Momentum Balance of Gravity Flows

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ABSTRACT

A unified scale analysis of the momentum balance of downslope gravity flows is developed to organize previous theories for the case of negligible ambient flow and fixed temperature deficit scale. The values of several nondimensional parameters are evaluated from observations in the literature and used to assess the validity of certain sets of assumptions required for existing theoretical approaches. A new, simple solution is presented which includes both advective and frictional effects. This solution, as well as several other previous solutions, are found to be useful approximations for certain geophysical situations.

1. Introduction

Previous models of cold-air drainage include different dynamics and thermodynamics. For example, in the conveniently simple treatment of Businger and Rao (1965), the buoyancy generation term is balanced by downslope advection of momentum, while turbulent momentum transport is neglected. In contrast, advection is omitted and the buoyancy term is balanced by the turbulent flux divergence in the studies of Prandtl (1942), Defant (1949) and others. Manins and Sawford (1979a) indicate that entrainment of momentum may be a main term in the momentum equation, although the observational study of Manins and Sawford (1979b) implies that much of the cold-air drainage is either nonturbulent or only weakly turbulent and thus not directly influenced by such entrainment.

Most theories omit a pressure gradient term due to downslope variations of the depth and strength of the cold air, which at least theoretically can become important (Ball, 1956; Ellison and Turner, 1959; Manins and Sawford, 1979a). Some analyses include Coriolis effects as a primary contribution (Holton, 1967; Mahrt and Schwerdtfeger, 1970; Paegle and Rasch, 1973; Brost and Wyngaard, 1978).

Clearly, previous attempts to model drainage flows have taken completely different directions, sometimes without knowledge of implied assumptions. The present study constructs a general scale analysis of the momentum equations to organize the previous treatments of drainage flows according to assumptions required (Sections 2 and 3). The criteria for the validity of various theories are then posed in terms of the values of several nondimensional numbers. In Sections 4 and 5, the physical properties of different classes of slope flows are discussed and a new solution is introduced, which seems applicable to some of the observed slope flow situations which are discussed in Section 6.

Analysis of the thermodynamics is more difficult, since turbulent heat transport in strongly stratified flow and interaction with clear-air radiational cooling is poorly understood and no simple parameterization seems appropriate. The latter depends on temperature distribution, moisture and sometimes blowing snow and ice crystals, in a complicated manner. Therefore, analysis of the thermodynamics will be considered outside the scope of this study, and the subsequent development will be expressed in terms of a specified temperature deficit.

2. Basic momentum equations

We partition the flow for any dependent variable as follows:

$$\phi^* = \phi_0 + \phi + \phi',$$

where ϕ_0 is the basic state flow representative of the flow away from the gravity current and is assumed to be motionless, homogeneous and stationary. ϕ is the flow associated with cooling over the slope which is time-averaged to eliminate "turbulence," and ϕ' represents remaining turbulent fluctuations. Dependent variables include the density ρ, pressure p, motion in the downslope (x) direction u, motion in the cross-slope (y) direction v, and motion perpendicular to the ground w. For mathematical convenience, θ is defined to be the deficit of potential temperature while θ_0 remains the basic state potential temperature representative of the potential temperature outside the gravity flow. We have also neglected the influence of moisture on the buoyancy, which is reasonable for most gravity flows.

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Assuming shallow convection (Dutton and Fichtl, 1969) and constant slope (no curvature), the equations for the downslope flow and motion perpendicular to the ground are, respectively,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + g\frac{\theta}{\theta_0}\sin\alpha + fv - \frac{\partial \overline{w'u'}}{\partial z}, \quad (1)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g\frac{\theta}{\theta_0}\cos\alpha, \quad (2)$$

where we have made the usual neglect of Coriolis and turbulent transport terms in the w-equation of motion and $(\partial/\partial x)\overline{u'^2}$ and $(\partial/\partial y)\overline{v'u'}$ in the downslope flow equation. The latter stress divergences are normally thought to be small, if the depth scale is sufficiently small compared to the horizontal length scale. t is time, f the Coriolis parameter and α the constant positive angle of the slope with respect to the horizontal. The overbar represents the time-average of the slope flow.

To scale the momentum equations, we define the following scales as U downslope velocity scale, L downslope length scale, V cross-slope velocity scale, $\mathcal{L}$ cross-slope length scale, H depth scale, $\Delta\theta$ scale value for potential temperature deficit of the layer.

Assuming that $V/\mathcal{L} \ll U/L$, and using the shallow convection approximation, the velocity scale for w can be estimated from incompressible mass continuity to be UH/L.

a. Hydrostatic approximation

By definition we require that the flow must be generated primarily by the buoyancy term, in which case other terms are smaller or act to retard the flow. Then (1) produces the following restriction on the downslope acceleration

$$\frac{du}{dt} \leqslant g\frac{\theta}{\theta_0}\sin\alpha. \quad (3)$$

We estimate the Lagrangian time scale to be L/U, which neglects, for example, influences due to gravity waves. Then from (3), we obtain the following restriction on the velocity scale

$$U \leqslant \left[g\frac{\theta}{\theta_0}\sin\alpha L\right]^{1/2}. \quad (4)$$

We can now use this relationship to determine when acceleration terms are important in the w-equation of motion. Using the above estimates of w and the Lagrangian time scale, the order of magnitude of the total acceleration perpendicular to the ground (2) becomes

$$\frac{dw}{dt} \approx U^2H/L^2. \quad (5)$$

Relationships (4) and (5) imply that the ratio of the acceleration terms to the buoyancy term in the w-equation of motion (2) is of the order of magnitude of

$$\frac{[g(\theta/\theta_0)\sin\alpha L](H/L^2)}{g(\theta/\theta_0)\cos\alpha} = \frac{H}{L}\tan\alpha. \quad (6)$$

