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Article (Published Version)

Mezzadri, F and Simm, N (2012) Moments of the transmission eigenvalues, proper delay times and random matrix theory II. *Journal of Mathematical Physics*, 53 (5). 053504-1-053504-42. ISSN 0022-2488

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Citation: [Journal of Mathematical Physics](#) **53**, 053504 (2012); doi: 10.1063/1.4708623

View online: <https://doi.org/10.1063/1.4708623>

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Moments of the transmission eigenvalues, proper delay times and random matrix theory II

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(Received 14 September 2011; accepted 9 March 2012; published online 16 May 2012)

We systematically study the first three terms in the asymptotic expansions of the moments of the transmission eigenvalues and proper delay times as the number of quantum channels n in the leads goes to infinity. The computations are based on the assumption that the Landauer-Büttiker scattering matrix for chaotic ballistic cavities can be modelled by the circular ensembles of random matrix theory. The starting points are the finite- n formulae that we recently discovered [F. Mezzadri and N. J. Simm, “Moments of the transmission eigenvalues, proper delay times and random matrix theory,” *J. Math. Phys.* **52**, 103511 (2011)]. Our analysis includes all the symmetry classes $\beta \in \{1, 2, 4\}$; in addition, it applies to the transmission eigenvalues of Andreev billiards, whose symmetry classes were classified by Zirnbauer [“Riemannian symmetric superspaces and their origin in random-matrix theory,” *J. Math. Phys.* **37**(10), 4986 (1996)] and Altland and Zirnbauer [“Random matrix theory of a chaotic Andreev quantum dot,” *Phys. Rev. Lett.* **76**(18), 3420 (1996); “Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures,” *Phys. Rev. B* **55**(2), 1142 (1997)]. Where applicable, our results are in complete agreement with the semiclassical theory of mesoscopic systems developed by Berkolaiko *et al.* [“Full counting statistics of chaotic cavities from classical action correlations,” *J. Phys. A: Math. Theor.* **41**(36), 365102 (2008)] and Berkolaiko and Kuipers [“Moments of the Wigner delay times,” *J. Phys. A: Math. Theor.* **43**(3), 035101 (2010); “Transport moments beyond the leading order,” *New J. Phys.* **13**(6), 063020 (2011)]. Our approach also applies to the Selberg-like integrals. We calculate the first two terms in their asymptotic expansion explicitly. © 2012 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4708623>]

I. INTRODUCTION

A quantum dot is often modelled by a two-dimensional billiard with holes on the boundary, whose sizes are proportional to the number of quantum channels in the leads. Because of the mesoscopic dimensions of the ballistic cavity, quantum mechanical phase coherence plays an important role in the dynamics of an electron inside it. Therefore, the electric current has an intrinsic stochastic nature, whose fluctuations are of theoretical and experimental interest. Furthermore, whenever the classical limit of the dynamics is chaotic, it is expected that such fluctuations should be characterized by universal features that are well described by *random matrix theory* (RMT).^{6–8, 15, 16, 39}

At low temperatures and voltage, the scattering inside the cavity is elastic and is described by the Landauer-Büttiker scattering matrix,

$$S := \begin{pmatrix} r_{m \times m} & t'_{m \times n} \\ t_{n \times m} & r'_{n \times n} \end{pmatrix}, \quad (1)$$

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where m and n are the number of quantum channels in the left and right leads, respectively. The sub-blocks $r_{m \times m}$, $t_{n \times m}$ and $r'_{n \times n}$ and $t'_{m \times n}$ are the reflection and transmission matrices through the incoming and outgoing lead. Without loss of generality we shall assume that $m > n$. The dimensionless quantum conductance at zero temperature is given by

$$\text{Tr } tt^\dagger = T_1 + \dots + T_n, \quad (2)$$

where T_1, \dots, T_n are the eigenvalues of the transmission matrix tt^\dagger . Since S is unitary, T_1, \dots, T_n lie in the interval $[0, 1]$.

The Wigner-Smith time delay matrix is defined by

$$Q := -i\hbar S^{-1} \frac{\partial S}{\partial E}. \quad (3)$$

The eigenvalues τ_1, \dots, τ_n of Q are called *proper delay times*, and their average

$$\tau_W := \frac{1}{n} \text{Tr } Q = \frac{1}{n} (\tau_1 + \dots + \tau_n) \quad (4)$$

is the *Wigner delay time*. The number of proper delay times n is the total number of quantum channels in the leads. As the name suggests, τ_W is a measure of the time that the electron spends in the ballistic cavity. The purpose of this paper is to provide a comprehensive asymptotic study in the limit as $n \rightarrow \infty$ of the moments of the densities of the transmission eigenvalues and of the delay times in chaotic quantum dots.

In two pioneering letters Blümel and Smilansky^{15,16} discovered that if the classical limit of the dynamics of a scattering system is chaotic, then the spectral correlations of the scattering matrix are well described by those of matrices in one of the circular Dyson's ensembles: the circular orthogonal ensemble (COE), the circular unitary ensemble (CUE) and the circular symplectic ensemble (CSE). If the dynamics is time-reversal invariant and $K^2 = 1$, where K is the time-reversal operator, then the appropriate ensemble is the COE; if $K^2 = -1$, it is the CSE; if the system does not have any symmetry, then the appropriate ensemble is the CUE.

Under the assumption that the quantum dynamics can be modelled by RMT, the transmission eigenvalues have the joint probability density function (*j.p.d.f.*) (Refs. 6, 8, 28, 34, and 39)

$$p^{(\beta, \delta)}(T_1, \dots, T_n) := \frac{1}{C} \prod_{j=1}^n T_j^\alpha (1 - T_j)^{\delta/2} \prod_{1 \leq j < k \leq n} |T_k - T_j|^\beta, \quad (5)$$

where $\beta \in \{1, 2, 4\}$, $\alpha = \frac{\beta}{2}(m - n + 1) - 1$, and $\delta \in \{-1, 0, 1, 2\}$. Throughout this paper, C refers to a normalization constant that may change at each occurrence. The right-hand side of (5) is the *j.p.d.f.* of the eigenvalues of matrices in the Jacobi ensembles,

$$p_{J_{a,b}}(x_1, \dots, x_n) := \frac{1}{C} \prod_{j=1}^n x_j^{\beta/2(b+1)-1} (1 - x_j)^{\beta/2(a+1)-1} \prod_{1 \leq j < k \leq n} |x_k - x_j|^\beta, \quad (6)$$

where $x_j \in [0, 1]$, $j = 1, \dots, n$,

$$a = \frac{2}{\beta} \left(1 + \frac{\delta}{2}\right) - 1 \quad \text{and} \quad b = m - n. \quad (7)$$

The parameter δ is different from zero only for *Andreev quantum dots*, which are ballistic cavities in contact with a superconductor. For such systems, the symmetry classes are not Dyson's ensembles, but those associated to symmetric spaces, which were classified by Zirnbauer,⁷⁴ Altland and Zirnbauer,³ and Dueñez.^{30,31} These symmetries are parametrized by the pair of integers,

$$(\beta, \delta) \in \{(1, -1), (2, -1), (4, 2), (2, 1)\}. \quad (8)$$

The moments of the density of the transmission eigenvalues are

$$M_{J_{a,b}}^{(\beta)}(k, n) := \left\langle \text{Tr}[(tt^\dagger)^k] \right\rangle = \langle T_1^k + \dots + T_n^k \rangle, \quad (9)$$

where the angle brackets denote the average taken with respect to the *j.p.d.f.* (6).

The assumption that the statistical fluctuations for the electric current in quantum dots are modelled by RMT leads to an explicit formula for the *j.p.d.f.* of the proper delay times too.²² We have

$$P_\beta(\gamma_1, \dots, \gamma_n) := \frac{1}{C} \prod_{j=1}^n \gamma_j^{n\beta/2} e^{-\beta\tau_H\gamma_j/2} \prod_{1 \leq j < k \leq n} |\gamma_k - \gamma_j|^\beta, \quad (10)$$

where $\gamma_j = \tau_j^{-1}$ and τ_H is the Heisenberg time. In our setting $\tau_H = n$. The *j.p.d.f.* (10) is the density of the Laguerre ensemble, namely,

$$p_{L_b}(x_1, \dots, x_n) := \frac{1}{C} \prod_{j=1}^n x_j^{\beta/2(b+1)-1} e^{-\beta x_j/2} \prod_{1 \leq j < k \leq n} |x_k - x_j|^\beta, \quad (11)$$

for $x_j \in [0, \infty)$, $j = 1, \dots, n$.

From (10), the moments of the density of proper delay times are the *negative* moments of the density of the Laguerre ensembles,

$$M_{L_b}^{(\beta)}(-k, n) := \frac{1}{n^k} \langle \text{Tr } Q^k \rangle = \frac{1}{n^k} \langle \gamma_1^{-k} + \dots + \gamma_n^{-k} \rangle, \quad k < n\beta/2 + 1. \quad (12)$$

The above average is taken with respect to the *j.p.d.f.* (11) and the parameter b is given by

$$b = n - 1 + 2/\beta. \quad (13)$$

For convenience, we have used the same notation to denote the parameters b in the Jacobi and Laguerre ensembles. In the rest of the article, it will be clear from the context to which ensembles it refers to.

We compute the first three terms of the asymptotic expansions of the averages (9) and (12) in the limit as $n \rightarrow \infty$. The starting points of our analysis are the finite- n formulae that we computed in a previous publication.⁵³ Explicit expressions for finite moments usually are rather involved and not suitable for an asymptotic analysis. However, our results⁵³ have the advantage of allowing an asymptotic study beyond the leading order.

Due to their applications to physics and multivariate analysis, the leading order asymptotics of the moments of the eigenvalue density for the Laguerre and Jacobi ensembles have been studied in detail both within RMT and using semiclassical techniques.^{4,9-11,21,23,24,27,42,56} The finite- n moments of the transmission eigenvalues for $\beta = 2$ were computed by Novaes,⁵⁷ and Vivo and Vivo.⁷² Besides the first part of this work,⁵³ recently another paper appeared⁴⁸ where the finite moments of the transmission eigenvalues for $\beta \in \{1, 2, 4\}$ were computed using a different approach. The leading order term of the density of the proper delay times was studied in Refs. 62, 63, and 68. However, in these articles our assumptions of ideal physical settings do not hold.

In a pioneering paper, Johansson⁴⁰ looked at the fluctuations of the linear statistics $\sum_j f(x_j)$ as $n \rightarrow \infty$ for ensembles whose *j.p.d.f.* is

$$p(x_1, \dots, x_n) = \frac{1}{C} \prod_{j=1}^n \exp(-nV(x_j)) \prod_{1 \leq j < k \leq n} |x_k - x_j|^\beta.$$

The potential $V(x)$ is a polynomial of even degree and $f(x)$ is a suitable test function. He proved that $\sum_j f(x_j)$ converges to a normal random variable with finite variance. In the same paper, he computed the next to leading order correction for arbitrary values of β . Subsequently, Dumitriu and Edelman³³ studied the global fluctuations of the spectra of the β -Hermite and β -Laguerre ensembles, that is, ensembles in which β can be any positive real number. They also computed the next to leading order correction to the Marčenko-Pastur law. Recent results on the global fluctuations of the spectra of β -ensembles include Refs. 43 and 19.

The statistical properties of quantum transport have been studied with semiclassical techniques too. The semiclassical approach is completely independent of a RMT analysis. Although the RMT conjecture for closed quantum systems with a chaotic classical limit has a long

history,^{13,14,17,54,55,67} semiclassical techniques have been applied to quantum transport relatively recently.^{9–12,20,38,45,46,59,61} Within this framework, the elements of the scattering matrix are approximated by sums over classical trajectories connecting incoming and outgoing channels in the leads. What is most important is that the outcome of semiclassical calculations are predictions for *individual* energy-averaged chaotic systems. This should be distinguished from the RMT approach, which involves calculating an average over an appropriate *ensemble* of different systems. The semiclassical limit $\hbar \rightarrow 0$ corresponds to the limit $n \rightarrow \infty$ in RMT.

Semiclassical calculations are perturbative in nature. The inverse channel number n^{-1} is proportional to \hbar and is the small parameter for the theory. Higher order terms are constructed by including successively more intricate families of classical trajectories into the sums and then performing the appropriate combinatorics. This task was completed to all orders for the conductance³⁸ and for the shot noise.²⁰ Higher moments of the density of the transmission eigenvalues are also known at leading order from semiclassical theory.⁹ Berkolaiko and Kuipers^{10,11} computed the leading order generating function of the moments of the proper delay times and the first two subleading corrections of the generating functions of the moments of both the transmission eigenvalues and the proper delay times for $\beta = 1$ and $\beta = 2$. Recently, Berkolaiko and Kuipers¹² and independently Novaes⁵⁹ announced two distinct combinatorial treatments of the correlations of the scattering trajectories that imply exact agreement at all orders in n^{-1} between the semiclassical and RMT calculations of the moments of the transmission eigenvalues. At present, however, it seems quite difficult to extract explicit expressions for the asymptotic expansions from the combinatorial formulae. The consistency of RMT with the semiclassical analysis of mesoscopic systems is a major success of these two approaches. Indeed, in closed systems we expect semiclassical theories to be consistent with RMT only when they describe the local fluctuations of the energy levels; the averages that we compute are affected by the global fluctuations of the spectra of random matrices. It is far from being obvious *a priori* that we should expect an exact agreement at all orders.

Not many semiclassical results are available for Andreev quantum dots. Adagideli and Beenakker² and subsequently Kuipers *et al.*^{44,47} developed a semiclassical theory of the energy gap in the density of states of Andreev billiards. Berkolaiko and Kuipers¹¹ computed the first two corrections to the density of states. The leading order term of the density of states was previously computed using RMT by Melsen *et al.*^{51,52} using a perturbative expansion developed by Brouwer and Beenakker,²¹ which is based on the computation of integrals of matrix elements of matrices in the circular ensembles.

Since the parameter b in the integrals (9) and (12) depends on the number of channels in the leads, we need to specify how the limit is taken. In the moments (9) both m and n tend to infinity, but the scaling parameter

$$u = \frac{m}{n} \tag{14}$$

remains finite. In other words, we fix a but let $b = n(u - 1)$ grow to infinity. For the Laguerre ensemble, it will be convenient to introduce a new variable w by defining

$$b = n(w - 1) + 2/\beta - 1. \tag{15}$$

The moments (12) are then recovered by setting $w = 2$. A similar generalization was also considered in Ref. 5.

Equations (7) and (15) contain n -independent terms which arise due to physical reasons and are rather unusual from a purely mathematical point of view. Indeed, they have nontrivial effects on the subleading order terms of the averages (9) and (12) (see, e.g., Remark 3.8). For this reason the moments of the proper delay times cannot be extracted from the results of Dumitriu and Edelman³³ beyond the leading order.

When $\beta > 0$, the density (6) is often referred to as the *Selberg density* and the averages

$$\frac{1}{C} \int_0^1 \cdots \int_0^1 \left(\sum_{j=1}^n x_j^k \right) \prod_{j=1}^n x_j^{\beta/2(b+1)-1} (1-x_j)^{\beta/2(a+1)-1} \times \prod_{1 \leq j < l \leq n} |x_l - x_j|^\beta dx_1 \cdots dx_n \quad (16)$$

as *Selberg-like* integrals (see, e.g., Refs. 35, 50, and 36). Their name originates from the fact that by setting $k = 0$ and $C = n$, the right-hand side of (16) becomes Selberg's integral, which was introduced for the first time in quantum transport by Savin and Sommers.⁶⁴ They computed the average of the shot noise and the Fano factor for arbitrary β nonperturbatively. Subsequently, Sommers *et al.*⁶⁹ and Savin *et al.*⁶⁵ applied this approach to calculate moments and cumulants up to the 4th order of linear and nonlinear statistics of the electric current. Novaes⁵⁷ used the Selberg-like integrals to study the moments of the transmission eigenvalues for $\beta = 2$ nonperturbatively. Khoruzhenko *et al.*⁴¹ combined such integrals with the theory of symmetric functions to compute higher order moments and cumulants of the conductance and shot noise.

One may ask what the asymptotic expansion of (16) is if both $a, b \rightarrow \infty$. This problem was studied in Refs. 24, 58, and 42 when both a and b are proportional to n . The scaling that they used was

$$a = (v - 1)n \quad \text{and} \quad b = (u - 1)n, \quad (17)$$

with u and v positive constants. The leading order term is known when $\beta = 2$ (Refs. 24 and 42) as well as for general β (Ref. 58). It turns out that at leading order, the integral (16) is independent of β . We compute the first subleading correction for $\beta \in \{1, 2, 4\}$. We also show that the leading order terms are consistent with the results in Refs. 27 and 58.

The structure of the article is as follows. In Sec. II, we outline some of the main ideas behind the proofs. In Sec. III, we treat the asymptotics of the moments for $\beta = 2$ ensembles. Section IV is devoted to $\beta = 1$ and $\beta = 4$ ensembles. In Sec. V, we study Selberg-like integrals and Sec. VI concludes the paper with comments and open problems.

II. PRELIMINARIES

As remarked in the Introduction, in the first part of this work⁵³ we computed the averages (9) and (12) for any number of open channels. Our results were expressed in terms of finite sums involving binomial coefficients and *Pochhammer Symbols* defined by the ratio

$$(x)_{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)}. \quad (18)$$

This simple fact has two important consequences. Firstly, the moments (9) and (12) admit the asymptotic expansions in integer powers of $1/n$,

$$n^{-1} M_{J_{a,b}}^{(\beta)}(k, n) \sim \sum_{p=0}^{\infty} T_{k,p}^{(\beta,\delta)}(u) n^{-p}, \quad (19a)$$

$$n^{k-1} M_{L_b}^{(\beta)}(-k, n) \sim \sum_{p=0}^{\infty} D_{k,p}^{(\beta)}(w) n^{-p}. \quad (19b)$$

Here and in the rest of the paper whenever the parameters a and b appear, they will depend implicitly on u or w according to (7) and (15).

