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MONADIC SECOND ORDER DEFINABLE RELATIONS ON THE BINARY TREE

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Abstract. Let S2S [WS2S] respectively be the strong [weak] monadic second order theory of the binary tree T in the language of two successor functions. An S2S-formula whose free variables are just individual variables defines a relation on T (rather than on the power set of T). We show that S2S and WS2S define the same relations on T, and we give a simple characterization of these relations.

§1. The infinite binary tree T is given by the set $\{0, 1\}^*$ of all finite (0, 1)-words, called the nodes of the tree. Every node x has two successor nodes, $s_0(x) := x0$ and $s_1(x) := x1$.

S2S is the monadic second order theory of (T, s_0, s_1) in the language of two successor functions: In addition to the first order theory there are set variables ranging over subsets of T, existential and universal quantifier over set variables and the membership relation. WS2S is the corresponding monadic *weak* second order theory: Set variables range only over *finite* subsets of T.

An S2S-formula with free set variables defines a relation on $P(T)$, the power set of T, while an S2S-formula with just free individual variables defines a relation on T.

The following results are due to M. O. Rabin (see [7] and [8]):

(I) *There are* S2S-*definable even one-place relations on* $P(T)$ *which are not* WS2S-*definable.*

(II) *A subset of* T *is* S2S-*definable iff it is regular*; *in particular*, S2S *and* WS2S *define the same one-place relations on* T.

A slightly simpler proof of (II), based on [4], is given in [10].

In this paper we give a simple characterization of the S2S-definable relations on T. In particular, we prove

THEOREM 1. *For* $n \in \omega$ *and* $R \subset T^n$, R *is* S2S-*definable iff it is* WS2S-*definable.*

The corresponding result for S1S, the monadic second order theory of the natural numbers with successor function, is due to J. R. Büchi [2] and answers a question raised by R. M. Robinson in [9]. Monadic second-order definability and weak monadic second-order definability are known to be equivalent even for relations on $P(\omega)$ (see W. Thomas [11]).

Our proof is based on Rabin's characterization of S2S-definability in terms of finite tree automata.

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A natural question (raised by Rabin upon communication of our result) is this: Does Theorem 1 generalize to the case where free variables are allowed to range over paths in T? We do not know the answer.

§2. We start by giving a characterization of the binary S2S-definable relations on T and some examples.

We use small letters $a, b, \ldots, x, y, \ldots$ for elements of T and capital letters $A, B, \ldots, X, Y, \ldots$ for subsets of T. Concatenation of words a, b is written ab. λ is the empty word, Λ the empty set. $AB := \{ab \mid a \in A \text{ and } b \in B\}$, $aB := \{a\}B = \{ab \mid b \in B\}$. Thus, $\lambda B = B$ and $\Lambda B = \Lambda$. $A^0 := \{\lambda\}$, $A^{n+1} := A^n A$, and $A^* := \bigcup_{n \in \omega} A^n$.

The *regular* subsets of T (in the sense of Kleene [5]) are given by the following formation rules:

a) Every finite subset of T is regular.

b) If A and B are regular subsets of T, then so are $A \cup B$, AB, and A^*.

For later use we state the following well-known fact.

PROPOSITION 1. *The class of regular sets is closed under Boolean operations; if aB is regular, then so is B.*

A relation $R \subset T^2$ is said to be *special* if $R = \{(ab, ac) \mid a \in A,\ b \in B,\ c \in C\}$ for some regular subsets A, B, C of T.

THEOREM 2. *For $R \subset T^2$, R is* S2S-*definable iff it is a finite union of special relations.*

EXAMPLES. Let us use the abbreviation $[A, B, C]$ for $\{(ab, ac) \mid a \in A, b \in B, c \in C\}$.

1. $[T, \{\lambda\}, T]$ is the partial order $\leq$ by initial segments, $[T, \{\lambda\}, T] \cup [T, 0T, 1T]$ is the lexicographical ordering, and $[T, \{\lambda\}, T \backslash \{\lambda\}] \cup [T, T \backslash \{\lambda\}, \{\lambda\}] \cup [T, 0T, 1T] \cup [T, 1T, 0T]$ is inequality.