Thus if the slope and/or aspect ratio are sufficiently small so that the above ratio is small compared to one, we can assume that the equation of motion perpendicular to the ground is hydrostatic, in which case

$$g\frac{\theta}{\theta_0}\cos\alpha = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}. \quad (7)$$

The ratio (6) is small compared to one for most geophysical slope flows. Virtually all previous theories of geophysical drainage flows have assumed the hydrostatic approximation for the equation of motion perpendicular to the ground (7). In some of the laboratory experiments of Ellison and Turner (1959), the ratio (6) is not small compared to one, in which case comparisons with their theory assuming (7) are not valid.

b. Equation for downslope motion

The hydrostatic Eq. (7) can be integrated in the z-direction from some arbitrary level to the top of the slope flow, h, where $p(h) = 0$, and then differentiated in the x-direction to derive an expression for the pressure gradient in the equation for downslope flow,

$$\left.\begin{aligned} \frac{1}{\rho_0}\frac{\partial p}{\partial x}(z) &= \cos\alpha\,\frac{g}{\theta_0}\frac{\partial}{\partial x}(\bar{\bar{\theta}}h) \\ \bar{\bar{\theta}} &\equiv \frac{1}{h}\int_z^h \theta dz \end{aligned}\right\}, \quad (8)$$

where $\bar{\bar{\theta}}$ becomes the vertically averaged deficit of potential temperature when $z \rightarrow 0$. Then the equation for downslope motion becomes

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$
$$=g\frac{\theta}{\theta_0}\sin\alpha-\cos\alpha\frac{g}{\theta_0}\frac{\partial(\bar{\bar{\theta}}h)}{\partial x}+fv-\frac{\partial\overline{u'w'}}{\partial z}. \quad (9)$$

For convenience the first term on the rhs will be referred to as the buoyancy term while the second term will be referred to as the thermal wind term, even though both terms are related to thermal wind effects. The second term becomes the classical thermal wind term as the slope magnitude vanishes. This would then become the same term that drives sea breeze circulations and related phenomena due to differential heating over flat terrain. If the temperature and depth of the cold air do not vary along the slope, then the thermal wind term in (9) vanishes. In this case, the isotherms are approximately parallel to the slope and flow is generated only by the buoyancy term. This buoyancy term is analogous to the thermal wind term appearing in Fleagle (1950).

In Fleagle's study, the coordinate system is not rotated, so that tilting of the isotherms due to cooling along the slope leads to a horizontal pressure gradient, *via* the usual thermal wind relationship in non-rotated coordinates. The horizontal pressure gradient appearing in Fleagle's treatment leads to horizontal acceleration. In his coordinate system, vertical motion and nonhydrostatic effects do not appear explicitly. However, since mass continuity demands that the flow near the surface be parallel to the slope, pressure adjustments and a nonhydrostatic vertical component of the acceleration is implied.

While the thermal wind term in (9) has been derived by using the hydrostatic approximation for flow perpendicular to the ground, the equation for flow parallel to the slope is not approximately hydrostatic except in the special condition

$$g\frac{\theta}{\theta_0}\sin\alpha=\cos\alpha\frac{g}{\theta_0}\frac{\partial}{\partial x}(\bar{\bar{\theta}}h). \quad (10)$$

In this trivial situation, downslope flow is not generated. The more usual situation where (10) is not valid, but (7) approximately holds, could be referred to as "quasi-hydrostatic." That is, the component of the gravitational force perpendicular to the ground is approximately balanced by the pressure gradient force, while the component of the gravitational force parallel to the slope is not balanced and leads to downslope acceleration. Most previous theories fall into this category, although a variety of terminology and explanations have been used.

3. Scale analysis for downslope momentum

Using the scales defined in (3), the terms in the downslope momentum equation, relative to the buoyancy term, assume the following order of magnitude:

$$\left.\begin{aligned}
\frac{\partial u}{\partial t}\Big/ g\frac{\theta}{\theta_0}\sin\alpha &= \mathrm{O}(\mathrm{F}\hat{H}/\hat{T})\\
u\frac{\partial u}{\partial x}\Big/ g\frac{\theta}{\theta_0}\sin\alpha &= \mathrm{O}(\mathrm{F}\hat{H})\\
v\frac{\partial u}{\partial y}\Big/ g\frac{\theta}{\theta_0}\sin\alpha &= \mathrm{O}(\mathrm{F}\hat{H}\hat{f}L/\mathcal{L})\\
w\frac{\partial u}{\partial z}\Big/ g\frac{\theta}{\theta_0}\sin\alpha &= \mathrm{O}(\mathrm{F}\hat{H})\\
\frac{g}{\theta_0}\frac{\partial}{\partial x}(\bar{\bar{\theta}}h)\cos\alpha/g\frac{\theta}{\theta_0}\sin\alpha &= \mathrm{O}(\hat{H})\\
fv/g\frac{\theta}{\theta_0}\sin\alpha &= \mathrm{O}(\mathrm{F}\hat{H}\hat{f}/\mathrm{Ro})\\
\frac{\partial\overline{w'u'}}{\partial z}\Big/ g\frac{\theta}{\theta_0}\sin\alpha &= \mathrm{O}[\mathrm{F}(C_D+k)/\sin\alpha]
\end{aligned}\right\}, \quad (11)$$

where

$$\left.\begin{aligned}
&\mathrm{F}\equiv\frac{U^2}{g'H},\quad \hat{T}\equiv TU/L\\
&C_D\equiv-(\overline{w'u'})_{\mathrm{sfc}}/U^2,\quad k\equiv(\overline{w'u'})_h/U^2\\
&\hat{H}\equiv H/\Delta Z_s,\quad \mathrm{Ro}\equiv U/fL,\quad \hat{f}\equiv fT
\end{aligned}\right\}, \quad (12)$$

g' is the reduced gravity $g\Delta\theta/\theta_0$, C_D is the drag coefficient, k is a mixing or entrainment coefficient to be discussed later, ΔZ_s is the surface elevation drop equal to $L\sin\alpha$, T is the Eulerian time scale, and we have used the fact that, with negligible ambient flow, the equation of motion for cross-slope flow implies that $V=\mathrm{O}(fUT)$.

