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Gabriele Camera
Chapman University, camera@chapman.edu

Dean Corbae
University of Wisconsin

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Money and Price Dispersion [(*)]

by

Gabriele Camera
Department of Economics
Purdue University
West Lafayette, IN 47907-1310
Gcamera@mgmt.purdue.edu

Dean Corbae
Department of Economics
University of Pittsburgh
Pittsburgh, PA 15260
corbae+@pitt.edu

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Abstract

We relax restrictions on the storage technology in a prototypical monetary search model to study price dispersion. When multiple units of currency can be stored, buyers and sellers enter matches with potentially different willingness to buy or sell. Across the distribution of possible bilateral matches, prices will generally differ even though agents have identical preferences and technologies. We provide existence conditions for a particularly simple equilibrium pattern of exchange and prices. We prove that in the limiting case where search frictions are eliminated, equilibrium prices are uniform. We also prove that a higher initial money stock raises the average price level and increases price dispersion in certain regions of the parameter space. Numerical examples are also provided.

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"…a change in the value of money, that is to say in the level of prices, is important only insofar as its incidence is unequal. Such changes have produced in the past, and are producing now, the vastest social consequences, because as we all know, when the value of money changes, it does not change equally for all persons for all purposes. A man's receipts and his outgoings are not modified in one uniform proportion. Thus, a change in prices and rewards, as measured in money, generally affects different classes unequally, transfers wealth from one to another, bestows affluence here and embarrassment there and redistributes Fortune's favors so as to frustrate design and disappoint expectation." Keynes (1932, p.80-81)

I. Introduction

Nearly four decades ago, Stigler (1961) documented price dispersion of homogeneous goods and offered a search theoretic explanation of its incidence.[1] Unlike previous "partial equilibrium" studies which focused on differences in production costs, consumer search costs, or private information, etc. as the source of price dispersion, here we examine price dispersion using a random bilateral matching model with divisible, nonstorable goods and costless storage of multiple units of currency. With pairwise trading, the amount of money held by both the buyer and seller affects their willingness to trade. This enables us to study how the quantity of money affects the distribution of prices across sales establishments.

Our work builds on the seminal contribution of Kiyotaki and Wright (1989). They showed that with random matching and private information of trading histories, money can ameliorate an absence of a double coincidence of wants. Due to the complexity of characterizing an equilibrium with random matching, the authors made stringent assumptions about production and storage technologies; only a single unit of the commodity could be produced and a single unit of a good (including money) could be held at any time. Their assumptions implied a trivial price where all trade is one-for-one.

The next "generation" of monetary search models were aimed at providing a richer theory of pricing. Retaining the unit inventory assumption, Shi (1995) and Trejos and Wright (1995) assumed agents bargain over the quantity of non-storable, divisible goods that trade for one unit of money. In a symmetric

[1] There have been numerous subsequent studies empirically documenting price dispersion. For instance, Pratt, Wise, and Zeckhauser (1979) obtained an average of twelve price quotations on each of thirty-nine randomly chosen standardized products in the Boston area and found a "surprising" amount of price differences. They report a large and significant positive relationship between percentage changes in the standard deviation of quotations

monetary equilibrium, a single price governs all unit currency exchange for goods.

Relaxing the unit storage technology allows one to study potentially rich patterns of monetary exchange and pricing. Matches take on a profuse degree of heterogeneity; rich buyers can meet poor sellers and poor buyers can meet rich sellers. In the former case, buyers may wish to take advantage of price discounts while in the latter case buyers may choose not to trade. This latter case clearly illustrates a difference for exchange patterns and prices arising from the unit restriction on storage employed in previous studies; buyers cannot walk away from goods they like if a pure strategy monetary equilibrium is to exist.

Recently, there are a number of "third generation" search models which relax the unit storage technology and track the economy-wide distribution of money holdings. These papers can address some of the distributional issues alluded to in the quote by Keynes (1932). Green and Zhou (1998) and Zhou (1998) consider an environment with perfectly divisible money but an indivisible, nonstorable good which may or may not be costly to produce. Wallace (1996) and Taber and Wallace (1998) study an environment similar to ours; goods are perfectly divisible but there is some indivisibility in money (e.g. pennies are the least divisible form of U.S. currency). Since there is a multiplicity of equilibria in this environment, Taber and Wallace attempt to find all monetary equilibria (both pure and mixed strategy) by numerical techniques.[2] Molico (1996) considers an environment where both money and goods are perfectly divisible, focusing on the critical issue of money growth and inflation in a search environment.

Our paper provides both analytic and numerical results for a generalized version of the environment in the papers by Shi and Trejos and Wright. Our analytic results are complementary to the papers of Green and Zhou. They conjecture a certain equilibrium pattern of exchange, one in which all trade takes place at one price, and then provide conditions under which such an equilibrium exists. In section IV of our paper, we follow a similar strategy; we conjecture a uniform pattern of exchange and then

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and the percentage change in average price for each product.

provide conditions under which such an equilibrium exists. In particular, we focus on an equilibrium exchange pattern where one unit of currency is traded in all matches for some quantity of goods (to be determined in each match through bargaining between the buyer and seller). This equilibrium exchange pattern is sufficient to carry out a fairly detailed analysis of endogenous price dispersion.[3] As in the papers of Green and Zhou, we do not attempt to characterize the entire set of equilibria analytically.[4]

While the equilibrium that we study generally exhibits price dispersion, we show analytically in the limiting case where search frictions are eliminated that prices are uniform across all possible matches. The intuition for such a result is clear; without costly search, a buyer would simply wait for the "best" price. We also show analytically that an economy with a higher initial money supply exhibits a higher average price level and a larger standard deviation of sellers' prices (in certain regions of the parameter space). The intuition for this result is also clear; an increase in the initial stock of money is associated with a larger proportion of rich traders who have a low valuation of money. The average seller will thus be willing to provide less goods for a given monetary offer. The less frequent encounters with sellers offering favorable terms of trade may also raise monetary offers by buyers as they try to maintain their consumption levels. Both these factors tend to increase the price level. Moreover, if a larger money stock generates larger dispersion in the distribution of money, this can be reflected in enhanced price dispersion. As in earlier search models, money need not be neutral depending on how it affects the distribution of holdings.[5]

The paper is organized as follows. Section II provides a specification of the environment. Section III describes equilibrium strategies and the distribution of money. Section IV provides existence conditions

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[2] Since Taber and Wallace have money-in-the-utility-function, they do not find non-monetary equilibria.

[3] In so doing, we show that the pattern of exchange which was exogenously imposed in earlier divisible goods search models such as Shi and Trejos and Wright can be an equilibrium in the more general environment.

[4] For instance, in a similar environment to ours, Wallace (1996) shows there is always an equilibrium where all people are either holding no money or the most possible currency (denoted N) and all trades involve an offer of N units. This is an example of a one-price equilibrium. Since our emphasis is on price dispersion, however, we focus our analytical work on an equilibrium with a simple form of heterogeneity. Our numerical work studies more complex patterns of exchange.

[5] It is simple to construct an equilibrium where money is completely neutral when the support of the distribution

for a particularly simple equilibrium exchange pattern and distribution of money holdings; an arbitrary unit of currency is traded in all coincident matches and the distribution of money holdings is censored geometric. In contrast, section V provides numerical examples of equilibria where there are heterogeneous exchange patterns across the economy. For instance, in some equilibria buyers walk away from high priced sellers in order to meet sellers with low reservation values since this enables them to buy goods at a discount.

II. Environment.

Time is continuous. There are k (≥3) types of agents with 1/k mass. Let N be a positive integer. The storage technology allows agents to costlessly hold up to N units of currency, in unit increments. For example, Shi (1995) and Trejos and Wright (1995) set N=1.

All agents have a production opportunity. An agent of type i can produce $q \in \mathbf{R}_+$ units of non-storable good j=i+1 (modulo k) with disutility of effort $c(q)=q$.[6]

Agent i likes good i only. That is, consumption of q units of good j gives agent i momentary utility $u = u \cdot \iota(j)$, where $u = \frac{q^{1-}}{1-}$, $q \in \mathbf{R}_+$, $\gamma \in (0,1)$, and $\iota(j)=\{1 \text{ if } j=i,\ 0 \text{ if } j \neq i\}$. The parameter $1/\gamma$ measures the intertemporal elasticity of substitution. Agents discount the future at rate ρ. Given the production technology, this specification of preferences is a convenient way to rule out barter as in Kiyotaki and Wright (1989).

Fiat money is randomly distributed (identically and independently across agents and types) in different amounts up to N at t=0. Given the storage technology, the support of the distribution of money is the set $\mathbf{N}=\{0,1,2,\ldots,N\}$. Define $m_n(t)$ as the proportion of agents of all types holding n units of currency at

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rises in proportion to the increase in the initial money stock. We discuss this in section IV.d.