2. The relation "$xy = z$" is not S2S-definable, not even the relation "$x = 0y$". Otherwise, the relation $x = 0y \wedge 1 \leq y$ could be represented as $\bigcup_k [A_k, B_k, C_k]$. This implies $A_k = \{\lambda\}$ and, since the relation is one-to-one, the B_k's and C_k's are singletons, which makes the relation finite, a contradiction.

3. "x and y are of the same length" is not S2S-definable.

It is even known that the theory WS2S (T, s_0, s_1, P) is undecidable if P is one of the predicates "$x = 0y$" or "x and y are of the same length". (See Savioz [10] and Buszkowski [3]. For the strong second order case, the following simple undecidability proof was pointed out to us by the referee: The domino problem on a quadrant of the plane can be formulated using "$x = 0y$", together with "$x = y1$", as grid successor functions on 0^*1^*.)

Incidentally, the reader who is familiar with the terminology of [1] will observe that the class of S2S-definable relations $R \subset T^2$ is properly included in the class of "rational" relations and properly contains the class of "recognizable" relations. (The relation "$x = 0y$" is rational but not S2S-definable, while the relation "$x \leq y$" is S2S-definable but not recognizable.)

§3. In order to state our result in more generality we need some additional notation and terminology.

If U is a word in $\{1, 2, \ldots, m\}^*$ and a_k, $k = 1, 2, \ldots, m$, are words in $T = \{0, 1\}^*$, then $U[\vec{a}]$ denotes the word in T obtained from U by substitution $k \to a_k$.

A finite sequence of words in $\{1,2,\ldots,m\}^*$ is said to be *admissible*, if it can be obtained according to the following rules:

a) The one-term sequence (k) whose entry is the one-letter word k is admissible.

b) If $(\vec{U}, V)$ is admissible ($\vec{U}$ possibly empty) and h, k do not occur in any word of this sequence and $h \neq k$, then $(\vec{U}, Vh, Vk)$ is admissible.

c) Any permutation of an admissible sequence is admissible.

EXAMPLE. $(371, 32, 374)$ is admissible; $(13, 23)$ is not.

If $(U_1, U_2, \ldots, U_n)$ is an admissible sequence of words in $\{1,2,\ldots,m\}^*$ and $A_1, A_2, \ldots, A_m$ are regular subsets of T, then

$$R = \{(U_1[\vec{a}], U_2[\vec{a}], \ldots, U_n[\vec{a}]) \mid a_i \in A_i, i = 1, \ldots m\}$$

is a *special* (n-ary) relation on T.

Let Th be the *first-order* theory of T in the following language and interpretation. Language: A constant λ, a binary function symbol $\wedge$ and, for each regular subset $A \subset T$, a binary predicate P_A. Interpretation: λ is the empty word, $x \wedge y$ is the maximal common initial segment of the words x and y, and $P_A(x, y)$ holds iff $x \in yA$ (that is, $x = ya$ for some $a \in A$). Thus the atomic formulas of Th are $P_A(t, s)$, where t, s are $\wedge$-terms built from individual variables and λ. $P_A(x, \lambda)$ means $x \in A$.

THEOREM. *Let $n \geq 1$. For relations $R \subset T^n$, the following are equivalent*:

(i) *R is* S2S-*definable.*

(ii) *R is* WS2S-*definable.*

(iii) *R is* Th-*definable by a finite disjunction of finite conjunctions of atomic formulas of* Th.

(iv) *R is a finite union of special relations.*

We note as a corollary:

COROLLARY. Th *admits quantifier elimination.*

We first prove the easy implications (ii) → (i), (iii) → (ii) and (iv) → (ii).

(ii) → (i). It is well known that the notion "X is finite" is S2S-definable.