F is the Froude number or inverse of the Richardson number, and here indicates the relative importance of the transport and Coriolis terms with respect to the buoyancy and thermal wind terms. Normally, the buoyancy term will be balanced by one of the terms proportional to the Froude number. However, if the Froude number is sufficiently small, the buoyancy term may be approximately balanced by a thermal wind term due to increasing depth and temperature deficit in the downslope direction. The resulting adverse pressure gradient causes the flow to be relatively weak.

The scale analysis of the thermal wind term assumes that the slope angle is small so that $\cos\alpha=\mathrm{O}(1)$. If the depth of the cold air at the slope origin is not small compared to the average flow depth H, or if the temperature variation along the slope is small compared to the average temperature deficit, then multiple depth and temperature scales are required and the relative magnitude of the thermal wind term will be smaller than $O(\hat{H})$. For example, if the slope flow is draining a large cold air supply from a plateau region, the cold air could be relatively thick at the top of the slope, in which case the gradient of flow depth over the slope might be much smaller than H/L.

In terms other than the thermal wind term, $\hat{H}$ appears with F. The product $[F\hat{H}]$ can be thought of as a modified Froude number where the flow depth is replaced by the surface elevation drop. $\hat{f}$ is the ratio of the flow time scale to the Coriolis time scale and indicates the importance of Coriolis effects, relative to temporal accelerations. $\hat{T}$ can be viewed as the ratio of the time scale T to the Lagrangian time L/U. Ro is the Rossby number, whose inverse indicates the importance of Coriolis effects relative to advective accelerations.

Relationships given in (11) allow us to construct criteria which must be satisfied before the dynamics can be simplified. For example, the flow can be assumed stationary with sufficiently small Froude number and large time scale that

$$F\hat{H}/\hat{T} \ll 1, \quad (13)$$

or in terms of dimensional time

$$T \gg U/g' \sin\alpha.$$

The advection of momentum can be neglected with Froude number sufficiently small that

$$F\hat{H} \ll 1. \quad (14)$$

Cross-slope advection can be neglected if the time scale is sufficiently small and the cross-slope length scale is sufficiently large so that

$$\mathcal{L}/\hat{f}L \gg F\hat{H}. \quad (15)$$

The thermal wind term can be neglected if the flow is thin relative to the elevation drop so that

$$\hat{H} \ll 1. \quad (16)$$

Although importance of the thermal wind term dramatically changes the flow behavior, this ratio is usually small as is discussed further in Section 5. Coriolis effects can be neglected if the Froude number and/or the time scale are sufficiently small that

$$F\hat{H}\hat{f}/R_0 \ll 1, \quad (17)$$

or in terms of dimensional time

$$T \ll g' \sin\alpha/Uf^2.$$

Conditions on the turbulent transport term are more obscure since turbulence observations in slope flows are limited and flux measurements are particularly difficult in weak turbulence characteristic of stably stratified flow. The stress divergence due to surface drag will be unimportant if

$$C_D F/\sin\alpha \ll 1. \quad (18)$$

Since C_D for stable flows is typically $O(10^{-3})$, the drag term is important only if the Froude number is much larger than the slope magnitude.

If the flow above the cold-air drainage is much weaker than the drainage flow, the momentum flux due to entrainment might be approximated in the conventional manner as $w_e U$ where w_e is the entrainment velocity or EU^2 where E is the entrainment coefficient. Then the momentum flux due to entrainment is unimportant if

$$EF/\sin\alpha \ll 1,$$

where

$$E \equiv w_e/U.$$

Estimates of w_e by Manins and Sawford (1979a) indicate that entrainment effects will be more important than surface drag.

Entrainment effects are difficult to determine in actual downslope flows since the transition between the drainage flow and overlying flow can be thick and the drainage flow is not necessarily fully turbulent (Manins and Sawford, 1979b; Mahrt and Larsen, 1982). The entrainment parameter can be replaced with a general exchange coefficient or drag coefficient k as in (12), representing the exchange of momentum between the slope flow and overlying ambient fluid.

Here nonzero k does not necessarily imply a net mass flux into the slope flow. In other words, it is not possible to define a direction to the mass entrainment (up or down) in an internal layer of turbulence embedded within nonturbulent flow. Then the criteria for neglecting stress at the top of the slope flow is

$$kF/\sin\alpha \ll 1. \quad (19)$$

The value of k and the importance of mixing in slope flows need to be estimated from future experimental work.

4. Idealized stationary flows

Eqs. (11–19) jointly suggest several constraints on the flow. For example, many theories of slope flows assume that the flow is stationary and Coriolis effects can be neglected, which can be shown from (13) and (17) to be valid only if

$$U/g' \sin\alpha \ll T \ll g' \sin\alpha/f^2 U, \quad (20)$$

where again g' is the reduced gravity $g\Delta\theta/\theta_0$. Condition (20) is possible only with sufficient separation of time scales which in turn requires