[6] For notational simplicity, we neglect referring to commodity type both in the technology and preferences.

time t for $n \in \mathbf{N}$, which implies

$$\sum_{n \in \mathbf{N}} m_n(t) = 1. \tag{1a}$$

Define $m(t)$ as the proportion of agents holding currency (i.e. $m(t)=1- m_0(t)$). We denote the per capita stock of money as M^s, assume it is constant over time, and is given by

$$M^s = \sum_{n \in \mathbf{N}} n\, m_n(t) \tag{1b}$$

It should be noted that the "indivisibility" of money in our inventory technology can be dealt with by appropriately choosing the "units" of money that agents can store.[7]

Meetings between traders are bilateral and random. All characteristics of one's partner are known (e.g. the quantity of money she is holding) except her trading history. At any given time, an agent meets one other trader with Poisson arrival rate α. Each agent of type i can be involved in two types of transactions. With probability α/k she can meet a partner of type i-1 from whom she would like to buy. We call type i a "buyer" in this match with type i-1. We index this buyer by $b \in \mathbf{N}$ holding b units of money. With probability α/k she can meet a partner of type i+1 who would like to buy her good. We call type i a "seller" in this match with type i+1. The seller may herself be holding $s \in \mathbf{N}$ units of money. All other matches (that is, between i and $i' \neq i+1$ or i-1) will involve no transaction given that goods are nonstorable and that it is costly to produce.

The buyer submits a take-it-or-leave-it offer of some quantity of money in $D_{b,s} = \{0,1, 2, \ldots, \min(b, N-s)\}$ for a specified quantity of goods to seller s.[8] The seller can accept or reject.

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[7] That is, suppose the unit of currency is dollars, agents can hold up to N dollars, and the supply of dollars is M^s. To address greater divisibility, the storage technology can allow agents to hold up to 2N half-dollars, 4N quarters, 10N dimes, or 100N pennies and the supply of money is still M^s dollars. We take this issue up in more detail in section V.

[8] Since the storage technology binds the seller's ability to accept d(t) units of currency, the buyer must take this into account when making her offer. This is why we have defined the set $D_{b,s}$ to depend on the $\min(b, N-s)$.

III. Equilibrium

We restrict attention to equilibria in which all agents with identical characteristics act alike and in which all of the k types are symmetric. An equilibrium is characterized by: a distribution of money holdings, a value function defined over money holdings, and a set of strategies for buyers and sellers. We discuss each in reverse order.

III.a Strategies.

For brevity we restrict attention to strategies defined for matches between i and i-1 or i+1.[9] We let the time t trading state in the match of the two agents be specified by their trading status (buyer with *b* units of money and seller with *s* units of money). Given the matching technology and informational assumptions, the prior trading history of each agent is unverifiable and so the relevant history upon which to define strategies is the trading state of the current match. For an agent holding n units of currency, her strategy is a complete contingent plan of actions specifying, for each possible period t trading state:

(i) as a buyer, a currency-quantity offer defining a quantity request for q(t) units of goods made to seller *s* for d(t) units of currency, and

(ii) as a seller, acceptance or rejection of the currency-quantity offer by buyer *b*.

Define the currency-quantity offer $q^d(b,s)(t) \in \mathbf{R}_+$ as the quantity of goods to be produced at time t by seller *s* for buyer *b* in exchange for $d(b,s)(t) \in D_{b,s}$ units of currency. For example, $q^2(4,0)(t)$ represents the quantity of goods to be produced by a seller with no money for 2 units of currency inventoried by a buyer who has 4 units of currency. Let $d^*(b,s)(t) \in D_{b,s}$ denote other agents' equilibrium currency offers and $q^{*d^*}(b,s)(t)$ the associated equilibrium quantity.

Since the set of offers may not be a singleton, let $\beta_d(b,s)(t)$ be the probability buyer *b* trades d units

[9] Recall that since goods are not storable, all other matches result in no transaction.

of money with seller s. In the match of buyer b with seller s, $\sum_{d \in D_{b,s}} \beta_d(b,s)(t) = 1$. Obviously, $\beta_0(b,s)(t)=1$ corresponds to the case where buyer b does not trade with seller s. Let all other buyers' strategies be denoted $\beta^*_d(b,s)(t)$, $\forall b$.

Let $\sigma(s,b)(t) \in \{0,1\}$ denote seller s's strategy of accepting or rejecting a buyer b's offer.[10] In particular, let $\sigma(s,b)(t)=1$ if she accepts the offer and 0 if she rejects. Let all other sellers' strategies be denoted $\sigma^*(b,s)(t)$, $\forall s$.

III.b Value functions.

An agent's optimal strategy is chosen with the objective of maximizing expected discounted utility. The value function for an agent holding $n \in \mathbf{N}$ units of currency is given by

$$\rho V_n(t) = (\alpha/k) \sum_{b \in \mathbf{N}} m_b(t) \sum_{d \in D_{b,s}} \beta^*_d(b,n)(t) \underset{\sigma(n,s)(t)}{Max}\, \sigma(n,b)(t)[V_{n+d^*}(t) - q^{*d^*}(b,n)(t) - V_n(t)]$$

$$+(\alpha/k) \sum_{s \in \mathbf{N}} m_s(t) \underset{\substack{\beta_d(n,s)(t), d(n,s)(t) \\ q^d(n,s)(t)}}{Max} \sum_{d \in D_{b,s}} \beta_d(n,s)(t)\, [V_{n-\sigma^*(s,n)d}(t) + \sigma^*(s,n)(t)u(q^d(n,s)(t)) - V_n(t)]$$

$$+ \dot{V}_n(t) \qquad (2)$$

Equation (2) describes the expected flow return to a trader holding n units of currency as the sum of the possible payoffs deriving from two types of trade: as a seller or a buyer. As a seller, with probability $\alpha m_b(t)/k$ she meets a trader holding b units of currency who likes her good. If the seller accepts the buyer's offer of $q^{*d^*}(b,n)(t)$ (i.e. she chooses $\sigma(n,b)(t)=1$), she nets the (flow) payoff $V_{n+d^*}(t) - q^{*d^*}(b,n)(t) - V_n(t)$. As a buyer, with probability $\alpha m_s(t)/k$ she meets another trader holding s units of currency who produces the good she likes. She submits the offer $q^d(n,s)(t)$ with probability $\beta_d(n,s)(t)$, which if accepted (i.e.

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[10]Restricting attention to pure strategies on the part of the seller is inconsequential. Due to the perfect divisibility of goods, a buyer can "sweeten" his offer by a very small amount so that the seller will strictly prefer to accept it.

$\sigma^*(s,\mathrm{n})(\mathrm{t})=1$), provides her with the (flow) payoff $V_{\mathrm{n}-\sigma^*(s,\mathrm{n})\mathrm{d}}(\mathrm{t}) + u(q^{\mathrm{d}}(\mathrm{n},s)(\mathrm{t})) - V_{\mathrm{n}}(\mathrm{t})$.

III.c Characterizing equilibrium strategies.

Next we characterize some properties of the symmetric equilibrium strategies $\beta^*_{\mathrm{d}}(b,s)(\mathrm{t})$, $\mathrm{d}^*(b,s)(\mathrm{t})$, $q^{*\mathrm{d}*}(b,s)(\mathrm{t})$, and $\sigma^*(\mathrm{s},b)(\mathrm{t})$, taking others' strategies as given. We first discuss seller s's optimal acceptance strategy $\sigma(\mathrm{s},b)(\mathrm{t})$ in a match with buyer b for any given $q^{\mathrm{d}}(b,s)(\mathrm{t})$.Offers $q^{\mathrm{d}}(b,s)(\mathrm{t})$ leaving less than zero net gain to the seller will not be accepted in equilibrium since the seller nets zero by rejecting. Hence

$$\sigma_*(\mathrm{s},b)(\mathrm{t})=\begin{cases}1 \text{ if } V_{s+\mathrm{d}}(\mathrm{t})-q^{\mathrm{d}}(b,s)(t)-V_s(\mathrm{t})\geq 0\\ 0 \text{ otherwise}\end{cases} \tag{3}$$

We will say that "no trade" occurs when $\mathrm{d}^*(b,s)(\mathrm{t})= q^{*\mathrm{d}*}(b,s)(\mathrm{t})=0$.

For exposition, we break up buyer b's strategies into sub-problems. Again, working backwards, consider a *given* $\mathrm{d}(b,s)(\mathrm{t})$. Optimal (subgame perfect) offers $q^{\mathrm{d}}(b,s)(\mathrm{t})$ by buyer b extract as much surplus from seller s as possible, subject to the seller's acceptance. Offers leaving more than zero net gain to the seller can be improved upon by the buyer; she can always ask for more goods, leaving the seller with an $\varepsilon>0$ (arbitrarily small) net payoff which is acceptable by the seller. Hence, in this specific case, the only subgame perfect offer, or "pricing rule", leaves the seller with a zero net payoff

$$q^{*\mathrm{d}}(b,s)(\mathrm{t})=V_{s+\mathrm{d}}(\mathrm{t})-V_s(\mathrm{t}) \tag{4}$$

By virtue of (4), seller s will not reject such offers since her best response to the pricing rule is $\sigma^*(s,b)(\mathrm{t})=1$ by (3). One can then define the (nominal) price in a match between buyer b and seller s as $p(b,s)(\mathrm{t})=\mathrm{d}(b,s)(\mathrm{t})/q^{\mathrm{d}}(b,s)(\mathrm{t})$.