(iii) → (ii). It is well known that λ and $x \wedge y$ are WS2S-definable. As to P_A: If A is finite, then $x \in yA$ if $\bigvee_{a \in A}(x = ya)$; for fixed a, ya is given by an (s_0, s_1)-term (the reader is reminded that s_0 and s_1 are the successor functions on T). Furthermore,

$$\begin{aligned}
x \in y(A \cup B) \quad &\text{iff} \quad (x \in yA \vee x \in yB),\\
x \in y(AB) \quad &\text{iff} \quad \exists z(z \in yA \wedge x \in zB),\\
x \in yA^* \quad &\text{iff} \quad \forall X^{\text{finite}}[(x \in X \wedge \forall u \forall v(u \in X \wedge u \in vA \rightarrow v \in X)) \rightarrow y \in X].
\end{aligned}$$

(iv) → (ii). By way of example, if

$$R = \{(ab, acd, ace) \mid a \in A, b \in B, c \in C, d \in D, e \in E\},$$

then $(x, y, z) \in R$ iff

$$\exists u[(x, u) \in \{(ab, ac) \mid a \in A, b \in B, c \in C\} \wedge y \in uD \wedge z \in uE]$$

iff

$$\exists u[\exists v(v \in A \wedge x \in vB \wedge u \in vC) \wedge y \in uD \wedge z \in uE],$$

which is WS2S-definable according to the last paragraph.

Next, we prove a simple proposition and state the main lemma, which settles the remaining implications (i) → (iii) and (i) → (iv).

For $n \geq 2$, $i, j \leq n$, $i \neq j$, let $\delta^n_{i,j}(x_1, \ldots, x_n)$ and $\varepsilon^n_{i,j}(x_1, \ldots, x_n)$ be S2S-formulas expressing the following:

$$\delta^n_{i,j}(\vec{x}): x_i \leq x_j \wedge \bigwedge_{k \neq i,j} \neg (x_i \leq x_k),$$

$$\varepsilon^n_{i,j}(\vec{x}): (x_i \wedge x_j)0 \leq x_i \wedge (x_i \wedge x_j)1 \leq x_j$$
$$\wedge \bigwedge_{k \neq i,j} \neg [(x_i \wedge x_j)0 \leq x_k \vee (x_i \wedge x_j)1 \leq x_k].$$

We write $T \vDash \varphi(\vec{x})$ if φ is identically true in the structure (T, s_0, s_1).

PROPOSITION 2. *For* $n \geq 2$,

$$T \vDash \bigvee_{i \neq j} [x_i = x_j \vee \delta^n_{i,j}(\vec{x}) \vee \varepsilon^n_{i,j}(\vec{x})].$$

PROOF (by induction). The assertion holds for $n = 2$. Given $(\vec{x}, x_{n+1})$, $n \geq 2$, assume that the x_i's are pairwise distinct, and, for instance, assume $\delta^n_{1,2}(\vec{x})$. If $\neg (x_1 \leq x_{n+1})$, then $\delta^{n+1}_{1,2}(\vec{x}, x_{n+1})$. If $x_1 < x_{n+1}$, then one of $\delta^{n+1}_{2,n+1}(\vec{x}, x_{n+1})$, $\delta^{n+1}_{n+1,2}(\vec{x}, x_{n+1})$, $\varepsilon^{n+1}_{2,n+1}(\vec{x}, x_{n+1})$ or $\varepsilon^{n+1}_{n+1,2}(\vec{x}, x_{n+1})$ holds. The case $\varepsilon^n_{1,2}(\vec{x})$ is analogous.

To avoid subscripts we consider $(n + 2)$-tuples $(x, y, \vec{z})$ and write $\delta(x, y, \vec{z})$ for $\delta^{n+2}_{1,2}(x, y, \vec{z})$, where x is x_1 and y is x_2.

