Monge-Ampère Operators, Lelong Numbers and Intersection Theory

Jean-Pierre Demailly

Université de Grenoble I,
Institut Fourier, BP 74,
URA 188 associée au C.N.R.S.
F-38402 Saint-Martin d'Hères

Abstract

This article is a survey on the theory of Monge-Ampère operators and Lelong numbers. The definition of complex Monge-Ampère operators is extended in such a way that wedge products of closed positive currents become admissible in a large variety of situations; the only basic requirement is that the polar set singularities have mutual intersections of the correct codimension. This makes possible to develope the intersection theory of analytic cycles by means of current theory and Lelong numbers. The advantage of this point of view, in addition to its wider generality, is to produce simpler proofs of previously known results, as well as to relate some of these results to other questions in analytic geometry or number theory. For instance, the generalized Lelong-Jensen formula provides a useful tool for studying the location and multiplicities of zeros of entire functions on $\mathbb{C}^n$ or on a manifold, in relation with the growth at infinity (Schwarz lemma type estimates). Finally, we obtain a general self-intersection inequality for divisors and positive $(1,1)$-currents on compact Kähler manifolds, based on a singularity attenuation technique for quasi-plurisubharmonic functions.

Contents

0. Introduction

This contribution is a survey article on the theory of Lelong numbers, viewed as a tool for studying intersection theory by complex differential geometry. We have not attempted to make an exhaustive compilation of the existing literature on the subject, nor to present a complete account of the state-of-the-art. Instead, we have tried to present a coherent unifying frame for the most basic results of the theory, based in part on our earlier works [De1,2,3,4] and on Siu's fundamental work [Siu]. To a large extent, the asserted results are given with complete proofs, many of them substantially shorter and simpler than their original counterparts. We only assume that the reader has some familiarity with differential calculus on complex manifolds and with the elementary facts concerning analytic sets and plurisubharmonic functions. The reader can consult Lelong's books [Le2,3] for an introduction to the subject. Most of our results still work on arbitrary complex analytic spaces, provided that suitable definitions are given for currents, plurisubharmonic functions, etc, in this more general situation. We have refrained ourselves from doing so for simplicity of exposition; we refer the reader to [De3] for the technical definitions required in the context of analytic spaces.

Let us first recall a few basic definitions. A *current* of degree q on an oriented differentiable manifold M is simply a differential q-form Θ with distribution coefficients. Alternatively, a current of degree q is an element Θ in the dual space $\mathcal{D}'_p(M)$ of the space $\mathcal{D}_p(M)$ of smooth differential forms of degree $p = \dim M - q$ with compact support; the duality pairing is given by

$$(0.1) \qquad \langle \Theta, \alpha \rangle = \int_M \Theta \wedge \alpha, \qquad \alpha \in \mathcal{D}_p(M).$$

A basic example is the *current of integration* $[S]$ over a compact oriented submanifold S of M :

$$(0.2) \qquad \langle [S], \alpha \rangle = \int_S \alpha, \qquad \deg\alpha = p = \dim_{\mathbb{R}} S.$$

Then $[S]$ is a current with measure coefficients, and Stokes' formula shows that $d[S] = (-1)^{q-1}[\partial S]$, in particular $d[S] = 0$ if S has no boundary. Because of this example, the integer p is said to be the dimension of Θ when $\Theta \in \mathcal{D}'_p(M)$. The current Θ is said to be *closed* if $d\Theta = 0$.

On a complex manifold X, we have similar notions of bidegree and bidimension. According to Lelong [Le1], a current T of bidimension (p,p) is said to be *(weakly) positive* if for every choice of smooth $(1,0)$-forms $\alpha_1, \ldots, \alpha_p$ on X the distribution

$$(0.3) \qquad T \wedge \mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha}_p \quad \text{is a positive measure.}$$

Then, the coefficients $T_{I,J}$ of T are complex measures, and up to constants, they are dominated by the trace measure $\sum T_{I,I}$ which is positive. With every closed analytic set $A \subset X$ of pure dimension p is associated a current of integration

$$(0.4) \qquad \langle [A], \alpha \rangle = \int_{A_{\rm reg}} \alpha, \qquad \alpha \in \mathcal{D}_{p,p}(X),$$

obtained by integrating over the regular points of A. It is easy to see that $[A]$ is positive. Lelong [Le1] has shown that $[A]$ has locally finite mass near $A_{\rm sing}$ and that $[A]$ is closed in X. This last result can be seen today as a consequence of the Skoda-El Mir extension theorem ([EM], [Sk3]; also [Sib]).

(0.5) Theorem. *Let E be a closed complete pluripolar set in X, and let Θ be a closed positive current on $X \smallsetminus E$ such that the coefficients $\Theta_{I,J}$ of Θ are measures with locally finite mass near E. Then the trivial extension $\widetilde{\Theta}$ obtained by extending the measures $\Theta_{I,J}$ by 0 on E is still closed.*

A *complete pluripolar* set is by definition a set E such that there is an open covering (Ω_j) of X and plurisubharmonic functions u_j on Ω_j with $E \cap \Omega_j = u_j^{-1}(-\infty)$. Any (closed) analytic set is of course complete pluripolar. Lelong's result $d[A] = 0$ is obtained by applying the El Mir-Skoda theorem to $\Theta = [A_{\rm reg}]$ on $X \smallsetminus A_{\rm sing}$. Another interesting consequence is

(0.6) Corollary. *Let T be a closed positive current on X and let E be a complete pluripolar set. Then $\mathbb{1}_E T$ and $\mathbb{1}_{X \smallsetminus E} T$ are closed positive currents. In fact $\mathbb{1}_E T = T \smallsetminus \widetilde{T}$, where $\widetilde{T}$ is the trivial extension of $T_{\restriction X \smallsetminus E}$ to X.*

The other main tool used in this paper is the theory of plurisubharmonic functions. If u is a plurisubharmonic function on X, we can associate with u a closed positive current $T = \mathrm{i}\partial\overline{\partial} u$ of bidegree $(1,1)$. Conversely, every closed positive current of bidegree $(1,1)$ can be written under this form if $H^2_{DR}(X, \mathbb{R}) = H^1(X, \mathcal{O}) = 0$. In the special case $u = \log|F|$ with a non zero holomorphic function $F \in \mathcal{O}(X)$, we have the important *Lelong-Poincaré* equation

$$(0.7) \qquad \frac{\mathrm{i}}{\pi} \partial\overline{\partial} \log|F| = [Z_F],$$

where $Z_F = \sum m_j Z_j$, $m_j \in \mathbb{N}$, is the zero divisor of F and $[Z_F] = \sum m_j [Z_j]$ is the associated current of integration.

Our goal is to develope the intersection theory of analytic cycles from this point of view. In particular, we would like to define the wedge product $T \wedge \mathrm{i}\partial\overline{\partial} u_1 \wedge \ldots \wedge \mathrm{i}\partial\overline{\partial} u_q$ of a closed positive current T by "generalized" divisors $\mathrm{i}\partial\overline{\partial} u_j$. In general this is not possible, because measures cannot be multiplied. However, we will show in sections 1,2 that Monge-Ampère operators of this type are well defined as soon as the set of poles of the u_j's have intersections of sufficiently low dimension. The proof rests on a procedure due to Bedford and Taylor [B-T1], [B-T2] and consists mostly in rather simple integration by parts. In spite of its simple nature, this result seems to be new.

Then, following [De2,4], we introduce the generalized Lelong numbers of a closed positive current $T \in \mathcal{D}'_{p,p}(X)$ with respect to a plurisubharmonic weight φ. Under suitable exhaustivity conditions for φ, we define $\nu(T,\varphi)$ as the residue

$$\nu(T,\varphi) = \int_{\varphi^{-1}(\infty)} T \wedge \Big(\frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\varphi\Big)^p. \tag{0.8}$$

The standard Lelong number $\nu(T,x)$ corresponds to the "isotropic" weight $\varphi(z) = \log|z-x|$; it can also be seen as the euclidean density of T at x, when T is compared to the current of integration over a p-dimensional vector subspace in $\mathbb{C}^n$. However the generalized definition is more flexible and allows us to give very simple proofs of several basic properties: in particular, the Lelong number $\nu(T,x)$ does not depend on the choice of coordinates, and coincides with the algebraic multiplicity in the case of a current of integration $T = [A]$ (Thie's theorem [Th]). These facts are obtained as a consequence of a comparison theorem for the Lelong numbers $\nu(T,\varphi)$ and $\nu(T,\psi)$ associated with different weights.

Next, we prove Siu's semicontinuity theorem in this general setting: if φ_y is a family of plurisubharmonic functions on X depending on a parameter $y \in Y$, such that $\varphi(x,y) := \varphi_y(x)$ is plurisubharmonic and satisfies some natural exhaustivity and continuity conditions, then $y \mapsto \nu(T,\varphi_y)$ is upper semicontinuous with respect to the (analytic) Zariski topology. Explicitly, the upperlevel sets

$$E_c(T) = \{y \in Y\,;\, \nu(T,\varphi_y) \ge c\}, \qquad c > 0 \tag{0.9}$$

are analytic in Y. The result of [Siu] concerning ordinary Lelong numbers is obtained for $\varphi(x,y) = \log|x-y|$, but the above result allows much more general variations of the weight. The proof uses ideas of Kiselman [Ki 1,2] and rests heavily on L^2 estimates for $\overline{\partial}$, specifically on the Hörmander-Bombieri-Skoda theorem ([Hö], [Bo], [Sk2]).

Next, following [De2], we investigate the behaviour of Lelong numbers under direct images by proper holomorphic maps. Let T be a closed positive current of bidimension (p,p) on X, let $F : X \to Y$ be a proper and finite analytic map and let $F_\star T$ be the direct image of T by F. We prove inequalities of the type

$$\nu(F_\star T, y) \ge \sum_{x \in \operatorname{Supp} T \cap F^{-1}(y)} \mu_p(F,x)\,\nu(T,x), \tag{0.10}$$

$$\nu(F_\star T, y) \le \sum_{x \in \operatorname{Supp} T \cap F^{-1}(y)} \overline{\mu}_p(F,x)\,\nu(T,x), \tag{0.10'}$$

where $\mu_p(F,x)$ and $\overline{\mu}_p(F,x)$ are suitable multiplicities attached to F at each point. In case $p = \dim X$, the multiplicity $\mu_p(F,x)$ coincides with the one introduced by Stoll [St] (see also Draper [Dr]).

As an application of these inequalities, we prove a general Schwarz lemma for entire functions in $\mathbb{C}^n$, relating the growth at infinity of such a function and the location of its zeros ([De1]). The proof is essentially based on the Lelong-Jensen formula. Finally, we use the Schwarz lemma to derive a simple proof of Bombieri's theorem [Bo] on algebraic values of meromorphic maps satisfying algebraic differential equations.

The above techniques are also useful for relating local intersection invariants to global ones, e.g. intersection numbers of analytic cycles in compact Kähler manifolds. In that case, it is very important to consider self-intersections or situations with excess dimension of intersection. Such situations can also be handled by the Monge-Ampère techniques, thanks to a general regularization process for closed positive $(1,1)$-currents on compact manifolds [De7]. The smooth approximations of a positive current are no longer positive, but they have a small negative part depending on the curvature of the tangent bundle of the ambient manifold. As an application, we obtain a general self-intersection inequality giving a bound for the degree of the strata of constant multiplicity in an effective divisor D, in terms of a polynomial in the cohomology class $\{D\} \in H^2_{DR}(X,\mathbb{R})$. This inequality can be seen as a generalization of the usual bound $d(d-1)/2$ for the number of multiple points of a plane curve of degree d. Other applications are presented in [De5,6].