Finally, the optimal quantity of currency offered $\mathrm{d}(b,s)(\mathrm{t})$ must yield the highest payoff to the buyer. Due to the discreteness of $D_{b,s}$ there may be multiple $\mathrm{d}(b,s)(\mathrm{t})$ which satisfy[11]

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[11] Since the buyer's payoff is concave in d, the indivisibility in currency may happen to generate the same payoff from two different (but adjacent) currency offers. In particular, if $\Pi(\mathrm{d})$ denotes the RHS of (5) for some d, then it is

$$\{d^*(b,s)(t)\}\in \underset{d(b,s)(t)\in D_{b,s}}{\arg\max}\ V_{b-d}(t)+\sigma^*(s,b)(t)u(q^{*d}(b,s)(t))-V_b(t) \tag{5}$$

Then $\beta^*_d(b,s)(t)$ can be associated with each $d^*(b,s)(t)$.

To summarize, if a buyer decides to trade, she will choose an offer ($d(b,s)(t)$) which leaves the seller with the minimum acceptable net payoff. The buyer may choose not to trade if she meets a rich seller who can only be induced to supply a small quantity of goods; instead she may wait to meet a poor seller who will provide more goods in the future.

III.d The distribution of money holdings.

The law of motion of money holdings, or flow of agents into and out of a given holding of n units of currency is given by

$$\dot{m}_n(t) = \frac{\alpha}{k}\sum_{s=0}^{n-1} m_s(t)\sum_{b=1}^{N}\beta^*_{n-s}(b,s)(t)\sigma^*(s,b)(t)\,m_b(t)+\frac{\alpha}{k}\sum_{b=n+1}^{N} m_b(t)\sum_{s=0}^{N-b+n}\beta^*_{b-n}(b,s)(t)\sigma^*(s,b)(t)m_s(t)$$

$$-\frac{\alpha}{k}m_n(t)\sum_{b=1}^{N} m_b(t)\sum_{d\in D_{b,n}}\beta^*_d(b,n)(t)\sigma^*(n,b)(t)-\frac{\alpha}{k}m_n(t)\sum_{s=0}^{N-1} m_s(t)\sum_{d\in D_{n,s}}\beta^*_d(n,s)(t)\sigma^*(s,n)(t) \tag{6}$$

for $n\in\{1,\ldots,N-1\}$, while the expressions for $n\in\{0,N\}$ are straightforward extensions of (6).[12] The first term reflects the sellers who move up to n since they receive payments $d(b,s)=n-s$. The second term reflects the buyers who move down to n since they make payments $d(b,s)=b-n$. The third term reflects sellers who were last holding n units of currency, receive a positive offer $d(b,n)$, choose to produce, and move up. The fourth term reflects buyers who were last holding n units of currency, choose to trade, make

possible to show that it is strictly concave. Since $\Pi(d)$ is defined over the discrete domain $D_{b,s}$ there may be at most two adjacent maxima, say d^* and d^{**}. This is because, for $z\geq 2$, if $d^{**}=d^*+z$ and $\Pi(d=d^*)$ is increasing in d, then strict concavity implies existence of $d^*<d'<d^{**}$ such that $\Pi(d')>\Pi(d^*)=\Pi(d^{**})$, which is a contradiction. A contradiction also occurs also if $\Pi(d=d^*)$ is decreasing in d, since then $\Pi(d^*)>\Pi(d^{**})$. A similar argument can be constructed for $z\leq -2$.

[12] In the case of n=0 the first term on the right hand side of (6) drops out and the fourth term needs to be eliminated (since individuals with no money holdings cannot buy). In the case of n=N the second term on the right hand side of (6) drops out and third needs to be eliminated (since individuals with N units of money do not sell due

offers d(n,*s*) and move down.

Two of the N+1 equations are redundant (given that inflows to one equation are coming from outflows from another). These are replaced by the adding up constraint on the population proportions (1a) and money supply equation (1b).

III.e Definition of symmetric steady state equilibrium.

A *symmetric steady state equilibrium* is defined to be $\{V_n, d^*(b,s), q^{*d^*}(b,s), \beta^*_d(b,s), \sigma^*(b,s), m_n, \forall n,b,s\}$ which satisfy (1a), (1b), (2), (3), (4), (5), (6) for $\dot{V}_n(t)=0$ and $\dot{m}_n(t)=0$.

IV. On Existence of a Simple Pattern of Exchange.

We now examine the existence of a steady state equilibrium where all buyers offer one unit of currency in exchange for some amount of goods (i.e. $d^*(b,s)=1$, $\beta_1^*(b,s)=1$, and $\sigma^*(s,b)=1$ for all integers b,s with $1\le b\le N$ and $0\le s\le N-1$). We conjecture an equilibrium with unit currency exchange and check under what conditions it exists. Given the variable cost component of production, this equilibrium admits price dispersion across sellers. [13]

IV.a The Distribution of Money Holdings.

Under the conjecture that all buyers offer one unit of currency in exchange for some amount of goods, the law of motion for money holdings in (6) for $n\in\{1,\ldots, N-1\}$ can be simplified to

$$\dot{m}_n(t)=\frac{\alpha}{k}m_{n-1}(t)(1-m_0(t))+\frac{\alpha}{k}m_{n+1}(t)(1-m_N(t))-\frac{\alpha}{k}m_n(t)(1-m_0(t))-\frac{\alpha}{k}m_n(t)(1-m_N(t)) \qquad (6')$$

As above, (6') shows that the proportion of traders holding n units of currency evolves according

to the upper bound on inventory).

[13] This case is also interesting since the storage technology of Shi (1995) and Trejos and Wright (1995) imposed

to four parts. The first is the inflow of the n-1 sellers (m_{n-1} is their proportion in the population) who are matched with buyers. Under the conjecture, the money holdings of these sellers grow by 1 unit. Similarly, the second term shows the inflow of n+1 buyers who expend one unit of currency trading with all willing sellers. The third and fourth component describe the outflows of traders holding n units of currency, who are either selling to buyers or buying from sellers, respectively.

In the steady state, (6') can be rearranged, for $m_N \neq 1$, to yield the recursion

$$m_n = m_0 \left(\frac{1 - m_0}{1 - m_N} \right)^n \tag{6''}$$

This censored geometric distribution is completely determined by (1b) and (6"). Green and Zhou consider a similar distribution of money holdings where in their case $m_N = 0$ since N=∞.

In the appendix (where all proofs are contained) we show

Proposition 1. *Given* $d^*(b,s)=1$, $\beta_1^*(b,s)=1$, $\sigma^*(s,b)=1$ $(\forall\ b,s)$*, there exists a unique steady-state distribution of money holdings that satisfies (1a)-(1b) and (6'). The proportion of agents without money* m_0 *is decreasing in* M^s *and the proportion at the upper bound* m_N *is increasing in* M^s.

IV.b Value functions.

Denote $\Delta V_s = V_s - V_{s-1}$. By (4), the equilibrium quantity of goods produced by seller *s* for a unit of currency is given by $q^*_s = \Delta V_{s+1}$.[14] Let $\phi = \alpha/\rho k$. Under the conjectured equilibrium, equation (2) for n>0 can be written

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unit currency transactions exogenously, and here we can see under what conditions unit currency transactions survive even if the storage technology allows multiple holdings of money.

[14] We drop the arguments d and *b* (where they are understood) and index the seller's money holdings by subscript *s*.

$$V_n=\phi\sum_{s=0}^{N-1} m_s\ u(\Delta V_{s+1})-\phi(1-m_N)\Delta V_n \tag{7}$$

and for n=0, $V_0=0$. Let $A=\phi(1-m_N)/[1+\phi(1-m_N)]$. For n>0, it is easy to show $V_n=\frac{1-A^n}{1-A}V_1$. Hence, if $V_1=0$, then $V_n=0$ for all n. This corresponds to the non-monetary equilibrium. Since we study monetary equilibria, we focus on the other, strictly positive solution for the value function. Its standard properties are listed below.

Lemma 1. $\{V_n\}_{n\in\mathbf{N}}$ *is bounded below and above.* $\{\Delta V_n\}_{n\in\mathbf{N}}$ *is positive and decreasing in* n.

The result on the value function has obvious implications for prices. Since $d^*=1$ in our conjectured equilibrium, $p^*_s=1/q^*_s$. Hence $\{p^*_s\}_{s\in\mathbf{N}\backslash N}$ is bounded below by $1/V_1$ and above by $1/\Delta V_N$. It is also increasing and concave in the seller's money holdings.

IV.c Existence of equilibrium with price dispersion.

Now we address optimality of the conjectured transaction strategy. In particular, $d^*(b,s)=1$ is optimal if trading more or less than one unit of currency does not provide any buyer with a larger flow payoff than the conjectured strategy. This amounts to checking (5) with d=1 for all b,s. That is, all buyers will choose to trade if

$$V_{b-1}+u(q^*_s)-V_b\geq 0,\ \forall b,s \tag{5a}$$

and will choose not to trade more than 1 unit of currency if

$$V_{b-1}+u(q^*_s)-V_b\geq V_{b-d}+u(q^d_s)-V_b,\ \forall d\in D_{b,s},\forall b,s \tag{5b}$$

Taken together, these constraints provide upper and lower bounds on V_n which must be satisfied for the conjectured equilibrium to exist.