MAIN LEMMA. *Let* $\varphi(x, y, \vec{z})$ *be an* S2S-*formula with* $n + 2$ *free individual variables.*

a) *If* $T \vDash \varphi(x, y, \vec{z}) \to \delta(x, y, \vec{z})$, *then there are regular sets* B_k *and* S2S-*formulas* $\varphi_k(x, \vec{z})$ *with* $n + 1$ *free individual variables*, $k = 1, \ldots, m$, *such that*

$$T \vDash \varphi(x, y, \vec{z}) \leftrightarrow \bigvee_k [\varphi_k(x, \vec{z}) \wedge y \in xB_k].$$

b) *If* $T \vDash \varphi(x, y, \vec{z}) \to \varepsilon(x, y, \vec{z})$, *then there are regular sets* A_k, B_k *and formulas* $\varphi_k(u, \vec{z})$ *with* $n + 1$ *variables such that*

$$T \vDash \varphi(x, y, \vec{z}) \leftrightarrow \bigvee_k [\varphi_k(x \wedge y, \vec{z}) \wedge x \in (x \wedge y)0A_k \wedge y \in (x \wedge y)1B_k].$$

For the following, call a formula $\varphi(\vec{x})$ *nice* if for some $i, j \leq n$, $i \neq j$, either $T \vDash \varphi(\vec{x}) \to x_i = x_j$ or $T \vDash \varphi(\vec{x}) \to \delta^n_{i,j}(\vec{x})$ or $T \vDash \varphi(\vec{x}) \to \varepsilon^n_{i,j}(\vec{x})$ holds.

Proof of (i) → (iii). We have to show that every S2S-formula $\varphi(x_1, \ldots, x_n)$ is equivalent to a disjunction of conjunctions of atomic formulas of Th. We do this by induction. For $n = 1$, $T \vDash \varphi(x) \leftrightarrow x \in A$ for some regular set A (Theorem (II)). "$x \in \lambda A$" is an atomic formula of Th.

Induction step. By Proposition 2, every formula is equivalent to a disjunction of nice formulas. The induction step for nice formulas is accomplished by the main lemma and by the obvious reduction: If $T \vDash \varphi(x, y, \vec{z}) \to x = y$, then

$$T \vDash \varphi(x, y, \vec{z}) \leftrightarrow (\varphi(x, x, \vec{z}) \wedge y \in x\{\lambda\}).$$

Proof of (i) → (iv) (by induction). For $n = 1$, again by Theorem (II), $\varphi(x)$ iff $x \in A = \{(a) \mid a \in A\}$, a special relation (we just define one-tuples this way: $(a) = a$).

Induction step. It again suffices to consider nice formulas.

Case 1. $T \models \varphi(x,y,\vec{z}) \to x = y$. By the induction hypothesis, $\varphi(x,x,\vec{z})$ iff $(x,\vec{z}) \in \bigcup_k R_k$ with R_k special. Thus, $\varphi(x,y,\vec{z})$ iff $\bigvee\!\!\!\bigvee [(x,\vec{z}) \in R_k$ and $y = x]$. But, by way of example,

$$(x,z) \in \{(ab,ac) \mid a \in A, b \in B, c \in C\} \quad \text{and} \quad y = x$$

iff

$$(x,y,z) \in \{(abd,abe,ac) \mid a \in A, b \in B, c \in C, d \in \{\lambda\}, e \in \{\lambda\}\}.$$

Case 2. $T \models \varphi(x,y,\vec{z}) \to \delta(x,y,\vec{z})$. By part a) of the main lemma and the induction hypothesis (and distributivity),

$$\varphi(x,y,\vec{z}) \quad \text{iff} \quad \bigvee\!\!\!\bigvee_{h,k} [(x,\vec{z}) \in R_{hk} \wedge y \in xB_k].$$

By way of example,

$$(x,z) \in \{(uv,uw) \mid u \in U, v \in V, w \in W\} \quad \text{and} \quad y \in xB$$

iff

$$(x,y,z) \in \{(uva,uvb,uw) \mid u \in U, v \in V, w \in W, a \in \{\lambda\}, b \in B\}.$$