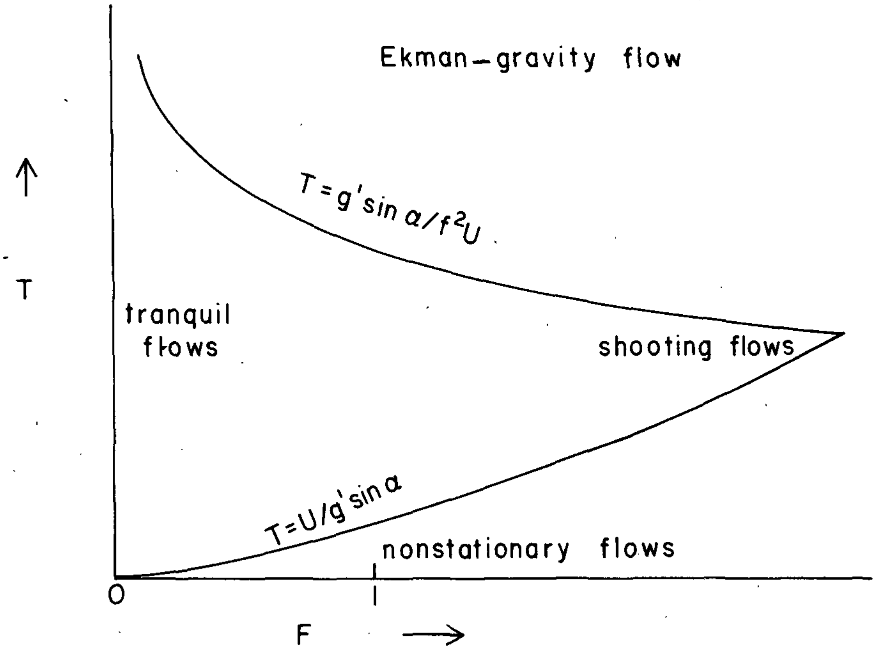

FIG. 1. Dynamical regimes for flow over two-dimensional slopes in time scale (T)–Froude number space for finite values of slope, Coriolis parameter and depth scale.

$$\frac{fU}{g'\sin\alpha} \ll 1. \tag{21}$$

This requirement is often satisfied in geophysical slope flows (Section 6). The region in time–Froude number space where the flow is stationary with negligible Coriolis effects is shown in Fig. 1. It can be shown from (20) and the definition of Froude number (12) that with increasing Froude number for a given slope, Coriolis parameter and flow depth, the minimum time scale for neglect of nonstationarity ($\sim U/g' \sin\alpha$) increases while the maximum time scale for neglect of Coriolis effects ($\sim g' \sin\alpha/f^2U$) decreases. For Froude number greater than $U \sin\alpha/fH$, there is no value of the time scale where stationary flows occur with negligible Coriolis effects.

When nonstationary and Coriolis terms are unimportant, then the scale analysis of (11) simplifies to

$$O(F\hat{H}) = O(1) - O(\hat{H}) - O[F(C_D + k)/\sin\alpha]. \tag{22}$$

This equation requires that either $\hat{H}$ is O(1) and/or F is sufficiently large that $F\hat{H}$ or $F(C_D + k)/\sin\alpha$ are O(1). Observations in Section 6 indicate that $\hat{H}$ is normally small, while the Froude number F generally is large (small Richardson number). Flows with Froude number large compared to one are referred to as shooting flows (Ball, 1956), since large Froude number implies that the flow speed is large compared to $[g'H]^{1/2}$. Relatively weak flows (Froude number small compared to one) are referred to as tranquil flows.

Since the importance of the thermal wind term relative to the acceleration is $O(F^{-1})$, the thermal wind term is potentially important only in tranquil flows; that is, the thermal wind term is most likely to be important in weak flows with large temperature deficit and flow depth. The thermal wind term resulting from increasing flow depth and/or temperature deficit in the direction of the flow acts to oppose the buoyancy acceleration causing the flow to be relatively weak and retain its small Froude number.

This feedback occurs for $F(C_D + k)/\sin\alpha < 1$ in Ball's (1956) analysis of the mass continuity and momentum equations for the case of constant temperature deficit, slope and frictional coefficient. This can be demonstrated by deriving an equation for change of the local Froude number where the latter is defined as

$$F^* \equiv \frac{u^2}{(g\theta/\theta_0)h}, \tag{23}$$

where again h is the local flow depth and u is the local downslope flow speed. Differentiating (23) and using the assumption of constant temperature deficit and the mass continuity equation (Ball's Eq. 3), we obtain

$$\frac{dF^*}{dx} = -3\frac{F^*}{h}\frac{dh}{dx}. \tag{24}$$

Solving for dh/dx and substituting into Ball's com-

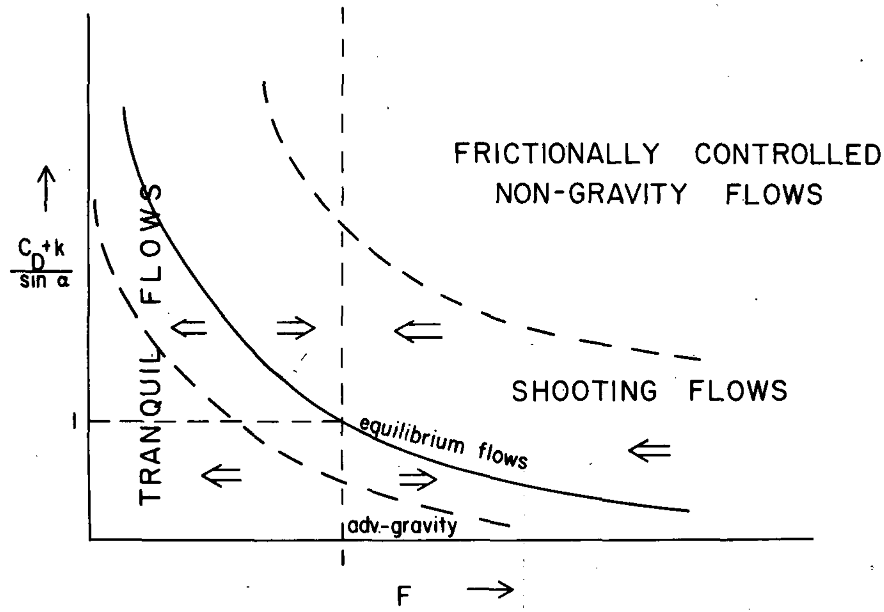

FIG. 2. Dynamical regimes for stationary slope flows with negligible Coriolis effects for a fixed value of the scaled depth $H/\Delta Z_s$. The vertical ordinate is the ratio of the drag coefficient to the terrain slope, F is Froude number, while arrows indicate the tendency of the local Froude number [Eqs. (23) and (25)].