The second important feature of our results⁵³ is that the finite- n formulae for (9) and (12) contain meromorphic functions that are particularly suitable to an asymptotic analysis. In order to understand this point, it is useful to consider an example for $\beta = 2$.

A. An example for $\beta = 2$

Let us define

$$\Delta M_{J_{a,b}}^{(2)}(k, n) := M_{J_{a,b}}^{(2)}(k, n) - M_{J_{a,b}}^{(2)}(k + 1, n). \tag{20}$$

In Ref. 53, we found that

$$\Delta M_{J_{a,b}}^{(2)}(k, n) = \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} U_{n,k,j}^{a,b}, \tag{21}$$

where

$$U_{n,k,j}^{a,b} = \frac{(a + b + 2n - 2j + k + 1)(a + b + n)_{(k-j+1)}(a + n - j + 1)_{(j)}(b + n)_{(k-j+1)}}{(a + b + 2n - j)_{(k+2)}(a + b + 2n - j + 1)_{(k)}(n + 1)_{(-j)}}. \tag{22}$$

The moments are given explicitly by

$$M_{J_{a,b}}^{(2)}(k, n) = M_{J_{a,b}}^{(2)}(1, n) - \sum_{j=1}^{k-1} \frac{1}{j} \sum_{i=0}^j \binom{j}{i} \binom{j}{i-1} U_{n,j,i}^{a,b}, \tag{23}$$

where

$$M_{J_{a,b}}^{(\beta)}(1, n) = \frac{n(b + n)}{a + b + 2n} \tag{24}$$

is a particular case of *Aomoto's integral* (see, e.g., Ref. 50, pp. 309–310).

We also found a formula for the negative moments in the Laguerre ensemble,

$$M_{L_b}^{(2)}(-k, n) = \frac{1}{k} \sum_{j=0}^{n-1} \binom{k+j}{k-1} \binom{k+j-1}{k-1} \frac{(b+n)_{(-k-j)}}{(1+n)_{(-j-1)}}. \tag{25}$$

The moments of the eigenvalue densities in ensembles with orthogonal and symplectic symmetries have a similar structure: the summands are products of two factors. The first one is n -independent and consists of binomial coefficients. This feature will obviously persist to all orders in the asymptotic expansions of the right-hand sides of Eqs. (21) and (25).

The coefficients

$$N(k, j) := \frac{1}{k} \binom{k}{j} \binom{k}{j-1} \tag{26}$$

appear frequently in enumerative combinatorics, where they are known as *Narayana numbers*. The n -independent factor in the summands of (25) are obtained from the Narayana numbers by extending them to negative integers,

$$N(-k, j) = \frac{1}{k} \binom{k+j}{k-1} \binom{k+j-1}{k-1}, \quad k > 0. \tag{27}$$

This identity follows from the elementary relation

$$\binom{-k}{j} = (-1)^j \binom{k+j-1}{k-1}. \tag{28}$$

The n -dependent part of the factors in the summands of (21) and (25) are ratios of Gamma functions, whose asymptotic expansion is⁷¹

$$\frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} \sim z^{\alpha-\beta} \sum_{i=0}^{\infty} (-1)^i \frac{(\beta - \alpha)_{(i)}}{i!} B_i^{(\alpha-\beta+1)}(\alpha) z^{-i}, \quad z \rightarrow \infty, \tag{29}$$

where $B_i^{(y)}(x)$ are the generalized Bernoulli polynomials. Now, substituting the leading order of (29) into, say, the j th term in the sum (21) leads to a power of n whose exponent is obtained by summing or subtracting the subscripts of the Pochhammer symbols in the ratio (22),

$$n^{1+k-j+1+j+k-j+1-k-2-k+j} = n. \tag{30}$$

Crucially the exponent is independent of j . Surprisingly this phenomenon happens at all orders of n for the moments of both the transmission eigenvalues and proper delay times in all the symmetry classes. Thus, at least in principle, to compute the asymptotic series (19a) up to a given order p we need to substitute into (21) the first p terms of the expansion (29).

Remark 2.1: In addition to the Narayana numbers, we will often need the *Narayana polynomials*

$$N_k(u) := \frac{1}{k} \sum_{j=1}^k \binom{k}{j} \binom{k}{j-1} u^j \tag{31}$$

as well as their generating function (see, e.g., Ref. 18)

$$\begin{aligned} \rho(u, s) &:= \sum_{k=1}^{\infty} N_k(u) s^k \\ &= \frac{1 - s(u + 1) - \sqrt{1 - 2s + s^2 - 2us - 2us^2 + u^2s^2}}{2s}. \end{aligned} \tag{32}$$

Indeed our results have a distinct combinatorial flavour. It is interesting to note that also the semiclassical approach to these problems relies heavily on combinatorics, although of a different kind. From the RMT point of view, it is not obvious *a priori* why combinatorics should play such an important role in these calculations.

B. Generating functions

The coefficients in the series (19a) and (19b) become increasingly involved as we go higher in the order of the expansion. Then, it becomes convenient to express our results in terms of *generating functions*.

Given the sets of coefficients $\mathcal{T}_{k,p}^{(\beta,\delta)}(u)$ and $\mathcal{D}_{k,p}^{(\beta)}(w)$, $k = 0, 1, \dots$, at a given order p in expansions (19a) and (19b), we define the generating functions

$$\mathcal{T}_p^{(\beta,\delta)}(u, s) := \sum_{k=1}^{\infty} \mathcal{T}_{k,p}^{(\beta,\delta)}(u) s^k, \tag{33a}$$

$$\mathcal{D}_p^{(\beta)}(w, s) := \sum_{k=1}^{\infty} \mathcal{D}_{k,p}^{(\beta)}(w) s^k. \tag{33b}$$

If these series are convergent in a neighbourhood of the origin B_ϵ , then $\mathcal{D}_p^{(\beta)}(w, s)$ and $\mathcal{T}_p^{(\beta,\delta)}(u, s)$ are analytic in B_ϵ and define the moments uniquely.

As already appeared in the remarks about formula (21), our results on the moments of the transmission eigenvalues are based on computing the differences between consecutive moments. It is therefore convenient to know their generating function too, which is defined by

$$\Delta \mathcal{T}_p^{(\beta,\delta)}(u, s) := \sum_{k=1}^{\infty} \Delta \mathcal{T}_{k,p}^{(\beta,\delta)}(u) s^k. \tag{34}$$

The coefficients of the powers of s in this formula are defined by

$$\Delta \mathcal{T}_{k,p}^{(\beta,\delta)}(u) := \mathcal{T}_{k,p}^{(\beta,\delta)}(u) - \mathcal{T}_{k+1,p}^{(\beta,\delta)}(u). \tag{35}$$

The moments at a given order p in the asymptotic expansion (19a) are then obtained from the formula

$$\mathcal{T}_{k,p}^{(\beta,\delta)}(u) = \mathcal{T}_{1,p}^{(\beta,\delta)}(u) - \sum_{j=1}^{k-1} \Delta \mathcal{T}_{j,p}^{(\beta,\delta)}(u). \tag{36}$$

The first moments $\mathcal{T}_{1,p}^{(\beta,\delta)}(u)$ can be computed directly from the asymptotic expansion of Aomoto’s integral (24), which gives

$$\mathcal{T}_{1,p}^{(\beta,\delta)}(u) = \frac{u}{(u+1)^{p+1}} \left[1 - \frac{2}{\beta} \left(\frac{\delta}{2} + 1 \right) \right]^p. \tag{37}$$

An immediate consequence of the definition (34) is the following functional relationship with (33a),

$$\mathcal{T}_p^{(\beta,\delta)}(u, s) = \frac{s}{s-1} \left(\Delta \mathcal{T}_p^{(\beta,\delta)}(u, s) - \mathcal{T}_{1,p}^{(\beta,\delta)}(u) \right). \tag{38}$$

III. UNITARY SYMMETRY

The techniques and tools that we use are broadly similar for all the symmetry classes. However, as is usual in RMT, the power β in the Vandermonde determinant of the $j.p.d.f.$ ’s (6) and (11) affects the complexity of the calculations. Although when $\delta \neq 0$ in (5) the matrix ensembles that model Andreev billiards do not belong to Dyson’s symmetry classes, for the sake of simplicity we shall use the usual terminology when referring to an ensemble characterized by a given β , namely, unitary, orthogonal, and symplectic if $\beta = 2$, $\beta = 1$, and $\beta = 4$, respectively.

A. Leading order

In Sec. IV A, Remark 4.1, we shall prove that the leading order contributions are the same for all symmetry classes. Therefore, we shall compute the limits

$$\lim_{n \rightarrow \infty} n^{-1} M_{J_{a,b}}^{(\beta)}(k, n) \quad \text{and} \quad \lim_{n \rightarrow \infty} n^{k-1} M_{L_b}^{(\beta)}(-k, n)$$

only when $\beta = 2$, since in this case the finite- n formulae simplify considerably.

Proposition 3.1: The leading order terms of the series (19a) and (19b) are independent of β and δ and are given by

$$\mathcal{D}_{k,0}^{(\beta)}(w) = \begin{cases} \frac{1}{w-1} & \text{if } k = 1 \\ \frac{(w-1)^{1-2k}}{k-1} \sum_{j=1}^{k-1} \binom{k-1}{j} \binom{k-1}{j-1} w^j & \text{if } k > 1, \end{cases} \tag{39a}$$

$$\mathcal{T}_{k,0}^{(\beta,\delta)}(u) = \frac{u}{u+1} - \sum_{j=1}^{k-1} \frac{1}{j} \sum_{i=1}^j \binom{j}{i} \binom{j}{i-1} \frac{u^{2i}}{(u+1)^{2j+1}}. \tag{39b}$$

Proof: We first prove (39a). We have

$$\frac{(b+n)_{(-k-j)}}{(1+n)_{(-1-j)}} = \frac{(wn)_{(-k-j)}}{(1+n)_{(-1-j)}} = n^{1-k} w^{-k-j} + O(n^{-k}), \quad n \rightarrow \infty. \tag{40}$$

Inserting the right-hand side of this equation into (25) gives

$$\mathcal{D}_{k,0}^{(\beta)}(w) = \frac{1}{k} \sum_{j=0}^{\infty} \binom{k+j-1}{k-1} \binom{k+j}{k-1} w^{-k-j}. \tag{41}$$

The infinite series on the right-hand side of (41) is convergent for $|w| > 1$. Indeed, since the ratio of successive terms is a rational function of the summation index j , it is a special case of a hypergeometric series. We find

$$\frac{1}{k} \sum_{j=0}^{\infty} \binom{k+j-1}{k-1} \binom{k+j}{k-1} w^{-k-j} = w^{-k} {}_2F_1(k, k+1; 2; w^{-1}), \tag{42}$$

where the hypergeometric function ${}_2F_1(a, b; c; z)$ is defined in Appendix B, Eq. (B4). If $k = 1$, this sum is a geometric series which converges to $1/(w - 1)$; if $w = 2$, it is the mean of the Wigner time delay. When $k > 1$ applying Euler’s transformation formula (see, e.g., Ref. 1, Eq. (15.3.3))

$${}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z)$$

converts the hypergeometric series in (42) into the finite sum

$$\begin{aligned} \mathcal{D}_{k,0}^{(\beta)}(w) &= \frac{w^{k-1}}{(w-1)^{2k-1}} {}_2F_1(2-k, 1-k; 2; w^{-1}) \\ &= \frac{1}{(w-1)^{2k-1}} \sum_{j=0}^{k-1} \frac{w^j}{(k-1)_{(j-k+1)}(k)_{(j-k+1)}(2)_{(k-j-1)}(k-j-1)!}, \end{aligned}$$

where we used that $(-x)_{(j)} = 1/(x+1)_{(-j)}$. Finally, the identity

$$\frac{1}{(k-1)_{(j-k+1)}(k)_{(j-k+1)}(2)_{(k-j-1)}(k-j-1)!} = \frac{1}{k-1} \binom{k-1}{j} \binom{k-1}{j-1} \tag{43}$$

yields the limit (39a).

In order to prove (39b), we first set $a = 0$ and $b = (u - 1)n$ in (22):

$$\begin{aligned} U_{n,k,j}^{a,b} &= \frac{((u+1)n - 2j + k + 1)(un)_{(k-j+1)}(n-j+1)_{(j)}(un)_{(k-j+1)}}{((u+1)n - j)_{(k+2)}((u+1)n - j + 1)_{(k)}(n+1)_{(-j)}} \\ &\sim n \frac{u^{2k-2j+2}}{(u+1)^{2k+1}} + O(1), \quad n \rightarrow \infty. \end{aligned} \tag{44}$$

Inserting this expression into (21) and relabelling the summation index gives

$$\lim_{n \rightarrow \infty} n^{-1} \Delta M_{j_{a,b}}^{(2)}(k, n) = \frac{1}{k} \sum_{j=1}^k \binom{k}{j} \binom{k}{j-1} \frac{u^{2j}}{(u+1)^{2k+1}}. \tag{45}$$

Finally, Eq. (39b) follows by taking the limit as $n \rightarrow \infty$ of the identity

$$n^{-1} M_{j_{a,b}}^{(2)}(k, n) = n^{-1} M_{j_{a,b}}^{(2)}(1, n) - n^{-1} \sum_{j=1}^{k-1} \Delta M_{j_{a,b}}^{(2)}(j, n), \tag{46}$$

where $M_{j_{a,b}}^{(2)}(1, n)$ is Aomoto’s integral (24). □

An immediate consequence of this proposition is the following.

Corollary 3.2: At leading order, the generating functions (33a) and (33b) are independent of β and δ and are given by

$$\mathcal{D}_0^{(\beta)}(w, s) = \frac{w - 1 - s - \sqrt{(w-1)^2 - 2s(w+1) + s^2}}{2}, \tag{47a}$$

$$\mathcal{T}_0^{(\beta,\delta)}(u, s) = \frac{1}{2} \left(\sqrt{1 + \frac{4us}{(u+1)^2(1-s)}} - 1 \right) (u+1). \tag{47b}$$

Proof: By definition and using formula (39a), we have

$$\begin{aligned} \mathcal{D}_0^{(\beta)}(w, s) &= \frac{s}{w-1} + \sum_{k=2}^{\infty} \frac{N_{k-1}(w)}{(w-1)^{2k-1}} s^k \\ &= \frac{s}{w-1} + \frac{s}{w-1} \rho((w, s/(w-1)^2), \end{aligned} \tag{48}$$

where $N_k(u)$ are the Narayana polynomials (31) and $\rho(u, s)$ is their generating function. Inserting formula (32) for $\rho(u, s)$ gives (47a).

We first compute the generating function $\Delta \mathcal{T}_0^{(\beta, \delta)}(u, s)$. From the explicit expression of the moments (39b), we obtain

$$\Delta \mathcal{T}_0^{(\beta, \delta)}(u, s) = \frac{1}{u+1} \rho(u^2, s/(u+1)^2), \tag{49}$$

where, as in the proof of Eq. (47a), $\rho(u, s)$ is the generating function (32). Elementary manipulations give

$$\Delta \mathcal{T}_0^{(\beta, \delta)}(u, s) = \frac{1}{2s} \left(\sqrt{1 + \frac{4us}{(u+1)^2(1-s)}} - 1 \right) (u+1)(s-1) + \frac{u}{u+1}. \tag{50}$$

Finally, Eq. (47b) follows from (38) and (37). □

Remark 3.3: Using Selberg’s integral Novaes⁵⁸ derived the formula

$$\mathcal{T}_{k,0}^{(\beta,0)}(u) = \sum_{j=0}^{k-1} \binom{k-1}{j} \binom{2j}{j} \frac{(-1)^j}{j+1} \frac{u^{j+1}}{(u+1)^{2j+1}}. \tag{51}$$

Taking the differences of this sum and comparing to (45) leads to the combinatorial identity

$$\frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} u^{2j} = \sum_{j=0}^k \binom{k-1}{j-1} \binom{2j}{j} \frac{(-1)^{j+1}}{j+1} u^{j+1} (u+1)^{2k-2j}. \tag{52}$$

This formula has already received attention in enumerative combinatorics, where it was proved using generating functions by Coker.²⁶ Subsequently, Chen *et al.*,²⁵ who referred to it as Coker’s *second* problem, gave a bijective proof. Combining Novaes’ expression and (39b) gives an independent proof of (52).

The identity (52) is intrinsically connected to averages in the Jacobi ensemble. Therefore, it seems natural to ask whether similar identities should appear when integrating over other ensembles, in particular the Laguerre ensembles. It is a straightforward consequence of our exact results⁵³ (see also Refs. 32 and 37) that

$$\lim_{n \rightarrow \infty} n^{-k-1} M_{L_b}^{(\beta)}(k, n) = \frac{1}{k} \sum_{j=1}^k \binom{k}{j} \binom{k}{j-1} w^j. \tag{53}$$

In the same paper, Novaes also computed this limit using Selberg’s integral and showed that

$$\lim_{n \rightarrow \infty} n^{-k-1} M_{L_b}^{(\beta)}(k, n) = \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{1}{j+1} \binom{2j}{j} \binom{k-1}{2j} w^{j+1} (1+w)^{k-2j-1}. \tag{54}$$

The equivalence between the sums (53) and (54) appeared in Ref. 25 too, where it was referred to as Coker’s *first* problem.

It is very interesting and surprising that these combinatorial identities can be proved solely using techniques from RMT.