Case 3. $T \models \varphi(x,y,\vec{z}) \to \varepsilon(x,y,\vec{z})$. Then, by part b) of the main lemma and the induction hypothesis,

$$\varphi(x,y,\vec{z}) \quad \text{iff} \quad \bigvee\!\!\!\bigvee_{h,k} [(x \wedge y,\vec{z}) \in R_{h,k} \quad \text{and} \quad x \in (x \wedge y)0A_k \quad \text{and} \quad y \in (x \wedge y)1B_k].$$

But, $(x \wedge y,z) \in \{(uv,uw) \mid u \in U, v \in V, w \in W\}$ and $x \in (x \wedge y)0A$ and $y \in (x \wedge y)1B$ iff $(x,y,z) \in (uva,uvb,uw) \mid u \in U, v \in V, w \in W, a \in 0A, b \in 1B\}$.

§4. In this section we prove the main lemma.

DEFINITION. An *n-automaton* is a system $\mathfrak{A} = (S, M, S_0, F)$, where S is a finite set, the set of states, $S_0 \subset S$, the set of initial states, $F \subset P(S)$, the set of designated subsets of S, and $M \subset S \times \{0,1\}^n \times S \times S$, the transition relation.

A *path* Π of T is a maximal (initial-segment-) totally ordered subset of T.

For a mapping $r\colon \Pi \to S$, define

$$\operatorname{In}(r) := \{s \in S \mid r^{-1}(s) \text{ is infinite}\}.$$

For an n-tuple $\vec{A} = (A_1, \ldots, A_n) \in P(T)^n$, define the *characteristic function* $\chi_{\vec{A}}\colon T \to \{0,1\}^n$ by

$$\chi_{\vec{A}}(x)(i) = 1 \quad \text{iff} \quad x \in A_i.$$

DEFINITION. Given an n-automaton $\mathfrak{A} = (S, M, S_0, F)$, an n-tuple $\vec{A} \in P(T)^n$ and a mapping $r\colon T \to S$. The pair $(\mathfrak{A}, r)$ *accepts* $\vec{A}$ if 1) $r(\lambda) \in S_0$, 2) In $(r \mid \Pi) \in F$ for every path $\Pi \subset T$, and

3) $$(r(x), \chi_{\vec{A}}(x), r(x0), r(x1)) \in M \quad \text{for all } x \in T.$$

We say $\mathfrak{A}$ accepts $\vec{A}$, if there is an r such that $(\mathfrak{A}, r)$ accepts $\vec{A}$.

The following theorem is due to Rabin [6].

(III) *Given an* S2S-*formula* $\varphi(X_1, \ldots, X_n)$, *there is an n- automaton* $\mathfrak{A}$ *such that for all* $\vec{A} \in P(T)^n$, $\mathfrak{A}$ *accepts* $\vec{A}$ *iff* $\varphi(\vec{A})$ *holds.*

In this case, the automaton $\mathfrak{A}$ is said to *represent* the formula φ. Individual variables are identified with singletons: $\mathfrak{A}$ is said to represent $\varphi(x, \ldots)$ if it represents the formula $\psi(X, \ldots) :\equiv \exists x(X = \{x\} \wedge \varphi(x, \ldots))$, and $\mathfrak{A}$ accepts $(a, \ldots)$ if it accepts $(\{a\}, \ldots)$.

The following lemma is a mild version of Rabin's grafting technique (see [7] or [8]).

LEMMA 1. *Let* $\mathfrak{A}$ *be an* $(n + 2)$-*automaton accepting only tuples of the form* $(a, aB, \vec{C})$, *where* $C_i \cap aT = \Lambda$, $i = 1, \ldots n$. *Suppose* $(\mathfrak{A}, r)$ *accepts* $(a, aB, \vec{C})$, $(\mathfrak{A}, r')$ *accepts* $(a', a'B', \vec{C}')$ *and* $r(a) = r'(a')$. *Then* $\mathfrak{A}$ *accepts* $(a, aB', \vec{C})$.