bination of the momentum and mass continuity equation (his Eq. 4), we obtain

$$\frac{hd\ln(\mathrm{F}^*)}{3dx}(1-\mathrm{F}^*) = -\sin\alpha\left[1-\frac{(C_D+k)}{\sin\alpha}\mathrm{F}^*\right], \quad (25)$$

where $x = 0$ is the position of the upstream boundary condition. This equation states that for $\mathrm{F}^*(C_D + k)/\sin\alpha < 1$ (stress divergence smaller than the buoyancy term), the local Froude number decreases down the slope for $\mathrm{F}^* < 1$ and increases down the slope for $\mathrm{F}^* > 1$. Thus tranquil flows remain weak tranquil flows and shooting flows remain strong shooting flows. This evolution is depicted with arrows in Fig. 2 for various parameter regimes.

For $\mathrm{F}^*(C_D + k)/\sin\alpha > 1$, the flow converges toward $\mathrm{F}^* = 1$ in which case the above equations approach a mathematical singularity. However, the frictional drag will normally not exceed the buoyancy term in gravity flows, except for the case of strong inflow or flow into a region of very small slope. Similar conclusions can be constructed from the equation for Richardson number derived in Ellison and Turner (1959, their Eq. 13) and Manins and Sawford (1979a, their Eq. 3.1).

Shooting flows can be further subdivided (Figs. 2–3) into "advective-gravity flow" where the buoyancy term leads primarily to acceleration down the slope [$\mathrm{F}^*(C_D + k)/\sin\alpha \ll 1$] and "equilibrium flow" (sometimes called normal flow) where the buoyancy acceleration is approximately balanced by frictional effects [$\mathrm{F}^*(C_D + k)/\sin\alpha \simeq 1$]. An equilibrium solution is unstable with respect to perturbations for $\mathrm{F}^* > 1$ (Fig. 2) and therefore not expected in the tranquil regime (Fig. 3).

The above flow types are summarized in Fig. 3 and Table 1. Fig. 3 was constructed by noting that the slope-buoyancy term $g' \sin\alpha$ is unimportant compared to the thermal wind term if $\hat{H} \gg 1$, and unimportant compared to advective accelerations if $\mathrm{F}\hat{H} \gg 1$ (Region VIII) and noting that advective accelerations are unimportant if $\mathrm{F}\hat{H} \ll 1$ (Regions I, III and VII), frictional effects are unimportant if $\mathrm{F}(C_D + k)/\sin\alpha \ll 1$ (Regions I, II, V, VII) and the thermal wind terms are unimportant if $\hat{H} \ll 1$ (Regions I, II, III and IV). Observational cases can be categorized by plotting the appropriate points on Fig. 3. Note that exact shapes of the regions in Fig. 3 will depend on the numerical value of $(C_D + k)/\sin\alpha$. Region VIII is identified as non-gravity flows although even a very small slope term may be able to completely alter the nature of the flow (Shirer and Wells, 1982).

5. Idealized solutions

In this section, we briefly examine various simplified flow solutions appearing in the literature in terms of required approximations. The results are summarized in Table 2. We first examine a series of solutions where Coriolis effects, cross-slope advection and nonstationarity are unimportant. Then, layer-in-

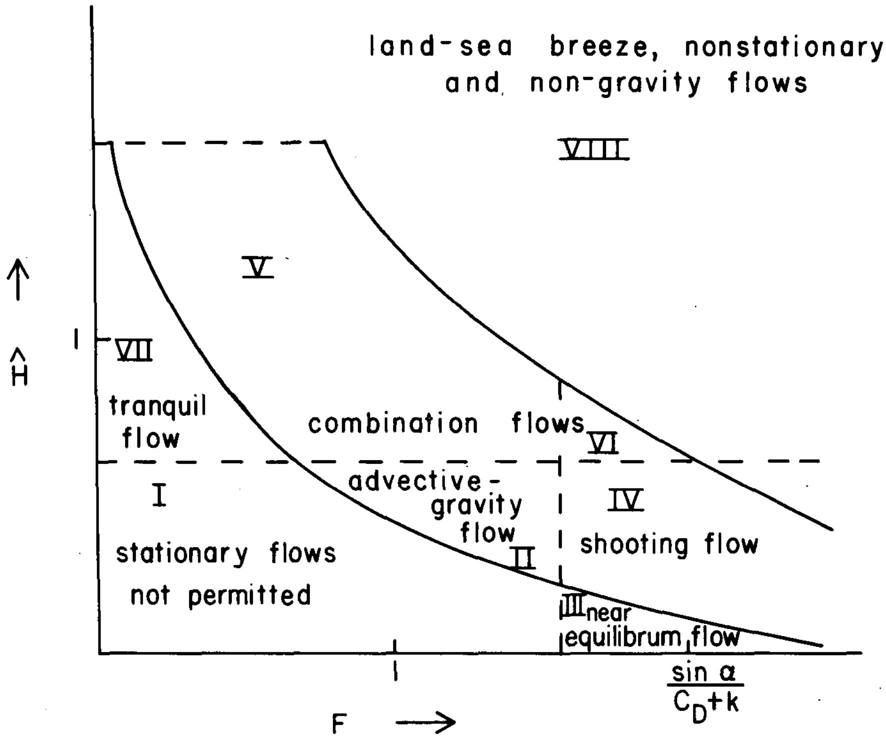

FIG. 3. Dynamical regimes for stationary slope flows with negligible Coriolis effects.

tegrating (9), using the incompressible mass continuity equation and assuming that the temperature deficit and flow speed both vanish at h and at the surface, we obtain

$$\frac{\partial}{\partial x} h\hat{u}^2 = hg\frac{\hat{\theta}}{\theta_0}\sin\alpha - \frac{g}{\theta_0}\cos\alpha\frac{\partial}{\partial x}\bar{\theta}h^2 - (C_D + k)\hat{u}^2, \quad (26)$$

where

$$\left.\begin{aligned} \hat{u}^2 &\equiv \frac{1}{h}\int_0^h u^2 dz \\ \bar{\theta} &\equiv \frac{1}{h^2}\int_0^h \left(\int_z^h \theta dz\right) dz \\ \hat{\theta} &\equiv \frac{1}{h}\int_0^h \theta dz \end{aligned}\right\},$$

and C_D and k assume the same meaning as in (12) by choosing the velocity scale U to be $\hat{u}$.