In the following proposition, we show that it is possible to find sufficient conditions under which unit currency trade remains optimal when the upper bound on inventory holdings is any finite number (which can be made arbitrarily large). The conditions for existence of such an equilibrium depend on the rate of intertemporal substitution (γ) and the degree of search frictions (captured by ϕ). Intuitively, as intertemporal substitution falls (or γ rises), buyers are less likely to postpone trade for a better price (i.e. violate (5a) with d=1) and less likely to run down money balances in discount matches (i.e. violate (5b) with d=1). Furthermore, as search frictions fall (or ϕ rises), the marginal utility of money approaches a constant and hence price dispersion falls. This makes it less likely that the agents will postpone trades for a better price (i.e. violate (5a) with d=1) or find discount matches (i.e. violate (5b) with d=1).

Proposition 2. *For any finite* N *and* $\gamma>\gamma(\mathrm{N})=1-\frac{1}{\mathrm{N}-1}$, *there exist* $\phi>\phi(\mathrm{N})$ *such that* $d^*(b,s)=1$, ($\forall$ *b,s*) *equilibria exist.*

The proof works off the fact that when d=1, a sufficient condition for (5a) is that the poorest buyer purchase from the richest seller (offering the highest price) and a sufficient condition for (5b) is that the richest buyer exchange only 1 unit of currency with the poorest seller (offering the lowest price). These conditions translate to constraints on V_1 which are made less binding as $\gamma \rightarrow 1$ and $\phi \rightarrow \infty$. It is also easy to see why the bounds on γ and ϕ should depend on N. For suppose that we fix γ and ϕ, and let N get arbitrarily large. In that case, the poor buyer (*b*=1) gets no utility from trading with a rich seller (*s*=N-1). In other words, inequality (5a), written as $u(A^{N-1}V_1) \geq V_1$, is violated since A<1.[15]

[15] In our working paper, we show that if there are fixed costs to bargaining, such d=1 equilibria can exist for all (*b,s*) less than an endogenous upper bound N'<N. In particular, we provide conditions such that, under the assumptions of an equilibrium in which $d^*=1$, no individual buys from a seller holding N' units of currency (or more). Since the output exchanged for money is decreasing in the seller's wealth, no individual buys from a seller who is "too" rich because he does not derive sufficient utility to cover the bargaining cost. As a consequence, rich

IV.d Properties of equilibrium

The next series of propositions characterize price dispersion as the search friction becomes arbitrarily small or the initial stock of money is varied. Our first result is that price dispersion vanishes even with heterogeneous buyers and sellers as the search frictions fall (i.e. $\phi \to \infty$). This occurs because all buyers face the same probability of meeting a seller and buyers are willing to wait for a sufficiently good price. As $\rho \to 0$ or $\alpha/k \to \infty$, any seller can be induced to produce the amount $\overline{q}$ satisfying $u(\overline{q}) = \overline{q}$.

Lemma 2. *For any finite* N *and* $\gamma > \gamma(N)$, $\Delta V_n \to (1-\gamma)^{-1/\gamma} = \overline{q}$, $\forall n$ *as* $\phi \to \infty$.

Proposition 3. *The variance of prices converges to zero as* $\phi \to \infty$.

In the next set of lemmas and propositions, we compare steady states in which the initial stock of money differs.[16] This enables us to study the effect of variations in money on the distribution of prices in a $d^*(b,s)=1$ equilibrium.[17] First we show that the average price level is increasing in the money supply.

Lemma 3. *Let* $\gamma > \gamma(N)$, $\phi > \phi(N)$. *Then* V_1 *is decreasing in* M^s.

Proposition 4. *The average price level* $P = \sum_{s=0}^{N-1} \frac{m_s}{1-m_N} \frac{1}{q_s}$ *is increasing in* M^s.

The proof uses the fact (from Proposition 1) that m_N is increasing and m_0 is decreasing in M^s. This

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sellers will not keep accumulating currency and an upper bound on money holdings is endogenously obtained.

16 It is simple to construct an equilibrium where money is completely neutral when the support of the distribution rises in proportion to the increase in the initial money stock. For instance, suppose M^s is doubled, N is doubled and we place the same mass of agents on even integers of the support of the distribution. A strategy where all agents trade two units of money for the same quantity of goods as the d=1 equilibrium leaves the value functions unchanged obviously implying neutrality. Since we do not change N in the experiments we undertake here, lead to non-neutrality.

17 Of course, since the model has multiple equilibria, there is nothing to assure us that we are comparing the only two distributions.

implies that as the level of liquidity in the economy grows the proportion of "discount" sellers (low *s* offering high q_s) diminishes, while the opposite occurs for rich sellers (which offer low q_s). This has two distinct effects. Buyers are able to consume less for each unit of money because the change in the distribution of money has caused a decrease in the proportion of poor sellers. Consequently the value function falls. Sellers are willing to offer less commodities than before since each unit of money can now buy less. Therefore, prices rise in each match. As a consequence the price level rises with M^s.

Next we consider the relationship between price dispersion and different initial liquidity levels. We show that price dispersion is increasing on a subset of the range of values of M^s.

Proposition 5. *Let* $\gamma > \gamma(N)$, $\phi > \phi(N)$, *then the variance of prices is increasing on* $M^s \in (0, N/2)$.

The proof is built around two features of the model. First, the dispersion of money holdings is maximized at $M^s = N/2$, where money is uniformly distributed. Second, even if individual prices all increase for higher initial levels of liquidity, prices offered by poor sellers change less relative to prices offered by rich sellers. The latter implies that both the price level, *P*, and the differential between high and low prices grow with M^s. This, paired with the increasing dispersion in money holdings, generates higher variance of prices as M^s moves from 0 towards N/2. However, as M^s increases above N/2, the dispersion of money holdings starts to drop since there is an increasing proportion of rich sellers. Eventually the latter effect can become so strong that it offsets the increasing differential between high and low prices, thereby causing the variance of prices to drop.

A natural question is whether welfare differs between two economies with dissimilar initial money stocks. To answer this question, we use the standard measure of ex-ante welfare

$$W = \sum_{n=0}^{N} m_n V_n \tag{8}$$

We next show that the two competing effects which generate hump shaped welfare in Trejos and Wright carry over here.

Proposition 6. *Suppose* N *is finite*, $\gamma > \gamma(N)$, $\phi > \phi(N)$. *Welfare is increasing in the money supply for* $M^s < \overline{M} < N$ *and decreasing for* $M^s > \overline{\overline{M}} \geq \overline{M}$.

On the one hand, a larger initial stock of money increases the measure of buyers in the economy and decreases the mass of sellers that have no money. For any given V_n, this increases ex-ante welfare since in a monetary equilibrium $V_n > V_0 = 0$, $n>0$. On the other hand, a decrease in the measure of poor sellers has a negative effect on both the frequency and amount of consumption for the average buyer. This decreases V_n, $n>0$. While the first effect dominates for low quantities of money, as M^s gets close to its (exogenous) upper bound N, the second effect starts to dominate.

V. On Steady State Equilibria with Other Exchange Patterns.

While we find the unit currency exchange equilibria an important benchmark for understanding price dispersion in a monetary search model, such equilibria exist only in certain regions of the parameter space. Here we provide numerical examples of pure-strategy equilibria where there are general exchange patterns in the economy. We focus on three particular issues. First, we show that price dispersion drops as search frictions drop. Second, we study how price dispersion is affected by changes in the initial stock of money. Our numerical results for general patterns of exchange are qualitatively similar regarding price dispersion to our analytic conclusions for the d=1 equilibrium. Finally we address sensitivity of the results to indivisibility in the inventory technology. Again, we find no qualitative change in the results. We stress, however, that our numerical results need to be understood in light of the multiplicity of equilibria for this model. That is, we provide one example equilibrium in each experiment out of what could be a large set of equilibria. Taber and Wallace (1998), on the other hand, undertake an exhaustive numerical search for all equilibria.[18]

A brief description of our numerical procedures (based on policy function iteration) is in order. We begin with an initial set of strategies. These imply an initial distribution of money holdings given by the solution of the nonlinear system of equations in (1a), (1b) and (6). They also imply, along with the distribution, an initial value function given by the solution of the nonlinear system of equations in (2).[19] Given the value function we then solve (3), (4), and (5) for a new set of optimal strategies. These determine a new distribution of money holdings and value functions. We stop when there is no change in optimal strategies. Our example equilibria were computed with several initial candidate strategies and converged to those reported in the text.

