly $EV_{3,t} = ET_{3,t} \leq ET_t^3 < \infty$. Moreover,

$$\begin{aligned} EV_{1,t'}^3 &= \sum_{j=1}^\infty \int_{[t'=j]} \sum_{n=1}^j (V_{1,n}^3 - V_{1,n-1}^3) \\ &\leq \sum_{j=1}^\infty \sum_{n=1}^j \int_{[t'=j]} (\vartheta_{1,n}^3 + 3V_{1,n-1}^2 \vartheta_{1,n} + 3V_{1,n-1} \vartheta_{1,n}^2) \\ &\leq O(1) + 6 \sum_{j=1}^\infty \sum_{n=1}^j \int_{[t'=j]} V_{1,n-1}^2 \vartheta_{1,n} \\ &= O(1) + 6 \sum_{n=1}^\infty \int_{[t' \geq n]} |x_n| V_{1,n-1}^2 \\ &\leq O(1) + 6 \sum_{n=1}^\infty \sum_{j=n}^\infty \int_{[t'=j]} |x_n| V_{1,t'}^2 \\ &\leq O(1) + 6 ET_{t'} V_{1,t'}^2 \\ &\leq O(1) + O(1)E^{2/3}V_{1,t'}^3, \end{aligned}$$

which implies, as earlier, that $EV_{1,t}^3 < \infty$ and completes the proof.

THEOREM 6. If $ET_t^3 < \infty$ and $E t^3 V_{3,t} < \infty$, $ES_t^3 = 3ES_t U_{2,t} + EU_{3,t} < \infty$.

PROOF. As in Theorem 5, after setting $p = \frac{3}{2}$, $r = 2$ in (23) of Section 3 to obtain

$$E |S_t U_{2,t}| \leq (E^3 |S_t|^3) (E^{2/3} U_{2,t}^{3/2}) \leq (E^3 |S_t|^3) (E^{2/3} t^3 V_{3,t}).$$

COROLLARY 1. Under the conditions of Theorem 6, $E(\sum_{j=1}^t x_j u_{2,j}) = 0$.

PROOF. Analogously, $EZ_t = 0$, whence $E(W_t - Z_t) = 0$.

COROLLARY 2. Under the conditions of Theorem 6, $ES_t U_{2,t} = E(\sum_{j=1}^t S_{j-1} u_{2,j})$.

The single requirement $ET_t^3 < \infty$, although equivalent to the two conditions of Theorem 5, is difficult to check. The following single condition is easily seen to imply all those of Theorems 5 and 6:

$$(36) \quad E(t^2 V_{3,t})' < \infty,$$

and in addition yields

$$\begin{aligned} ET_t^3 &= 3ET_t^2 V_{1,t} + 3ET_t(V_{2,t} - 2 \sum_{j=1}^t V_{1,j} \vartheta_{1,j}) + EV_{3,t} \\ &\quad - 3E(\sum_{j=1}^t V_{1,j} \vartheta_{2,j}) - 3E(\sum_{j=1}^t V_{2,j} \vartheta_{1,j}) + 6E(\sum_{j=1}^t \vartheta_{1,j} \sum_{i=1}^j V_{1,i} \vartheta_{1,i}). \end{aligned}$$

5. Sums of independent random variables. In this section, the random variables x_1, x_2, \dots will be supposed independent. If $Ex_n = 0$, all prior theorems are,

of course, applicable but may be reformulated in especially simple terms with conditions that are susceptible of immediate verification. For example, from Theorems 3 and 6, we obtain:

THEOREM 7. *If x_1, x_2, \dots are independent with $Ex_n = 0, Ex_n^2 = \sigma^2, Ex_n^3 = \gamma, Ex_n^4 = \beta < \infty$ and t is a stopping rule with $Et^2 < \infty$, then $ES_t^4 < \infty$ and*

$$ES_t^4 = 6\sigma^2EtS_t^2 + 4\gamma EtS_t + \beta Et - 3\sigma^4Et(t + 1).$$

THEOREM 8. *If x_1, x_2, \dots are independent with $Ex_n = 0, Ex_n^2 = \sigma^2, Ex_n^3 = \gamma, E|x_n|^3 \leq C < \infty$, and if t is a stopping variable with $Et^3 < \infty$, then $ES_t^3 = \gamma Et + 3\sigma^2EtS_t < \infty$.*

PROOF. According to Theorem 6 and Lemma 10, it suffices to verify that

$$EV_{3,t} \leq E(t^{\frac{1}{2}}V_{3,t}) \leq CEt^{3/2} < \infty,$$

$$EV_{1,t}^3 \leq E[t(1 + C)]^3 < \infty.$$

In the final theorem, the requirement of Theorem 8 that $Et^3 < \infty$ will be relaxed at the expense of increasing the moment assumptions on x_n .

THEOREM 9. *If x_1, x_2, \dots are independent with $Ex_n = 0, Ex_n^2 = \sigma^2, Ex_n^3 = \gamma, Ex_n^4 \leq C < \infty$, and if t is a stopping variable with $Et^2 < \infty$, then $ES_t^3 = \gamma Et + 3\sigma^2EtS_t$.*

PROOF. Here, the martingale Y_n of (30) simplifies to $Y_n = S_n^3 - 3\sigma^2nS_n - n\gamma$. The theorem will follow from Lemmas 1 and 2 once it is established that $E \sum_1^t |Y_{n+1} - Y_n| = E \sum_1^t E(|Y_{n+1} - Y_n| | \mathcal{F}_n) < \infty$. Now if B and D are finite constants (not necessarily the same in each appearance),

$$E(|S_{n+1}^3 - S_n^3| | \mathcal{F}_n) \leq 6E(|x_{n+1}|^3 + S_n^2|x_{n+1}| | \mathcal{F}_n) \leq BS_n^2 + D,$$

$$E(|(n + 1)S_{n+1} - nS_n| | \mathcal{F}_n) = E(|S_n + (n + 1)x_{n+1}| | \mathcal{F}_n) \leq S_n^2 + nD,$$

whence

$$E(|Y_{n+1} - Y_n| | \mathcal{F}_n) \leq BS_n^2 + nD.$$

Next, Lemma 9 is applicable below since (17) insures $ES_t^2 < \infty$ while Lemmas 6 and 2 guarantee (10). Consequently,

$$\begin{aligned} E \sum_1^t E(|Y_{n+1} - Y_n| | \mathcal{F}_n) &\leq B \cdot E(\sum_1^t S_n^2) + D \cdot Et(t + 1) \\ &\leq B \cdot EtS_t^2 + D \cdot Et(t + 1) \\ &\leq B \cdot (E^{\frac{1}{2}}t^2)(E^{\frac{1}{2}}S_t^4) + D \cdot Et(t + 1) < \infty. \end{aligned}$$

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