This paper is an expanded version of a course made in Nice in July 1989, at a summer school on Complex Analysis organized by the ICPAM. Since then, the author has benefited from many valuable remarks made by several mathematicians, in particular M.S. Narasimhan at the Tata Institute, Th. Peternell and M. Schneider at Bayreuth University, and Z. Błocki, S. Kołodziej, J. Siciak and T. Winiarski at the Jagellonian University in Cracow. The author expresses his warm thanks to these institutions for their hospitality.

1. Definition of Monge-Ampère Operators

Let X be a n-dimensional complex manifold. We denote by $d = d' + d''$ the usual decomposition of the exterior derivative in terms of its $(1,0)$ and $(0,1)$ parts, and we set

$$d^c = \frac{1}{2i\pi}(d' - d'').$$

It follows in particular that d^c is a real operator, i.e. $\overline{d^c u} = d^c\overline{u}$, and that $dd^c = \frac{\mathrm{i}}{\pi}d'd''$. Although not quite standard, the $1/2\mathrm{i}\pi$ normalization is very convenient for many purposes, since we may then forget 2π almost everywhere (e.g. in the Lelong-Poincaré equation (0.7)). In this context, we have the following integration by part formula.

(1.1) Formula. *Let $\Omega \subset\subset X$ be a smoothly bounded open set in X and let f, g be forms of class C^2 on $\overline{\Omega}$ of pure bidegrees (p,p) and (q,q) with $p+q = n-1$. Then*

$$\int_\Omega f \wedge dd^c g - dd^c f \wedge g = \int_{\partial\Omega} f \wedge d^c g - d^c f \wedge g.$$

Proof. By Stokes' theorem the right hand side is the integral over Ω of

$$d(f \wedge d^c g - d^c f \wedge g) = f \wedge dd^c g - dd^c f \wedge g + (df \wedge d^c g + d^c f \wedge dg).$$

As all forms of total degree $2n$ and bidegree $\neq (n,n)$ are zero, we get

$$df \wedge d^c g = \frac{1}{2\mathrm{i}\pi}(d''f \wedge d'g - d'f \wedge d''g) = -d^c f \wedge dg. \qquad \square$$

Let u be a plurisubharmonic function on X and let T be a closed positive current of bidimension (p,p), i.e. of bidegree $(n-p, n-p)$. Our desire is to define the wedge product $dd^c u \wedge T$ even when neither u nor T are smooth. A priori, this product does not make sense because $dd^c u$ and T have measure coefficients and measures cannot be multiplied. The discussion made in section 9 shows that there is no way of defining $dd^c u \wedge T$ as a closed positive current without further hypotheses (see also [Ki3] for interesting counterexamples). Assume however that u is a *locally bounded* plurisubharmonic function. Then the current uT is well defined since u is a locally bounded Borel function and T has measure coefficients. According to Bedford-Taylor [B-T2] we define

$$dd^c u \wedge T = dd^c(uT)$$

where $dd^c(\ \ )$ is taken in the sense of distribution (or current) theory.

(1.2) Proposition. *The wedge product $dd^c u \wedge T$ is again a closed positive current.*

Proof. The result is local. In an open set $\Omega \subset \mathbb{C}^n$, we can use convolution with a family of regularizing kernels to find a decreasing sequence of smooth plurisubharmonic functions $u_k = u \star \rho_{1/k}$ converging pointwise to u. Then $u \le u_k \le u_1$ and Lebesgue's dominated convergence theorem shows that $u_k T$

converges weakly to uT; thus $dd^c(u_kT)$ converges weakly to $dd^c(uT)$ by the weak continuity of differentiations. However, since u_k is smooth, $dd^c(u_kT)$ coincides with the product $dd^cu_k \wedge T$ in its usual sense. As $T \geq 0$ and as dd^cu_k is a positive $(1,1)$-form, we have $dd^cu_k \wedge T \geq 0$, hence the weak limit $dd^cu \wedge T$ is ≥ 0 (and obviously closed). □

Given locally bounded plurisubharmonic functions $u_1, \ldots, u_q$, we define inductively

$$dd^cu_1 \wedge dd^cu_2 \wedge \ldots \wedge dd^cu_q \wedge T = dd^c(u_1 dd^cu_2 \ldots \wedge dd^cu_q \wedge T).$$

By (1.2) the product is a closed positive current. In particular, when u is a locally bounded plurisubharmonic function, the bidegree (n,n) current $(dd^cu)^n$ is well defined and is a positive measure. If u is of class C^2, a computation in local coordinates gives

$$(dd^cu)^n = \det\big(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\big) \cdot \frac{n!}{\pi^n}\, \mathrm{i}dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \mathrm{i}dz_n \wedge d\overline{z}_n.$$

The expression "Monge-Ampère operator" classically refers to the non-linear partial differential operator $u \longmapsto \det(\partial^2 u/\partial z_j \partial \overline{z}_k)$. By extension, all operators $(dd^c)^q$ defined above are also called Monge-Ampère operators.

Now, let Θ be a current of order 0. When $K \subset\subset X$ is an arbitrary compact subset, we define a *mass* semi-norm

$$||\Theta||_K = \sum_j \int_{K_j} \sum_{I,J} |\Theta_{I,J}|$$

by taking a partition $K = \bigcup K_j$ where each $\overline{K}_j$ is contained in a coordinate patch and where $\Theta_{I,J}$ are the corresponding measure coefficients. Up to constants, the semi-norm $||\Theta||_K$ does not depend on the choice of the coordinate systems involved. When K itself is contained in a coordinate patch, we set $\beta = dd^c|z|^2$ over K; then, if $\Theta \geq 0$, there are constants $C_1, C_2 > 0$ such that

$$C_1||\Theta||_K \leq \int_K \Theta \wedge \beta^p \leq C_2||\Theta||_K.$$

We denote by $L^1(K)$, resp. by $L^\infty(K)$, the space of integrable (resp. bounded measurable) functions on K with respect to any smooth positive density on X.

(1.3) Chern-Levine-Nirenberg inequalities ([C-L-N]). *For all compact subsets K, L of X with $L \subset K^\circ$, there exists a constant $C_{K,L} \geq 0$ such that*

$$||dd^cu_1 \wedge \ldots \wedge dd^cu_q \wedge T||_L \leq C_{K,L}\ ||u_1||_{L^\infty(K)} \ldots ||u_q||_{L^\infty(K)}\ ||T||_K.$$

Proof. By induction, it is sufficient to prove the result for $q = 1$ and $u_1 = u$. There is a covering of L by a family of balls $B'_j \subset\subset B_j \subset K$ contained in coordinate patches of X. Let $\chi \in \mathcal{D}(B_j)$ be equal to 1 on $\overline{B}'_j$. Then

$$||dd^c u \wedge T||_{L\cap\overline{B}'_j} \le C \int_{\overline{B}'_j} dd^c u \wedge T \wedge \beta^{p-1} \le C \int_{B_j} \chi\, dd^c u \wedge T \wedge \beta^{p-1}.$$

As T and β are closed, an integration by parts yields

$$||dd^c u \wedge T||_{L\cap\overline{B}'_j} \le C \int_{B_j} u\, T \wedge dd^c\chi \wedge \beta^{p-1} \le C' ||u||_{L^\infty(K)} ||T||_K$$

where C' is equal to C multiplied by a bound for the coefficients of the smooth form $dd^c\chi \wedge \beta^{p-1}$. □

(1.4) *Remark.* With the same notations as above, any plurisubharmonic function V on X satisfies inequalities of the type

(a) $\quad ||dd^c V||_L \le C_{K,L}\, ||V||_{L^1(K)}$.

(b) $\quad \sup_L V_+ \le C_{K,L}\, ||V||_{L^1(K)}$.

In fact the inequality

$$\int_{L\cap\overline{B}'_j} dd^c V \wedge \beta^{n-1} \le \int_{B_j} \chi dd^c V \wedge \beta^{n-1} = \int_{B_j} V dd^c\chi \wedge \beta^{n-1}$$

implies (a), and (b) follows from the mean value inequality.

(1.5) *Remark.* Products of the form $\Theta = \gamma_1 \wedge \ldots \wedge \gamma_q \wedge T$ with mixed $(1,1)$-forms $\gamma_j = dd^c u_j$ or $\gamma_j = dv_j \wedge d^c w_j + dw_j \wedge d^c v_j$ are also well defined whenever u_j, v_j, w_j are locally bounded plurisubharmonic functions. Moreover, for $L \subset K^\circ$, we have

$$||\Theta||_L \le C_{K,L} ||T||_K \prod ||u_j||_{L^\infty(K)} \prod ||v_j||_{L^\infty(K)} \prod ||w_j||_{L^\infty(K)}.$$

To check this, we may suppose $v_j, w_j \ge 0$ and $||v_j|| = ||w_j|| = 1$ in $L^\infty(K)$. Then the inequality follows from (1.3) by the polarization identity

$$2(dv_j \wedge d^c w_j + dw_j \wedge d^c v_j) = dd^c(v_j + w_j)^2 - dd^c v_j^2 - dd^c w_j^2 - v_j dd^c w_j - w_j dd^c v_j$$

in which all dd^c operators act on plurisubharmonic functions.

(1.6) Corollary. *Let $u_1, \ldots, u_q$ be continuous (finite) plurisubharmonic functions and let $u_1^k, \ldots, u_q^k$ be sequences of plurisubharmonic functions converging locally uniformly to $u_1, \ldots, u_q$. If T_k is a sequence of closed positive currents converging weakly to T, then*

(a) $u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T_k \longrightarrow u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$ *weakly.*

(b) $dd^c u_1^k \wedge \ldots \wedge dd^c u_q^k \wedge T_k \longrightarrow dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ *weakly.*

Proof. We observe that (b) is an immediate consequence of (a) by the weak continuity of dd^c. By using induction on q, it is enough to prove result (a) when $q = 1$. If (u^k) converges locally uniformly to a finite continuous plurisubharmonic function u, we introduce local regularizations $u_\varepsilon = u \star \rho_\varepsilon$ and write

$$u^k T_k - uT = (u^k - u)T_k + (u - u_\varepsilon)T_k + u_\varepsilon(T_k - T).$$

As the sequence T_k is weakly convergent, it is locally uniformly bounded in mass, thus $||(u^k - u)T_k||_K \le ||u^k - u||_{L^\infty(K)} ||T_k||_K$ converges to 0 on every compact set K. The same argument shows that $||(u - u_\varepsilon)T_k||_K$ can be made arbitrarily small by choosing ε small enough. Finally u_ε is smooth, so $u_\varepsilon(T_k - T)$ converges weakly to 0. □

Now, we prove a deeper monotone continuity theorem due to Bedford-Taylor [B-T2], according to which the continuity and uniform convergence assumptions can be dropped if each sequence (u_j^k) is decreasing and T_k is a constant sequence.

(1.7) Theorem. *Let $u_1, \ldots, u_q$ be locally bounded plurisubharmonic functions and let $u_1^k, \ldots, u_q^k$ be decreasing sequences of plurisubharmonic functions converging pointwise to $u_1, \ldots, u_q$. Then*

(a) $u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \longrightarrow u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$ *weakly.*

(b) $dd^c u_1^k \wedge \ldots \wedge dd^c u_q^k \wedge T \longrightarrow dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ *weakly.*

Proof. Again by induction, observing that (a) $\Longrightarrow$ (b) and that (a) is obvious for $q = 1$ thanks to Lebesgue's bounded convergence theorem. To proceed with the induction step, we first have to make some slight modifications of our functions u_j and u_j^k.