Remark 3.4: When $w = 2$, the generating function (47a) reduces to

$$\mathcal{D}_0^{(\beta)}(2, s) = \frac{1 - s - \sqrt{1 - 6s + s^2}}{2}, \quad (55)$$

which was computed in Ref. 10 using both periodic orbit theory and RMT.

Remark 3.5: The leading order generating function (47b) agrees with the semiclassical formula computed by Berkolaiko *et al.*⁹ Their notation is slightly different from ours. Their main asymptotic parameter is the total number of quantum channels $N = n + m = (u + 1)n$; they also introduce the variable $\xi = u/(u + 1)^2$. From our point of view, it is more convenient to use the dimension of the ensemble as asymptotic variable.

Remark 3.6: Instead of deriving exact formulae for the moments and extracting their asymptotics, one could instead study the asymptotic expansion for the *mean eigenvalue density*,

$$\rho_n(x) := \left\langle \sum_{j=1}^n \delta(x - x_j) \right\rangle. \quad (56)$$

The moments are by definition $\int x^k \rho_n(x) dx$. For example, if the average is taken with respect to the *j.p.d.f* of the Laguerre ensemble, then the limit

$$\lim_{n \rightarrow \infty} n^{-1} \rho_n(x) = \frac{\sqrt{(x - v_-)(v_+ - x)}}{2\pi x} \quad (57)$$

is the Marčenko-Pastur law.⁴⁹ The support of (57) is $v_{\pm} = (\sqrt{w} \pm 1)^2$, where w is defined as in (15). This fact was exploited to obtain the leading contribution to the density of the delay times.²² The positive moments of this density are known too,^{37,60} from which the corresponding negative moments may be obtained from a substitution $x \rightarrow x^{-1}$ in (57). They are easily seen to be in agreement with formula (39a).

B. Beyond leading order

We now compute the next two terms in the asymptotic expansion of the moments of the transmission eigenvalues and proper delay times when $\beta = 2$.

The starting point to compute higher order corrections to formulae (39a) and (39b) is to expand the n -dependent factor

$$\frac{(b+n)_{(-k-j)}}{(1+n)_{(-1-j)}} \quad (58)$$

for the moments of the proper delay times and the coefficients (22) for those of the transmission eigenvalues. For this purpose we need the first three terms in the expansion (29) (see, e.g., Ref. 1, Eq. (6.1.47)),

$$\begin{aligned} \frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} &\sim z^{\alpha - \beta} \left(1 + \frac{1}{2z} (\alpha - \beta)(\alpha + \beta - 1) \right. \\ &\quad \left. + \frac{1}{12z^2} \left(\frac{\alpha - \beta}{2} \right) (3(\alpha + \beta - 1)^2 - (\alpha - \beta + 1)) + O(z^{-3}) \right), \quad z \rightarrow \infty. \end{aligned} \quad (59)$$

The coefficients of $1/z$ and $1/z^2$ will give the terms of order $O(n^{-1})$ and $O(n^{-2})$, respectively, in the asymptotic expansions (19a) and (19b). Such terms will be polynomials in u (or w), whose coefficients will be the Narayana numbers (26) or will closely resemble them.

In the Wigner-Dyson symmetry classes, there is no contribution at the next to leading order when $\beta = 2$. However, in Andreev quantum dots ($\delta \neq 0$), there is always a non-zero correction.

Proposition 3.7: Let $\beta = 2$. The next to leading order terms in the asymptotic expansions (19a) and (19b) are

$$\mathcal{D}_{k,1}^{(2)}(w) = 0, \quad (60a)$$

$$\Delta \mathcal{T}_{k,1}^{(2,\delta)}(u) = \frac{\delta}{2(u-1)} \left(\frac{u-1}{u+1} \right)^{2k+2} \mathcal{T}_{k+1,0}^{(\beta,\delta)}(-u), \quad (60b)$$

where $\mathcal{T}_{k,0}^{(\beta,\delta)}(u)$ is independent of β and δ and is given explicitly in (39b).

Proof: For the negative Laguerre moments, we set $b = (w-1)n$ and insert the term of order $O(n^{-k})$ in Eq. (40) into (25) leading to

$$\mathcal{D}_{k,1}^{(2)}(w) = \frac{1}{2k} \sum_{j=0}^{\infty} \binom{k+j-1}{k-1} \binom{k+j}{k-1} w^{-k-j-1} ((k+j)(k+j+1) - wj(1+j)) \quad (61)$$

$$= \frac{k}{2} \sum_{j=0}^{\infty} \binom{k+j}{k} \binom{k+j+1}{k} w^{-k-j-1} \quad (61')$$

$$- \frac{k}{2} \sum_{j=0}^{\infty} \binom{k+j-1}{k} \binom{k+j}{k} w^{-k-j} = 0.$$

Equation (60a) follows from shifting the summation index $j \rightarrow j-1$ in the sum (61').

We now prove (60b). We first use (59) to expand (21) in an asymptotic series and obtain

$$\Delta \mathcal{T}_{k,1}^{(2,\delta)}(u) = \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} \frac{u^{2k-2j+1}}{(u+1)^{2k+2}} F_{k,1}^{(2,\delta)}(u, j), \quad (62)$$

where

$$F_{k,1}^{(2,\delta)}(u, j) = j(\delta/2 + 1 - j)u^2 - uk\delta/2 + (k-j+1)(k-j+\delta/2).$$

Collecting the coefficients of δ in (62) gives

$$\begin{aligned} \Delta \mathcal{T}_{k,1}^{(2,\delta)}(u) &= \frac{\delta}{2k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} \frac{u^{2k-2j+1}}{(u+1)^{2k+2}} ((k-j+1) - uk + u^2j) \\ &+ \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} \frac{u^{2k-2j+1}}{(u+1)^{2k+2}} ((k-j+1)(k-j) + u^2j(1-j)). \end{aligned} \quad (63)$$

Elementary manipulations show that the second sum in the right-hand side of (63) vanishes, while the first simplifies to

$$\begin{aligned} \Delta \mathcal{T}_{k,1}^{(2,\delta)}(u) &= \frac{\delta}{2(u+1)^{2k+2}} \left(\sum_{j=0}^k \binom{k}{j}^2 u^{2j+1} - \sum_{j=0}^k \binom{k}{j} \binom{k}{j+1} u^{2j+2} \right) \\ &= -\frac{\delta}{2(u+1)^{2k+2}} \sum_{j=0}^{2k} \binom{k}{\lfloor j/2 \rfloor} \binom{k}{\lceil j/2 \rceil} (-u)^{j+1}. \end{aligned} \quad (64)$$

Finally, the statement of the proposition follows from the combinatorial identity (see Ref. 24, Proposition V7)

$$\frac{u}{(u+1)^{2k+1}} \sum_{j=0}^{2k} \binom{k}{\lfloor j/2 \rfloor} \binom{k}{\lceil j/2 \rceil} u^j = \sum_{j=0}^k \binom{k}{j} \binom{2j}{j} \frac{(-1)^j}{j+1} \frac{u^{j+1}}{(u+1)^{2j+1}}, \quad (65)$$

which allows the polynomial in (64) to be expressed in terms of the leading order term $\mathcal{T}_{k,0}^{(\beta,\delta)}(u)$ given by (51). \square

Remark 3.8: At first it is quite surprising that the next to leading order term (60b) is different from zero. The fact that the correction at order $O(1/n)$ of the density of the eigenvalues of $\beta = 2$ ensembles is usually zero is a general phenomenon due to the symmetry of the ensembles. It was proved by Johansson⁴⁰ for ensembles of random matrices with an even degree polynomial potential. It also appeared in the work of Dumitriu and Edelman³³ for the β -Hermite and β -Laguerre ensembles. Dumitriu and Edelman³³ showed that for these ensembles, the term of order $O(1/n^p)$ is multiplied by a *palindromic* polynomial in $-2/\beta$ that is independent of n . At the next to leading order this polynomial is zero at $\beta = 2$. For the Jacobi ensembles this property is still unproved, but given that it is a direct consequence of the symmetries of the Jack polynomials, one would expect the same behaviour.

Why, then, is formula (60b) not zero? The answer is that for physical reasons the exponent of the factors $(1 - x_j)$ in the *j.p.d.f.* of the eigenvalues of the Jacobi ensembles *does not scale* with n (see Eq. (7)). If we adopt the scaling (17), then the next to leading order term is zero for $\beta = 2$. Indeed, this is the content of Proposition 5.2 for the Selberg-like integrals.

Remark 3.9: For higher order terms, it is no longer possible to simplify the expressions which would appear in place of Eqs. (61) and (62) through elementary manipulation. Instead, one requires a more systematic procedure to handle more complex polynomials, which become increasingly involved as we go further in the asymptotic expansion. However, as we shall see, often the coefficients in such polynomials are products of two binomial coefficients similar to those appearing in the first line of (64). Thus, in many instances they can be conveniently expressed in terms of Jacobi polynomials (see Definition (B1)), whose properties can be used to obtain manageable formulae. A first example of this procedure is given in the proof of Proposition 3.10.

The next to leading order generating functions will be discussed in Sec. IV, where we shall derive a formula for general δ with $\beta \in \{1, 2, 4\}$.

Proposition 3.10: For $\beta = 2$ the third coefficient in the expansion (19b) of the moments of the proper delay times is

$$\mathcal{D}_{k,2}^{(2)}(w) = \frac{(k + 1)(k + 2)w}{12(w - 1)^{k+3}} P_{k-2}^{(2,2)}(\tilde{w}), \tag{66}$$

where $P_j^{(\alpha,\beta)}(x)$ refers to the Jacobi polynomial of degree j and parameters α and β defined in Eq. (B1) and

$$\tilde{w} := \frac{w + 1}{w - 1}. \tag{67}$$

Proof: As in previous proofs, we begin by inserting (59) into (58) and extract the component of order $O(n^{-k-1})$. Then, by substituting it into (25), we arrive at

$$\mathcal{D}_{k,2}^{(2)}(w) = \frac{1}{k} \sum_{j=0}^{\infty} \binom{k+j}{k-1} \binom{k+j-1}{k-1} w^{-k-j-2} G_{k,2}^{(2)}(w, j), \tag{68}$$

where

$$G_{k,2}^{(2)}(w, j) = \frac{w^2}{24} j(j-1)(j+1)(3j+2) - \frac{w}{4} (k+j)(k+j+1)j(1+j) + \frac{1}{24} (k+j)(k+j+1)(k+j+2)(3k+3j+1).$$

The demonstration now proceeds with lengthy and systematic, although elementary, algebraic manipulations. It is useful to give an example that clarifies the pattern of the calculations. Consider

the contribution to (68) coming from the linear term in $G_{k,2}^{(2)}(w, j)$,

$$\begin{aligned}
 & -\frac{1}{4k} \sum_{j=0}^{\infty} \binom{k+j}{k-1} \binom{k+j-1}{k-1} w^{-k-j-1} (k+j)(k+j+1)j(1+j) \\
 & = -\frac{k(1+k)^2}{4} \sum_{j=0}^{\infty} \binom{k+j}{k+1} \binom{k+j+1}{k+1} w^{-k-j-1}. \quad (69)
 \end{aligned}$$

Using the definition of hypergeometric function (B4) and the relation (B5b) between hypergeometric functions and Jacobi polynomials, the right-hand side of (69) becomes

$$\begin{aligned}
 & -\frac{k(k+1)^2(k+2)}{4} w^{-k-2} ({}_2F_1(k+2, k+3, 2, w^{-1})) \\
 & = -\frac{k(k+1)(k+2)w P_k^{(1,1)}(\tilde{w})}{4(w-1)^{k+3}}. \quad (70)
 \end{aligned}$$

Repeating the same procedure for the other powers of w in $G_{k,2}^{(2)}(w, j)$ eventually gives

$$\begin{aligned}
 \frac{24(w-1)^{k+3}}{w(k+1)(k+2)} \mathcal{D}_{k,2}^{(2)}(w) & = 3k(P_k^{(0,2)}(\tilde{w}) + P_k^{(2,0)}(\tilde{w})) - 6kP_k^{(1,1)}(\tilde{w}) \\
 & + (3k-2)(P_{k-1}^{(1,2)}(\tilde{w}) - P_{k-1}^{(2,1)}(\tilde{w})). \quad (71)
 \end{aligned}$$

Let us subtract the right-hand side of Eq. (66) from (71); then, the statement of the proposition is equivalent to the identity

$$\begin{aligned}
 & 3k(P_k^{(0,2)}(\tilde{w}) + P_k^{(2,0)}(\tilde{w})) - 6kP_k^{(1,1)}(\tilde{w}) \\
 & + (3k-2)(P_{k-1}^{(1,2)}(\tilde{w}) - P_{k-1}^{(2,1)}(\tilde{w})) - 2P_{k-2}^{(2,2)}(\tilde{w}) = 0. \quad (72)
 \end{aligned}$$

The proof follows from the orthogonality of the Jacobi polynomials and the following general procedure:

1. Write each Jacobi polynomial in (72) in terms of the polynomials $P_j^{(2,2)}(\tilde{w})$ using the connection formulae (B3).
2. Apply the three-term recurrence relation (B2) to each Jacobi polynomial until only the polynomials $P_k^{(2,2)}(\tilde{w})$ and $P_{k-1}^{(2,2)}(\tilde{w})$ appear in the formula.
3. The resulting expression is of the form

$$A_k(w)P_k^{(2,2)}(\tilde{w}) + B_k(w)P_{k-1}^{(2,2)}(\tilde{w}),$$

for some rational functions $A_k(w)$ and $B_k(w)$.

4. A direct computation shows

$$A_k(w) = B_k(w) = 0.$$

□

Remark 3.11: Note that step (1) of the above proof amounts to writing each Jacobi polynomial in a consistent linearly independent basis. By formula (B3), the number of non-zero coefficients in this basis is typically quite small and independent of k . This property is essential for our procedure

to work. Indeed, consider the identities

$$\begin{aligned}
 P_k^{(0,2)}(\tilde{w}) + P_k^{(2,0)}(\tilde{w}) &= \frac{(k+3)(k+4)P_k^{(2,2)}(\tilde{w})}{(2k+3)(k+2)} + \frac{(k+2)P_{k-2}^{(2,2)}(\tilde{w})}{2k+3} \\
 P_{k-1}^{(1,2)}(\tilde{w}) - P_{k-1}^{(2,1)}(\tilde{w}) &= -P_{k-2}^{(2,2)}(\tilde{w}) \\
 P_k^{(1,1)}(\tilde{w}) &= \frac{1}{2} \left(\frac{(k+3)(k+4)P_k^{(2,2)}(\tilde{w})}{(2k+3)(k+2)} - \frac{(k+1)P_{k-2}^{(2,2)}(\tilde{w})}{2k+3} \right),
 \end{aligned}$$

which are a direct consequence of (B3). When substituted into (72), they immediately yield the desired cancellations.

In some of the proofs of this paper (more specifically in Lemma 3.13 and Propositions 3.14, 4.8, 4.9, and 5.2), we will use this method to justify the equivalence of two apparently different combinations of Jacobi polynomials. This task is most conveniently performed using a computer algebra package such as MAPLE or MATHEMATICA, whereby the rather heavy algebra involved in steps (1)–(4) may be executed algorithmically.

In order to keep the notation as simple as possible, we now introduce a convention that we shall often use when proving statements about generating functions. If $p(x)$ is a polynomial or a function analytic in a neighbourhood of $x = 0$, then we shall write $[x^k]p(x)$ to indicate the Taylor coefficient of x^k in $p(x)$. This formalism can be trivially extended to a function of several variables. Now, when in a proof we write $[s^k]\mathcal{T}_p^{(\beta,\delta)}(u, s)$ or $[s^k]\mathcal{D}_p^{(\beta)}(w, s)$, we do not refer to $\mathcal{T}_{k,p}^{(\beta,\delta)}(u)$ or $\mathcal{D}_{k,p}^{(\beta)}(w)$, but to the Taylor coefficients of a function, usually an algebraic function, whose explicit expression is the statement we intend to prove, like, for example, the right-hand sides of Eqs. (47a) and (47b) in Corollary 3.2. In other words, adopting this slight abuse of notation, Corollary 3.2 is equivalent to

$$[s^k]\mathcal{D}_0^{(\beta)}(w, s) = \mathcal{D}_{k,0}^{(\beta)}(w) \quad \text{and} \quad [s^k]\mathcal{T}_0^{(\beta,\delta)}(u, s) = \mathcal{T}_{k,0}^{(\beta,\delta)}(u).$$

We now have

Corollary 3.12: The generating function of the moments (66) is given by

$$\mathcal{D}_2^{(2)}(w, s) = \frac{ws^2}{(s^2 - 2(w+1)s + (w-1)^2)^{5/2}}. \tag{73}$$

Proof: The idea of proof is straightforward. We first show that the Taylor coefficients of (73) satisfy a particular three term recurrence relation. Then, we demonstrate that the moments (66) satisfy exactly the same recurrence relation. Finally, we are left to verify the initial conditions

$$\begin{aligned}
 [s]\mathcal{D}_2^{(2)}(w, s) &= 0, \\
 [s^2]\mathcal{D}_2^{(2)}(w, s) &= \frac{w}{(w-1)^5},
 \end{aligned}$$

which can be easily checked.