Solutions for (26) can be obtained only if the depth of the cold air can be determined. The depth of the cold air is changed by horizontal divergence through mass continuity and by heat transfer due to radiative flux and entrainment or intermittent mixing near level h. In this section, we make the simplest approximation of constant flow depth in order to produce several existing solutions in the literature.

a. Advective-gravity flow

If the flow is sufficiently thin that $\hat{H} \ll 1$ and if turbulent transport is negligible [$F(C_D + k)/\sin\alpha \ll 1$], then the buoyancy term accelerates the flow down the slope without significant opposition. From an Eulerian point of view, the buoyancy acceleration is balanced by downslope advection of weaker momentum.

The flow solution for this two-term balance was presented by Businger and Rao (1965). Including vertical advection, the solution with the present coordinate system and notation of (26) becomes

$$\hat{u} = \left[g\frac{\hat{\theta}}{\theta_0}(\sin\alpha)x\right]^{1/2} \leqslant \hat{u}_a \equiv \left[g\frac{\hat{\theta}}{\theta_0}\Delta Z_s\right]^{1/2}, \quad (27)$$

where x is the distance downstream from the virtual source where $u = 0$. With advective–gravity flow, the downslope flow increases down the slope according to the square root of the distance along the slope and according to the square root of the temperature deficit.

b. Equilibrium flow

If the buoyancy acceleration is balanced by the turbulent-stress divergence ($\hat{H} \ll 1$, $F\hat{H} \ll 1$), then

TABLE 1. Classes of stationary flows with negligible Coriolis effects.

Region (Fig. 3)	Parameter restrictions	Flow type
I	$\hat{H} \ll 1$ $F\hat{H} \ll 1$ $F(C_D + k)/\sin\alpha \ll 1$	Nonstationary flows
II	$\hat{H} \ll 1$ $F\hat{H} = O(1)$ $F(C_D + k)/\sin\alpha \ll 1$	Advective-gravity flow
III	$\hat{H} \ll 1$ $F\hat{H} \ll 1$ $F(C_D + k)/\sin\alpha = O(1)$	Near equilibrium flow
IV	$\hat{H} \ll 1$ $F\hat{H} \approx O(1)$ $F(C_D + k)/\sin\alpha = O(1)$	Shooting flow
V	$\hat{H} = O(1)$ $F\hat{H} = O(1)$ $F(C_D + k)/\sin\alpha \ll 1$	Combination flows
VI	$\hat{H} = O(1)$ $F\hat{H} = O(1)$ $F(C_D + k)/\sin\alpha = O(1)$	Combination flow with friction
VII	$F\hat{H} \ll 1$ $\hat{H} = O(1)$ $F(C_D + k)/\sin\alpha \ll 1$	Tranquil flow
VIII	$F\hat{H} \gg 1$	Non-gravity flows

$$\hat{u} = \hat{u}_e \equiv \left[hg \frac{\hat{\theta}}{\theta_0} \sin\alpha/(C_D + k) \right]^{1/2}. \tag{28}$$

A solution of this form has been examined by Ball (1956). In the log-linear theory applied by Munro and Davies (1977), $(C_D + k)$ is replaced by $k/[\ln(z/z_0) + \alpha z/L]$ where k in this relationship is the von Kármán constant, z_0 the surface roughness, L the Obukhov length and α an empirical coefficient.

With the above force balance (28), the Froude number is equal to $\sin\alpha/(C_D + k)$ (Fig. 3) if we choose $U = \hat{u}$ and $\Delta\theta = \theta$. The flow descends the slope with constant speed for the case of constant temperature deficit and depth. As in the advective-gravity flow case, the flow speed is proportional to the square root of both the potential temperature deficit and slope. In the case where the Reynolds stress divergence is assumed to be linearly proportional to the flow speed, the equilibrium flow is linearly proportional to the slope and buoyancy deficit (Petkovsek and Hocevar, 1971). The analysis of gravity flow over a glacier by Munro and Davies (1977) implies that the coefficient of proportionality implied by (28) is sensitive to stability. The early solutions of Prandtl (1942) and Defant (1949) are also a form of the equilibrium solution which employ eddy diffusivities and a simple thermodynamic relationship where diffusion of heat is

TABLE 2. Classes of stationary drainage flows ($F\hat{H}/\hat{T} \ll 1$) with negligible cross-slope advection $F\hat{H}\hat{f}L/\mathcal{L} \ll 1$.