As the sequence (u_j^k) is decreasing and as u_j is locally bounded, the family $(u_j^k)_{k \in \mathbb{N}}$ is locally uniformly bounded. The results are local, so we can work on a Stein open set $\Omega \subset\subset X$ with strongly pseudoconvex boundary. We use the following notations:

(1.8) Let ψ be a strongly plurisubharmonic function of class C^∞ near $\overline{\Omega}$ with $\psi < 0$ on Ω and $\psi = 0$, $d\psi \ne 0$ on $\partial\Omega$.

(1.8′) We set $\Omega_\delta = \{z \in \Omega\,;\, \psi(z) < -\delta\}$ for all $\delta > 0$.

After addition of a constant we can assume that $-M \le u_j^k \le -1$ near $\overline{\Omega}$. Let us denote by $(u_j^{k,\varepsilon})$, $\varepsilon \in]0,\varepsilon_0]$, an increasing family of regularizations converging to u_j^k as $\varepsilon \to 0$ and such that $-M \le u_j^{k,\varepsilon} \le -1$ on $\overline{\Omega}$. Set $A = M/\delta$ with $\delta > 0$ small and replace u_j^k by $v_j^k = \max\{A\psi, u_j^k\}$ and $u_j^{k,\varepsilon}$ by $v_j^{k,\varepsilon} = \max_\varepsilon\{A\psi, u_j^{k,\varepsilon}\}$ where $\max_\varepsilon = \max \star \rho_\varepsilon$ is a regularized max function (the construction of v_j^k is described by Fig. 1).

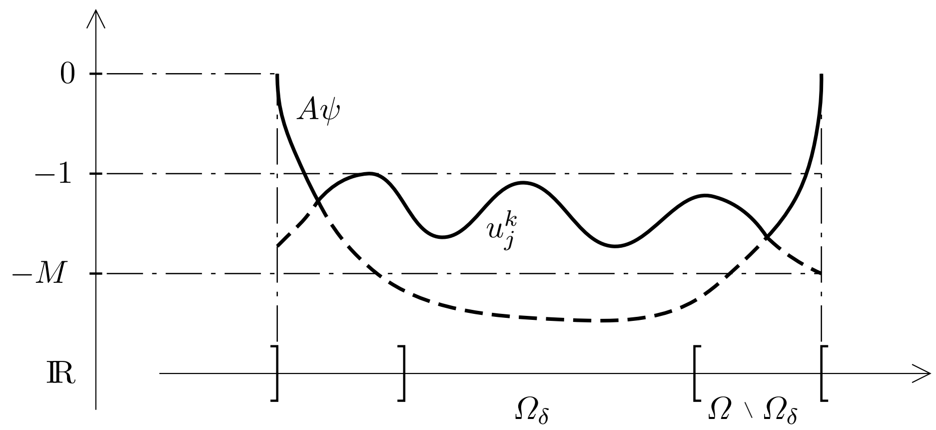

Fig. 1. Construction of v_j^k

Then v_j^k coincides with u_j^k on Ω_δ since $A\psi < -A\delta = -M$ on Ω_δ, and v_j^k is equal to $A\psi$ on the corona $\Omega \setminus \Omega_{\delta/M}$. Without loss of generality, we can therefore assume that all u_j^k (and similarly all $u_j^{k,\varepsilon}$) coincide with $A\psi$ on a fixed neighborhood of $\partial\Omega$. We need a lemma.

(1.9) Lemma. *Let f_k be a decreasing sequence of upper semi-continuous functions converging to f on some separable locally compact space X and μ_k a sequence of positive measures converging weakly to μ on X. Then every weak limit ν of $f_k\mu_k$ satisfies $\nu \le f\mu$.*

Indeed if (g_p) is a decreasing sequence of continuous functions converging to f_{k_0} for some k_0, then $f_k\mu_k \le f_{k_0}\mu_k \le g_p\mu_k$ for $k \ge k_0$, thus $\nu \le g_p\mu$ as $k \to +\infty$. The monotone convergence theorem then gives $\nu \le f_{k_0}\mu$ as $p \to +\infty$ and $\nu \le f\mu$ as $k_0 \to +\infty$. □

End of proof of Theorem 1.7. Assume that (a) has been proved for $q-1$. Then

$$S^k = dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \longrightarrow S = dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T.$$

By 1.3 the sequence $(u_1^k S^k)$ has locally bounded mass, hence is relatively compact for the weak topology. In order to prove (a), we only have to show that every weak limit Θ of $u_1^k S^k$ is equal to $u_1 S$. Let (m,m) be the bidimension of S and let γ be an arbitrary smooth and strongly positive form of bidegree (m,m). Then the positive measures $S^k \wedge \gamma$ converge weakly to $S\wedge\gamma$ and Lemma 1.9 shows that $\Theta\wedge\gamma \le u_1 S\wedge\gamma$, hence $\Theta \le u_1 S$. To get the equality, we set $\beta = dd^c\psi > 0$ and show that $\int_\Omega u_1 S\wedge\beta^m \le \int_\Omega \Theta\wedge\beta^m$, i.e.

$$\int_\Omega u_1 dd^c u_2\wedge\ldots\wedge dd^c u_q\wedge T\wedge\beta^m \le \liminf_{k\to+\infty}\int_\Omega u_1^k dd^c u_2^k\wedge\ldots\wedge dd^c u_q^k\wedge T\wedge\beta^m.$$

As $u_1 \le u_1^k \le u_1^{k,\varepsilon_1}$ for every $\varepsilon_1 > 0$, we get

$$\begin{aligned}\int_\Omega & u_1 dd^c u_2\wedge\ldots\wedge dd^c u_q\wedge T\wedge\beta^m\\ &\le \int_\Omega u_1^{k,\varepsilon_1} dd^c u_2\wedge\ldots\wedge dd^c u_q\wedge T\wedge\beta^m\\ &= \int_\Omega dd^c u_1^{k,\varepsilon_1}\wedge u_2 dd^c u_3\wedge\ldots\wedge dd^c u_q\wedge T\wedge\beta^m\end{aligned}$$

after an integration by parts (there is no boundary term because u_1^{k,ε_1} and u_2 both vanish on $\partial\Omega$). Repeating this argument with $u_2,\ldots,u_q$, we obtain

$$\begin{aligned}\int_\Omega & u_1 dd^c u_2\wedge\ldots\wedge dd^c u_q\wedge T\wedge\beta^m\\ &\le \int_\Omega dd^c u_1^{k,\varepsilon_1}\wedge\ldots\wedge dd^c u_{q-1}^{k,\varepsilon_{q-1}}\wedge u_q T\wedge\beta^m\\ &\le \int_\Omega u_1^{k,\varepsilon_1} dd^c u_2^{k,\varepsilon_2}\wedge\ldots\wedge dd^c u_q^{k,\varepsilon_q}\wedge T\wedge\beta^m.\end{aligned}$$

Now let $\varepsilon_q \to 0,\ldots,\varepsilon_1 \to 0$ in this order. We have weak convergence at each step and $u_1^{k,\varepsilon_1} = 0$ on the boundary; therefore the integral in the last line converges and we get the desired inequality

$$\int_\Omega u_1 dd^c u_2\wedge\ldots\wedge dd^c u_q\wedge T\wedge\beta^m \le \int_\Omega u_1^k dd^c u_2^k\wedge\ldots\wedge dd^c u_q^k\wedge T\wedge\beta^m. \quad \square$$

(1.10) Corollary. *The product $dd^c u_1\wedge\ldots\wedge dd^c u_q\wedge T$ is symmetric with respect to $u_1,\ldots,u_q$.*

Proof. Observe that the definition was unsymmetric. The result is true when $u_1,\ldots,u_q$ are smooth and follows in general from Th. 1.7 applied to the sequences $u_j^k = u_j \star \rho_{1/k}$, $1 \le j \le q$. $\square$

(1.11) Proposition. *Let K, L be compact subsets of X such that $L \subset K^\circ$. For any plurisubharmonic functions $V, u_1, \ldots, u_q$ on X such that $u_1, \ldots, u_q$ are locally bounded, there is an inequality*

$$||V dd^c u_1 \wedge \ldots \wedge dd^c u_q||_L \le C_{K,L}\, ||V||_{L^1(K)} ||u_1||_{L^\infty(K)} \ldots ||u_q||_{L^\infty(K)}.$$

Proof. We may assume that L is contained in a strongly pseudoconvex open set $\Omega = \{\psi < 0\} \subset K$ (otherwise we cover L by small balls contained in K). A suitable normalization gives $-2 \le u_j \le -1$ on K; then we can modify u_j on $\Omega \setminus L$ so that $u_j = A\psi$ on $\Omega \setminus \Omega_\delta$ with a fixed constant A and $\delta > 0$ such that $L \subset \Omega_\delta$. Let $\chi \ge 0$ be a smooth function equal to $-\psi$ on Ω_δ with compact support in Ω. If we take $||V||_{L^1(K)} = 1$, we see that V_+ is uniformly bounded on Ω_δ by 1.4 (b); after subtraction of a fixed constant we can assume $V \le 0$ on Ω_δ. First suppose $q \le n-1$. As $u_j = A\psi$ on $\Omega \setminus \Omega_\delta$, we find

$$\begin{aligned}
&\int_{\Omega_\delta} -V\, dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q} \\
&= \int_\Omega V\, dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q-1} \wedge dd^c\chi - A^q \int_{\Omega \setminus \Omega_\delta} V\, \beta^{n-1} \wedge dd^c \chi \\
&= \int_\Omega \chi\, dd^c V \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q-1} - A^q \int_{\Omega \setminus \Omega_\delta} V\, \beta^{n-1} \wedge dd^c \chi.
\end{aligned}$$

The first integral of the last line is uniformly bounded thanks to 1.3 and 1.4 (a), and the second one is bounded by $||V||_{L^1(\Omega)} \le$ constant. Inequality 1.11 follows for $q \le n-1$. If $q = n$, we can work instead on $X \times \mathbb{C}$ and consider $V, u_1, \ldots, u_q$ as functions on $X \times \mathbb{C}$ independent of the extra variable in $\mathbb{C}$. □

2. Case of Unbounded Plurisubharmonic Functions

We would like to define $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ also in some cases when $u_1, \ldots, u_q$ are not bounded below everywhere, especially when the u_j have logarithmic poles. Consider first the case $q = 1$ and let u be a plurisubharmonic function on X. The *pole set* of u is by definition $P(u) = u^{-1}(-\infty)$. We define the *unbounded locus* $L(u)$ to be the set of points $x \in X$ such that u is unbounded in every neighborhood of x. Clearly $L(u)$ is closed and we have $L(u) \supset \overline{P(u)}$ but in general these sets are different: in fact, $u(z) = \sum k^{-2} \log(|z - 1/k| + e^{-k^3})$ is everywhere finite in $\mathbb{C}$ but $L(u) = \{0\}$.

(2.1) Proposition. *We make two additional assumptions:*

(a) *T has non zero bidimension (p,p) (i.e. degree of $T < 2n$).*

(b) *X is covered by a family of Stein open sets $\Omega \subset\subset X$ whose boundaries $\partial\Omega$ do not intersect $L(u) \cap \operatorname{Supp} T$.*

Then the current uT has locally finite mass in X.

For any current T, hypothesis 2.1 (b) is clearly satisfied when u has a discrete unbounded locus $L(u)$; an interesting example is $u = \log|F|$ where $F = (F_1, \ldots, F_N)$ are holomorphic functions having a discrete set of common zeros. Observe that the current uT need not have locally finite mass when T has degree $2n$ (i.e. T is a measure); example: $T = \delta_0$ and $u(z) = \log|z|$ in $\mathbb{C}^n$. The result also fails when the sets Ω are not assumed to be Stein; example: $X =$ blow-up of $\mathbb{C}^n$ at 0, $T = [E] =$ current of integration on the exceptional divisor and $u(z) = \log|z|$.