PROOF. Define the run $\bar{r}$ by $\bar{r}(x) := r(x)$ if $x \notin aT$ and $\bar{r}(ay) := r'(a'y)$. Then it is easy to see that $(\mathfrak{A}, \bar{r})$ accepts $(a, aB', \vec{C})$.

LEMMA 2. *Let* $\alpha(x, Y, Z_1, \ldots, Z_n)$ *be an* S2S-*formula such that*

(1) $$T \models \alpha(x, Y, \vec{Z}) \rightarrow Y \subset xT \wedge \bigwedge\!\!\!\bigwedge (Z_i \cap xT = \Lambda)$$

and

(2) $$T \models \alpha(x, Y, \vec{Z}) \wedge \alpha(x, Y', \vec{Z}) \rightarrow Y = Y'.$$

Then there are finitely many regular sets $B_k \subset T$, $k = 1, \ldots, m$, *such that*

$$T \models \alpha(x, Y, \vec{Z}) \rightarrow \bigvee\!\!\!\bigvee_k (Y = xB_k).$$

PROOF. α is represented by an $(n + 2)$-automaton $\mathfrak{A}$ satisfying the hypothesis of Lemma 1 (because of (1)). Let S be the set of states of $\mathfrak{A}$. Let $\phi(s, a, B, \vec{C})$ be the following statement: $s \in S$ and there is a run r: $T \rightarrow S$ such that $r(a) = s$ and $(\mathfrak{A}, r)$ accepts $(a, aB, \vec{C})$. In particular, $\phi(s, a, B, \vec{C})$ implies $\alpha(a, aB, \vec{C})$. If $\phi(s, a, B, \vec{C})$ and $\phi(s, a', B', \vec{C}')$, then, by Lemma 1, $\mathfrak{A}$ accepts $(a, aB', \vec{C})$, so $\alpha(a, aB', \vec{C})$ holds. By (2) we get $aB = aB'$; hence $B = B' =: B_s$. If $\alpha(a, D, \vec{C})$, then $D = aB$ and $\phi(s, a, B, \vec{C})$ for some s and B; that is, $D = aB_s$ for some $s \in S$.

It remains to show that the B_s's are regular. Fix s, a, and $\vec{C}$ such that $\alpha(a, aB_s, \vec{C})$ holds. The formula $\psi(Y) :\equiv \exists \vec{Z} \alpha(a, Y, \vec{Z})$ defines a finite relation on $P(T)$, and the set aB_s belongs to it.

Choose "discriminators" $d_i, e_j \in T$ such that

$$T \models \psi(Y) \wedge \bigwedge\!\!\!\bigwedge (d_i \in Y) \wedge \bigwedge\!\!\!\bigwedge \neg (e_j \in Y) \leftrightarrow Y = aB_s.$$

By (II), aB_s is regular; hence, by Proposition 1, B_s is regular.

LEMMA 3. *Let* $\psi(u, Y_0, Y_1, \vec{z})$ *be an* S2S-*formula such that*

(1′) $$T \models \psi(u, Y_0, Y_1, \vec{z}) \rightarrow Y_0 \subset u0T \wedge Y_1 \subset u1T \wedge \bigwedge\!\!\!\bigwedge \neg (z_i \in (u0T \cup u1T))$$

and

(2′) $$T \models \psi(u, Y_0, Y_1, \vec{z}) \wedge \psi(u, Y'_0, Y'_1, \vec{z}) \rightarrow (Y_0 = Y'_0 \leftrightarrow Y_1 = Y'_1).$$

Then there are finitely many regular sets $A_k, B_k, k = 1, \ldots m$, *such that*

$$T \models \psi(u, Y_0, Y_1, \vec{z}) \rightarrow \bigvee\!\!\!\bigvee_k (Y_0 = u0A_k \wedge Y_1 = u1B_k).$$