We provide an illustration of Proposition 3 by considering the effect of changes in the extent of search frictions on the price distribution across sellers s in Figure 1 and the coefficient of variation of prices in Table 1. The parameterization is M^s=4, γ=0.5, k=5 and N=30. In particular, in Figure 1 we report the (normalized) average price of each seller for three different degrees of search friction (respectively ϕ=200,ϕ=20,ϕ=2.22) and it is apparent that the dispersion drops as the search frictions decrease[20]. This is confirmed by Table 1 which provides a numerical summary of the effect of search frictions on the price level and the coefficient of variation. Thus, the results from Proposition 3 carry over to equilibria where there are diverse transactions (i.e. $d \neq 1$). A decrease in search frictions has a dampening effect on the terms of trade variance for two reasons. As the extent of the search friction falls, buyers tend to adopt similar patterns of exchange where only the minimum denomination of currency is offered (which is the case from

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[18] Given the large number of potential equilibria, Taber and Wallace must restrict the upper bound of money holdings to 3 units.

[19] In (2), we only consider pure strategies.

[20] The normalization, for each ϕ, is obtained by taking the ratio of the average price charged by seller *s* to the general price level (the normalizing factor). A distribution of normalized prices close to one implies similarity of prices across different sellers. Recall that for the chosen parameters, in a d=1 equilibrium q converges to 4 as ρ tends to zero (the price converges to 0.25 according to Proposition 2). The price levels in Figure 2 are 0.256, 0.408, and 4.677 for ϕ=200,ϕ=20,ϕ=2.22.

Proposition 2). The fall in search frictions also directly raise the value function, which via the take-it-or-leave-it bargaining protocol results in all sellers' willingness to produce larger amounts of goods. The two effects combine to help drive the variance of prices to zero.

Table 1 and Figure 1 about here

Table 1 also provides information on the average propensity to consume and its dispersion. Notice that average propensity to consume falls as search frictions decrease (i.e. as ϕ rises). As search frictions drop so does the size of average monetary payments and the accumulation of currency. The combination of these two effects explains the declining average propensity to consume and the higher coefficient of variation of the propensity to consume.

Next we illustrate the effect that changes in the amount of initial liquidity has on the average price level and the variance of prices across steady states for a similar parameterization as above (the only difference is γ=0.9 and ϕ=10). Figures 2a and 2b present the patterns of monetary exchange, d*, for two different levels of initial money supply. In Figure 2a, where the M^s is small, the distribution of money has few rich buyers and sellers. As suggested by lemma 3 for the d=1 case, poor buyers (with a high marginal valuation of currency) will not buy from rich sellers (with a low marginal valuation of currency who are unwilling to produce much output). On the other hand poor sellers are willing to produce a lot even for small amounts of currency. Notice also that the dispersion in the pattern of transaction is very limited. At higher levels of liquidity, the marginal valuation of currency drops, the dispersion of currency offers across matches increases and less agents forego consumption. This is evidenced in Figure 2b. All buyers buy and rich buyers are able to offer large amounts of currency to take advantage of price "discounts" from poor sellers.

Figures 2a,2b,3 about here

These results are illustrated in Figure 3, which reports the price level and the standard deviation of prices for different initial money supplies. It is observed that the price level is monotonically increasing in the money supply, in accordance with Proposition 4 for the d=1 case. Also, the standard deviation is hump shaped as suggested by Proposition 5 for the d=1 case.

Finally, we address the issue of the indivisibility of our inventory technology by illustrating the behavior of value functions, the distribution of money holdings, and the distribution of prices for the same parameterization as above except that we progressively relax the indivisibility associated with our inventory technology. This is achieved by amending the storage technology in three different experiments so that agents can to store up to N units of currency in divisions of η=1,2,4 units (e.g. dollars, half-dollars, quarters). With this specification we can approximate the perfectly divisible money case by letting η grow large.

Consistent with Lemma 1, Figure 4 shows the value function (defined on $n'=n/(N\eta)$) is increasing (and appears to be converging to some concave function) as η increases from 1 through 4. By increasing divisibility, individuals have a higher degree of flexibility in choosing currency offers. Hence, one would expect the lifetime utility of an individual to increase at a decreasing rate. This is also evidenced in the distribution of money holdings which becomes bell shaped as divisibility is increased (Figure 5). With the increased divisibility buyers can choose less dramatic movements *out* of their inventory position (and sellers may experience less dramatic movements *into* higher inventory positions). Hence the majority of traders will tend to be amassed around the median of the money holdings distribution. Table 2 shows that the results on the distribution of prices across different initial money stocks is again qualitatively similar to what was found in Propositions 4 and 5 for the d=1 case. In particular, the average price level as well as its variance is increasing in the initial stock of money.

Figure 4,5 and Table 2 about here

The fifth and sixth columns of Table 2 on the propensity to consume also illustrate two interesting features of the model played by money's important insurance role. For any given division of currency, the average propensity to consume decreases with the money supply. With more liquidity, buyers can spend a smaller fraction of their money and still consumption smooth. Also noteworthy is that as divisibility rises (for a constant money stock), the average propensity to spend falls.

Finally, the last column shows an instance in which welfare actually falls with increased divisibility. Moving from η=3 to η=4, for M^s=8, decreases welfare. Two competing effects are at work. Increasing η exerts a positive effect on the value function, but it also affects the distribution of money holdings. In particular, the steady state fraction of rich individuals may drop significantly relative to poor traders as indivisibility is lessened. This is more likely to occur for large quantities of money since as buyers optimally choose money outlays on a finer set, their average propensity to spend drops. This generates increased dispersion of money holdings, because traders are not entirely concentrated on the right-end of the distribution.

VI. Conclusion

We have relaxed the unit storage technology in a prototypical monetary search environment in order to examine equilibrium price dispersion. In particular, for a certain class of equilibria we prove that there are uniform prices in the limit as search frictions are ameliorated. We also show that in certain regions of the parameter space more initial liquidity leads to higher dispersion of money holdings and, since buyers' and sellers' willingness to spend and produce depends on their liquidity positions, an increase in the money stock raises not only prices in each match (the standard result in unit storage models) but also the variance of prices across matches. While we have not undertaken an exhaustive numerical investigation of all equilibria, our qualitative results carry over in several numerical examples where there are diverse patterns of exchange.

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Appendix

Proof of Proposition 1

Substituting (6'') into (1b) yields $m_N = \dfrac{1-m_0(1+M^s)}{(1-m_0)(N+1)-M^s}$. Substituting this expression into (6') for m_N yields a single equation in m_0 that must be satisfied

$$F(m_0)\equiv\frac{m_0[(1-m_0)(N+1)-M^s]^{N+1}}{1-m_0(1+M^s)}=(N-M^s)^N\equiv f$$

Note that $F(m_0)$ is a continuous function and that $F(0)=0$ and $F(m_0)=\infty$ as $m_0\rightarrow(1+M^s)^{-1}$ from below. Furthermore, the set of m_0 that satisfies $F'(m_0)=0$ is given by $\{(1+N-M^s)[(1+N)(1+M^s)]^{-1}, (1+N)^{-1}\}$. When $M^s=N/2$, then $(1+N-M^s)[(1+N)(1+M^s)]^{-1}=(1+N)^{-1}$ and $F((1+N)^{-1})=f$. This corresponds to the uniform distribution. When $M^s<N/2$, then $(1+N-M^s)[(1+N)(1+M^s)]^{-1}>(1+N)^{-1}$ and it can be shown that $F'(m_0)>0$ for $m_0\in(0,(1+N)^{-1})$, $F'(m_0)<0$ for $m_0\in(1+N)^{-1}, (1+N-M^s)[(1+N)(1+M^s)]^{-1})$, and $F'(m_0)>0$ for $m_0\in((1+N-M^s)[(1+N)(1+M^s)]^{-1}, (1+M^s)^{-1})$. Thus, since F is a continuous function of m_0, $F(m_0)=f$ at $m_0^*\in((1+N-M^s)[(1+N)(1+M^s)]^{-1}, (1+M^s)^{-1})$, and $m_0^*=(1+N)^{-1}$. But the latter root is inconsistent with the restrictions on the distribution of money holdings. The case when $M^s>N/2$ parallels that above.

Let $\{m_n\}$ identify the distribution of money holdings $\{m_0, m_1\ldots, m_N\}$, whose elements satisfy (1a)-(1b) and (6'). We first notice that $\{m_n\}$ is decreasing for $M^s\in(0,N/2)$, is increasing for $M^s\in(N/2, N)$ and uniform with $m_n=(N+1)^{-1}$ $\forall$n for $M^s=N/2$. The latter is shown by observing that (1b) is $N/2=\sum_{n\in\mathbf{N}} n m_n$, which is uniquely satisfied by $m_n=(N+1)^{-1}$, $\forall$n, since $\sum_{n=1}^{N} n = N(N+1)/2$. Next, (6') implies $\{m_n\}$ is an increasing (decreasing) sequence when $m_0<m_N$ ($m_0>m_N$). Now suppose that $M^s>N/2$ and $m_0> m_1>\ldots>m_N$. Since m_n is constant for $M^s=N/2$ then $\{m_n\}$ cannot be decreasing for $M^s>N/2$ and still satisfy (1b). A similar contradiction can be constructed for $M^s<N/2$ when one assumes $m_0<m_1<\ldots<m_N$. Thus $\{m_n\}$ is an increasing sequence for $M^s\in(N/2,N)$ and $\{m_n\}$ is a decreasing sequence for $M^s\in(0, N/2)$.