Proof. By shrinking Ω slightly, we may assume that Ω has a smooth strongly pseudoconvex boundary. Let ψ be a defining function of Ω as in (1.8). By subtracting a constant to u, we may assume $u \le -\varepsilon$ on $\overline{\Omega}$. We fix δ so small that $\overline{\Omega} \setminus \Omega_\delta$ does not intersect $L(u) \cap \operatorname{Supp} T$ and we select a neighborhood ω of $(\overline{\Omega} \setminus \Omega_\delta) \cap \operatorname{Supp} T$ such that $\overline{\omega} \cap L(u) = \emptyset$. Then we define

$$u_s(z) = \begin{cases} \max\{u(z), A\psi(z)\} & \text{on } \omega, \\ \max\{u(z), s\} & \text{on } \Omega_\delta = \{\psi < -\delta\}. \end{cases}$$

By construction $u \ge -M$ on ω for some constant $M > 0$. We fix $A \ge M/\delta$ and take $s \le -M$, so

$$\max\{u(z), A\psi(z)\} = \max\{u(z), s\} = u(z) \quad \text{on } \omega \cap \Omega_\delta$$

and our definition of u_s is coherent. Observe that u_s is defined on $\omega \cup \Omega_\delta$, which is a neighborhood of $\overline{\Omega} \cap \operatorname{Supp} T$. Now, $u_s = A\psi$ on $\omega \cap (\Omega \setminus \Omega_{\varepsilon/A})$, hence Stokes' theorem implies

$$\begin{aligned}\int_\Omega dd^c u_s \wedge T \wedge (dd^c\psi)^{p-1} &- \int_\Omega A dd^c\psi \wedge T \wedge (dd^c\psi)^{p-1} \\ &= \int_\Omega dd^c\big[(u_s - A\psi)T \wedge (dd^c\psi)^{p-1}\big] = 0\end{aligned}$$

because the current $[\ldots]$ has a compact support contained in $\overline{\Omega}_{\varepsilon/A}$. Since u_s and ψ both vanish on $\partial\Omega$, an integration by parts gives

$$\begin{aligned}\int_\Omega u_s T \wedge (dd^c\psi)^p &= \int_\Omega \psi dd^c u_s \wedge T \wedge (dd^c\psi)^{p-1} \\ &\ge -||\psi||_{L^\infty(\Omega)} \int_\Omega T \wedge dd^c u_s \wedge (dd^c\psi)^{p-1} \\ &= -||\psi||_{L^\infty(\Omega)} A \int_\Omega T \wedge (dd^c\psi)^p.\end{aligned}$$

Finally, take $A = M/\delta$, let s tend to $-\infty$ and use the inequality $u \geq -M$ on ω. We obtain

$$\begin{aligned}\int_\Omega u\,T\wedge(dd^c\psi)^p &\geq -M\int_\omega T\wedge(dd^c\psi)^p + \lim_{s\to-\infty}\int_{\Omega_\delta} u_sT\wedge(dd^c\psi)^p\\ &\geq -\big(M+||\psi||_{L^\infty(\Omega)}M/\delta\big)\int_\Omega T\wedge(dd^c\psi)^p.\end{aligned}$$

The last integral is finite. This concludes the proof. □

(2.2) *Remark.* If Ω is smooth and strongly pseudoconvex, the above proof shows in fact that

$$||uT||_{\overline{\Omega}} \leq \frac{C}{\delta}||u||_{L^\infty((\overline{\Omega}\setminus\Omega_\delta)\cap\operatorname{Supp}T)}||T||_{\overline{\Omega}}$$

when $L(u)\cap\operatorname{Supp}T \subset \Omega_\delta$. In fact, if u is continuous and if ω is chosen sufficiently small, the constant M can be taken arbitrarily close to $||u||_{L^\infty((\overline{\Omega}\setminus\Omega_\delta)\cap\operatorname{Supp}T)}$. Moreover, the maximum principle implies

$$||u_+||_{L^\infty(\overline{\Omega}\cap\operatorname{Supp}T)} = ||u_+||_{L^\infty(\partial\Omega\cap\operatorname{Supp}T)},$$

so we can achieve $u < -\varepsilon$ on a neighborhood of $\overline{\Omega}\cap\operatorname{Supp}T$ by subtracting $||u||_{L^\infty((\overline{\Omega}\setminus\Omega_\delta)\cap\operatorname{Supp}T)} + 2\varepsilon$ [Proof of maximum principle: if $u(z_0) > 0$ at $z_0 \in \Omega\cap\operatorname{Supp}T$ and $u \leq 0$ near $\partial\Omega\cap\operatorname{Supp}T$, then

$$\int_\Omega u_+T\wedge(dd^c\psi)^p = \int_\Omega \psi dd^cu_+\wedge T\wedge(dd^c\psi)^{p-1} \leq 0,$$

a contradiction]. □

(2.3) Corollary. *Let $u_1,\ldots,u_q$ be plurisubharmonic functions on X such that X is covered by Stein open sets Ω with $\partial\Omega\cap L(u_j)\cap\operatorname{Supp}T = \emptyset$. We use again induction to define*

$$dd^cu_1\wedge dd^cu_2\wedge\ldots\wedge dd^cu_q\wedge T = dd^c(u_1dd^cu_2\ldots\wedge dd^cu_q\wedge T).$$

Then, if $u_1^k,\ldots,u_q^k$ are decreasing sequences of plurisubharmonic functions converging pointwise to $u_1,\ldots,u_q$, $q \leq p$, properties (1.7 a, b) *hold.*

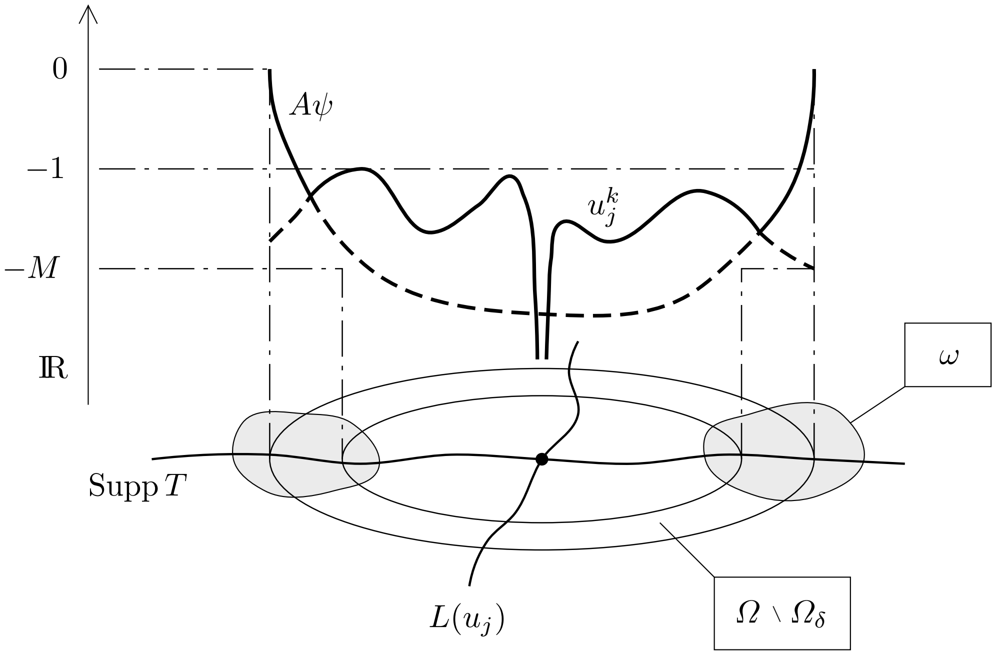

Fig. 2. Modified construction of v_j^k

Proof. Same proof as for Th. 1.7, with the following minor modification: the max procedure $v_j^k := \max\{u_j^k, A\psi\}$ is applied only on a neighborhood ω of $\operatorname{Supp} T \cap (\overline{\Omega} \smallsetminus \Omega_\delta)$ with $\delta > 0$ small, and u_j^k is left unchanged near $\operatorname{Supp} T \cap \overline{\Omega}_\delta$ (see Fig. 2). Observe that the integration by part process requires the functions u_j^k and $u_j^{k,\varepsilon}$ to be defined only near $\overline{\Omega} \cap \operatorname{Supp} T$. □

(2.4) Proposition. *Let $\Omega \subset\subset X$ be a Stein open subset. If V is a plurisubharmonic function on X and $u_1, \ldots, u_q$, $1 \le q \le n-1$, are plurisubharmonic functions such that $\partial\Omega \cap L(u_j) = \emptyset$, then $V dd^c u_1 \wedge \ldots \wedge dd^c u_q$ has locally finite mass in Ω.*

Proof. Same proof as for 1.11, when $\delta > 0$ is taken so small that $\Omega_\delta \supset L(u_j)$ for all $1 \le j \le q$. □

Finally, we show that Monge-Ampère operators can also be defined in the case of plurisubharmonic functions with non compact pole sets, provided that the mutual intersections of the pole sets are of sufficiently small Hausdorff dimension with respect to the dimension p of T.

(2.5) Theorem. *Let $u_1, \ldots, u_q$ be plurisubharmonic functions on X. The currents $u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$ and $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ are well defined and have locally finite mass in X as soon as $q \le p$ and*

$$\mathcal{H}_{2p-2m+1}\big(L(u_{j_1}) \cap \ldots \cap L(u_{j_m}) \cap \operatorname{Supp} T\big) = 0$$

for all choices of indices $j_1 < \ldots < j_m$ in $\{1, \ldots, q\}$.

The proof is an easy induction on q, thanks to the following improved version of the Chern-Levine-Nirenberg inequalities.

(2.6) Proposition. *Let $A_1, \ldots, A_q \subset X$ be closed sets such that*

$$\mathcal{H}_{2p-2m+1}\big(A_{j_1} \cap \ldots \cap A_{j_m} \cap \operatorname{Supp} T\big) = 0$$

for all choices of $j_1 < \ldots < j_m$ in $\{1, \ldots, q\}$. Then for all compact sets K, L of X with $L \subset K^\circ$, there exist neighborhoods V_j of $K \cap A_j$ and a constant $C = C(K, L, A_j)$ such that the conditions $u_j \le 0$ on K and $L(u_j) \subset A_j$ imply

(a) $||u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T||_L \le C ||u_1||_{L^\infty(K \setminus V_1)} \ldots ||u_q||_{L^\infty(K \setminus V_q)} ||T||_K$

(b) $||dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T||_L \le C ||u_1||_{L^\infty(K \setminus V_1)} \ldots ||u_q||_{L^\infty(K \setminus V_q)} ||T||_K$.

Proof. We need only show that every point $x_0 \in K^\circ$ has a neighborhood L such that (a), (b) hold. Hence it is enough to work in a coordinate open set. We may thus assume that $X \subset \mathbb{C}^n$ is open, and after a regularization process $u_j = \lim u_j \star \rho_\varepsilon$ for $j = q, q-1, \ldots, 1$ in this order, that $u_1, \ldots, u_q$ are smooth. We proceed by induction on q in two steps:

Step 1. $(\mathrm{b}_{q-1}) \Longrightarrow (\mathrm{b}_q)$,
Step 2. (a_{q-1}) and $(\mathrm{b}_q) \Longrightarrow (\mathrm{a}_q)$,

where (b_0) is the trivial statement $||T||_L \le ||T||_K$ and (a_0) is void. Observe that we have $(\mathrm{a}_q) \Longrightarrow (\mathrm{a}_\ell)$ and $(\mathrm{b}_q) \Longrightarrow (\mathrm{b}_\ell)$ for $\ell \le q \le p$ by taking $u_{\ell+1}(z) = \ldots = u_q(z) = |z|^2$. We need the following elementary fact.