Differentiating (73) once with respect to s leads to the differential equation

$$\begin{aligned}
 s(s^2 - 2(w+1)s + (w-1)^2) \frac{d\mathcal{D}_2^{(2)}(w, s)}{ds} \\
 + (5s(s-w-1) + 4(w+1)s - 2(w-1)^2 - 2s^2)\mathcal{D}_2^{(2)}(w, s) = 0.
 \end{aligned}$$

Inserting the power series expansion

$$\mathcal{D}_2^{(2)}(w, s) = \sum_{k=1}^{\infty} p_k(w)s^k$$

and equating coefficients of s^k gives

$$(w - 1)^2(k - 2)p_k(w) - (w + 1)(2k - 1)p_{k-1}(w) + (k + 1)p_{k-2}(w) = 0. \tag{74}$$

We now have to prove that the moments (66) satisfy the recurrence relation (74). Inserting (66) into the left-hand side of (74) and simplifying leads to

$$(k + 2)(k - 2)P_{k-2}^{(2,2)}(\tilde{w}) - (2k - 1)k\tilde{w}P_{k-3}^{(2,2)}(\tilde{w}) + (k - 1)kP_{k-4}^{(2,2)}(\tilde{w}),$$

which is just the three term recurrence relation (B2) for Jacobi polynomials with $\alpha = \beta = 2$ and $n = k - 3$. □

As we go beyond the next to leading order in the expansion (19a), the corrections to the moments of the density of the transmission eigenvalues become increasingly involved. Therefore, although we need the explicit expression of such coefficients in the proof, it is more convenient to state our results only in terms of generating functions.

We first need the following lemma.

Lemma 3.13: The Taylor coefficients of

$$f(s; u) = \frac{u^2 s}{(1 - s)^{1/2}((u + 1)^2 - s(u - 1)^2)^{5/2}} \tag{75}$$

as a function of s are

$$[s^k]f(s; u) = u^2 \frac{k(k + 1)(u - 1)^{k-2}}{(u + 1)^{k+5}} \left((1 - u^2)((6k)^{-1}P_{k-1}^{(1,1)}(\tilde{u}) - \frac{2}{3}P_{k-1}^{(0,0)}(\tilde{u}) - \frac{2u}{3}P_{k-2}^{(1,1)}(\tilde{u})) \right), \tag{76}$$

where

$$\tilde{u} := \frac{u^2 + 1}{u^2 - 1}. \tag{77}$$

Proof: The idea of the proof is the same as that one of Corollary 3.12 and consists of two parts. First we obtain a recurrence relation for the Taylor coefficients of the generating function (75); then we argue that the right-hand side of (76) satisfies the same recurrence equation.

One easily sees from (75) that

$$[s^1]f(s; u) = \frac{u^2}{(u + 1)^5},$$

$$[s^2]f(s; u) = \frac{u^2(3u^2 - 4u + 3)}{(u + 1)^7}. \tag{78}$$

Differentiating (75) once with respect to s leads to the differential equation

$$2s(1 - s)(A - Bs)\frac{df(s; u)}{ds} + (4Bs^2 - 3Bs + As - 2A)f(s; u) = 0, \tag{79}$$

where

$$A = (u + 1)^2 \quad \text{and} \quad B = (u - 1)^2.$$

Substituting the expansion

$$f(s; u) = \sum_{k=1}^{\infty} p_k(u)s^k$$

into (79) and equating the coefficients of s^k gives

$$2A(k-1)p_k(u) + (3A - B - 2(A+B)k)p_{k-1}(u) + 2Bkp_{k-2}(u) = 0, \tag{80}$$

with initial conditions given by (78).

Inserting the right-hand side of (76) into the recurrence relation (80) shows that the statement of the lemma is equivalent to an identity among Jacobi polynomials, which can be proved using the strategy outlined at the end of Proposition 3.10 and in Remark 3.11. For this particular case, it is more convenient to express each Jacobi polynomial in the basis $P_j^{(1,1)}(\tilde{u})$. \square

It turns out that $f(s; u)$ is the generating function $\Delta\mathcal{T}_2^{(2,0)}(u, s)$. This result was first obtained using periodic orbit theory by Berkolaiko and Kuipers¹¹—more precisely, they computed $\mathcal{T}_2^{(2,0)}(u, s)$, which is related to $\Delta\mathcal{T}_2^{(2,0)}(u, s)$ by (38). The RMT proof of this fact is a particular case of the next proposition, where we obtain the same generating function for Andreev billiards.

Proposition 3.14: Let $\beta = 2$ and $\delta > -2$. The generating function for the coefficients $\mathcal{T}_{k,2}^{(2,\delta)}(u)$ is

$$\mathcal{T}_2^{(2,\delta)}(u, s) = \frac{\delta^2 us((u+1)^2 - s(u-1)^2) - 4s^2 u^2}{4(1-s)^{3/2}((u+1)^2 - s(u-1)^2)^{5/2}}. \tag{81}$$

Proof: Our techniques to compute the moments of the transmission eigenvalues are best suited to compute the differences of the moments. Therefore, we shall study

$$\Delta\mathcal{T}_2^{(2,\delta)}(u, s) = \frac{4su^2 + \delta^2 u(s(u-1)^2 - (u+1)^2)}{4\sqrt{1-s}((u+1)^2 - s(u-1)^2)^{5/2}} + \frac{u\delta^2}{4(u+1)^3}. \tag{82}$$

Expanding the coefficient (22) to order $O(n^{-1})$ and inserting it into (21) gives

$$\Delta\mathcal{T}_{k,2}^{(2,\delta)}(u) = \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} \frac{u^{2k-2j}}{(u+1)^{2k+3}} F_{k,2}^{(2,\delta)}(u, j), \tag{83}$$

where $F_{k,2}^{(2,\delta)}(u, j)$ is a polynomial quadratic in δ and of 4th order in u ; the definition of $F_{k,2}^{(2,\delta)}(u, j)$ is reported in Appendix D, Eq. (D2). Thus, the right-hand side of (83) is a quadratic form in δ too.

In order to prove this proposition, we need to show that Eq. (83) coincides with the k th Taylor coefficient of the right-hand side of (82) in a neighborhood of $s = 0$ and for any $u > 1$. The proof is systematic and we shall outline the main steps.

Take, for example,

$$[\delta^0][u^1]F_{k,2}^{(2,\delta)}(u, j) = -(k-j)(k-j+1)(2k-2j+1)/3$$

and consider its contribution to the right-hand side of Eq. (83),

$$-\frac{2k(k-1)}{3} \sum_{j=0}^k \binom{k-1}{j} \binom{k-2}{j-1} \frac{u^{2k-2j}}{(u+1)^{2k+3}} - \frac{k}{3} \sum_{j=0}^k \binom{k-1}{j} \binom{k-1}{j-1} \frac{u^{2k-2j}}{(u+1)^{2k+3}}. \tag{84}$$

The ratio of consecutive terms in these two sums are rational functions of the summation indices. Thus, they are special cases of hypergeometric functions, which, in turn, can be expressed in terms of Jacobi polynomials using the correspondence (B5a). Therefore, Eq. (84) becomes

$$-\frac{u^2(u^2-1)^{k-2}}{(u+1)^{2k+3}} \left(2k(k-1)P_{k-2}^{(1,0)}(\tilde{u})/3 + kP_{k-2}^{(1,1)}(\tilde{u})/3 \right).$$

Repeating this procedure for each monomial in u and δ in $F_{k,2}^{(2,\delta)}(u, j)$, we see that (83) becomes a sum of different Jacobi polynomials, leading to an explicit (though complicated) formula for each coefficient of δ . For convenience we report these formulae in Appendix D, Eqs. (D3), (D4), and (D5).

Up to now we have proved that $\Delta\mathcal{T}_{k,2}^{(2,\delta)}(u)$ can be written as a particular combination of Jacobi polynomials. The rest of the proof proceeds as follows: (a) we shall express $[s^k]\Delta\mathcal{T}_2^{(2,\delta)}(u, s)$ as a linear combination of Jacobi polynomials too; (b) by using the same procedure as in the proof of Proposition 3.10—points (1)–(4)—it follows that

$$[s^k]\Delta\mathcal{T}_2^{(2,\delta)}(u, s) - \Delta\mathcal{T}_{k,2}^{(2,\delta)}(u) = 0.$$

We shall only outline how to achieve (a), as we have already discussed how to prove (b).

First note that $\Delta\mathcal{T}_2^{(2,\delta)}(u, s)$ can be written in terms of $\Delta\mathcal{T}_2^{(2,0)}(u, s)$,

$$\Delta\mathcal{T}_2^{(2,\delta)}(u, s) = \Delta\mathcal{T}_2^{(2,0)}(u, s) \left(1 + \frac{\delta^2(u-1)^2}{4u} - \frac{\delta^2(u+1)^2}{4us} \right) + \frac{u\delta^2}{4(u+1)^3}.$$

Thus, the Taylor coefficients of (82) are

$$\begin{aligned} [s^k]\Delta\mathcal{T}_2^{(2,\delta)}(u, s) &= [s^k]\Delta\mathcal{T}_2^{(2,0)}(u, s) + \delta^2 \left(\frac{(u-1)^2}{4u} [s^k]\Delta\mathcal{T}_2^{(2,0)}(u, s) \right. \\ &\quad \left. - \frac{(u+1)^2}{4u} [s^{k+1}]\Delta\mathcal{T}_2^{(2,0)}(u, s) \right). \end{aligned} \quad (85)$$

Now, from Lemma 3.13 and Eq. (82), we have $f(u; s) = \Delta\mathcal{T}_2^{(2,0)}(u, s)$; therefore,

$$\frac{[s^k]6\Delta\mathcal{T}_2^{(2,0)}(u, s)(u+1)^{k+5}}{u^2k(k+1)(u-1)^{k-2}} = (1-u^2) \left(k^{-1}P_{k-1}^{(1,1)}(\tilde{u}) - 4P_{k-1}^{(0,0)}(\tilde{u}) \right) - 4uP_{k-2}^{(1,1)}(\tilde{u}).$$

Inserting this expression into the right-hand side of (85) gives $[s^k]\Delta\mathcal{T}_2^{(2,\delta)}(u, s)$ in terms of Jacobi polynomials. \square

IV. ORTHOGONAL AND SYMPLECTIC SYMMETRIES

Our aim in this section is to compute the first two corrections to the leading order terms (39b) and (39a) when $\beta = 1$ and $\beta = 4$. The finite n formulae for the averages (9) and (12) that we obtained⁵³ for ensembles with both orthogonal and symplectic symmetries have a similar structure. Thus, the techniques and calculations used to study their asymptotic behaviour are almost identical for both symmetry classes. Although we will state the results for $\beta = 1$ and $\beta = 4$, most of the proofs will focus on $\beta = 1$ ensembles, as they contain the essential details of the arguments.

A. From finite- n to asymptotics—Preliminaries

The finite- n moments of the transmission eigenvalues and proper delay times are

$$\begin{aligned} M_{J_{a,b}}^{(1)}(k, n) &= M_{J_{a,b}}^{(2)}(k, n-1) \\ &\quad - 2 \sum_{j=1}^{\lfloor k/2 \rfloor} \sum_{i=0}^{k-2j} \binom{k}{i} \binom{k}{i+2j} S_{i,j}^{a/2, b/2}(k, (n-1)/2) \\ &\quad + I_{J_{a,b}}(k, n) \end{aligned} \quad (86a)$$

and

$$\begin{aligned}
 M_{L_b}^{(1)}(-k, n) &= M_{L_b}^{(2)}(-k, n - 1) \\
 &- 2^{1-k} \sum_{j=1}^{n/2-1} \sum_{i=0}^{n-2j} \binom{k+j-1}{k-1} \binom{k+i+2j-1}{k-1} S_{i,j}^{b/2}(-k, (n-1)/2) \\
 &+ I_{L_b}(-k, n).
 \end{aligned}
 \tag{86b}$$

We give the explicit expressions of $S_{i,j}^{a/2,b/2}(k, n)$ and $S_{i,j}^{b/2}(-k, n)$ in Eqs. (A3) and (A1a); the terms $I_{J_{a,b}}(k, n)$ and $I_{L_b}(-k, n)$ are defined in (A4) and (A1b), respectively. The quantities $M_{J_{a,b}}^{(2)}(k, n - 1)$ and $M_{L_b}^{(2)}(-k, n - 1)$ are the moments for $\beta = 2$, whose formulae are given in Eqs. (23) and (25).

Remark 4.1: Consider the asymptotic expansion of $S_{i,j}^{a/2,b/2}(k, (n - 1)/2)$ and $I_{J_{a,b}}(k, n)$. Inserting the leading order of (29) gives

$$2S_{i,j}^{a/2,b/2}(k, (n - 1)/2) \sim \frac{u^{2k-2j-2i}}{(1+u)^{2k}} + O(n^{-1}), \quad n \rightarrow \infty,
 \tag{87a}$$

$$I_{J_{a,b}}(k, n) \sim \sum_{j=0}^k \binom{2k}{2j} \frac{u^{2j}}{(1+u)^{2k}} + O(n^{-1}), \quad n \rightarrow \infty.
 \tag{87b}$$

The computation of the right-hand sides requires an analysis similar to that one used to obtain (44) for $\beta = 2$. It follows immediately that $S_{i,j}^{a/2,b/2}(k, n)$ and $I_{J_{a,b}}(k, n)$ do not contribute at leading order. As a consequence, the leading order term in the expansion (19a) is independent of β and δ . The same is true for the moments of the proper delay times (19b). In other words, the contributions made by the orthogonal and symplectic symmetries of the ensembles affect only the higher order terms. This is a general property of the density of the eigenvalues in random matrix ensembles.

Higher order coefficients in the right-hand sides of (19a) and (19b) are computed by determining the contributions of higher order terms in the expansions of $S_{i,j}^{a/2,b/2}(k, n)$, $S_{i,j}^{b/2}(k, n)$, $I_{J_{a,b}}(k, n)$ and $I_{L_b}(-k, n)$. Thus, for example, in order to compute a given contribution in the expansion of (86a), we shall separately study the series

$$\frac{1}{n} M_{J_{a,b}}^{(2)}(k, n - 1) \sim \sum_{p=0}^{\infty} \mathcal{U}_{k,p}^{J_{a,b}}(u) n^{-p},
 \tag{88a}$$

$$\sum_{j=1}^{\lfloor k/2 \rfloor} \sum_{i=0}^{k-2j} \binom{k}{i} \binom{k}{i+2j} 2S_{i,j}^{a/2,b/2}(k, (n-1)/2) \sim \sum_{p=0}^{\infty} \mathcal{S}_{k,p}^{J_{a,b}}(u) n^{-p},
 \tag{88b}$$

and

$$I_{J_{a,b}}(k, n) \sim \sum_{p=0}^{\infty} \mathcal{T}_{k,p}^{J_{a,b}}(u) n^{-p}
 \tag{88c}$$

as $n \rightarrow \infty$. It follows that the coefficients in the expansion (19a) can be represented as

$$\mathcal{T}_{k,p}^{(1,\delta)}(u) = \mathcal{U}_{k,p}^{(1,\delta)}(u) - \mathcal{S}_{k,p}^{(1,\delta)}(u) + \mathcal{I}_{k,p}^{(1,\delta)}(u).
 \tag{89}$$

For convenience, we have expressed the parameters a and b of the Jacobi ensemble in terms of β and δ (see Eq. (7)); thus, in (89) we have replaced the superscript $J_{a,b}$ with (β, δ) . We shall adopt this notation in the rest of the paper. Similarly, for the proper delay times we shall use the notation (β) instead of L_b (see Eq. (15)). The contributions to other quantities can be broken down in the

same way. Therefore, we shall write

$$\Delta \mathcal{T}_{k,p}^{(1,\delta)}(u) = \Delta \mathcal{U}_{k,p}^{(1,\delta)}(u) - \Delta \mathcal{S}_{k,p}^{(1,\delta)}(u) + \Delta \mathcal{I}_{k,p}^{(1,\delta)}(u) \tag{90a}$$

$$\mathcal{D}_{k,p}^{(1)}(w) = \mathcal{U}_{-k,p}^{(1)}(w) - \mathcal{S}_{-k,p}^{(1)}(w) + \mathcal{I}_{-k,p}^{(1)}(w). \tag{90b}$$

Without loss of generality, we shall refer to $\mathcal{U}_{k,p}^{(1,\delta)}(u)$, $\Delta \mathcal{U}_{k,p}^{(1,\delta)}(u)$ and $\mathcal{U}_{-k,p}^{(1)}(w)$ as *unitary contributions*; similarly, we shall call $\mathcal{S}_{k,p}^{(1,\delta)}(u)$, $\Delta \mathcal{S}_{k,p}^{(1,\delta)}(u)$ and $\mathcal{S}_{-k,p}^{(1)}(w)$ *symplectic contributions*.

B. Next to leading order — Negative moments in the Laguerre ensembles

Proposition 4.2: Let $\beta \in \{1, 2, 4\}$, the next to leading order term in the expansion (19a) is

$$\mathcal{D}_{k,1}^{(\beta)}(w) = \frac{1}{2(w-1)^{2k}} \left(\frac{2}{\beta} - 1 \right) \sum_{j=0}^k \left(\binom{2k}{2j} - \binom{k}{j}^2 \right) w^j. \tag{91}$$

Proof: When $\beta = 2$ the right-hand side of (91) is zero, which is consistent with Proposition 3.7; we shall only discuss the proof for $\beta = 1$, as it is identical to that for $\beta = 4$.