To show that m_0 is decreasing in M^s let $m=1-m_0$ and $a=m/(1-m_N)$. Using (1b) and (6') we obtain $m=\sum_{n=1}^{N} a^n / [1+\sum_{n=1}^{N} a^n]$ and $M^s=(1-m)\sum_{n=1}^{N} n a^n$. Rearrange the latter as $M^s=(1-m) a d(\sum_{n=1}^{N} a^n)/da$. Since $m/(1-m)=\sum_{n=1}^{N} a^n$ then $d(\sum_{n=1}^{N} a^n)/da=dm/da$, and $M^s=dm/da[a/(1-m)]$. Denote the partial derivatives with respect to M^s with a prime ('). Since $m' = (dm/da)a'>0$ and $dm/da>0$, either both $a'<0$ and $m'<0$, or $a'>0$ and $m'>0$. It is easy to see that the conjecture $a'>0$ and $m'>0$ is consistent with $dm/da>0$. Conversely, the assumption that $a'<0$ and $m'<0$ produces a contradiction. In fact the right hand side of the equation $m=(1-m)\sum_{n=1}^{N} a^n$ is decreasing in M^s. Together with $a'<0$, this implies that $(1-m)\sum_{n=1}^{N} n a^n$ (i.e. M^s) must be decreasing in M^s, which is a contradiction. Hence m_0 is decreasing in M^s.

Finally we show that m_N is increasing in M^s. Assume it is not and observe that this is inconsistent with (1b) and (6') since m_0 is decreasing in M^s. □

Proof of Lemma 1

The sequence $\{V_n\}$ is bounded below and above, if $0 \le V_1 < \infty$, since $0<A<1$. V_1 is derived from (7) with n=1 using (6'') and $\Delta V_{s+1}=A^s V_1$ to yield

$$V_1=\frac{m_0}{(1-m_N)(1-\gamma)}\sum_{s=0}^{N-1}\left(\frac{1-m_0}{1-m_N}\right)^s V_1^{(1-\gamma)} A^{s(1-\gamma)+1} \tag{7'}$$

which has two solutions. One solution to (7') is zero. The other is positive and given by

$$V_1=\left[\frac{1}{(\frac{1}{\phi}+1-m_N)(1-\gamma)}\sum_{s=0}^{N-1} m_s A^{s(1-\gamma)}\right]^{1/\gamma} \tag{7''}$$

Clearly (7'') shows that $V_1>0$. Then V_1 is finite if the summation on the right hand side of (7'') is finite. The series $\sum m_n$ converges in **R** since the sequence of its partials sums is bounded by 1. Similarly, the series

$\sum | A^{n(1- \ )} - A^{(n+1)(1- \ )} |$ is convergent in **R**. Then V_1 is seen to be finite by applying Abel's test for the convergence of series.

Next, $\Delta V_n = A^{n-1} V_1 > 0$. $\Delta V_{n+1} - \Delta V_n = - A^{n-1} (1-A)V_1 < 0$. ☐

<u>Proof of Proposition 2</u>

Expression (5.a) with d=1 can be written $u(\Delta V_{s+1}) \geq \Delta V_b$ or $u(A^s V_1) \geq A^{b-1} V_1$, which can be rearranged to

$$V_1 \leq \left(\frac{A^{s(1-\gamma)}}{A^{b-1}(1-\gamma)} \right)^{1/\gamma} \tag{a.1}$$

The minimum of the right hand side of (a.1) with respect to b and s is achieved when s=N-1 and b=1. This is because the value function is increasing and concave, so that momentary utility is smallest in trades with large s and the lifetime utility loss is largest for small b. After substituting for V_1 using (7"), rearrange (a.1) with s=N-1 and b=1 as

$$1 \leq \ ^{(N-1)(1-\gamma)}(1/\phi + 1 - m_N) \left[m_0 \sum_{s=0}^{N-1} \left(\frac{(1-m_0) A^{1-}}{1-m_N} \right)^s \right]^{-1} \tag{a.1'}$$

Let the right hand side of (a.1') be identified by $g(\phi,\bullet)$ (where "•" indicates all the other parameters: γ, M^s, N) which is continuous in $\phi \in [0,\infty)$ and $\lim_{\phi \to \infty} g(\phi,\bullet)=1$. After some tedious algebra it can be shown that the sign of $g(\phi,\bullet)/ \ \phi$ is given by

$$\sum_{s=0}^{N-1} (N-1) \left(\frac{1-m_0}{1-m_N} A^{1-} \ \right)^s - \sum_{s=0}^{N-1} \left(s + \frac{1}{1-\gamma} \right) \left(\frac{1-m_0}{1-m_N} A^{1-\gamma} \right)^s .$$

A sufficient condition for $g(\phi,\bullet)/ \ \phi < 0$ is $N-1 < 1/(1-\gamma)$. Therefore $g(\phi,\bullet)$ is decreasing in ϕ $\forall \gamma \geq \gamma(N) = 1 - 1/(N-1)$. Consequently (a.1) is satisfied $\forall \phi$, $\forall \gamma \geq \gamma(N)$.

Expression (5.b) with d=1 can be written $u(\Delta V_{s+1})$-$u(V_{s+d'}-V_s) \geq -(V_{b-1} - V_{b-d'})$ or $u(A^s V_1)$-$u(\sum_{i=1}^{d'} A^{s+i-1} V_1) \geq -V_1 \sum_{i=0}^{d'-2} A^{b-d'+i}$, since $V_{b-1} - V_{b-d'} = (V_{b-1} - V_{b-2}) + (V_{b-2}-V_{b-3}) + \ldots + (V_{b-d'+1} - V_{b-d'})$.

Rearranging this expression yields

$$V_1 \geq \left\{ \frac{A^{s(1-\gamma)}\left[\left(\sum_{i=0}^{d'-1} A^i\right)^{1-\gamma} - 1\right]}{(1-\gamma) A^b \sum_{i=1}^{d'-1} A^{-i-1}} \right\}^{\frac{1}{\gamma}} \qquad \forall d' \in D_{N,0} \backslash \{1\} \tag{a.2}$$

The right hand side is increasing in *b* and decreasing in *s*. Its maximum with respect to *s* and *b* (for a given d') is achieved at *s*=0 and *b*=N. Expanding the summations, substituting for V_1 and evaluating it at *s*=0 and *b*=N leads to the following inequality $1 \geq \frac{(1-A)^{\gamma}[(1-A^{d'})^{1-\gamma} - (1-A)^{1-\gamma}]}{A^{N-d'+1}(1-A^{d'-1})}$ and define the right hand side of this expression as $h(\phi,\bullet)$ (where "•" indicates all the other parameters: γ, M^s N, d').[21] For all parameterizations $h(\phi,\bullet)$ is a continuous function on $\phi \in [0,\infty)$. Since $\lim_{\phi\to\infty} h(\phi,\bullet)=0/0$, use l'Hopital's rule to obtain

$$\lim_{\phi\to\infty} \frac{-\gamma(1-A)^{\gamma-1}(1-A^{d'})^{1-\gamma} - (1-\gamma)(1-A)^{\gamma}(1-A^{d'})^{-\gamma} d' A^{d'-1} + 1}{(N-d'+1)A^{N-d'}(1-A^{d'-1}) - A^{N-1}(d'-1)} = \frac{1-d'^{1-\gamma}}{1-d'}$$

which is seen to be less than one $\forall$d'. Also, since $\lim_{\phi\to 0} A'=\infty$, it is easy to see that $\lim_{\phi\to 0} h(\phi,\bullet)=\infty$. By the intermediate value theorem there exists one $\phi(N)<\infty$ such that $\forall\ \phi \geq \phi(N)$ (a.2) is satisfied. □

Proof of Lemma 2

Since $\lim_{\phi\to\infty} A=1$ and $\sum_{s=0}^{N-1} m_s = 1 - m_N$, then from (7") $\lim_{\phi\to\infty} V_1=(1-\gamma)^{-1/\gamma}$. Furthermore since $\Delta V_{s+1}=A^s V_1$

[21] We can treat d' as a parameter since we are considering a one-shot deviation.

then $\lim_{\phi\to\infty} \Delta V_{s+1} = (1-\gamma)^{-1/\gamma}\ \forall s$.. □

Proof of Proposition 3

Since the price for a transaction with seller s is $1/\Delta V_{s+1}$, then by Lemma 2 as $\phi\to\infty$ the price converges to $(1-\gamma)^{1/\gamma}$, $\forall s$. That is, the price in each transaction converges to a constant $\overline{q}$ satisfying $u(\overline{q}) = \overline{q}$ .Also, by Proposition 2, the $d^*=1$ equilibrium will still be supported as $\phi\to\infty$. Consequently the variance of prices converges to zero in the limit. □