(2.7) Lemma. *Let $F \subset \mathbb{C}^n$ be a closed set such that $\mathcal{H}_{2s+1}(F) = 0$ for some integer $0 \le s < n$. Then for almost all choices of unitary coordinates $(z_1, \ldots, z_n) = (z', z'')$ with $z' = (z_1, \ldots, z_s)$, $z'' = (z_{s+1}, \ldots, z_n)$ and almost all radii of balls $B'' = B(0, r'') \subset \mathbb{C}^{n-s}$, the set $\{0\} \times \partial B''$ does not intersect F.*

Proof. The unitary group $U(n)$ has real dimension n^2. There is a proper submersion

$$\Phi : U(n) \times \big(\mathbb{C}^{n-s} \smallsetminus \{0\}\big) \longrightarrow \mathbb{C}^n \smallsetminus \{0\}, \qquad (g, z'') \longmapsto g(0, z''),$$

whose fibers have real dimension $N = n^2 - 2s$. It follows that the inverse image $\Phi^{-1}(F)$ has zero Hausdorff measure $\mathcal{H}_{N+2s+1} = \mathcal{H}_{n^2+1}$. The set of pairs $(g, r'') \in U(n) \times \mathbb{R}_+^\star$ such that $g(\{0\} \times \partial B'')$ intersects F is precisely the image of $\Phi^{-1}(F)$ in $U(n) \times \mathbb{R}_+^\star$ by the Lipschitz map $(g, z'') \mapsto (g, |z''|)$. Hence this set has zero $\mathcal{H}_{n^2+1}$-measure. □

Proof of step 1. Take $x_0 = 0 \in K^\circ$. Suppose first $0 \in A_1 \cap \ldots \cap A_q$ and set $F = A_1 \cap \ldots \cap A_q \cap \operatorname{Supp} T$. Since $\mathcal{H}_{2p-2q+1}(F) = 0$, Lemma 2.7 implies that there are coordinates $z' = (z_1, \ldots, z_s)$, $z'' = (z_{s+1}, \ldots, z_n)$ with $s = p - q$ and a ball $\overline{B}''$ such that $F \cap \big(\{0\} \times \partial B''\big) = \emptyset$ and $\{0\} \times \overline{B}'' \subset K^\circ$. By compactness of K, we can find neighborhoods W_j of $K \cap A_j$ and a ball $B' = B(0, r') \subset \mathbb{C}^s$ such that $\overline{B}' \times \overline{B}'' \subset K^\circ$ and

$$\overline{W}_1 \cap \ldots \cap \overline{W}_q \cap \operatorname{Supp} T \cap \Big(\overline{B}' \times \big(\overline{B}'' \smallsetminus (1-\delta)B''\big)\Big) = \emptyset \tag{2.8}$$

for $\delta > 0$ small. If $0 \notin A_j$ for some j, we choose instead W_j to be a small neighborhood of 0 such that $\overline{W}_j \subset (\overline{B}' \times (1-\delta)B'') \smallsetminus A_j$; property (2.8) is then automatically satisfied. Let $\chi_j \geq 0$ be a function with compact support in W_j, equal to 1 near $K \cap A_j$ if $A_j \ni 0$ (resp. equal to 1 near 0 if $A_j \not\ni 0$) and let $\chi(z') \geq 0$ be a function equal to 1 on $1/2\, B'$ with compact support in B'. Then

$$\int_{B' \times B''} dd^c(\chi_1 u_1) \wedge \ldots \wedge dd^c(\chi_q u_q) \wedge T \wedge \chi(z')\,(dd^c|z'|^2)^s = 0$$

because the integrand is dd^c exact and has compact support in $B' \times B''$ thanks to (2.8). If we expand all factors $dd^c(\chi_j u_j)$, we find a term

$$\chi_1 \ldots \chi_q \chi(z') dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T \geq 0$$

which coincides with $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ on a small neighborhood of 0 where $\chi_j = \chi = 1$. The other terms involve

$$d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j + u_j dd^c \chi_j$$

for at least one index j. However $d\chi_j$ and $dd^c\chi_j$ vanish on some neighborhood V'_j of $K \cap A_j$ and therefore u_j is bounded on $\overline{B}' \times \overline{B}'' \smallsetminus V'_j$. We then apply the induction hypothesis (b_{q-1}) to the current

$$\Theta = dd^c u_1 \wedge \ldots \wedge \widehat{dd^c u_j} \wedge \ldots \wedge dd^c u_q \wedge T$$

and the usual Chern-Levine-Nirenberg inequality to the product of Θ with the mixed term $d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j$. Remark 1.5 can be applied because χ_j is smooth and is therefore a difference $\chi_j^{(1)} - \chi_j^{(2)}$ of locally bounded plurisubharmonic functions in $\mathbb{C}^n$. Let K' be a compact neighborhood of $\overline{B}' \times \overline{B}''$ with $K' \subset K^\circ$, and let V_j be a neighborhood of $K \cap A_j$ with $\overline{V}_j \subset V'_j$. Then with $L' := (\overline{B}' \times \overline{B}'') \smallsetminus V'_j \subset (K' \smallsetminus V_j)^\circ$ we obtain

$$||(d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j) \wedge \Theta||_{\overline{B}' \times \overline{B}''} = ||(d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j) \wedge \Theta||_{L'}$$
$$\le C_1 ||u_j||_{L^\infty(K' \smallsetminus V_j)} ||\Theta||_{K' \smallsetminus V_j},$$
$$||\Theta||_{K' \smallsetminus V_j} \le ||\Theta||_{K'} \le C_2 ||u_1||_{L^\infty(K \smallsetminus V_1)} \ldots \widehat{||u_j||} \ldots ||u_q||_{L^\infty(K \smallsetminus V_q)} ||T||_K.$$

Now, we may slightly move the unitary basis in $\mathbb{C}^n$ and get coordinate systems $z^m = (z^m_1, \ldots, z^m_n)$ with the same properties as above, such that the forms

$$(dd^c|z^{m\prime}|^2)^s = \frac{s!}{\pi^s} \mathrm{i}\, dz^m_1 \wedge d\overline{z}^m_1 \wedge \ldots \wedge \mathrm{i}\, dz^m_s \wedge d\overline{z}^m_s, \qquad 1 \le m \le N$$

define a basis of $\bigwedge^{s,s}(\mathbb{C}^n)^\star$. It follows that all measures

$$dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \mathrm{i}\, dz^m_1 \wedge d\overline{z}^m_1 \wedge \ldots \wedge \mathrm{i}\, dz^m_s \wedge d\overline{z}^m_s$$

satisfy estimate (b_q) on a small neighborhood L of 0.

Proof of Step 2. We argue in a similar way with the integrals

$$\int_{B' \times B''} \chi_1 u_1 dd^c(\chi_2 u_2) \wedge \ldots dd^c(\chi_q u_q) \wedge T \wedge \chi(z')(dd^c|z'|^2)^s \wedge dd^c|z_{s+1}|^2$$
$$= \int_{B' \times B''} |z_{s+1}|^2 dd^c(\chi_1 u_1) \wedge \ldots dd^c(\chi_q u_q) \wedge T \wedge \chi(z')(dd^c|z'|^2)^s.$$

We already know by (b_q) and Remark 1.5 that all terms in the right hand integral admit the desired bound. For $q = 1$, this shows that $(\mathrm{b}_1) \Longrightarrow (\mathrm{a}_1)$. Except for $\chi_1 \ldots \chi_q \chi(z')\, u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$, all terms in the left hand integral involve derivatives of χ_j. By construction, the support of these derivatives is disjoint from A_j, thus we only have to obtain a bound for

$$\int_L u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha$$

when $L = \overline{B}(x_0, r)$ is disjoint from A_j for some $j \ge 2$, say $L \cap A_2 = \emptyset$, and α is a constant positive form of type $(p-q, p-q)$. Then $\overline{B}(x_0, r+\varepsilon) \subset K^\circ \smallsetminus \overline{V}_2$ for some $\varepsilon > 0$ and some neighborhood V_2 of $K \cap A_2$. By the max construction used e.g. in Prop. 2.1, we can replace u_2 by a plurisubharmonic function $\widetilde{u}_2$ equal to u_2 in L and to $A(|z - x_0|^2 - r^2) - M$ in $\overline{B}(x_0, r+\varepsilon) \smallsetminus B(x_0, r+\varepsilon/2)$, with $M = ||u_2||_{L^\infty(K \smallsetminus V_2)}$ and $A = M/\varepsilon r$. Let $\chi \ge 0$ be a smooth function equal to 1 on $B(x_0, r+\varepsilon/2)$ with support in $B(x_0, r)$. Then

$$\int_{B(x_0, r+\varepsilon)} u_1 dd^c(\chi \widetilde{u}_2) \wedge dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha$$
$$= \int_{B(x_0, r+\varepsilon)} \chi \widetilde{u}_2 dd^c u_1 \wedge dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha$$
$$\le O(1)\ ||u_1||_{L^\infty(K \smallsetminus V_1)} \ldots ||u_q||_{L^\infty(K \smallsetminus V_q)} ||T||_K$$

where the last estimate is obtained by the induction hypothesis (b_{q-1}) applied to $dd^c u_1 \wedge dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T$. By construction

$$dd^c(\chi \widetilde{u}_2) = \chi\, dd^c \widetilde{u}_2 + (\text{smooth terms involving } d\chi)$$

coincides with $dd^c u_2$ in L, and (a_{q-1}) implies the required estimate for the other terms in the left hand integral. □

Proposition 2.9. *With the assumptions of Th.* 2.5*, the analogue of the monotone convergence Theorem* 1.7 (a,b) *holds.*

Proof. By the arguments already used in the proof of Th. 1.7 (e.g. by Lemma 1.9), it is enough to show that

$$\begin{aligned}\int_{B' \times B''} & \chi_1 \ldots \chi_q\, u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha \\ & \leq \liminf_{k \to +\infty} \int_{B' \times B''} \chi_1 \ldots \chi_q\, u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \wedge \alpha\end{aligned}$$

where $\alpha = \chi(z')(dd^c|z'|^2)^s$ is closed. Here the functions χ_j, χ are chosen as in the proof of Step 1 in 2.7, especially their product has compact support in $B' \times B''$ and $\chi_j = \chi = 1$ in a neighborhood of the given point x_0. We argue by induction on q and also on the number m of functions $(u_j)_{j \geq 1}$ which are unbounded near x_0. If u_j is bounded near x_0, we take $W_j'' \subset\subset W_j' \subset\subset W_j$ to be small balls of center x_0 on which u_j is bounded and we modify the sequence u_j^k on the corona $W_j \setminus W_j''$ so as to make it constant and equal to a smooth function $A|z - x_0|^2 + B$ on the smaller corona $W_j \setminus W_j'$. In that case, we take χ_j equal to 1 near $\overline{W}_j'$ and $\operatorname{Supp} \chi_j \subset W_j$. For every $\ell = 1, \ldots, q$, we are going to check that

$$\begin{aligned} & \liminf_{k \to +\infty} \int_{B' \times B''} \chi_1 u_1^k dd^c(\chi_2 u_2^k) \wedge \ldots \\ & \qquad dd^c(\chi_{\ell-1} u_{\ell-1}^k) \wedge dd^c(\chi_\ell u_\ell) \wedge dd^c(\chi_{\ell+1} u_{\ell+1}) \ldots dd^c(\chi_q u_q) \wedge T \wedge \alpha \\ \leq & \liminf_{k \to +\infty} \int_{B' \times B''} \chi_1 u_1^k dd^c(\chi_2 u_2^k) \wedge \ldots \\ & \qquad dd^c(\chi_{\ell-1} u_{\ell-1}^k) \wedge dd^c(\chi_\ell u_\ell^k) \wedge dd^c(\chi_{\ell+1} u_{\ell+1}) \ldots dd^c(\chi_q u_q) \wedge T \wedge \alpha. \end{aligned}$$