Take the next to leading order term as $n \rightarrow \infty$ of formulae (A1a) and (A1b). Then, by proceeding as outlined in Sec. IV A, we arrive at

$$\mathcal{U}_{-k,1}^{(1)}(w) = - \sum_{j=0}^{\infty} \binom{k+j-1}{k-1} \binom{k+j}{k} w^{-k-j}, \tag{92a}$$

$$\mathcal{I}_{-k,1}^{(1)}(w) = \sum_{j=0}^{\infty} \binom{2k+2j-1}{2k-1} w^{-k-j}. \tag{92b}$$

To obtain the next to leading order symplectic contribution, we need to work a little bit more. We obtain

$$\mathcal{S}_{-k,1}^{(1)}(w) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} \binom{k+i-1}{k-1} \binom{k+2j+i-1}{k-1} w^{-k-i-j} \tag{93}$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^j \binom{k+j-i-1}{k-1} \binom{k+j+i+1}{k-1} w^{-k-j-1} \tag{93'}$$

$$= \frac{1}{2} \left(\sum_{j=0}^{\infty} \binom{2k+2j+1}{2k-1} w^{-k-j-1} - \sum_{j=0}^{\infty} \binom{k+j}{k-1}^2 w^{-k-j-1} \right).$$

The inner sum in line (93') was evaluated using Chu-Vandermonde's summation (see Example C.3, Appendix C). Combining Eqs. (92a), (92b), and (93) into (90b) yields

$$\mathcal{D}_{k,1}^{(1)}(w) = \frac{1}{2} \left(\sum_{j=0}^{\infty} \binom{2k+2j-1}{2k-1} w^{-k-j} - (w-1) \sum_{j=0}^{\infty} \binom{k+j}{k}^2 w^{-k-j-1} \right). \tag{94}$$

The identity

$$(w-1)^{-2k} \sum_{j=0}^k \binom{2k}{2j} w^j = \sum_{j=0}^{\infty} \binom{2k+2j-1}{2k-1} w^{-k-j}, \tag{95}$$

which is a simple consequence of the binomial theorem, takes care of the first series in (94); the second sum is a hypergeometric function,

$$\begin{aligned} \sum_{j=0}^{\infty} \binom{k+j}{k}^2 w^{-k-j-1} &= w^{-k-1} {}_2F_1(k+1, k+1, 1, w^{-1}) \\ &= (w-1)^{-k-1} P_k^{(0,0)}(\tilde{w}). \end{aligned}$$

The explicit representation of the Jacobi polynomials (B1) combined with (95) completes the proof. \square

Corollary 4.3: The generating function of the next to leading order corrections of the moments (91) is

$$\begin{aligned} \mathcal{D}_1^{(\beta)}(w, s) &= \frac{1}{2} \left(\frac{2}{\beta} - 1 \right) \left(\frac{(w-1)^2 - (w+1)s}{s^2 - 2s(w+1) + (w-1)^2} \right. \\ &\quad \left. - \frac{(w-1)\sqrt{s^2 - 2s(w+1) + (w-1)^2}}{s^2 - 2s(w+1) + (w-1)^2} \right). \end{aligned} \quad (96)$$

Proof: We now show that the moments (91) are the Taylor coefficients of the function (96). Consider the identity

$$\sum_{j=0}^k \binom{2k}{2j} w^j = \frac{1}{2} \left((1 - \sqrt{w})^{2k} + (1 + \sqrt{w})^{2k} \right). \quad (97)$$

For s sufficiently small we can write

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^{\infty} \sum_{j=0}^k \binom{2k}{2j} w^j s^k &= \frac{1}{4} \left(\frac{s(1 - \sqrt{w})^2}{1 - s(1 - \sqrt{w})^2} + \frac{s(1 + \sqrt{w})^2}{1 - s(1 + \sqrt{w})^2} \right) \\ &= \frac{1 - s(w+1)}{2(s^2(w-1)^2 - 2s(w+1) + 1)} - \frac{1}{2}. \end{aligned} \quad (98)$$

A direct inspection of the definition of the Jacobi polynomials (B1) gives the generating function

$$\frac{1}{2} \sum_{k=1}^{\infty} \sum_{j=0}^k \binom{k}{j}^2 w^j s^k = \frac{1}{2} \sum_{k=1}^{\infty} (w-1)^k P_k^{(0,0)}(\tilde{w}) s^k. \quad (99)$$

Recall that $\tilde{w} = (w+1)/(w-1)$. Furthermore, the generating function of the Jacobi polynomials $P_k^{(\alpha,\beta)}(w)$ when $\alpha = \beta = 0$ is (see Ref. 1, Eq. (22.9.1))

$$\sum_{k=1}^{\infty} P_k^{(0,0)}(w) s^k = \frac{1}{\sqrt{s^2 - 2ws + 1}} - 1. \quad (100)$$

Combining Eqs. (99) and (100) leads to

$$\frac{1}{2} \sum_{k=1}^{\infty} \sum_{j=0}^k \binom{k}{j}^2 w^j s^k = \frac{1}{2\sqrt{s^2(w-1)^2 - 2(w+1)s + 1}} - \frac{1}{2}. \quad (101)$$

Finally, to complete the proof we introduce the scaling $s \rightarrow (w-1)^{-2}s$ and subtract (101) from (98). \square

Remark 4.4: When $w = 2$ and $\beta = 1$, Eq. (96) reduces to

$$\mathcal{D}_1^{(1)}(2, s) = \frac{1 - 3s - \sqrt{1 - 6s + s^2}}{2(1 - 6s + s^2)},$$

which was computed from periodic orbit theory.¹¹

C. Next to leading order—Transmission eigenvalues

The following proposition extends formula (60b) to the cases $\beta = 1$ and $\beta = 4$. In Ref. 28 (Eq. (9)), they obtain the next to leading order correction of the eigenvalue density, though no proof was given. For the Wigner-Dyson symmetries, $\delta = 0$; the corresponding result first appeared in Ref. 21 and was derived via integration over matrix elements combined with a perturbative approach based on diagrammatic methods. We will present a derivation using a completely alternative method.

Proposition 4.5: For $\beta \in \{1, 2, 4\}$ and $\delta > -2$ one has

$$\begin{aligned} \Delta \mathcal{T}_{k,1}^{(\beta,\delta)}(u) &= \left(\frac{2}{\beta} - 1\right) \frac{u}{(u+1)^2} \left(\frac{u-1}{u+1}\right)^{2k} \\ &+ \frac{\delta}{\beta} \frac{1}{(u+1)^{2k+2}} \left(\sum_{j=0}^k \binom{k}{j}^2 u^{2j+1} - \sum_{j=0}^k \binom{k}{j} \binom{k}{j+1} u^{2j+2} \right). \end{aligned} \quad (102)$$

Proof: As previously we shall discuss only the symmetry $\beta = 1$. Let us set $a = \delta + 1$, $b = (u - 1)n$ in formula (86a).

Following the general strategy outlined in Sec. IV A, we see that the unitary contribution in (88a) is

$$\Delta \mathcal{U}_{k,1}^{(1,\delta)}(u) = \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} \frac{u^{2k-2j+1}}{(u+1)^{2k+2}} F_{k,1}^{(1,\delta)}(u, j),$$

where

$$F_{k,1}^{(1,\delta)}(u, j) = (1 + k - j)(k + \delta - j - 1) + uk(1 - \delta) + u^2 j(\delta - j).$$

Simplifying, we find

$$\Delta \mathcal{U}_{k,1}^{(1,\delta)}(u) = \frac{(\delta - 1)}{(u+1)^{2k+2}} \left(\sum_{j=0}^k \binom{k}{j}^2 u^{2j+1} - \sum_{j=0}^{k-1} \binom{k}{j} \binom{k}{j+1} u^{2j+2} \right), \quad (103)$$

which is obtained using the same method which led from (62) to Eq. (64), provided that we substitute $a = \delta + 1$ and $n \rightarrow n - 1$ in (22).

The symplectic contribution at order $p = 1$ is obtained by replacing (87a) into (88b). Using the Chu-Vandermonde summation of Lemma C.1, we obtain

$$\begin{aligned} \mathcal{S}_{k,1}^{(1,\delta)}(u) &= \sum_{j=1}^{k-1} \sum_{i=1}^j \binom{k}{j-i} \binom{k}{j+i} \frac{u^{2j}}{(1+u)^{2k}} \\ &= \frac{1}{2} \sum_{j=1}^{k-1} \left(\binom{2k}{2j} - \binom{k}{j}^2 \right) \frac{u^{2j}}{(1+u)^{2k}}. \end{aligned} \quad (104)$$

Finally, we must examine the term $I_{a,b}(k, n)$ in (86a). Straightforward application of (87b) gives

$$\Delta \mathcal{I}_{k,1}^{(1,\delta)}(u) = 2u(u-1)^{2k}(u+1)^{-2k-2}. \quad (105)$$

Inserting (104), (103), and (105) into (90a) shows that Eq. (102) is equivalent to proving the combinatorial identity

$$u(u-1)^{2k} = (u+1)^{2k+2} \Delta \mathcal{S}_{k,1}^{(1,\delta)}(u) + \left(\sum_{j=0}^k \binom{k}{j}^2 u^{2j+1} - \sum_{j=0}^k \binom{k}{j} \binom{k}{j+1} u^{2j+2} \right), \tag{106}$$

where $\Delta \mathcal{S}_{k,1}^{(\beta,\delta)}(u) := \mathcal{S}_{k,1}^{(1,\delta)}(u) - \mathcal{S}_{k+1,1}^{(1,\delta)}(u)$ is given in (104).

Equation (106) can be proved simply by comparing the coefficients of the monomials $u^{2j} (u^{2j} + 1)$ in the polynomials on both sides of the equation. It turns out that the right-hand side can be reduced to the left-hand side by repeated applications of Pascal’s rule,

$$\binom{k}{j} + \binom{k}{j+1} = \binom{k+1}{j+1}.$$

□

Using the identity (65), we can write the statement of Proposition 4.5 in terms of the leading order (51). Thus, we have

Corollary 4.6: The first correction for the differences of moments can be written in terms of formula (51) as

$$\Delta \mathcal{T}_{k,1}^{(\beta,\delta)}(u) = \frac{u}{(u+1)^2} \left(\frac{u-1}{u+1} \right)^{2k} \left(\frac{2}{\beta} - 1 + \frac{\delta(u-1)}{\beta u} \mathcal{T}_{k+1,0}^{(\beta,\delta)}(-u) \right). \tag{107}$$

From Corollary 4.6, we can compute the generating function of (107).

Corollary 4.7: The next to leading order generating function of the moments of the transmission eigenvalues is

$$\begin{aligned} \mathcal{T}_1^{(\beta,\delta)}(u, s) &= \left(\frac{2}{\beta} - 1 \right) \left(\frac{us}{(s-1)((u+1)^2 - s(u-1)^2)} \right) \\ &+ \frac{\delta}{2\beta} \left(\frac{(u+1)}{\sqrt{1-s}\sqrt{(u+1)^2 - s(u-1)^2}} + \frac{1}{s-1} \right). \end{aligned} \tag{108}$$

Proof: The generating function of the factor $\mathcal{T}_{k+1,0}^{(\beta,\delta)}(-u)$ in Eq. (107) is easily obtained from (47b). Then, the substitution

$$s \rightarrow s \left(\frac{u-1}{u+1} \right)^2$$

and some algebra lead to

$$\begin{aligned} \Delta \mathcal{T}_1^{(\beta,\delta)}(u, s) &= \left(\frac{2}{\beta} - 1 \right) \left(\frac{u}{(u+1)^2 - s(u-1)^2} - \frac{u}{(u+1)^2} \right) \\ &- \frac{\delta}{2\beta} \left(\frac{(u+1)\sqrt{1-s}}{s\sqrt{(u+1)^2 - s(u-1)^2}} + \frac{2u}{(u+1)^2} - \frac{1}{s} \right). \end{aligned} \tag{109}$$

Finally, formula (108) follows from (38).

□

D. Second corrections

As we go higher in the order of the expansions (19a) and (19b), the formulae for the coefficients become very cumbersome. Thus, the results are better expressed only in terms of generating functions. For the sake of simplicity, we shall only discuss the symmetry classes with $\beta = 1$.

1. Negative moments in the Laguerre orthogonal ensemble

Proposition 4.8: The generating function of the second corrections to the moments of the proper delay times is

$$\begin{aligned} \mathcal{D}_2^{(1)}(w, s) = & s \frac{(w + 1)s^2 - (2w^2 - 3w + 2)s + (w - 1)^2(w + 1)}{(s^2 - 2(w + 1)s + (w - 1)^2)^{5/2}} \\ & + s \frac{(w - 1)s + 1 - w^2}{(s^2 - 2(w + 1)s + (w - 1)^2)^2}. \end{aligned} \tag{110}$$

Proof: The proof is split into two parts: first we compute the third coefficients in the asymptotic expansion of formula (86b); then we show that they coincide with the Taylor coefficients of the right-hand side of Eq. (110).

We start from the coefficients of the unitary contribution $\mathcal{U}_{k,2}^{(1)}(w)$. Inserting $n \rightarrow n - 1$ and then $b = 1 + (w - 1)n$ in (25), we find

$$\begin{aligned} \mathcal{U}_{k,2}^{(1)}(w) = & \frac{w(k + 1)}{(w - 1)^{k+3}} \left((w - 1)(P_{k-1}^{(1,1)}(\tilde{w}) - (k/2)P_k^{(0,1)}(\tilde{w})) \right. \\ & \left. - \frac{2 + k}{12}P_{k-1}^{(1,2)}(\tilde{w}) + \frac{7(k + 2)}{12}P_{k-1}^{(2,1)}(\tilde{w}) \right). \end{aligned} \tag{111}$$

The derivation of this formula is almost identical to the discussion of Eqs. (69) and (70) in the proof of Proposition 3.10.

The term involving the asymptotics of $I_{L_b}(-k, n)$ is the easiest to compute; it is an application of the expansion (59) to (A1b). Its contribution is

$$\mathcal{I}_{k,2}^{(1)}(w) = -\frac{k}{2(w - 1)^{2k+1}} \sum_{j=0}^k \binom{2k + 2}{2j + 1} w^j. \tag{112}$$

The coefficient of the symplectic contribution leads to the double sum

$$\mathcal{S}_{k,2}^{(1)}(w) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} \binom{k + i - 1}{k - 1} \binom{k + 2j + i - 1}{k - 1} w^{-k-i-j-1} G_{k,2}^{(1)}(w, i, j), \tag{113}$$

where

$$G_{k,2}^{(1)}(w, i, j) = (k + i + 2j - 1)(k + i + 2j)/2 - j(j - 1) - w(j^2 + i(4j + i + 1)/2).$$

Now consider, for example, the contribution from the coefficient of w in $G_{k,2}^{(1)}(w, i, j)$. We have

$$\begin{aligned} & \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} \binom{k + i - 1}{k - 1} \binom{k + 2j + i - 1}{k - 1} w^{-k-i-j} (j^2 + i(4j + i + 1)/2) \\ & = \sum_{j=1}^{\infty} \sum_{i=1}^j \binom{k + j - i - 1}{k - 1} \binom{k + j + i - 1}{k - 1} w^{-k-j} (i^2 + (j - i)(4i + (j - i) + 1)/2). \end{aligned}$$

The finite sum in this equation can be computed exactly in terms of binomial coefficients using Chu-Vandermonde’s summation techniques discussed in Appendix C. The remaining infinite series is a combination of hypergeometric sums which may be evaluated in terms of Jacobi polynomials

using the identity (B5b). Repeating this procedure and inserting the contributions (111), (112), and $\mathcal{S}_{k,2}^{(1)}(w)$ into (90b), we arrive at

$$\begin{aligned}
 2(w-1)^{k+2}\mathcal{D}_{k,2}^{(1)}(w) &= -k(w-1)P_{k-1}^{(1,0)}(\tilde{w}) - k(k+1)(P_k^{(1,0)}(\tilde{w}) + wP_k^{(0,1)}(\tilde{w})) \\
 &\quad + (w/6)(w-1)^{-1}(k+1)(k+2)(13P_{k-1}^{(2,1)}(\tilde{w}) - P_{k-1}^{(1,2)}(\tilde{w})) \\
 &\quad - \frac{k}{2(w-1)^{k-1}} \sum_{j=0}^k \binom{2k+2}{2j+1} w^j \\
 &\quad - kwP_{k-1}^{(0,2)}(\tilde{w}) - (1-3w(k+1))P_{k-1}^{(1,1)}(\tilde{w}).
 \end{aligned} \tag{114}$$

We need to show that the Taylor coefficients of the right-hand side of Eq. (110) are given by $\mathcal{D}_{k,2}^{(1)}(w)$ in (114).

First notice that

$$- \sum_{k=0}^{\infty} \frac{k}{4(w-1)^{2k+1}} \sum_{j=0}^k \binom{2k+2}{2j+1} w^j s^k = s \frac{(w-1)s + 1 - w^2}{(s^2 - 2(w+1)s + (w-1)^2)^2}, \tag{115}$$

which follows from elementary manipulations involving the binomial theorem and geometric series. Then, a straightforward application of Corollary 3.12 gives

$$\begin{aligned}
 [s^k] \frac{(w+1)s^3 - (2w^2 - 3w + 2)s^2 + (w-1)^2(w+1)s}{(s^2 - 2(w+1)s + (w-1)^2)^{5/2}} \\
 = \frac{(w+1)}{w} \mathcal{D}_{k-1,2}^{(2)}(w) - \frac{(2w^2 - 3w + 2)}{w} \mathcal{D}_{k,2}^{(2)}(w) + \frac{(w-1)^2(w+1)}{w} \mathcal{D}_{k+1,2}^{(2)}(w).
 \end{aligned} \tag{116}$$

Inserting the explicit formula (66) for $\mathcal{D}_{k,2}^{(2)}(w)$ into the right-hand side of Eq. (116) and applying the three term recurrence relation (B2) to remove $P_{k-3}^{(2,2)}(\tilde{w})$, we obtain

$$\frac{(w+1)P_{k-1}^{(2,2)}(\tilde{w})(k+3)}{4(w-1)^{k+2}} + \frac{(8w+7wk-3w^2-3)(k+1)P_{k-2}^{(2,2)}(\tilde{w})}{12(w-1)^{k+3}}.$$

Combining this expression with (115) leads to

$$\begin{aligned}
 [s^k] \mathcal{D}_2^{(1)}(w, s) &= \frac{(w+1)P_{k-1}^{(2,2)}(\tilde{w})(k+3)}{4(w-1)^{k+2}} - \frac{k}{4(w-1)^{2k+1}} \sum_{j=0}^k \binom{2k+2}{2j+1} w^j \\
 &\quad + \frac{(8w+7wk-3w^2-3)(k+1)P_{k-2}^{(2,2)}(\tilde{w})}{12(w-1)^{k+3}}.
 \end{aligned} \tag{117}$$

The demonstration that coefficients (114) are equivalent to the right-hand side of Eq. (117) involves cumbersome algebraic manipulations, but it is otherwise straightforward. It can be carried out systematically following the steps (1)–(4) of the procedure outlined at the end of the proof of Proposition 3.10 and further discussed in Remark 3.11. \square

2. Moments in the Jacobi orthogonal ensemble

As for the second correction of the moments of the proper delay times the results for the transmission eigenvalues are better expressed only in terms of generating functions. For simplicity, we first discuss the moments of the transmission for the classical Dyson’s ensemble, i.e., when $\delta = 0$.