Proof of Lemma 3

From (7') $(1-\gamma)V_1^{\gamma}=(1-m_N)^{-1}\sum_{s=0}^{N-1} m_s A^{s(1-\gamma)+1}$. Let $\mu_s=m_s(1-m_N)^{-1}$ so $\sum_{s=0}^{N-1}\mu_s=1$ and $V_1\propto\sum_{s=0}^{N-1}\mu_s A^{s(1-\gamma)}$. Denote partial derivatives with respect to M^s with a prime ('). Recall that $m'_0<0$, $m'_N>0$, and $\{m_s\}$ is a decreasing (increasing) sequence for $M^s<N/2$ ($M^s>N/2$) satisfying (1a)-(1b) and (6') and notice that $\sum_{s=0}^{N-1}\mu'_s=0$. Consequently there exists an integer $s(M^s)<N-1$ such that $\mu_s'<0$ for $s< s(M^s)$, and $\mu_s'>0$ for $s>s(M^s)$. Recall also that $A^{s(1-\gamma)}$ is decreasing in s (since $A<1$) and that $A'<0$. We show that V_1 is decreasing in M^s by noticing that $V'_1\propto d\left(\sum_{s=0}^{N-1}\mu_s A^{s(1-\gamma)}\right)/dM^s=\sum_{s=0}^{N-1}\mu'_s A^{s(1-\gamma)}+(1-\gamma)\sum_{s=0}^{N-1}\mu_s\, sA^{s(1-\gamma)-1}A'$, a finite quantity. The second summation in the last expression is negative, since $A'<0$, and clearly finite. The first summation is also negative since $\sum_{s=0}^{N-1}\mu'_s A^{s(1-\gamma)}<\sum_{s=0}^{N-1}\mu'_s=0$ because $A^{s(1-\gamma)}$ is decreasing in s and $\mu_s'<0$ for $s<s(M^s)$, while $\mu_s'>0$ for $s>s(M^s)$. It is also finite since the μ'_s are finite, $A^{s(1-\gamma)}$ converges to zero and the sequence of partial sums of $|A^{s(1-\gamma)}- A^{(s+1)(1-\gamma)}|$ converges. Each μ'_s is finite since m'_s is. This is easily seen since (1a) implies $\sum_{s=0}^{N} m'_s = 0 \; \forall M^s \in (0,N)$, which requires finite $m'_s \; \forall s$.□

Proof of Proposition 4

In a $d^*=1$ equilibrium the average price for seller s is q_s^{-1}. Retain the notation and assumptions used in the proof of lemma 3 and recall that $q_s = A^s V_1$, decreasing in s, and $q'_s<0$ (since both A and V_1 decrease in M^s). Define the probability of being paired to a seller offering a price q_s as $\mu_s = m_s(1-m_N)^{-1}$ so that the price level P is the average price across sellers, where $P = \sum_{s=0}^{N-1} \mu_s \frac{1}{q_s}$ and $P' = \sum_{s=0}^{N-1} \mu'_s \frac{1}{q_s} - \sum_{s=0}^{N-1} \frac{q'_s \mu_s}{q_s^2}$. The second summation in the last expression is negative because $q'_s<0$. The first summation in P' is positive since (as in the proof of lemma 3) there exists an $s(M^s)$ such that $\mu'_s<0$ for $s<s(M^s)$, $\mu'_s>0$ for $s>s(M^s)$ and q_s^{-1} is increasing in s. Hence $P' > 0$. □

Proof of Proposition 5

We start the proof by recalling that when $d^*(b,s)=1$ the distribution of money holdings is uniform at $M^s=N/2$. Notice from (6") and (1b) that M^s is the mean of the censored geometric distribution of money. Also recall that $m_n > m_{n+1}$ if $M^s<N/2$ while $m_n \leq m_{n+1}$ if $M^s \geq N/2$, i.e. higher M^s implies a redistribution of weight from low to high money holders.

Next, we observe that the variance of the distribution of money holdings is zero for $M^s \in \{0,N\}$ (the distribution is degenerated with $m_0=1$ or $m_N = 1$) and it is maximized at $M^s=N/2$ (uniform distribution), so that it is increasing on $M^s \in (0,N/2)$ and decreasing on $M^s \in (N/2, N)$ and it is quasiconcave on M^s.

Finally, observe that prices charged by high sellers respond to changes in M^s more than prices charged by low sellers, so that the dispersion of prices is increasing in M^s. Recall that q_s^{-1} is the price offered by seller s, $q_s = A^s V_1$, let a prime (') denote the partial derivative with respect to M^s, and let $1/q_N - 1/q_0$ be a measure of the dispersion of prices. Recall—from previous proofs—that $A'<0$, $(1/q_s)'>0$ and $V'_1<0$.

We confirm that $(1/q_s)'>(1/q_{s-1})'$ for all $i\in\{1,2,...N-s\}$, by rearranging the inequality as $-(sA^{s-1}A'V_1+A^sV'_1)/(A^sV_1)^2 > -[(s-1)A^{s-2}A'V_1+A^{s-1}V'_1)]/(A^{s-1}V_1)^2$ which simplifies to $s>s-1$. This last inequality implies that, since P is an average of all prices, P is more (less) responsive to changes in M^s than $1/q_s$ for low (high) s, thus the dispersion of prices increases with increments in M^s. Recall that both prices and μ_s are continuous and differentiable functions of M^s, so that the variance of prices, $\sum_{s=0}^{N-1}\mu_s\left(\frac{1}{q_s}-P\right)^2$, is continuous and differentiable on $M^s\in(0,N)$. Let $M^s=N/2$ and consider a marginal drop in M^s. Assume this leads to an increased variance of prices. The change in M^s decreases the price level and increases the probability of trades with low sellers (since $m_n>m_{n+1}$ for $M^s<N/2$). Since the price offered by a low seller is less responsive to changes in M^s than P, the deviation from P decreases for low sellers and is also more heavily weighted. For a similar reason high sellers experience lower deviations from P which are less weighted than before (as M^s drops there are less high sellers). But this is in contradiction with the assumption of increased variance. This argument may be replicated for all $M^s\in(0,N/2]$, hence the variance of prices must be increasing on $M^s\in(0,N/2)$. □

Proof of Proposition 6

Use a prime (') to denote the partial derivative with respect to M^s. From (8) it is seen that W is a continuously differentiable function on $M^s\in[0,N]$, since both m_n and V_n are continuously differentiable and

$$W'=\sum_{n=1}^{N}\left(m'_n V_n + m_n V'_n\right)$$

a continuous function on M^s.

Notice that $V_n=(V_n-V_{n-1}) + (V_{n-1}-V_{n-2}) + \ldots + (V_1-V_0)$, since $V_0=0$. In a $d^*(b,s)=1$ equilibrium $q_s=V_{s+1}-V_s$ and $V_n=\sum_{s=0}^{n-1}q_s$. From the proof of proposition 4, $q'_n<0$ $\forall n$ implies $V'_n<0$, $\forall M^s$. From the proof of proposition 1, and (1a)-(1b), as $M^s\to0$ then $m_0\to1$ and $m_n\to0$ $\forall n\neq0$, while as $M^s\to N$ then $m_N\to1$ and

$m_n \to 0$ $\forall$n$\neq$N. This implies $\lim_{M^s\to 0} A = \overline{A} = \frac{\phi}{1+\phi}$, $\lim_{M^s\to N} A=0$, $\lim_{M^s\to 0} m'_n \geq 0$ $\forall$n$\neq$0, and $\lim_{M^s\to N} m'_n \leq 0$ $\forall$n$\neq$N. Also, both $\lim_{M^s\to 0} m'_n$ and $\lim_{M^s\to N} m'_n$ are finite, from the proof of lemma 3. Using (7"), let $\overline{V} = \lim_{M^s\to 0} V_1 = \lim_{M^s\to 0} q_1 = \left[\frac{\phi}{(1+\phi)(1-\gamma)}\right]^{\frac{1}{\gamma}}$ and $\lim_{M^s\to N} V_1 = \lim_{M^s\to N} q_1 = 0$. Since $V_n = \frac{1-A^n}{1-A} V_1$ then $\lim_{M^s\to 0} V_n = \frac{1-\overline{A}^n}{1-\overline{A}} \overline{V} > 0$ and $\lim_{M^s\to N} V_n = 0$. Finally notice—from the proof of lemma 3—that V'_n is finite $\forall M^s \in [0,N]$ since V'_1 is finite.