In order to do this, we integrate by parts $\chi_1 u_1^k dd^c(\chi_\ell u_\ell)$ into $\chi_\ell u_\ell dd^c(\chi_1 u_1^k)$ for $\ell \geq 2$, and we use the inequality $u_\ell \leq u_\ell^k$. Of course, the derivatives $d\chi_j$, $d^c\chi_j$, $dd^c\chi_j$ produce terms which are no longer positive and we have to take care of these. However, $\operatorname{Supp} d\chi_j$ is disjoint from the unbounded locus of u_j when u_j is unbounded, and contained in $W_j \setminus \overline{W}_j'$ when u_j is bounded. The number m of unbounded functions is therefore replaced by $m - 1$ in the first case, whereas in the second case $u_j^k = u_j$ is constant and smooth on $\operatorname{Supp} d\chi_j$, so q can be replaced by $q - 1$. By induction on $q + m$ (and thanks to the polarization technique 1.5), the limit of the terms involving derivatives of χ_j is equal on both sides to the corresponding terms obtained by suppressing all indices k. Hence these terms do not give any contribution in the inequalities. $\square$

We finally quote the following simple consequences of Th. 2.5 when T is arbitrary and $q = 1$, resp. when $T = 1$ has bidegree $(0, 0)$ and q is arbitrary.

(2.10) Corollary. *Let T be a closed positive current of bidimension (p, p) and let u be a plurisubharmonic function on X such that $L(u) \cap \operatorname{Supp} T$ is contained in an analytic set of dimension at most $p - 1$. Then uT and $dd^c u \wedge T$ are well defined and have locally finite mass in X.* $\square$

(2.11) Corollary. *Let $u_1, \ldots, u_q$ be plurisubharmonic functions on X such that $L(u_j)$ is contained in an analytic set $A_j \subset X$ for every j. Then $dd^c u_1 \wedge \ldots \wedge dd^c u_q$ is well defined as soon as $A_{j_1} \cap \ldots \cap A_{j_m}$ has codimension at least m for all choices of indices $j_1 < \ldots < j_m$ in $\{1, \ldots, q\}$.* $\square$

In the particular case when $u_j = \log |f_j|$ for some non zero holomorphic function f_j on X, we see that the intersection product of the associated zero divisors $[Z_j] = dd^c u_j$ is well defined as soon as the supports $|Z_j|$ satisfy $\operatorname{codim} |Z_{j_1}| \cap \ldots \cap |Z_{j_m}| = m$ for every m. Similarly, when $T = [A]$ is an analytic p-cycle, Cor. 2.10 shows that $[Z] \wedge [A]$ is well defined for every divisor Z such that $\dim |Z| \cap |A| = p - 1$. These observations easily imply the following

(2.12) Proposition. *Suppose that the divisors Z_j satisfy the above codimension condition and let $(C_k)_{k \geq 1}$ be the irreducible components of the point set intersection $|Z_1| \cap \ldots \cap |Z_q|$. Then there exist integers $m_k > 0$ such that*

$$[Z_1] \wedge \ldots \wedge [Z_q] = \sum m_k [C_k].$$

The number m_k is called the multiplicity of intersection of $Z_1, \ldots, Z_q$ along C_k.

Proof. The wedge product has bidegree (q, q) and support in $C = \bigcup C_k$ where $\operatorname{codim} C = q$, so it must be a sum as above with $m_k \in \mathbb{R}_+$. We check by induction on q that m_k is a positive integer. If we denote by A some irreducible component of $|Z_1| \cap \ldots \cap |Z_{q-1}|$, we need only check that $[A] \wedge [Z_q]$ is an integral analytic cycle of codimension q with positive coefficients on each component C_k of the intersection. However $[A] \wedge [Z_q] = dd^c(\log |f_q| \, [A])$. First suppose that no component of $A \cap f_q^{-1}(0)$ is contained in the singular part A_{sing}. Then the Lelong-Poincaré equation applied on A_{reg} shows that $dd^c(\log|f_q|\,[A]) = \sum m_k[C_k]$ on $X \smallsetminus A_{\mathrm{sing}}$, where m_k is the vanishing order of f_q along C_k in A_{reg}. Since $C \cap A_{\mathrm{sing}}$ has codimension $q+1$ at least, the equality must hold on X. In general, we replace f_q by $f_q - \varepsilon$ so that the divisor of $f_q - \varepsilon$ has no component contained in A_{sing}. Then $dd^c(\log|f_q-\varepsilon|\,[A])$ is an integral codimension q cycle with positive multiplicities on each component of $A \cap f_q^{-1}(\varepsilon)$ and we conclude by letting ε tend to zero. □

3. Generalized Lelong Numbers

The concepts we are going to study mostly concern the behaviour of currents or plurisubharmonic functions in a neighborhood of a point at which we have for instance a logarithmic pole. Since the interesting applications are local, we assume from now on (unless otherwise stated) that X is a Stein manifold, i.e. that X has a strictly plurisubharmonic exhaustion function. Let $\varphi : X \longrightarrow [-\infty, +\infty[$ be a continuous plurisubharmonic function (in general φ may have $-\infty$ poles, our continuity assumption means that e^{φ} is continuous). The sets

$$S(r) = \{x \in X\,;\ \varphi(x) = r\}, \tag{3.1}$$

$$B(r) = \{x \in X\,;\ \varphi(x) < r\}, \tag{3.1'}$$

$$\overline{B}(r) = \{x \in X\,;\ \varphi(x) \le r\} \tag{3.1''}$$

will be called *pseudo-spheres* and *pseudo-balls* associated with φ. Note that $\overline{B}(r)$ is not necessarily equal to the closure of $B(r)$, but this is often true in concrete situations. The most simple example we have in mind is the case of the function $\varphi(z) = \log|z-a|$ on an open subset $X \subset \mathbb{C}^n$; in this case $B(r)$ is the euclidean ball of center a and radius e^r ; moreover, the forms

$$\frac{1}{2}dd^c e^{2\varphi} = \frac{\mathrm{i}}{2\pi}d'd''|z|^2, \qquad dd^c\varphi = \frac{\mathrm{i}}{\pi}d'd''\log|z-a| \tag{3.2}$$

can be interpreted respectively as the flat hermitian metric on $\mathbb{C}^n$ and as the pull-back over $\mathbb{C}^n$ of the Fubini-Study metric of $\mathbb{P}^{n-1}$, translated by a.

(3.3) Definition. *We say that φ is semi-exhaustive if there exists a real number R such that $B(R) \subset\subset X$. Similarly, φ is said to be semi-exhaustive on a closed subset $A \subset X$ if there exists R such that $A \cap B(R) \subset\subset X$.*

We are interested especially in the set of poles $S(-\infty) = \{\varphi = -\infty\}$ and in the behaviour of φ near $S(-\infty)$. Let T be a closed positive current of bidimension (p,p) on X. Assume that φ is semi-exhaustive on $\operatorname{Supp} T$ and that $B(R) \cap \operatorname{Supp} T \subset\subset X$. Then $P = S(-\infty) \cap \mathrm{SuppT}$ is compact and the results of §2 show that the measure $T \wedge (dd^c\varphi)^p$ is well defined. Following [De2], [De4], we introduce:

(3.4) Definition. *If φ is semi-exhaustive on* $\operatorname{Supp} T$ *and if R is such that $B(R) \cap \operatorname{Supp} T \subset\subset X$, we set for all $r \in]-\infty, R[$*

$$\nu(T,\varphi,r) = \int_{B(r)} T \wedge (dd^c\varphi)^p,$$

$$\nu(T,\varphi) = \int_{S(-\infty)} T \wedge (dd^c\varphi)^p = \lim_{r\to-\infty} \nu(T,\varphi,r).$$

The number $\nu(T,\varphi)$ will be called the (generalized) Lelong number of T with respect to the weight φ.

If we had not required $T \wedge (dd^c\varphi)^p$ to be defined pointwise on $\varphi^{-1}(-\infty)$, the assumption that X is Stein could have been dropped: in fact, the integral over $B(r)$ always makes sense if we define

$$\nu(T,\varphi,r) = \int_{B(r)} T \wedge \big(dd^c \max\{\varphi, s\}\big)^p \quad \text{with} \quad s < r.$$

Stokes' formula shows that the right hand integral is actually independent of s. The example given after (2.1) shows however that $T \wedge (dd^c\varphi)^p$ need not exist on $\varphi^{-1}(-\infty)$ if $\varphi^{-1}(-\infty)$ contains an exceptional compact analytic subset. We leave the reader consider by himself this more general situation and extend our statements by the $\max\{\varphi, s\}$ technique. Observe that $r \longmapsto \nu(T,\varphi,r)$ is always an increasing function of r. Before giving examples, we need a formula.

(3.5) Formula. *For any convex increasing function $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ we have*

$$\int_{B(r)} T \wedge (dd^c\chi\circ\varphi)^p = \chi'(r-0)^p\, \nu(T,\varphi,r)$$

where $\chi'(r-0)$ denotes the left derivative of χ at r.

Proof. Let χ_ε be the convex function equal to χ on $[r-\varepsilon, +\infty[$ and to a linear function of slope $\chi'(r-\varepsilon-0)$ on $]-\infty, r-\varepsilon]$. We get $dd^c(\chi_\varepsilon \circ \varphi) = \chi'(r-\varepsilon-0) dd^c\varphi$ on $B(r-\varepsilon)$ and Stokes' theorem implies

$$\begin{aligned}\int_{B(r)} T \wedge (dd^c\chi\circ\varphi)^p &= \int_{B(r)} T \wedge (dd^c\chi_\varepsilon\circ\varphi)^p \\ &\geq \int_{B(r-\varepsilon)} T \wedge (dd^c\chi_\varepsilon\circ\varphi)^p \\ &= \chi'(r-\varepsilon-0)^p \nu(T,\varphi,r-\varepsilon).\end{aligned}$$

Similarly, taking $\widetilde{\chi}_\varepsilon$ equal to χ on $]-\infty, r-\varepsilon]$ and linear on $[r-\varepsilon, r]$, we obtain

$$\int_{B(r-\varepsilon)} T \wedge (dd^c\chi\circ\varphi)^p \le \int_{B(r)} T \wedge (dd^c\widetilde{\chi}_\varepsilon\circ\varphi)^p = \chi'(r-\varepsilon-0)^p \nu(T,\varphi,r).$$

The expected formula follows when ε tends to 0. $\square$

We get in particular $\int_{B(r)} T\wedge(dd^c e^{2\varphi})^p = (2e^{2r})^p\nu(T,\varphi,r)$, whence the formula

$$\nu(T,\varphi,r) = e^{-2pr}\int_{B(r)} T\wedge\Big(\frac{1}{2}dd^c e^{2\varphi}\Big)^p. \tag{3.6}$$

Now, assume that X is an open subset of $\mathbb{C}^n$ and that $\varphi(z)=\log|z-a|$ for some $a\in X$. Formula (3.6) gives

$$\nu(T,\varphi,\log r) = r^{-2p}\int_{|z-a|<r} T\wedge\Big(\frac{\mathrm{i}}{2\pi}d'd''|z|^2\Big)^p.$$

The positive measure $\sigma_T = \frac{1}{p!}T\wedge(\frac{\mathrm{i}}{2}d'd''|z|^2)^p = 2^{-p}\sum T_{I,I}\,.\,\mathrm{i}^n dz_1\wedge\ldots\wedge d\overline{z}_n$ is called the *trace measure* of T. We get

$$\nu(T,\varphi,\log r) = \frac{\sigma_T\big(B(a,r)\big)}{\pi^p r^{2p}/p!} \tag{3.7}$$

and $\nu(T,\varphi)$ is the limit of this ratio as $r\to 0$. This limit is called the (*ordinary*) *Lelong number* of T at point a and is denoted $\nu(T,a)$. This was precisely the original definition of Lelong (cf. [Le3]). Let us mention a simple but important consequence.