Proposition 4.9: The second correction of the moments of the transmission eigenvalues for $\beta = 1$ and $\delta = 0$ is

$$\mathcal{T}_2^{(1,0)}(u, s) = -\frac{us(s^2(u-1)^2 + 3us - (u+1)^2)}{((u+1)^2 - s(u-1)^2)^{5/2}(1-s)^{3/2}}. \tag{118}$$

Proof: The proof follows the usual strategy: first, we compute the coefficient at $p = 2$ of the expansion (19a); then we show that they coincide with the Taylor coefficients of (118). Furthermore, rather than computing (118) directly, we will look at

$$\Delta \mathcal{T}_2^{(1,0)}(u, s) = \frac{u(s^2(u-1)^2 + 3us - (u+1)^2)}{((u+1)^2 - s(u-1)^2)^{5/2}\sqrt{1-s}} + \frac{u}{(u+1)^3}. \tag{119}$$

According to (90a), to begin with we split $\Delta \mathcal{T}_{k,2}^{(1,0)}(u)$ into the contributions of $\Delta \mathcal{L}_{k,2}^{(1,0)}(u)$, $\Delta \mathcal{S}_{k,2}^{(1,0)}(u)$, and $\Delta \mathcal{T}_{k,2}^{(1,0)}(u)$. First we note that straightforward calculation analogous to the proof that $\Delta \mathcal{T}_{k,1}^{(2,0)}(u) = 0$ in Proposition 3.7 shows that $\Delta \mathcal{T}_{k,2}^{(1,0)}(u) = 0$; therefore, we need not worry about this term.

The contribution of the symplectic term involving the double summation in (88b) can be written as

$$\mathcal{S}_{k,2}^{(1,0)}(u) = \frac{1}{(u+1)^{2k+1}} \sum_{j=1}^{\lfloor k/2 \rfloor} \sum_{i=0}^{k-2j} \binom{k}{i} \binom{k}{i+2j} u^{2k-2i-2j-1} F_{k,2}^{(1)}(u, i, j), \tag{120}$$

where

$$F_{k,2}^{(1)}(u, i, j) = (j + i^2 + 4ij + 2j^2 - 2ki + k^2 - 4kj) + 2ukj + u^2(j - i^2 - 2j^2 - 4ij).$$

For example, taking the coefficient of u^2 in $F_{k,2}^{(1)}(u, i, j)$ leads to

$$\begin{aligned} & \frac{1}{(u+1)^{2k+1}} \sum_{j=1}^{\lfloor k/2 \rfloor} \sum_{i=0}^{k-2j} \binom{k}{i} \binom{k}{i+2j} u^{2k-2i-2j+1} (j - i^2 - 2j^2 - 4ij) \\ &= \frac{1}{(u+1)^{2k+1}} \sum_{j=1}^{k-1} \sum_{i=1}^j \binom{k}{j+i} \binom{k}{j-i} u^{2k-2j+1} (i - (j-i)^2 - 2i^2 - 4(j-i)i). \end{aligned} \tag{121}$$

The inner sums in this equation are of the form

$$\sum_{i=1}^j \binom{k}{j+i} \binom{k}{j-i} i^p, \quad p = 0, 1, 2. \tag{122}$$

They are Chu-Vandermonde’s summations discussed in Appendix C and can be evaluated explicitly in terms of binomial coefficients. This allows us to write (121) as a sum of Jacobi polynomials. Finally, including the contributions of the linear and constant terms in u in $F_{k,2}^{(1)}(u, i, j)$, we arrive at

$$\begin{aligned} \frac{(u+1)^{k+3}}{(u-1)^{k-2}u} \mathcal{S}_{k,2}^{(1,0)}(u) &= k(k-1)(1-u^2)P_{k-2}^{(0,1)}(\tilde{u}) + (k/2)(1-2k)P_{k-2}^{(1,1)}(\tilde{u}) \\ &+ (k/2)u^2P_{k-2}^{(1,1)}(\tilde{u}) + k^2uP_{k-2}^{(1,1)}(\tilde{u}), \end{aligned} \tag{123}$$

where \tilde{u} is defined in (77)

Now, $(u+1)^{2k+3} \Delta \mathcal{T}_{k,2}^{(1,0)}(u)$ is a polynomial in u of degree $2k + 1$; for convenience, we separate it into the sum of two polynomials containing only monomials of even and odd degree, respectively. We write

$$(u+1)^{2k+3} \Delta \mathcal{T}_{k,2}^{(1,0)}(u) = \Delta \mathcal{T}_{k,2}^{(1,0),o}(u) + \Delta \mathcal{T}_{k,2}^{(1,0),e}(u). \tag{124}$$

Let us focus first on the term containing only odd monomials. We have

$$\Delta\mathcal{T}_{k,2}^{(1,0),o}(u) = \Delta\mathcal{U}_{k,2}^{(1,0),o}(u) - \Delta\mathcal{S}_{k,2}^{(1,0),o}(u). \tag{125}$$

The contribution of the unitary term gives

$$\Delta\mathcal{U}_{k,2}^{(1,0),o}(u) = -u(u^2 - 1)^k(k + 1) \left((k/6)(\tilde{u}^2 - 1)P_{k-2}^{(1,1)}(\tilde{u}) + (k + 1)P_k^{(0,0)}(\tilde{u}) \right). \tag{126}$$

This formula is obtained by inserting (59) into (21), by shifting $n \rightarrow n - 1$, and setting $a = 1$ and $b = (u - 1)n$.

Since the numerator of \tilde{u} is quadratic in u , the monomials of odd degree in

$$(u + 1)^{2k+3} \Delta\mathcal{S}_{k,2}^{(1,0)}(u)$$

can be extracted from (123) by simple inspection; then, Eqs. (125) and (126) give

$$\begin{aligned} \frac{\Delta\mathcal{T}_{k,2}^{(1,0),o}(u)}{u(u^2 - 1)^{k-2}} &= -(k/2)(1 - 2k + 2u^2 + 2ku^2 + u^4)P_{k-2}^{(1,1)}(\tilde{u}) \\ &+ (u^4 - 1)k(k - 1)P_{k-2}^{(0,1)}(\tilde{u}) - k(k + 1)(1 - u^2)^2P_{k-1}^{(0,1)}(\tilde{u}) \\ &- (1/2)(u^2 - 1)(k + 1)(2k + 1 - u^2)P_{k-1}^{(1,1)}(\tilde{u}) \\ &- (u^2 - 1)^2(k + 1) \left((k/6)(\tilde{u}^2 - 1)P_{k-2}^{(1,1)}(\tilde{u}) + P_k^{(0,0)}(\tilde{u}) \right). \end{aligned} \tag{127a}$$

The contribution of $\Delta\mathcal{T}_{k,2}^{(1,0),e}(u)$ can be computed in the same way. We obtain

$$\begin{aligned} \frac{\Delta\mathcal{T}_{k,2}^{(1,0),e}(u)}{u^2(u^2 - 1)^{k-2}} &= (k - 1) \left((k/2)(\tilde{u} + 1)P_{k-3}^{(1,2)}(\tilde{u}) + u^2P_{k-2}^{(0,2)}(\tilde{u}) - (k/3)P_{k-2}^{(1,0)}(\tilde{u}) \right) \\ &+ (k/6)(5u^2 - 1)P_{k-2}^{(1,1)}(\tilde{u}) + (2k + 1)(u^2 - 1)P_{k-1}^{(0,1)}(\tilde{u}) \\ &+ (2k/3)(k - 1)u^2P_{k-2}^{(0,1)}(\tilde{u}) - 2k(k - 1)(1 - u^2)P_{k-2}^{(0,1)}(\tilde{u}) \\ &- (k + 1)^2(1 - u^2)P_{k-1}^{(1,1)}(\tilde{u}) - k(k + 1)u^2P_{k-2}^{(1,1)}(\tilde{u}) \\ &+ k(k - 1)P_{k-2}^{(1,1)}(\tilde{u}). \end{aligned} \tag{127b}$$

Inserting Eqs. (127a) and (127b) into (124) gives the explicit expression of the second corrections of the moments. We now need to demonstrate that they coincide with the Taylor coefficients of the right-hand side of (119).

By comparing (119) with $\Delta\mathcal{T}_2^{(2,0)}(u, s)$ in Eq. (82), we write

$$\begin{aligned} \Delta\mathcal{T}_2^{(1,0)}(u, s) &= 3\Delta\mathcal{T}_2^{(2,0)}(u, s) - \frac{4(u + 1)^2}{u} \Delta\mathcal{T}_2^{(2,0)}(u, s) \\ &+ \frac{s(u + 1)^2}{u} \Delta\mathcal{T}_2^{(2,0)}(u, s) - \frac{(u + 1)^2}{us} \Delta\mathcal{T}_2^{(2,0)}(u, s). \end{aligned} \tag{128}$$

From Proposition 3.14, we know the Taylor coefficients of $\Delta\mathcal{T}_2^{(2,0)}(u, s)$. Thus, we end up with the formulae

$$\begin{aligned} \frac{[s^k]^o \Delta\mathcal{T}_2^{(1,0)}(u, s)}{u(u^2 - 1)^k} &= (1/6)(k + 2)P_k^{(1,1)}(\tilde{u}) - (2/3)(k + 1)(k + 2)P_k^{(0,0)}(\tilde{u}) \\ &- (k/6)P_{k-2}^{(1,1)}(\tilde{u}) + (2k/3)(k - 1)P_{k-2}^{(0,0)}(\tilde{u}) \\ &- (k/2)(k + 1)(\tilde{u} - 1)^2u^2P_{k-2}^{(1,1)}(\tilde{u}) \end{aligned} \tag{129a}$$

for the component including only monomials of odd powers of u and

$$\begin{aligned} \frac{[s^k]^e \Delta \mathcal{T}_2^{(1,0)}(u, s)}{u^2(u^2 - 1)^{k-1}} &= (2/3)(k + 2)(k + 1)P_{k-1}^{(1,1)}(\tilde{u}) - (1/2)(k + 1)P_{k-1}^{(1,1)}(\tilde{u}) \\ &+ 2k(k + 1)P_{k-1}^{(0,0)}(\tilde{u}) - (2/3)(k - 1)kP_{k-3}^{(1,1)}(\tilde{u}) \end{aligned} \tag{129b}$$

for the monomials of even degree.

Finally, the equivalence of the pair of Eqs. (127) to (129) is proved systematically through steps (1)–(4) discussed in the proof of Proposition 3.10. \square

Remark 4.10: Formula (118) coincides with the generating function computed with semiclassical techniques by Berkolaiko and Kuipers.¹¹

The computation of the second correction for Andreev billiards ($\delta \neq 0$) is almost identical to the proof of Proposition 4.9, but obviously the algebraic manipulations are much more involved. We did not carry out the proof in its entirety. The expression of such a generating function, however, can be inferred from general methods and by inspecting lower order terms.

Conjecture 4.11: The generating function of the second correction of the moments of the transmission eigenvalues for Andreev billiards with $\beta = 1$ and $\delta > -2$ is

$$\begin{aligned} \mathcal{T}_2^{(1,\delta)}(u, s) &= -\frac{us(s^2(u - 1)^2 + 3us - (u + 1)^2)}{((u + 1)^2 - s(u - 1)^2)^{5/2}(1 - s)^{3/2}} + \frac{3\delta us}{2(u + 1)^3(s - 1)} \\ &+ \delta s \frac{((u + 1)^2 - s(u - 1)^2)^2 + 2u\delta((u + 1)^2 - s(u - 1)^2)}{2((u + 1)^2 - s(u - 1)^2)^{5/2}(1 - s)^{3/2}} \\ &+ \delta s \frac{(u + 1)^2(u^2 - 5u + 1) - s(u - 1)^2(u^2 - 4u + 1)}{2(s - 1)(u + 1)((u + 1)^2 - s(u - 1)^2)^2} \\ &- \delta \frac{3s^3u(u - 1)^4}{2(s - 1)(u + 1)^3((u + 1)^2 - s(u - 1)^2)^2}. \end{aligned}$$

V. ASYMPTOTICS OF SELBERG-LIKE INTEGRALS

In this section, we will be interested in asymptotics as $n \rightarrow \infty$ of the integrals

$$\begin{aligned} \mathcal{M}_k^{(\beta)}(u, v) &= \frac{1}{C} \int_0^1 \cdots \int_0^1 \left(\sum_{j=1}^n x_j^k \right) \prod_{j=1}^n x_j^{\beta/2(b+1)-1} (1 - x_j)^{\beta/2(a+1)-1} \\ &\times \prod_{1 \leq j < k \leq n} |x_k - x_j|^\beta dx_1 \cdots dx_n, \quad k = 1, 2, \dots \end{aligned} \tag{130}$$

with the assumption that $\beta \in \{1, 2, 4\}$ and

$$a = (v - 1)n \quad \text{and} \quad b = (u - 1)n, \quad u, v \geq 1. \tag{131}$$

The integrals (130) have been referred to *Selberg-like* by several authors.^{24,42,58} Indeed, the normalization constant

$$\begin{aligned} C_n(\beta, a, b) &= \int_0^1 \cdots \int_0^1 \prod_{j=1}^n x_j^{\beta/2(b+1)-1} (1 - x_j)^{\beta/2(a+1)-1} \\ &\times \prod_{1 \leq j < k \leq n} |x_k - x_j|^\beta dx_1 \cdots dx_n \end{aligned} \tag{132}$$

was evaluated for any $\beta > 0$ by Selberg in 1944 (Ref. 66), who obtained the formula

$$C_n(\beta, a, b) = \prod_{j=0}^{n-1} \frac{\Gamma(\beta/2(b+1+j))\Gamma(\beta/2(a+1+j))\Gamma(1+(j+1)\beta/2)}{\Gamma(\beta/2(a+b+1+n+j))\Gamma(1+\beta/2)}. \tag{133}$$

The integral (132) has since been named *Selberg’s integral*.

In the first part of this work,⁵³ we gave finite- n formulae for (130) which imply the existence of an asymptotic expansion in powers of $1/n$,

$$\frac{1}{n} \mathcal{M}_k^{(\beta)}(u, v) \sim \sum_{p=0}^{\infty} \mathcal{M}_{k,p}^{(\beta)}(u, v) n^{-p}, \quad n \rightarrow \infty. \tag{134}$$

We now compute the first two coefficients in this expansion.

Expressions for the β -independent leading order term appeared in Refs. 24 and 42 with the assumption that $\beta = 2$; a different formula is also available in the literature,⁵⁸ whose proof was valid for any β . We introduce a different approach, which gives a simpler characterization of the leading order term and allows to compute the next to leading order coefficient.

A. Leading order term

As for the moments of the transmission eigenvalues, we introduce the differences

$$\Delta \mathcal{M}_k^{(\beta)}(u, v) := \mathcal{M}_k^{(\beta)}(u, v) - \mathcal{M}_{k+1}^{(\beta)}(u, v); \tag{135a}$$

similarly, for the coefficients of their asymptotic series, we write

$$\Delta \mathcal{M}_{k,p}^{(\beta)}(u, v) := \mathcal{M}_{k,p}^{(\beta)}(u, v) - \mathcal{M}_{k+1,p}^{(\beta)}(u, v). \tag{135b}$$

Furthermore, we shall set

$$\mathcal{M}_0^{(\beta)}(u, v) = \Delta \mathcal{M}_0^{(\beta)}(u, v) = 0.$$

Proposition 5.1: Let $\beta \in \{1, 2, 4\}$, $a = (v - 1)n$ and $b = (u - 1)n$. We have

$$\mathcal{M}_{k,0}^{(\beta)}(u, v) = \frac{u}{u+v} - \sum_{j=1}^{k-1} \frac{1}{j} \sum_{i=0}^j \binom{j}{i} \binom{j}{i-1} \frac{v^i u^{j-i+1} (u+v-1)^{j-i+1}}{(u+v)^{2j+1}}. \tag{136}$$

Proof: It is sufficient to consider $\beta = 2$, because, as for the moments of the transmission eigenvalues, a simple calculation discussed in Remark 4.1 shows that the terms in (86a) that do not come from the unitary contributions contribute at subleading order.