The above imply that $\lim_{M^s\to 0} W' = \overline{V} \lim_{M^s\to 0} \sum_{n=1}^{N} \frac{1-\overline{A}^n}{1-\overline{A}} m'_n \geq 0$ (since the lim of the product is the product of the lim) and $\lim_{M^s\to N} W' = \lim_{M^s\to N} \sum_{n=1}^{N} m_n V'_n = \lim_{M^s\to N} V'_N < 0$. By applying the intermediate value theorem, there must exist at least an $\overline{M} \in (0,N)$ such that W'=0 for $M^s = \overline{M}$ and W'>0 for $M^s < \overline{M}$. There also must exist an $\overline{\overline{M}} \in [\overline{M}, N)$ such that W'=0 for $M^s = \overline{\overline{M}}$ and W'<0 for $M^s > \overline{\overline{M}}$.☐

Figure 2b. Transaction Patterns (M^S=27)

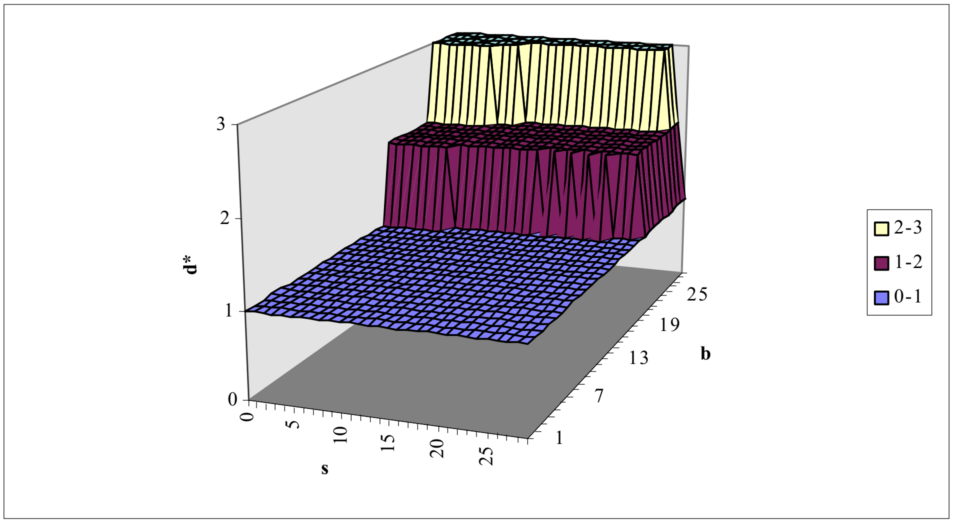

Figure 3. Price Level and its Standard Deviation Across M^S

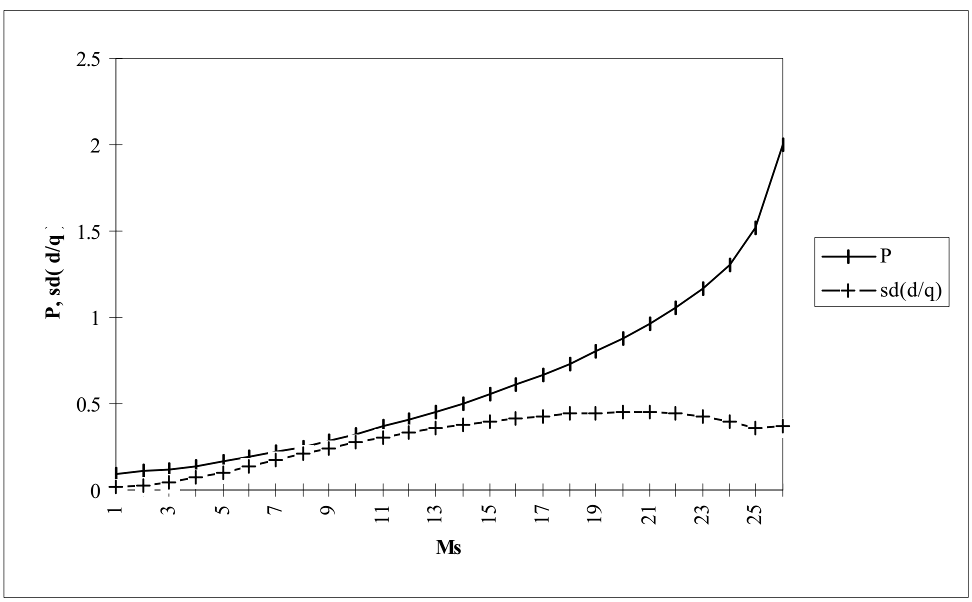

Figure 4. Value Functions ($V_{n'}$, $n'=n/(N\eta)$) for η=1,2,4.

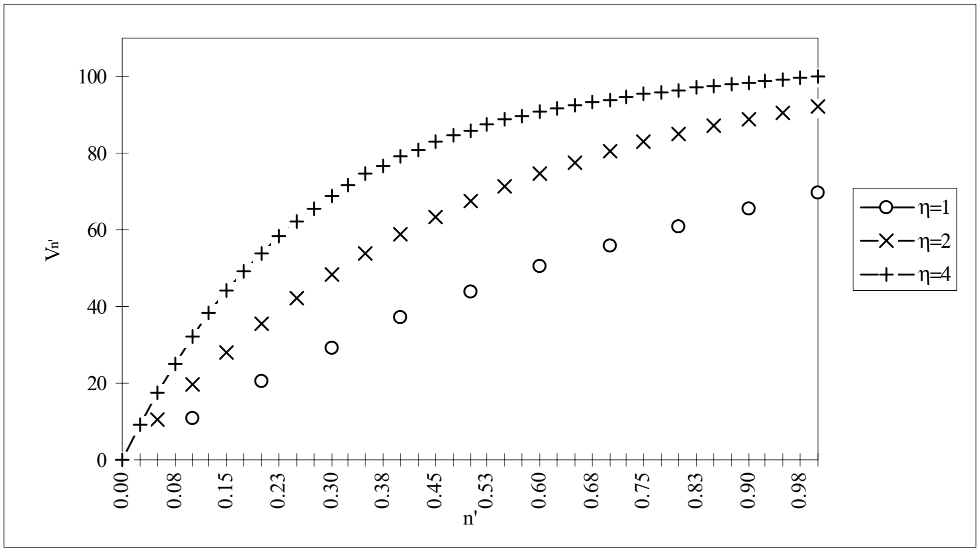

Figure 5. Distribution of money holdings ($m_{n'}$, $n'=n/(N\eta)$) for η=1,2,4.

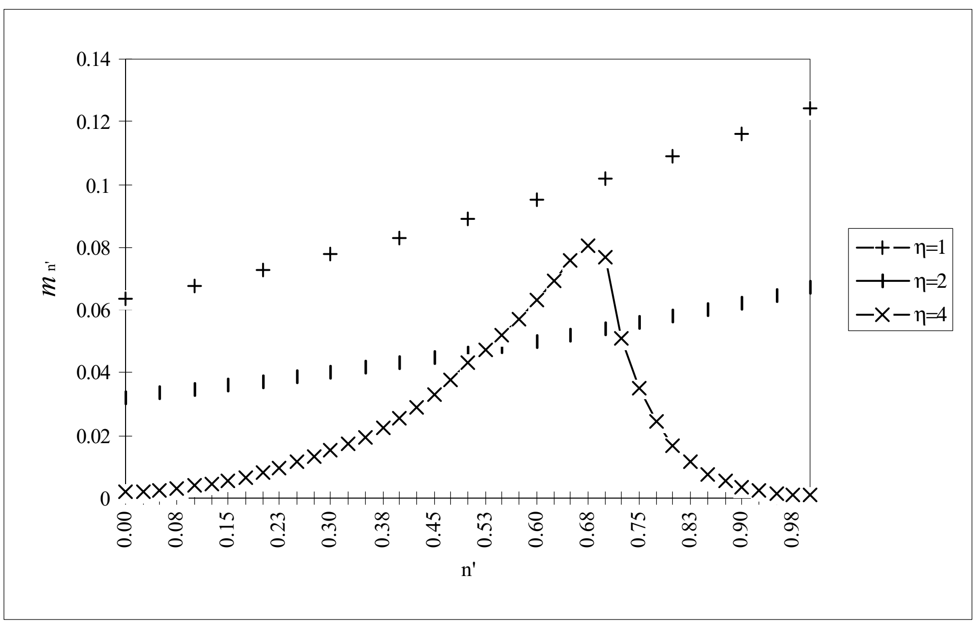

Table 1

ϕ	*P*	sd(d/q)/*P*	E(d/*b*)	sd(d/*b*)/E(d/*b*)
200	0.256	0.007	0.309	0.6214
20	0.408	0.15	0.309	0.621
2.22	4.677	0.479	0.46	0.343

Note: M^S=4, γ=0.5, N=30, k=5, *P*=price level, sd(d/q)/*P*=coefficient of variation of price level, E(d/b)=average propensity to consume, sd(d/b)=standard deviation of propensity to consume.

Table 2

η	M^s	*P*	sd(d/q)/*P*	E(d/*b*)	sd(d/*b*)	W
1	1	0.096	0.145	0.693	0.319	10.544
	5	0.153	0.294	0.293	0.263	41.158
	8	0.274	0.244	0.147	0.111	50.369
2	1	0.054	0.277	0.546	0.336	19.362
	5	0.145	0.558	0.180	0.218	61.436
	8	0.338	0.360	0.074	0.071	69.683
3	1	0.041	0.397	0.452	0.327	26.973
	5	0.185	0.717	0.12	0.154	73.552
	8	0.436	0.307	0.051	0.021	82.655
4	1	0.035	0.514	0.387	0.31	33.541
	5	0.217	0.571	0.074	0.097	83.038
	8	0.580	0.227	0.060	0.01	82.2

Note: ρ=0.02, γ=0.9, N=10, k=5, *P*=average price level, sd(d/q)=standard deviation of price level, E(d/*b*)=average propensity to consume, sd(d/*b*)=standard deviation of propensity to consume. Also, prices are adjusted (divided by η). The price is d/q so, for instance, when η=1, d=1 implies a whole unit of currency has been traded, but when η=4, d=1 implies that only a quarter of the unit of currency has been traded.