(3.8) Consequence. *The ratio $\sigma_T\big(B(a,r)\big)/r^{2p}$ is an increasing function of r. Moreover, for every compact subset $K\subset X$ and every $r_0 < d(K,\partial X)$ we have*

$$\sigma_T\big(B(a,r)\big) \le Cr^{2p} \quad \text{for } a\in K \text{ and } r\le r_0,$$

where $C = \sigma_T\big(K+\overline{B}(0,r_0)\big)/r_0^{2p}$.

All these results are particularly interesting when $T=[A]$ is the current of integration over an analytic subset $A\subset X$ of pure dimension p. Then $\sigma_T\big(B(a,r)\big)$ is the euclidean area of $A\cap B(a,r)$, while $\pi^p r^{2p}/p!$ is the area of a ball of radius r in a p-dimensional subspace of $\mathbb{C}^n$. Thus $\nu(T,\varphi,\log r)$ is the ratio of these areas and the Lelong number $\nu(T,a)$ is the limit ratio.

(3.9) *Remark.* It is immediate to check that

$$\nu([A],x) = \begin{cases} 0 & \text{for } \ x \notin A, \\ 1 & \text{when } x \in A \text{ is a regular point.} \end{cases}$$

We will see later that $\nu([A],x)$ is always an integer (Thie's theorem 5.8).

(3.10) *Remark.* When $X = \mathbb{C}^n$, $\varphi(z) = \log|z-a|$ and $A = X$ (i.e. $T = 1$), we obtain in particular $\int_{B(a,r)}(dd^c \log|z-a|)^n = 1$ for all r. This implies

$$(dd^c \log|z-a|)^n = \delta_a.$$

This fundamental formula can be viewed as a higher dimensional analogue of the usual formula $\Delta \log|z-a| = 2\pi\delta_a$ in $\mathbb{C}$. □

We next prove a result which shows in particular that the Lelong numbers of a closed positive current are zero except on a very small set.

(3.11) Proposition. *If T is a closed positive current of bidimension (p,p), then for each $c > 0$ the set $E_c = \{x \in X\,;\,\nu(T,x) \geq c$ is a closed set of locally finite $\mathcal{H}_{2p}$ Hausdorff measure in X.*

Proof. By (3.7), we infer $\nu(T,a) = \lim_{r\to 0} \sigma_T\big(\overline{B}(a,r)\big)p!/\pi^p r^{2p}$. The function $a \mapsto \sigma_T\big(\overline{B}(a,r)\big)$ is clearly upper semicontinuous. Hence the decreasing limit $\nu(T,a)$ as r decreases to 0 is also upper semicontinuous in a. This implies that E_c is closed. Now, let K be a compact subset in X and let $\{a_j\}_{1\leq j\leq N}$, $N = N(\varepsilon)$, be a maximal collection of points in $E_c \cap K$ such that $|a_j - a_k| \geq 2\varepsilon$ for $j \neq k$. The balls $B(a_j, 2\varepsilon)$ cover $E_c \cap K$, whereas the balls $B(a_j,\varepsilon)$ are disjoint. If $K_{c,\varepsilon}$ is the set of points which are at distance $\leq \varepsilon$ of $E_c \cap K$, we get

$$\sigma_T(K_{c,\varepsilon}) \geq \sum \sigma_T\big(B(a_j,\varepsilon)\big) \geq N(\varepsilon)\, c\pi^p\varepsilon^{2p}/p!,$$

since $\nu(T,a_j) \geq c$. By the definition of Hausdorff measure, we infer

$$\begin{aligned} \mathcal{H}_{2p}(E_c \cap K) &\leq \liminf_{\varepsilon\to 0} \sum \big(\operatorname{diam} B(a_j,2\varepsilon)\big)^{2p} \\ &\leq \liminf_{\varepsilon\to 0} N(\varepsilon)(4\varepsilon)^{2p} \leq \frac{p!4^{2p}}{c\pi^p}\sigma_T(E_c \cap K). \end{aligned}$$

□

Finally, we conclude this section by proving two simple semi-continuity results for Lelong numbers.

(3.12) Proposition. *Let T_k be a sequence of closed positive currents of bidimension (p,p) converging weakly to a limit T. Suppose that there is a closed set A such that* $\operatorname{Supp} T_k \subset A$ *for all k and such that φ is semi-exhaustive on A with $A \cap B(R) \subset\subset X$. Then for all $r < R$ we have*

$$\begin{aligned}\int_{B(r)} T\wedge(dd^c\varphi)^p &\le \liminf_{k\to+\infty}\int_{B(r)} T_k\wedge(dd^c\varphi)^p\\ &\le \limsup_{k\to+\infty}\int_{\overline{B}(r)} T_k\wedge(dd^c\varphi)^p \le \int_{\overline{B}(r)} T\wedge(dd^c\varphi)^p.\end{aligned}$$

When r tends to $-\infty$, we find in particular

$$\limsup_{k\to+\infty}\nu(T_k,\varphi)\le\nu(T,\varphi).$$

Proof. Let us prove for instance the third inequality. Let φ_ℓ be a sequence of smooth plurisubharmonic approximations of φ with $\varphi\le\varphi_\ell<\varphi+1/\ell$ on $\{r-\varepsilon\le\varphi\le r+\varepsilon\}$. We set

$$\psi_\ell=\begin{cases}\varphi & \text{on } \overline{B}(r),\\ \max\{\varphi,(1+\varepsilon)(\varphi_\ell-1/\ell)-r\varepsilon\} & \text{on } X\smallsetminus B(r).\end{cases}$$

This definition is coherent since $\psi_\ell=\varphi$ near $S(r)$, and we have

$$\psi_\ell=(1+\varepsilon)(\varphi_\ell-1/\ell)-r\varepsilon \quad \text{near}\quad S(r+\varepsilon/2)$$

as soon as ℓ is large enough, i.e. $(1+\varepsilon)/\ell\le\varepsilon^2/2$. Let χ_ε be a cut-off function equal to 1 in $B(r+\varepsilon/2)$ with support in $B(r+\varepsilon)$. Then

$$\begin{aligned}\int_{\overline{B}(r)} T_k\wedge(dd^c\varphi)^p &\le \int_{B(r+\varepsilon/2)} T_k\wedge(dd^c\psi_\ell)^p\\ &=(1+\varepsilon)^p\int_{B(r+\varepsilon/2)} T_k\wedge(dd^c\varphi_\ell)^p\\ &\le(1+\varepsilon)^p\int_{B(r+\varepsilon)}\chi_\varepsilon T_k\wedge(dd^c\varphi_\ell)^p.\end{aligned}$$

As $\chi_\varepsilon(dd^c\varphi_\ell)^p$ is smooth with compact support and as T_k converges weakly to T, we infer

$$\limsup_{k\to+\infty}\int_{\overline{B}(r)} T_k\wedge(dd^c\varphi)^p\le(1+\varepsilon)^p\int_{B(r+\varepsilon)}\chi_\varepsilon T\wedge(dd^c\varphi_\ell)^p.$$

We then let ℓ tend to $+\infty$ and ε tend to 0 to get the desired inequality. The first inequality is obtained in a similar way, we define ψ_ℓ so that $\psi_\ell=\varphi$ on $X\smallsetminus B(r)$ and $\psi_\ell=\max\{(1-\varepsilon)(\varphi_\ell-1/\ell)+r\varepsilon\}$ on $\overline{B}(r)$, and we take $\chi_\varepsilon=1$ on $B(r-\varepsilon)$ with $\operatorname{Supp}\chi_\varepsilon\subset B(r-\varepsilon/2)$. Then for ℓ large

$$\begin{aligned}\int_{B(r)} T_k\wedge(dd^c\varphi)^p &\ge \int_{B(r-\varepsilon/2)} T_k\wedge(dd^c\psi_\ell)^p\\ &\ge(1-\varepsilon)^p\int_{B(r-\varepsilon/2)}\chi_\varepsilon T_k\wedge(dd^c\varphi_\ell)^p. \qquad\square\end{aligned}$$

(3.13) Proposition. *Let φ_k be a (non necessarily monotone) sequence of continuous plurisubharmonic functions such that e^{φ_k} converges uniformly* *to e^φ on every compact subset of X. Suppose that $\{\varphi < R\} \cap \operatorname{Supp} T \subset\subset X$. Then for $r < R$ we have*

$$\limsup_{k\to+\infty} \int_{\{\varphi_k \le r\}\cap\{\varphi<R\}} T \wedge (dd^c\varphi_k)^p \le \int_{\{\varphi\le r\}} T \wedge (dd^c\varphi)^p.$$

In particular $\limsup_{k\to+\infty} \nu(T,\varphi_k) \le \nu(T,\varphi)$.

When we take $\varphi_k(z) = \log|z-a_k|$ with $a_k \to a$, Prop. 3.13 implies the upper semicontinuity of $a \mapsto \nu(T,a)$ which was already noticed in the proof of Prop. 3.11.

Proof. Our assumption is equivalent to saying that $\max\{\varphi_k, t\}$ converges locally uniformly to $\max\{\varphi, t\}$ for every t. Then Cor. 1.6 shows that $T \wedge (dd^c \max\{\varphi_k, t\})^p$ converges weakly to $T \wedge (dd^c \max\{\varphi, t\})^p$. If χ_ε is a cut-off function equal to 1 on $\{\varphi \le r + \varepsilon/2\}$ with support in $\{\varphi < r+\varepsilon\}$, we get

$$\lim_{k\to+\infty} \int_X \chi_\varepsilon T \wedge (dd^c \max\{\varphi_k, t\})^p = \int_X \chi_\varepsilon T \wedge (dd^c \max\{\varphi, t\})^p.$$

For k large, we have $\{\varphi_k \le r\} \cap \{\varphi < R\} \subset \{\varphi < r + \varepsilon/2\}$, thus when ε tends to 0 we infer

$$\limsup_{k\to+\infty} \int_{\{\varphi_k \le r\}\cap\{\varphi<R\}} T \wedge (dd^c \max\{\varphi_k, t\})^p \le \int_{\{\varphi\le r\}} T \wedge (dd^c \max\{\varphi, t\})^p.$$

When we choose $t < r$, this is equivalent to the first inequality in statement (3.13). □