Summary of possible additional restrictions

Term neglected	*Required criteria*
1) $u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z}$	$F\hat{H} \ll 1$
2) $\dfrac{g}{\theta_0}\cos\alpha \dfrac{\partial}{\partial x}(\bar{\bar{\theta}}h)$	$\hat{H} \ll 1$
3) fv	$F\hat{H}\hat{f}/\mathrm{Ro} \ll 1$
4) $\dfrac{\partial \overline{w'u'}}{\partial z}$	$F(C_D + k)/\sin\alpha \ll 1$

Special flows

Name	*Theories in literature*	*Restrictions*	*Additional parameter relationships*
Advective-gravity	Businger and Rao (1965)	(2) (3) (4)	$F\hat{H} = O(1)$
Equilibrium flow	Prandtl (1942), Defant (1949), Petkovsek and Hocevar (1971), Manins and Sawford (1979b), Munro and Davies (1977), Lettau (1966)	(1) (2) (3)	$F(C_D + k)/\sin\alpha = O(1)$
Shooting flows	Ball (1956), Ellison and Turner (1959), Tang (1976), Manins and Sawford (1979a)	(2) (3)	$F \gg 1$
Tranquil flows	Ball (1956), Ellison and Turner (1959), Manins and Sawford (1979a)	(3)	$\hat{H} \gtrsim O(1)$, $F \lesssim O(1)$
Ekman-gravity flow	Mahrt and Schwerdtfeger (1970)	(1) (2)	$F\hat{H}\hat{f}/\mathrm{Ro} = O(1)$

balanced by temperature advection associated with the basic state stratification. With such thermodynamics, the flow strength becomes linearly proportional to the temperature deficit.

c. Shooting flow

If both downslope advection and turbulent transport are important, but the thermal wind term remains unimportant $\hat{H} \ll 1$, then (26) reduces to the three-term balance analogous to the shooting flows studied by Ball (1956), Ellison and Turner (1959) and Manins and Sawford (1979a). Tang (1976) numerically obtained solutions for such flow over periodic terrain.

Neglecting the thermal wind term and again assuming constant h, the solution to (26) is

$$\left.\begin{aligned} \hat{u} &= \{\hat{u}_e^2[1 - \exp(-x/L_e)] \\ &\qquad + \hat{u}^2(0)\exp(-x/L_e)\}^{1/2} \\ \hat{u}_e &\equiv \left[hg\frac{\hat{\theta}}{\theta_0}\sin\alpha/(C_D + k)\right]^{1/2} \\ L_e &\equiv h/(C_D + k) \end{aligned}\right\}, \quad (29)$$

where $u(0)$ is the flow at some arbitrary point on the slope where the integration is initiated ($x = 0$). This solution indicates that the flow adjusts to the equilibrium flow solution (28) after a horizontal adjustment length scale of

$$h/(C_D + k).$$

Since the adjustment rate is proportional to the stress divergence, the adjustment length scale is inversely proportional to the drag coefficient and proportional to the depth of the flow. That is, the greater the drag coefficient, the faster the flow adjusts to an equilibrium between buoyancy and frictional effects. A numerical analysis of shooting flows where temperature adjustments and flow depth are changed by entrainment appears in Manins and Sawford (1979a).

d. Ekman-gravity flow

With sufficiently large length scale and resulting large parcel time scale, Coriolis influences become important. Ekman-gravity flow results from a balance between the Coriolis, Reynolds stress and buoyancy terms (Mahrt and Schwerdtfeger, 1970). The thermal wind term in their study is analogous to the buoyancy term in this study, while the thermal wind term due to variation of flow temperature and depth was not included. Such flow attempts to orient itself parallel to the terrain contours, except for a frictionally driven downslope component. The large scale studies of slope flow by Paegle and Rasch (1973) and others, additionally include synoptic scale pressure gradients and inertial oscillations, both beyond the scope of this study.

6. Analysis of existing data

We now analyze existing data in the literature to identify the criteria and idealized flow types which are likely to approximate actual geophysical situations. To evaluate the necessary criteria, observations of structure of both wind and temperature are required. Previous observations which seem suitable are listed in Table 3 along with subsequent calculations.

The depth H is chosen to be the depth of significant temperature deficit which coincides with the layer of enhanced thermal stratification; that is, there are no cases of well-mixed slope flows in Table 3. The velocity scale is chosen to be the layer-averaged downslope speed, and the temperature deficit scale is chosen to be the layer-averaged deficit of potential temperature.

The time scale T is chosen to be 10^4 s to estimate the order of magnitude of temporal accelerations associated with the diurnal evolution of drainage circulations. Then the computed value of $F\hat{H}/\hat{T}$ is small for all the flows except Lettau's (Table 3), implying that temporal accelerations due to the diurnal variation are generally unimportant. However, this calculation does not rule out the importance of nonstationarity due to pulsation of the gravity flow and variations of the ambient flow.

In the observational studies cited here, the cross-slope flow was generally negligible so that cross-slope advections are probably unimportant. The Lagrangian time scale was too small for the Coriolis term to be significant, except in the study of Lettau where the Coriolis and buoyancy terms were the same order of magnitude.

Table 3 indicates that for $C_D = 2.5 \times 10^{-3}$, surface drag is only marginally important except in the flows of Martin, Munro and Davies, and Lettau where it appears to be a primary term. The value of 2.5×10^{-3} is equal or comparable to values suggested by several of the authors in Table 3, although this value seems too large in the case of Lettau. Advection appears to be of some importance, especially in the flows of Ohata and Higuchi, Mahrt and Larsen, and Munro and Davies where advection appears to be a primary term and the advective-gravity solution becomes a reasonable approximation ($U/\hat{u}_a \sim 1$). The thermal wind term is potentially important only in the flow of Mahrt and Larsen.

Internal drag due to mixing at the top of the flow is of unknown importance, although its importance was suggested in the study of Ball (1956), and inferred in the analyses of Manins and Sawford (1979b), and Mahrt and Larsen (1982). By a process of elimination,

TABLE 3. Scale variables computed from previous observational studies.