PROOF. Let $\alpha(x, Y_0, Y_1, \vec{z})$ be the formula

$$\exists u[x = u0 \wedge \psi(u, Y_0, Y_1, \vec{z})].$$

Then

$$T \models \alpha(x, Y_0, Y_1, \vec{z}) \to Y_0 \subset xT \wedge (Y_1 \cap xT = \Lambda) \wedge \bigwedge\!\!\!\bigwedge(\{z_i\} \cap xT = \Lambda) \tag{1}$$

and

$$T \models \alpha(x, Y_0, Y_1, \vec{z}) \wedge \alpha(x, Y'_0, Y_1, \vec{z}) \to Y_0 = Y'_0. \tag{2}$$

By Lemma 2, there are regular sets C_r such that $\alpha(x, Y_0, Y_1, \vec{z})$ implies $\bigvee\!\!\!\bigvee(Y_0 = xC_r)$. Since $\psi(u, Y_0, Y_1, \vec{z})$ implies $\alpha(u0, Y_0, Y_1, \vec{z})$, we have

$$T \models \psi(u, Y_0, Y_1, \vec{z}) \to \bigvee\!\!\!\bigvee(Y_0 = u0C_r).$$

By symmetry, there are regular sets D_s such that

$$T \models \psi(u, Y_0, Y_1, \vec{z}) \to \bigvee\!\!\!\bigvee(Y_1 = u1D_s).$$

This concludes the proof: Just let k run over the pairs (r, s) and let $A_{(r,s)} := C_r$ and $B_{(r,s)} := D_s$.

PROOF OF THE MAIN LEMMA. a) Assume $T \models \varphi(x, y, \vec{z}) \to \delta(x, y, \vec{z})$. Let

$$\alpha(x, Y, \vec{z}) :\equiv Y \neq \Lambda \wedge \forall v[v \in Y \leftrightarrow \varphi(x, v, \vec{z})].$$

Then α satisfies hypotheses (1) and (2) of Lemma 2 (since $\alpha(x, Y, \vec{z})$ implies $Y \neq \Lambda$, it implies $\neg(x \leq z_i)$, i.e. $\{z_i\} \cap xT = \Lambda$). Therefore, $T \models \alpha(x, Y, \vec{z}) \to \bigvee\!\!\!\bigvee(Y = xB_k)$ for some regular B_k's. We conclude that

$$\begin{aligned} T \models \varphi(x, y, \vec{z}) &\leftrightarrow \exists Y[\alpha(x, Y, \vec{z}) \wedge y \in Y] \\ &\leftrightarrow \bigvee\!\!\!\bigvee_k [\alpha(x, xB_k, \vec{z}) \wedge y \in xB_k]. \end{aligned}$$

We are done with $\varphi_k(x, \vec{z}) :\equiv \alpha(x, xB_k, \vec{z})$.

b) Assume $T \models \varphi(x, y, \vec{z}) \to \varepsilon(x, y, \vec{z})$. Let $\psi(u, Y_0, Y_1, \vec{z})$ be the following formula:

$$\begin{aligned} &Y_0 \neq \Lambda \wedge Y_1 \neq \Lambda \wedge Y_0 \subset u0T \wedge Y_1 \subset u1T \\ &\quad \wedge \forall x \in Y_0 \forall y \in Y_1 \varphi(x, y, \vec{z}) \\ &\quad \wedge \forall x \in (u0T \setminus Y_0) \exists y \in Y_1 \neg \varphi(x, y, \vec{z}) \\ &\quad \wedge \forall y \in (u1T \setminus Y_1) \exists x \in Y_0 \neg \varphi(x, y, \vec{z}). \end{aligned}$$

Then ψ satisfies hypotheses (1′) and (2′) of Lemma 3.

(1′). Assume $\psi(u, Y_0, Y_1, \vec{z})$. Then, by definition, $Y_0 \subset u0T$ and $Y_1 \subset u1T$. Y_0 and Y_1 are nonempty. Let $x \in Y_0$ and $y \in Y_1$. Then $\varphi(x, y, \vec{z})$, and therefore $\varepsilon(x, y, \vec{z})$ holds, and $u = x \wedge y$. Thus

$$\bigwedge\!\!\!\bigwedge \neg(z_i \in (u0T \cup u1T)).$$

(2′). Assume $\psi(u, Y_0, Y_1, \vec{z})$, $\psi(u, Y'_0, Y'_1, \vec{z})$, $Y_0 = Y'_0$ and, for a contradiction, $y \in Y'_1 \setminus Y_1$. Then there is $x \in Y_0$ with $\neg\varphi(x, y, \vec{z})$, contradicting $\psi(u, Y'_0, Y'_1, \vec{z})$.