The first term in (136) is the limit of Aomoto’s integral, namely,

$$\lim_{n \rightarrow \infty} n^{-1} M_{J_{a,b}}^{(\beta)}(1, n) = \lim_{n \rightarrow \infty} \frac{b+n}{a+b+2n} = \frac{u}{u+v}.$$

Inserting the leading order term of the asymptotic expansion (29) into (22) gives

$$\begin{aligned} U_{n,k,j}^{a,b} &= \frac{((u+v)n - 2j + k + 1) \binom{(u+v-1)n}{(k-j+1)} \binom{vn - j + 1}{(j)} \binom{un}{(k-j+1)}}{\binom{(u+v)n - j}{(k+2)} \binom{(u+v)n - j + 1}{(k)} \binom{(n+1)}{(-j)}} \\ &\sim n \frac{v^j u^{k-j+1} (u+v-1)^{k-j+1}}{(u+v)^{2k+1}} + O(1), \quad n \rightarrow \infty, \end{aligned} \tag{137}$$

where a and b depend on u and v through the scaling (131). Substituting the right-hand side of (137) into (21) leads to a formula for the differences,

$$\Delta \mathcal{M}_{k,0}^{(\beta)}(u, v) = \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} \frac{v^j u^{k-j+1} (u+v-1)^{k-j+1}}{(u+v)^{2k+1}}. \tag{138}$$

□

B. Next to leading order term

Proposition 5.2: Let $\beta \in \{1, 2, 4\}$, $a = (v - 1)n$ and $b = (u - 1)n$. The next to leading order coefficient in the expansion (134) is

$$\mathcal{M}_{k,1}^{(\beta)}(u, v) = \left(\frac{2}{\beta} - 1\right) \frac{v^k}{2(u+v)^{2k}} \sum_{j=0}^k \left(\binom{2k}{2j} - \binom{k}{j}^2 \right) \left(\frac{u(u+v-1)}{v} \right)^j.$$

Proof: We discuss only the case when $\beta = 1$, as when $\beta = 2$ and $\beta = 4$, the proof is analogous. We can write down a relation analogous to that one in (89), where now $a = (v - 1)n$ grows with n instead of being independent of n . For the first subleading correction, it reads

$$\mathcal{M}_{k,1}^{(1)}(u, v) = \mathcal{U}_{k,1}^{J_{a,b}}(u, v) - \mathcal{S}_{k,1}^{J_{a,b}}(u, v) + \mathcal{I}_{k,1}^{J_{a,b}}(u, v). \tag{139}$$

Insert the leading order of (29) into the symplectic contribution in (86a); then, a calculation similar to the derivation of (104) in Proposition 4.5 shows that

$$\begin{aligned} \mathcal{S}_{k,1}^{J_{a,b}}(u, v) &= \sum_{j=1}^{\lfloor k/2 \rfloor} \sum_{i=0}^{k-2j} \binom{k}{i} \binom{k}{i+2j} \frac{v^{i+j} u^{k-i-j} (u+v-1)^{k-i-j}}{(u+v)^{2k}} \\ &= \frac{v^k}{2(u+v)^{2k}} \sum_{j=0}^{k-1} \left(\binom{2k}{2j} - \binom{k}{j}^2 \right) \left(\frac{u(u+v-1)}{v} \right)^j. \end{aligned}$$

In the same way, starting from formula (A4) we arrive at

$$\mathcal{I}_{k,1}^{J_{a,b}}(u, v) = \frac{v^k}{(u+v)^{2k}} \sum_{j=0}^k \binom{2k}{2j} \left(\frac{u(u+v-1)}{v} \right)^j.$$

Therefore, it is sufficient to prove that

$$\mathcal{U}_{k,1}^{J_{a,b}}(u, v) = -\frac{v^k}{(u+v)^{2k}} \sum_{j=0}^k \binom{k}{j}^2 \left(\frac{u(u+v-1)}{v} \right)^j = -\frac{v^k}{(u+v)^{2k}} (y-1)^k P_k^{(0,0)}(\tilde{y}),$$

where we have introduced the new variables

$$y = \frac{u}{v}(u+v-1) \quad \text{and} \quad \tilde{y} = \frac{y+1}{y-1}. \tag{140}$$

Taking the differences, we see that the statement of the proposition is equivalent to

$$\Delta \mathcal{U}_{k,1}^{J_{a,b}}(u, v) = -\frac{v^k (y-1)^k}{(u+v)^{2k}} P_k^{(0,0)}(\tilde{y}) + \frac{v^{k+1} (y-1)^{k+1}}{(u+v)^{2k+2}} P_{k+1}^{(0,0)}(\tilde{y}). \tag{141}$$

Replace $n \rightarrow n - 1$ in (21) and insert the scaling (131). Then, use (59) to expand (141) up to the next to leading order correction. We obtain

$$\begin{aligned} \Delta \mathcal{U}_{k,1}^{a,b}(u, v) &= \frac{1}{k} \sum_{j=1}^k \binom{k}{j} \binom{k}{j-1} \frac{u^{k-j+1} v^j (u+v-1)^{k-j+1}}{2(u+v)^{2k+1}} \\ &\times \left(\frac{(k-j+1)(k-j-2)}{(u+v-1)} - \frac{j(j+1)}{v} + \frac{(k-j+1)(k-j-2)}{u} \right. \\ &\left. + \frac{6k+4-2k^2+4kj}{(u+v)} - j(1+j) \right), \end{aligned} \tag{142}$$

which may be written in terms of Jacobi polynomials as

$$\begin{aligned} \Delta \mathcal{U}_{k,1}^{a,b}(u, v) &= \frac{(y-1)^{k-2} u(u+v-1) v^k}{(u+v)^{2k+1}} \left(\left(\frac{2k}{u+v} - \frac{1}{v} - 1 \right) (y-1) P_{k-1}^{(0,1)}(\tilde{y}) \right. \\ &+ \frac{k(u-1)}{v} P_{k-2}^{(1,1)}(\tilde{y}) - \left(\frac{1}{u+v-1} + \frac{1}{u} \right) (y-1) P_{k-1}^{(1,0)}(\tilde{y}) \\ &\left. + \frac{(6k+4-2k^2)(y-1)}{2k(u+v)} P_{k-1}^{(1,1)}(\tilde{y}) \right). \end{aligned} \tag{143}$$

By subtracting (141) from (143), we obtain an expression involving Jacobi polynomials. In order to complete the proof, we need to show that it vanishes. As previously, we achieve it systematically, following the steps (1)–(4) outlined at the end of the proof of Proposition 3.10. \square

C. Leading order generating function

Let us define the generating function

$$H(u, v; s) = \sum_{k=0}^{\infty} \mathcal{M}_{k,0}^{(\beta)}(u, v) s^k. \tag{144}$$

We now present a new proof of the following.

Theorem 5.3 (Novaes⁵⁸): *The generating function $H(u, v; s)$ obeys the quadratic equation*

$$H = \frac{us}{u+v-(1+u)s} - \frac{1-s}{u+v-(1+u)s} H^2. \tag{145}$$

Proof: By rearranging summation indices as $j \rightarrow k - j$, we can write formula (136) in terms of a Narayana polynomial of degree k (see (31)):

$$\begin{aligned} \Delta \mathcal{M}_{k,0}^{(\beta)}(u, v) &= \frac{v^{k+1}}{(u+v)^{2k+1}} \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} \left(\frac{u(u+v-1)}{v} \right)^j \\ &= \frac{v^{k+1}}{(u+v)^{2k+1}} N_k \left(\frac{u(u+v-1)}{v} \right). \end{aligned}$$

Thus, the generating function of $\Delta \mathcal{M}_{k,0}^{(\beta)}(u, v)$ can be easily related to the generating function of the Narayana polynomials, which we report in (32). More explicitly

$$\tilde{H}(u, v; s) := \sum_{k=1}^{\infty} \Delta \mathcal{M}_{k,0}^{(\beta)}(u, v) s^k = \frac{v}{u+v} \rho \left(\frac{u(u+v-1)}{v}, \frac{sv}{(u+v)^2} \right). \tag{146}$$

The generating functions H and \tilde{H} are related by

$$\tilde{H} = (1 - s^{-1})H + \mathcal{M}_{1,0}^{(\beta)}(u, v). \quad (147)$$

It follows immediately that

$$H = \frac{s}{s-1} \left(\frac{v}{u+v} \rho \left(\frac{u(u+v-1)}{v}, \frac{sv}{(u+v)^2} \right) - \frac{u}{u+v} \right). \quad (148)$$

The generating function $\rho(x, s)$ satisfies the quadratic equation (see, e.g., Ref. 29),

$$\rho = (x + \rho)(1 + \rho)s,$$

which leads to

$$(us + (s-1)H)(s + (s-1)H) = (u+v)(s-1)H + us.$$

Rearranging this expression gives (145). \square

Formula (145) is equivalent (in our notation) to Eq. (58) in Ref. 58 obtained by a completely different approach.

D. The limiting eigenvalue density of the Jacobi ensembles

The leading order term of the density (56) averaged over the Jacobi ensembles first appeared in the multivariate statistics literature^{4,73} and later in the context of chaotic quantum transport, first when $a = 0$ and $b = 0$ (Refs. 6 and 7) and then only when $a = 0$ (Ref. 21). More recently it has attracted attention in the free probability literature^{23,27} with the same scaling as that adopted in this section. It was found that

$$\rho_\infty(x) = \lim_{n \rightarrow \infty} n^{-1} \rho_n(x) = (u+v) \frac{\sqrt{(x-\lambda_-)(\lambda_+ - x)}}{2\pi x(1-x)}, \quad (149)$$

with support

$$\lambda_\pm = \left(\sqrt{\frac{u}{u+v} \left(1 - \frac{1}{u+v} \right)} \pm \sqrt{\frac{1}{u+v} \left(1 - \frac{u}{u+v} \right)} \right)^2. \quad (150)$$

The density (149) is normalized to one; in Refs. 23 and 27, a different normalization and notation is used. We now show that the leading order contributions of the Selberg-like integrals $\mathcal{M}_k^{(\beta)}(u, v)$ are the moments of the limiting density.

Formula (149) makes it apparent why it is helpful to consider the differences between the moments, rather than the moments themselves: multiplication by

$$x^k - x^{k+1} = (1-x)x^k$$

cancels the poles in the denominator of (149), making the resulting integration tractable.

Our method is similar in style to that used in Ref. 37 for a similar problem involving the Laguerre ensembles.

Theorem 5.4: *The difference of the moments of the limiting density $\rho_\infty(x)$ are*

$$\begin{aligned} \int_{\lambda_-}^{\lambda_+} x^k (1-x) \rho_\infty(x) dx &= \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} \frac{v^j u^{k-j+1} (u+v-1)^{k-j+1}}{(u+v)^{2k+1}} \\ &= \Delta \mathcal{M}_{k,0}^{(\beta)}(u, v). \end{aligned} \quad (151)$$

Proof: Define

$$\begin{aligned}
 I_k &= \int_{\lambda_-}^{\lambda_+} x^k (1-x) \rho_\infty(x) dx \\
 &= \frac{u+v}{2\pi} \int_{\lambda_-}^{\lambda_+} x^{k-1} \left(\sqrt{x(\lambda_- + \lambda_+) - \lambda_- \lambda_+ - x^2} \right) dx.
 \end{aligned}
 \tag{152}$$

The substitution

$$x = \frac{\lambda_- + \lambda_+}{2} + \frac{\lambda_+ - \lambda_-}{2} \cos(\theta)$$

transforms the integral in the right-hand side of (152) into

$$I_k = \frac{\alpha_2^2(u+v)}{4\pi} \int_{-\pi}^{\pi} (\alpha_1 + \alpha_2 \cos(\theta))^{k-1} \sin^2(\theta) d\theta,$$

where $\alpha_1 = \frac{\lambda_+ + \lambda_-}{2}$ and $\alpha_2 = \frac{\lambda_+ - \lambda_-}{2}$.

Let p and q and $z(\theta)$ be such that

$$\alpha_1 = p^2 + q^2, \quad \alpha_2 = 2pq, \quad z(\theta) = (p + qe^{i\theta})^{k-1}.
 \tag{153}$$

Then, we have

$$|z(\theta)|^2 = (\alpha_1 + \alpha_2 \cos(\theta))^{k-1}.$$

Set

$$\zeta = e^{i\theta} z(\theta) \quad \text{and} \quad \eta = e^{-i\theta} z(\theta).$$

Using

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} = \frac{1}{2} \Re(1 - e^{2i\theta}),$$

we see that

$$\begin{aligned}
 \frac{I_k}{u+v} &= \frac{\alpha_2^2}{8\pi} \int_{-\pi}^{\pi} \Re(1 - e^{2i\theta}) |z(\theta)|^2 d\theta \\
 &= \frac{\alpha_2^2}{8\pi} \left(\int_{-\pi}^{\pi} |z(\theta)|^2 d\theta - \Re \left(\int_{-\pi}^{\pi} \zeta(\theta) \overline{\eta(\theta)} d\theta \right) \right).
 \end{aligned}$$

These integrals can be evaluated by calculating the Fourier modes of ζ , η and z and applying Parseval's identity. The Fourier coefficients are

$$z_j = p^{k-1-j} q^j \binom{k-1}{j}, \quad \zeta_j = p^{k-j} q^{j-1} \binom{k-1}{j-1}, \quad \eta_j = p^{k-j-2} q^{j+1} \binom{k-1}{j+1}.$$

Parseval's identity gives

$$\begin{aligned}
 \frac{I_k}{u+v} &= \frac{\alpha_2^2}{4} \sum_{j=0}^{k-1} p^{2k-2j-2} q^{2j} \binom{k-1}{j}^2 - \frac{\alpha_2^2}{4} \sum_{j=0}^{k-1} p^{2k-2j-2} q^{2j} \binom{k-1}{j-1} \binom{k-1}{j+1} \\
 &= \frac{\alpha_2^2}{4k} \sum_{j=0}^{k-1} p^{2k-2j-2} q^{2j} \binom{k}{j+1} \binom{k}{j}.
 \end{aligned}
 \tag{154}$$

Solving p and q in terms of α_1 and α_2 in Eq. (153) and using (150) allows us to write

$$p = \sqrt{\frac{u(u+v-1)}{(u+v)^2}}, \quad q = \sqrt{\frac{v}{(u+v)^2}}, \quad \alpha_2^2 = \frac{4uv(u+v-1)}{(u+v)^4}.$$

Substituting these expressions into (154) and replacing $j \rightarrow j - 1$ gives Eq. (151). □

VI. CONCLUSIONS AND OUTLOOK

In this article, we have computed the first three terms of the asymptotic expansions in the limit as $n \rightarrow \infty$ of the moments of the density of the transmission eigenvalues and delay times in chaotic ballistic cavities. The fundamental assumption is that the chaotic dynamics allows us to model the scattering matrix (1) with the circular ensembles from RMT. Our formulae are available for all symmetry classes $\beta \in \{1, 2, 4\}$ —with a few exceptions for the second corrections to the leading order terms, for which we did not perform the calculations for $\beta = 4$. For the moments of the transmission eigenvalues, we treat Andreev billiards too. Finally, we studied the asymptotics of the Selberg-like integrals as well.

Our results on the moments of the transmission eigenvalues and proper time delays for $\beta = 1$ and $\beta = 2$ symmetry classes agree with those computed using semiclassical techniques.^{9–12,59} It would be interesting to know if the results that we obtain for Andreev billiards beyond the leading order agree with semiclassics too.

In recent announcements Berkolaiko and Kuipers¹² and independently Novaes⁵⁹ show that semiclassical computations lead to the same asymptotic expansions of the moments of the transmission eigenvalues as RMT. Their results involve combinatorial expressions for the correlations of the scattering trajectories. At this stage, however, it is not clear how to extract explicit formulae from the combinatorics. Their formulae include unsolved combinatorial problems too. It is a challenging project to go even further and to obtain the full asymptotic expansions for the moments of the transmission eigenvalues and proper delay times using RMT techniques. Nevertheless, it would be interesting pursue this program, since combinatorics would not appear in the calculations and in the final formulae. Furthermore, the equivalence of the two approaches could answer unsolved combinatorial problems.

In principle such asymptotic series can be obtained because the asymptotic properties discussed in Sec. II A hold at all orders in negative powers of n . The finite- n formulae that we computed in the first part of this project⁵³ contain products of the form

$$\prod_{j=1}^p \frac{\Gamma(a_j z + c_j)}{\Gamma(b_j z + d_j)}. \quad (155)$$

One could insert the asymptotic series (29) into (155) and repeat the systematic procedures developed in this article to find further corrections. In practice, however, when we go beyond the second correction it is not clear how to organise the resulting asymptotic terms in a way that would lead to results in closed form, as those obtained in this paper.

We believe that, as for the asymptotic of the ratio (29), it may be possible to establish recursion relations for the terms in the asymptotic series of (155). This would lead to recursion relations for the full expansions of the moments of the proper delay times and of the transmission eigenvalues.

ACKNOWLEDGMENTS

We would like to express our gratitude to Gregory Berkolaiko, Peter Forrester, Jonathan Keating, Jack Kuipers, and Dmitry Savin for stimulating and helpful discussions. This research is partially supported by the Engineering and Physical Sciences Research Council (UK) (EPSRC (GB)) Grant No. EP/G019843/1 and the Leverhulme Trust, Research Fellowship No. RF/4/RFG/2009/0092.

APPENDIX A: EXACT RESULTS FOR $\beta = 1$ AND $\beta = 4$ MATRIX ENSEMBLES

In this appendix, we report formulae for finite- n that we derived in the first part of this work⁵³ and that are the starting points for our asymptotic analysis.

Equations (86a) and (86b) for $\beta = 1$ were expressed in terms of certain coefficients which are defined below. We also give the corresponding results for $\beta = 4$. The expressions for $\beta = 2$ are

presented at Eqs. (21) and (25). Recall that

$$(x)_{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)}$$

is the Pochhammer symbol.

In the following, we assume that the channel number satisfies $n > k\beta/2$. This allows for some simplifications; furthermore, a small n is not required for an asymptotic analysis.