Data source	H (m)	$\bar{U}$ (m s^{-1})	$\overline{\Delta\theta}$ (K)	L (m)	Slope $\sin\alpha$ (%)	ΔZ_s (m)	F	Ri ≡ F^{-1}	$\hat{H}$	F$\hat{H}$	F$\hat{H}/\hat{T}$	C_DF/$\sin\alpha$	$U/\hat{u}_a$	$\sin\alpha$/F (10^{-2})
Martin (1975, Fig. 3) 20 September 1970	2	2.4	2.5	1100	12	132	31.10	0.03	0.015	0.47	0.02	0.65	0.69	0.39
6 September 1970 (Figs. 6, 8)	2	2.8	3.5	1100	12	132	30.24	0.03	0.015	0.45	0.02	0.63	0.68	0.40
Ohata and Higuchi (1979) 5 September 1975	3	1.6	3.25	200	15	30	7.09	0.14	0.100	0.71	0.01	0.12	0.84	2.12
3 July 1975	3	2.9	2	700	17	120	37.85	0.03	0.025	0.95	0.02	0.56	0.98	0.45
Manins and Sawford (1979b, Fig. 6)	45	2.7	4	4000	6.1	244	1.09	0.92	0.184	0.20	0.03	0.04	0.45	5.60
Mahrt and Larsen (1982 Fig. 3)	10	1.0	2.2	400	3.5	14	1.23	0.81	0.714	0.87	0.04	0.09	0.94	2.85
Munro and Davies (1977, Fig. 3)	6	4.4	3.2	5000	5	250	27.23	0.04	0.024	0.65	0.08	1.36	0.81	0.18
Lettau (1966)	8	2.7	4.5	2×10^5	0.17	340	5.47	0.18	0.024	0.13	0.96	8.04	0.36	0.03

the present scale analysis indicates that mixing at the top of the gravity flow is important in the flow of Manins and Sawford in agreement with their analysis of the momentum budget. Mixing at the top of the gravity flow studied by Manins and Sawford is independently suggested by low Richardson numbers at these levels due to extension of the gravity flow and shear above the region of strong stratification.

An estimate of the upper bound of the internal mixing coefficient k_{max} can be constructed by assuming the buoyancy acceleration to be balanced by internal drag. From (28) this estimate is

$$k_{max} = \sin\alpha/\mathrm{F}. \quad (30)$$

If we neglect the influence of the sometimes significant flow aloft, k_{max} can be estimated from (30) with the data in Table 3. The upper bound values of k (Table 2) are generally O(10^{-2}), which is the same order of magnitude as the entrainment coefficient used by Manins and Sawford (1979b).

7. Conclusions

The above scale analysis indicates that atmospheric gravity flows can sometimes be approximated by a balance between the buoyancy acceleration and downslope advection of weaker momentum. In other situations, surface drag and/or turbulent transport of momentum associated with mixing at the top of the gravity flow appears to be important. Both of these circumstances are included in the simple shooting flow solution (29). The thermal wind term due to downslope variation of flow depth and temperature deficit was estimated to be generally unimportant.

The above solutions and scale analysis can serve as useful instructive tools for organizing previous theories and guidelines for future observational programs, even though such solutions fail to account for possible atmospheric complications due to three-dimensionality, surface inhomogeneity and flow aloft.

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REFERENCES

Ball, F. Kl, 1956: The theory of strong katabatic winds. *Aust. J. Phys.*, **9**, 373–386.

Brost, R. A., and J. C. Wyngaard, 1978: A model study of the stably stratified planetary boundary layer. *J. Atmos. Sci.*, **35**, 1427–1440.

Businger, J. A., and K. Ramana Rao, 1965: The formation of drainage wind on a snow-dome. *J. Glaciol.*, **4**, 833–841.

Defant, F., 1949: Zur Theorie der Hangwinde, nebst bemerkungen zur Theorie der Berg- und Talwinde. *Arch. Meteor. Geophys. Bioklim.*, **A1**, 421–450.

Dutton, J. A., and G. H. Fichtl, 1969: Approximate equations of motion for gases and liquids. *J. Atmos. Sci.*, **26,** 241–254.

Ellison, T. H., and J. S. Turner, 1959: Turbulent entrainment in stratified flows. *J. Fluid Mech.*, **6,** 423–448.

Fleagle, R. G., 1950: A theory of air drainage. *J. Meteor.*, **7,** 227–232.

Holton, J. R., 1967: The diurnal boundary layer wind oscillation above sloping terrain. *Tellus*, **19,** 199–205.

Lettau, H. H., 1966: A case study of katabatic flow on the south polar plateau. *Antarct. Res. Ser.*, **9,** Amer. Geophys. Union, 1–11.

Mahrt, L., and S. Larsen, 1982: Small scale drainage front. *Tellus* (in press).

——, and W. S. Schwerdtfeger, 1970: Ekman spirals for exponential thermal wind. *Bound.-Layer Meteor.*, **1,** 137–145.

Manins, P. C., and B. L. Sawford, 1979a: A model of katabatic winds. *J. Atmos. Sci.*, **36,** 619–630.

——, and ——, 1979b: Katabatic winds: A field case study. *Quart. J. Roy. Meteor. Soc.*, **105,** 1011–1025.

Martin, S., 1975: Wind regimes and heat exchange on Glacier de Saint-Sorlin. *J. Glaciol.*, **14,** 91–105.

Munro, D. S., and J. A. Davies, 1977: An experimental study of the glacier boundary layer over melting ice. *J. Glaciol.*, **18,** 425–436.

Ohata, T., and K. Higuchi, 1979: Gravity wind on a snow patch. *J. Meteor. Soc. Japan*, **57,** 254–263.

Paegle, J., and G. Rasch, 1973: Three-dimensional characteristics of diurnally varying boundary-layer flows. *Mon. Wea. Rev.*, **101,** 746–756.

Petkovsek, Z., and A. Hocevar, 1971: Night drainage winds. *Arch. Meteor. Geophys. Bioklim.*, **A20,** 353–360.

Prandtl, L., 1942: *Strömungslehre.* Verlag Vieweg & Sohn, Braunschweig, 373–375.

Shirer, H. N., and R. Wells, 1982: Improving spectral models by unfolding their singularities, *J. Atmos. Sci.*, **39,** 610–621.

Tang, W., 1976: Theoretical study of cross-valley wind circulation. *Arch. Meteor. Geophys. Bioklim.*, **A25,** 1–18.