By Lemma 3, there are regular sets A_k and B_k such that

$$(*) \qquad T \models \psi(u, Y_0, Y_1, \vec{z}) \to \bigvee_k (Y_0 = u0A_k \wedge Y_1 = u1B_k).$$

Next, we show

$$(**) \qquad T \models \varphi(x, y, \vec{z}) \leftrightarrow \exists Y_0 \exists Y_1 [\psi(x \wedge y, Y_0, Y_1, \vec{z}) \wedge x \in Y_0 \wedge y \in Y_1].$$

Assume $\varphi(x, y, \vec{z})$. Let $Y_0 := \{v \in (x \wedge y)0T \mid \varphi(v, y, \vec{z})\}$ and $Y_1 := \{w \in (x \wedge y)1T \mid \varphi(v, w, \vec{z})$ for all $v \in Y_0\}$. Then $\psi(x \wedge y, Y_0, Y_1, \vec{z})$ and $x \in Y_0$ and $y \in Y_1$. The converse implication is trivial.

By (*) and (**) we conclude that

$$T \models \varphi(x, y, \vec{z}) \leftrightarrow \bigvee_k [\psi(x \wedge y, (x \wedge y)0A_k, (x \wedge y)1B_k, \vec{z}) \wedge x \in (x \wedge y)0A_k \wedge y \in (x \wedge y)1B_k].$$

Setting $\varphi_k(u, \vec{z}) :\equiv \psi(u, u0A_k, u1B_k, \vec{z})$, we are done.

REFERENCES

[1] J. Berstel, ***Transductions and context-free languages***, Teubner, Stuttgart, 1979.

[2] J. R. Büchi, *On a decision method in restricted second order arithmetic*, ***Proceedings of the international conference on logic, methodology and philosophy of science, 1960***, Stanford University Press, Stanford, California, 1962, pp. 1–11.

[3] W. Buszkowski, *Logical complexity of some classes of tree languages generated by multiple-tree-automata*, ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik***, vol. 26 (1980), pp. 41–49.

[4] Y. Gurevich and L. Harrington, *Trees, automata and games*, ***Proceedings of the fourteenth ACM symposium on the theory of computing (San Francisco, 1982)***, ACM, New York, 1982, pp. 60–65.

[5] S. C. Kleene, *Representation of events in nerve nets and finite automata*, ***Automata studies***, Annals of Mathematics Studies, no. 34, Princeton University Press, Princeton, New Jersey, 1956, pp. 3–41.

[6] M. O. Rabin, *Decidability of second order theories and automata on infinite trees*, ***Transactions of the American Mathematical Society***, vol. 141 (1969), pp. 1–35.

[7] ———, *Weakly definable relations and special automata*, ***Mathematical logic and foundations of set theory (proceedings of an international colloquium, Jerusalem, 1968)***, North-Holland, Amsterdam, 1970, pp. 1–23.

[8] ———, ***Automata on infinite objects and Church's problem***, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, no. 13, American Mathematical Society, Providence, Rhode Island, 1972.

[9] R. M. Robinson, *Restricted set-theoretical definitions in arithmetic*, ***Proceedings of the American Mathematical Society***, vol. 9 (1958), pp. 238–242.

[10] Ch. Savioz, ***Extensions décidables et indécidables de la théorie monadique du deuxième ordre de deux fonctions de successeur***, Dissertation No. 7898, ETH, Zürich, 1985.

[11] W. Thomas, *On the bounded monadic theory of well-ordered structures*, this Journal, vol. 45 (1980), pp. 334–338.

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