1. Moments of the proper delay times

We need the following expressions for the Laguerre ensemble:

$$S_{i,j}^b(-k, n) = \frac{(2b+2n)_{(-k-i-2j+1)}(2n-i-2j+1)_{(i)}}{2^{-k-2j+2}(n+1)_{(-j)}(b+n)_{(1-j)}}, \quad (\text{A1a})$$

$$I_{L_b}(-k, n) = 2^{-k} \sum_{j=0}^{n/2-1} \binom{2k+2j-1}{2j} \frac{(\frac{1}{2}(b+n))_{(-k-j)}}{(\frac{1}{2}(1+n))_{(-j)}} + \phi_{-k,n}^L, \quad (\text{A1b})$$

where $\phi_{-k,n}^L$ is an exponentially decaying term which has no contribution to the asymptotic expansions (19b) for any asymptotic order p . Its explicit expression can be found in Ref. 53, Sec. 6.3, Eq. (6.29). The exact moments for $\beta = 1$ are then given by Eq. (86b); for $\beta = 4$, we have

$$M_{L_b}^{(4)}(-k, n) = 2^{k-1} M_{L_{2b}}^{(2)}(-k, 2n) - \sum_{j=1}^n \sum_{i=0}^{2n-2j} \binom{k+j-1}{k-1} \binom{k+i+2j-1}{k-1} S_{i,j}^b(-k, n). \quad (\text{A2})$$

2. Moments of the transmission eigenvalues

The formulae we need for the Jacobi ensemble are

$$S_{i,j}^{a,b}(k, n) = \frac{2^{4j-3}(2a+2n-i-2j+1)_{(i)}(2b+2n)_{(k-i-2j+1)}(2a+2b+2n)_{(k-i-2j+1)}}{(2n-2j+1)_{(-i)}(n+1)_{(-j)}(a+n+1)_{(-j)}(b+n)_{(1-j)}(a+b+n)_{(1-j)}} \times \frac{(2a+2b+4n-4j+1)(2a+2b+4n-2i-4j+k+1)}{(2a+2b+4n-i-2j+1)_{(1+k)}(2a+2b+4n-i-4j+1)_{(1+k)}} \quad (\text{A3})$$

and

$$I_{J_{a,b}}(k, n) = 4^k \sum_{j=0}^k \binom{2k}{2j} \frac{(a+b+2n-4j-1+2k)(\frac{1}{2}(a+b+n))_{(k-j)}(\frac{1}{2}(b+n))_{(k-j)}}{(a+b+2n-2j-1)_{(2k+1)}(\frac{1}{2}(a+n+1))_{(-j)}(\frac{1}{2}(1+n))_{(-j)}}. \quad (\text{A4})$$

The exact moments for $\beta = 1$ are given in (19a); for $\beta = 4$, we have

$$M_{J_{a,b}}^{(4)}(k, n) = \frac{1}{2} M_{J_{2a,2b}}^{(2)}(k, n) - \sum_{j=1}^{\lfloor k/2 \rfloor} \sum_{i=0}^{k-2j} \binom{k}{i} \binom{k}{i+2j} S_{i,j}^{a,b}(k, n). \quad (\text{A5})$$

APPENDIX B: JACOBI POLYNOMIALS AND HYPERGEOMETRIC FUNCTIONS

This appendix contains definitions and identities of Jacobi polynomials and Gauss hypergeometric functions that are needed throughout the paper. They can be found in standard references, in particular in the books by Abramowitz and Stegun,¹ Chap. 15, and Szegő,⁷⁰ Chap. 4. However, because of their extensive use in many proofs, for the convenience of the reader we list them here.

The Jacobi polynomials may be defined explicitly by the formula

$$P_n^{(\alpha,\beta)}(x) := \sum_{j=0}^n \binom{n+\alpha}{n-j} \binom{n+\beta}{j} \left(\frac{x-1}{2}\right)^j \left(\frac{x+1}{2}\right)^{n-j}. \tag{B1}$$

If $\alpha > -1$ and $\beta > -1$, they are orthogonal on $[-1, 1]$ with respect to the measure

$$d\mu = (1-x)^\alpha(1+x)^\beta dx$$

and satisfy the three-term recurrence equation

$$\begin{aligned} &2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{(\alpha,\beta)}(x) \\ &= (2n+\alpha+\beta-1)((2n+\alpha+\beta)(2n+\alpha+\beta-2)x + \alpha^2 - \beta^2)P_{n-1}^{(\alpha,\beta)}(x) \\ &\quad - 2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{(\alpha,\beta)}(x), \quad n = 2, 3, \dots, \end{aligned} \tag{B2}$$

with initial conditions

$$P_0^{(\alpha,\beta)}(x) = 1 \quad \text{and} \quad P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta).$$

They have connection coefficients of the form ($p \in \mathbb{Z}$)

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \sum_{j=0}^p C_{j,p,n}^{\alpha,\beta} P_{n-j}^{(\alpha,\beta+p)}(x), \\ P_n^{(\alpha,\beta)}(x) &= \sum_{j=0}^p (-1)^j C_{j,p,n}^{\beta,\alpha} P_{n-j}^{(\alpha+p,\beta)}(x), \end{aligned} \tag{B3}$$

where

$$C_{j,p,n}^{\alpha,\beta} = \binom{p}{j} \frac{(\alpha+\beta+n+1)_{(p-j)}(\alpha+\beta+2n-2j+1+p)}{(\alpha+\beta+2n-j+1)_{(1+p)}(\alpha+n+1)_{(-j)}}.$$

The Gauss hypergeometric function is defined in the unit circle by the series

$${}_2F_1(a, b; c; z) := \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!}; \tag{B4}$$

it can be analytically continued in the rest of the complex plane. For special values of a and b , this series truncates and becomes a polynomial. In particular, for $\alpha, \beta \in \mathbb{Z}$ it is related to the Jacobi polynomials by the identities

$${}_2F_1(-n, -n+\beta; \alpha+1; x) = \frac{\Gamma(\alpha+1)\Gamma(n-\beta+1)}{\Gamma(n+\alpha-\beta+1)} (x-1)^{n-\beta} P_{n-\beta}^{(\beta,\alpha)}\left(\frac{x+1}{x-1}\right), \tag{B5a}$$

$${}_2F_1(n, n+\beta; \alpha+1; x^{-1}) = \left(\frac{x}{x-1}\right)^{\beta+n} \frac{\Gamma(\alpha+1)\Gamma(n-\alpha)}{\Gamma(n)} P_{n-\alpha-1}^{(\alpha,\beta)}\left(\frac{x+1}{x-1}\right). \tag{B5b}$$

APPENDIX C: GENERALIZATIONS OF CHU-VANDERMONDE’S SUMMATIONS

Many proofs in Sec. IV require computing sums of the form

$$\sum_{i=1}^j \binom{k}{j-i} \binom{k}{j+i} i^p, \tag{C1a}$$

$$\sum_{i=1}^j \binom{k+j-i-1}{k-1} \binom{k+j+i-1}{k-1} i^p, \tag{C1b}$$

where p is an integer. These may be interpreted as generalizations of the classical Chu-Vandermonde convolution identities

$$\sum_{i=-m}^l \binom{r}{m+i} \binom{s}{l-i} = \binom{r+s}{m+l}, \tag{C2a}$$

$$\sum_{i=-m}^l \binom{m+i}{r} \binom{l-i}{s} = \binom{m+l+1}{r+s+1}. \tag{C2b}$$

We now present two lemmas which show how the asymptotics of the double sums in Eqs. (86a) and (86b) are related to sums (C1a) and (C1b), respectively. We then give two examples describing the basic strategy used to evaluate (C1a) and (C1b).

Lemma C.1: Set $\theta = \lfloor k/2 \rfloor$. For any finite set of coefficients $\{C_{i,j}\}$ we have the identity

$$\sum_{j=1}^{\theta} \sum_{i=0}^{k-2j} \binom{k}{i} \binom{k}{i+2j} C_{i,j} = \sum_{j=1}^{k-1} \sum_{i=1}^j \binom{k}{j-i} \binom{k}{i+j} C_{k-i,j,i}. \tag{C3}$$

Proof: Let S_k denote the left-hand side of Eq. (C3). Shifting the summation index in the inner sum as $i \rightarrow k - i - j$ gives

$$\begin{aligned} S_k &= \sum_{j=1}^{\theta} \sum_{i=j}^{k-j} \binom{k}{k-i-j} \binom{k}{k-i+j} C_{k-i,j,j} \\ &= \sum_{i=1}^{k-1} \sum_{j=1}^{\theta-|\theta-i|+1} \binom{k}{i+j} \binom{k}{i-j} C_{k-i,j,j}, \end{aligned} \tag{C4}$$

where in the last passage we interchanged the order of summation. By decomposing the outer sum according to whether $\theta - i \leq 0$ or $\theta - i > 0$, Eq. (C4) becomes

$$\begin{aligned} S_k &= \sum_{i=1}^{\theta} \sum_{j=1}^{i+1} \binom{k}{i+j} \binom{k}{i-j} C_{k-i,j,j} \\ &\quad + \sum_{i=\theta+1}^{k-1} \sum_{j=1}^{2\theta-i+1} \binom{k}{i+j} \binom{k}{i-j} C_{k-i,j,j}. \end{aligned}$$

Now note that the upper limits in both inner sums can be replaced by i : in the first one because if $j = i + 1$, then

$$\binom{k}{i-j} = \binom{k}{-1} = 0;$$

in the second inner sum $i \geq \theta + 1$, thus $2\theta - i + 1 \leq i$. Since j lies in the range

$$2\theta - i + 1 < j \leq i,$$

then

$$j + i > 2\theta + 1 > k \quad \text{and} \quad \binom{k}{i+j} = 0,$$

yielding no contribution to the sum. Finally, relabelling $(i, j) \rightarrow (j, i)$ gives the right-hand side of Eq. (C3). □

Lemma C.2: Suppose that $\{C_{i,j}\}$ is any set of coefficients such that the double series

$$\Theta = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} \binom{k+i-1}{k-1} \binom{k+2j+i-1}{k-1} w^{-k-j-i} C_{i,j} \tag{C5}$$

is absolutely convergent. Then

$$\Theta = \sum_{j=1}^{\infty} \sum_{i=1}^j \binom{k+j-i-1}{k-1} \binom{k+j+i-1}{k-1} w^{-k-j} C_{j-i,i}. \tag{C6}$$

Proof: First, shift the summation index $i \rightarrow i - j$ in (C5); then, (C6) is obtained by interchanging the order of summation and relabelling the indices. \square

Lemmas C.1 and C.2 are important because the inner sums on the right-hand side of (C3) and in (C6) can be explicitly computed whenever $C_{i,j}$ is a polynomial in i and j . This technique is better illustrated with two examples.

Example C.3: Set $p = 0$ in (C1b). From (C2b), we have

$$\sum_{i=-j}^j \binom{k+j-i-1}{k-1} \binom{k+j+i-1}{k-1} = \binom{2k+2j-1}{2k-1},$$

which implies

$$\sum_{i=1}^j \binom{k+j-i-1}{k-1} \binom{k+j+i-1}{k-1} = \frac{1}{2} \left(\binom{2k+2j-1}{2k-1} - \binom{k+j-1}{k-1} \right)^2.$$

This formula was used to obtain (among others) Eq. (93) in Proposition 4.2.

Example C.4: Now take $p = 2$ in the sum (C1a). We have

$$\begin{aligned} \sum_{i=-j}^j \binom{k}{j-i} \binom{k}{j+i} i^2 &= k \sum_{i=-j}^j \binom{k}{j-i} \binom{k-1}{j+i-1} i - j \sum_{i=-j}^j \binom{k}{j-i} \binom{k}{j+i} i \\ &= -k^2 \sum_{i=-j}^j \binom{k-1}{j-i-1} \binom{k-1}{j+i-1} + kj \sum_{i=-j}^j \binom{k-1}{j-i-1} \binom{k}{j+i} \\ &\quad - j^2 \sum_{i=-j}^j \binom{k}{j-i} \binom{k}{j+i} + kj \sum_{i=-j}^j \binom{k}{j-i} \binom{k-1}{j+i-1}. \end{aligned}$$

Evaluating each sum using the identity (C2a) and simplifying give

$$\sum_{i=1}^j \binom{k}{j-i} \binom{k}{j+i} i^2 = \frac{k}{4} (2k-2).$$

To obtain this formula we used the symmetry of the summands in (C1a) under the substitution $i \rightarrow -i$.

Completely analogous considerations allow the computation of these sums for higher values of p .

APPENDIX D: AN EXPLICIT EXAMPLE

Many proofs in this article are based on a systematic approach outlined in Proposition 3.10 and further discussed in Remark 3.11. The algebra involved is rather cumbersome, but can be easily

done with a symbolic algebra computer package like those contained in MAPLE or MATHEMATICA. In order to give the reader a flavour of what such formulae look like, in this appendix we report the explicit expressions that occur in the computation of the second correction of the differences of the moments of the transmission eigenvalues $\Delta\mathcal{T}_{k,2}^{(2,\delta)}(u)$.

In the proof of Proposition 3.14, we showed that

$$\Delta\mathcal{T}_{k,2}^{(2,\delta)}(u) = \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} \frac{u^{2k-2j}}{(u+1)^{2k+3}} F_{k,2}^{(2,\delta)}(u, j). \tag{D1}$$

The coefficient $F_{k,2}^{(2,\delta)}(u, j)$ is polynomial of 2nd order in δ and of 4th in u . It is given explicitly by

$$F_{k,2}^{(2,\delta)}(u, j) = A_j + \frac{\delta}{2} B_j + \frac{\delta^2}{4} C_j, \tag{D2}$$

where

$$\begin{aligned} A_j &= (j-k)(j-1-k)(3j^2 - 6jk - j + k + 3k^2 - 1)/6 \\ &\quad + (j-1-k)(2j-1-2k)(j-k)u/3 \\ &\quad - j(j-1-k)(-jk + k + 1 + j^2 - j)u^2 \\ &\quad - j(2j-1)(j-1)u^3/3 + j(j-1)(3j^2 - 5j + 1)u^4/6, \\ \frac{2B_j}{(u-1)} &= (j-k)(j-1-k)(2j-1-2k) \\ &\quad - j(2j-1)(j-1)u^3 + (1+2j)(j-k)(j-1-k)u \\ &\quad - j(j-1)(2j-2k-3)u^2 \end{aligned}$$

and

$$\begin{aligned} C_j &= 1/2(j-k)(j-1-k) + u(k+1)(j-1-k) \\ &\quad + u^2(j(k-j+1) + k(k+1)/2) - j(k+1)u^3 + j(j-1)u^4/2. \end{aligned}$$

One of the main steps in the demonstration of Proposition 3.14 was to turn formula (D1) into a linear combination of Jacobi polynomials. Below we give the final expression.

Define the transform of a sequence $\xi = (\xi_l)_{l=0}^\infty$ (which may also depend on parameters u and k) by the formula

$$\mathcal{N}[\xi] := \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \binom{k}{j-1} \frac{u^{2k-2j}}{(u+1)^{2k+3}} \xi_j.$$

The second correction $\Delta\mathcal{T}_{k,2}^{(2,\delta)}(u)$ can be written in terms of the sequences $A = (A_l)_{l=0}^\infty$, $B = (B_l)_{l=0}^\infty$ and $C = (C_l)_{l=0}^\infty$ as

$$\Delta\mathcal{T}_{k,2}^{(2,\delta)}(u) = \mathcal{N}[A] + \frac{\delta}{2} \mathcal{N}[B] + \frac{\delta^2}{4} \mathcal{N}[C].$$

By the definition of Jacobi polynomials (B1) and minor algebraic manipulations, we arrive at

$$\begin{aligned} (u+1)^{k+6} \mathcal{N}[A] &= k(1-5k+3k^2)P_{k-2}^{(1,1)}(\tilde{u})(u-1)^{k-2}(u^2+1)u^2/6 \\ &\quad + k(k-1)(2-6k)P_{k-3}^{(2,1)}(\tilde{u})(u-1)^{k-3}u^2/6 + (k-1)^2kP_{k-3}^{(1,1)}(\tilde{u})(u-1)^{k-1}u^2(u+1)^2/2 \\ &\quad - 2k(k-1)P_{k-2}^{(1,0)}(\tilde{u})(u-1)^{k-2}u^3(u+1)/3 + kP_{k-1}^{(0,0)}(\tilde{u})(u-1)^{k-1}u^2(u+1)^2 \\ &\quad - 2k^2P_{k-2}^{(1,1)}(\tilde{u})(u-1)^{k-2}u^3(u+1)/3 + 2k(k-1)P_{k-3}^{(1,2)}(\tilde{u})(u-1)^{k-3}u^5(1-u(3k-1)/2). \tag{D3} \end{aligned}$$

The coefficient of $\frac{\delta}{2}$ is

$$(u+1)^{k+6}\mathcal{N}[B] = -k(k-1)P_{k-2}^{(1,0)}(\tilde{u})(u-1)^{k-1}(u+1)u^2 \\ + kP_{k-2}^{(1,1)}(\tilde{u})(u-1)^{k-1}(u+1)u^2 + k(k-1)P_{k-3}^{(1,2)}(\tilde{u})(u-1)^{k-2}u^4 = 0. \quad (\text{D4})$$

This identity is obtained using formula (B3) and the three-term recurrence relation (B2). The coefficient of $\frac{\delta^2}{4}$ is given by

$$(u+1)^{k+5}\mathcal{N}[C] = kP_{k-2}^{(1,1)}(\tilde{u})(u-1)^{k-2}u^2(1+u^2)/2 \\ - (1+k)uP_{k-1}^{(1,0)}(\tilde{u})(u-1)^{k-1}(u+1)/2 + ku^2P_{k-1}^{(0,0)}(\tilde{u})(u-1)^{k-1}(u+1) \\ - (k+1)u^3P_{k-1}^{(0,1)}(\tilde{u})(u-1)^{k-1}(u+1) + (k+1)u^2P_{k-1}^{(1,1)}(\tilde{u})(u-1)^{k-1}(u+1)/2. \quad (\text{D5})$$

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