4. The Lelong-Jensen Formula

We assume in this section that X is Stein, that φ is *semi-exhaustive* on X and that $B(R) \subset\subset X$. We set for simplicity $\varphi_{\ge r} = \max\{\varphi, r\}$. For every $r \in \,]-\infty, R[$, the measures $dd^c(\varphi_{\ge r})^n$ are well defined. By Cor. 1.6, the map $r \longmapsto (dd^c\varphi_{\ge r})^n$ is continuous on $]-\infty, R[$ with respect to the weak topology. As $(dd^c\varphi_{\ge r})^n = (dd^c\varphi)^n$ on $X \setminus \overline{B}(r)$ and as $\varphi_{\ge r} \equiv r$, $(dd^c\varphi_{\ge r})^n = 0$ on $B(r)$, the left continuity implies $(dd^c\varphi_{\ge r})^n \ge \mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n$. Here $\mathbb{1}_A$ denotes the characteristic function of any subset $A \subset X$. According to the definition introduced in [De3], the collection of *Monge-Ampère measures* associated with φ is the family of positive measures μ_r such that

$$\mu_r = (dd^c\varphi_{\ge r})^n - \mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n, \qquad r \in \,]-\infty, R[. \tag{4.1}$$

The measure μ_r is supported on $S(r)$ and $r \longmapsto \mu_r$ is weakly continuous on the left by the bounded convergence theorem. Stokes' formula shows that $\int_{B(s)}(dd^c\varphi_{\geq r})^n - (dd^c\varphi)^n = 0$ for $s > r$, hence the total mass $\mu_r(S(r)) = \mu_r(B(s))$ is equal to the difference between the masses of $(dd^c\varphi)^n$ and $\mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n$ over $B(s)$, i.e.

$$\mu_r\big(S(r)\big) = \int_{B(r)}(dd^c\varphi)^n. \tag{4.2}$$

(4.3) Example. When $(dd^c\varphi)^n = 0$ on $X \setminus \varphi^{-1}(-\infty)$, formula (4.1) can be simplified into $\mu_r = (dd^c\varphi_{\geq r})^n$. This is so for $\varphi(z) = \log|z|$. In this case, the invariance of φ under unitary transformations implies that μ_r is also invariant. As the total mass of μ_r is equal to 1 by 3.10 and (4.2), we see that μ_r is the invariant measure of mass 1 on the euclidean sphere of radius e^r.

Proposition 4.4. *Assume that φ is smooth near $S(r)$ and that $d\varphi \neq 0$ on $S(r)$, i.e. r is a non critical value. Then $S(r) = \partial B(r)$ is a smooth oriented real hypersurface and the measure μ_r is given by the $(2n-1)$-volume form $(dd^c\varphi)^{n-1} \wedge d^c\varphi_{\restriction S(r)}$.*

Proof. Write $\max\{t, r\} = \lim_{k\to+\infty}\chi_k(t)$ where χ is a decreasing sequence of smooth convex functions with $\chi_k(t) = r$ for $t \leq r - 1/k$, $\chi_k(t) = t$ for $t \geq r + 1/k$. Corollary 1.6 shows that $(dd^c\chi_k\circ\varphi)^n$ converges weakly to $(dd^c\varphi_{\geq r})^n$. Let h be a smooth function with compact support near $S(r)$. We first compute the limit, noting that $\lim\chi'_k\circ\varphi = \mathbb{1}_{X\setminus B(r)}$ a.e., and then apply Stokes' theorem on $X \setminus B(r)$ (the boundary is $S(r)$ with opposite orientation):

$$\begin{aligned}
\int_X h(dd^c\varphi_{\geq r})^n &= \lim_{k\to+\infty}\int_X h(dd^c\chi_k\circ\varphi)^n \\
&= \lim_{k\to+\infty}\int_X -dh\wedge(dd^c\chi_k\circ\varphi)^{n-1}\wedge d^c(\chi_k\circ\varphi) \\
&= \lim_{k\to+\infty}\int_X -(\chi'_k\circ\varphi)^n\, dh\wedge(dd^c\varphi)^{n-1}\wedge d^c\varphi \\
&= \int_{X\setminus B(r)} -dh\wedge(dd^c\varphi)^{n-1}\wedge d^c\varphi \\
&= \int_{S(r)} h\,(dd^c\varphi)^{n-1}\wedge d^c\varphi + \int_{X\setminus B(r)} h\,(dd^c\varphi)^{n-1}\wedge d^c\varphi.
\end{aligned}$$

Near $S(r)$ we thus have an equality of measures

$$(dd^c\varphi_{\geq r})^n = (dd^c\varphi)^{n-1}\wedge d^c\varphi_{\restriction S(r)} + \mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n. \qquad \square$$

(4.5) Lelong-Jensen formula. *Let V be any plurisubharmonic function on X. Then V is μ_r-integrable for every $r \in]-\infty, R[$ and*

$$\mu_r(V) - \int_{B(r)} V(dd^c\varphi)^n = \int_{-\infty}^{r} \nu(dd^cV, \varphi, t)\,dt.$$

Proof. Proposition 1.11 shows that V is integrable with respect to $(dd^c\varphi_{\geq r})^n$, hence V is μ_r-integrable. By definition

$$\nu(dd^cV, \varphi, t) = \int_{\varphi(z)<t} dd^cV \wedge (dd^c\varphi)^{n-1}$$

and the Fubini theorem gives

$$\begin{aligned}\int_{-\infty}^{r} \nu(dd^cV, \varphi, t)\,dt &= \iint_{\varphi(z)<t<r} dd^cV(z) \wedge (dd^c\varphi(z))^{n-1}\,dt \\ &= \int_{B(r)} (r-\varphi) dd^cV \wedge (dd^c\varphi)^{n-1}. \qquad (4.6)\end{aligned}$$

We first show that formula 4.5 is true when φ and V are smooth. As both members of the formula are left continuous with respect to r and as almost all values of φ are non critical by Sard's theorem, we may assume r non critical. Formula 1.1 applied with $f = (r-\varphi)(dd^c\varphi)^{n-1}$ and $g = V$ shows that integral (4.6) is equal to

$$\int_{S(r)} V(dd^c\varphi)^{n-1} \wedge d^c\varphi - \int_{B(r)} V\,(dd^c\varphi)^n = \mu_r(V) - \int_{B(r)} V\,(dd^c\varphi)^n.$$

Formula 4.5 is thus proved when φ and V are smooth. If V is smooth and φ merely continuous and finite, one can write $\varphi = \lim \varphi_k$ where φ_k is a decreasing sequence of smooth plurisubharmonic functions (because X is Stein). Then $dd^cV \wedge (dd^c\varphi_k)^{n-1}$ converges weakly to $dd^cV \wedge (dd^c\varphi)^{n-1}$ and (4.6) converges, since $\mathbb{1}_{B(r)}(r-\varphi)$ is continuous with compact support on X. The left hand side of formula 4.5 also converges because the definition of μ_r implies

$$\mu_{k,r}(V) - \int_{\varphi_k<r} V(dd^c\varphi_k)^n = \int_X V\big((dd^c\varphi_{k,\geq r})^n - (dd^c\varphi_k)^n\big)$$

and we can apply again weak convergence on a neighborhood of $\overline{B}(r)$. If φ assumes $-\infty$ as a value, we replace φ by $\varphi_{\geq -k}$ where $k \to +\infty$. Then $\mu_r(V)$ is unchanged, $\int_{B(r)} V(dd^c\varphi_{\geq -k})^n$ converges to $\int_{B(r)} V(dd^c\varphi)^n$ and the right hand side of formula 4.5 is replaced by $\int_{-k}^{r} \nu(dd^cV, \varphi, t)\,dt$. Finally, for V arbitrary, write $V = \lim \downarrow V_l$ with a sequence of smooth functions V_l. Then $dd^cV_l \wedge (dd^c\varphi)^{n-1}$ converges weakly to $dd^cV \wedge (dd^c\varphi)^{n-1}$ by Prop. 2.4, so

$$\liminf_{k\to+\infty} \int_{B(r)} (r-\varphi) dd^cV_l \wedge (dd^c\varphi)^{n-1} \geq \int_{B(r)} (r-\varphi) dd^cV \wedge (dd^c\varphi)^{n-1},$$

with equality if φ is finite. As $\mu_r(V_l)$ and $\int_{B(r)} V_l(dd^c\varphi)^n$ converge to the expected limits by the monotone convergence theorem, we get

$$\mu_r(V) - \int_{B(r)} V(dd^c\varphi)^n \geq \int_{B(r)} (r-\varphi) dd^cV \wedge (dd^c\varphi)^{n-1},$$

with equality if φ does not assume $-\infty$ as a value. In particular, replacing φ by $\varphi_{\geq -k}$, we find

$$\mu_r(V) - \int_{B(r)} V(dd^c\varphi_{\geq -k})^n = \int_k^r \nu(dd^cV, \varphi, t)\, dt,$$

and from the equality $-V = \sup -V_l$ we easily get

$$\int_{B(r)} -V(dd^c\varphi)^n \leq \liminf_{k\to+\infty} \int_{B(r)} -V(dd^c\varphi_{\geq -k})^n.$$

A limit as $k \to +\infty$ yields the expected converse inequality

$$\mu_r(V) - \int_{B(r)} V(dd^c\varphi)^n \leq \int_{-\infty}^r \nu(dd^cV, \varphi, t)\, dt. \qquad \square$$

For $r < r_0 < R$, the Lelong-Jensen formula implies

$$\mu_r(V) - \mu_{r_0}(V) + \int_{B(r_0)\setminus B(r)} V(dd^c\varphi)^n = \int_{r_0}^r \nu(dd^cV, \varphi, t)\, dt. \tag{4.7}$$

(4.8) Corollary. *Assume that $(dd^c\varphi)^n = 0$ on $X \setminus S(-\infty)$. Then $r \longmapsto \mu_r(V)$ is a convex increasing function of r and the lelong number $\nu(dd^cV, \varphi)$ is given by*

$$\nu(dd^cV, \varphi) = \lim_{r\to-\infty} \frac{\mu_r(V)}{r}.$$

Proof. By (4.7) we have

$$\mu_r(V) = \mu_{r_0}(V) + \int_{r_0}^r \nu(dd^cV, \varphi, t)\, dt.$$

As $\nu(dd^cV, \varphi, t)$ is increasing and nonnegative, it follows that $r \longmapsto \mu_r(V)$ is convex and increasing. The formula for $\nu(dd^cV, \varphi) = \lim_{t\to-\infty} \nu(dd^cV, \varphi, t)$ is then obvious. $\square$

(4.9) Example. Let X be an open subset of $\mathbb{C}^n$ equipped with the semi-exhaustive function $\varphi(z) = \log|z-a|$, $a \in X$. Then $(dd^c\varphi)^n = \delta_a$ and the Lelong-Jensen formula becomes

$$\mu_r(V) = V(a) + \int_{-\infty}^r \nu(dd^cV, \varphi, t)\, dt.$$

As μ_r is the mean value measure on the sphere $S(a, e^r)$, we make the change of variables $r \mapsto \log r$, $t \mapsto \log t$ and obtain the more familiar formula

$$\text{(4.9 a)} \qquad \mu(V, S(a,r)) = V(a) + \int_0^r \nu(dd^cV, a, t)\,\frac{dt}{t}$$

where $\nu(dd^cV, a, t) = \nu(dd^cV, \varphi, \log t)$ is given by (3.7):

$$\text{(4.9 b)} \qquad \nu(dd^cV, a, t) = \frac{1}{\pi^{n-1}t^{2n-2}/(n-1)!}\int_{B(a,t)} \frac{1}{2\pi}\Delta V.$$

In this setting, Cor. 4.8 implies

$$\text{(4.9 c)} \qquad \nu(dd^cV, a) = \lim_{r\to 0}\frac{\mu\big(V, S(a,r)\big)}{\log r} = \lim_{r\to 0}\frac{\sup_{S(a,r)} V}{\log r}.$$

To prove the last equality, we may assume $V \le 0$ after subtraction of a constant. Inequality $\ge$ follows from the obvious estimate $\mu(V, S(a,r)) \le \sup_{S(a,r)} V$, while inequality $\le$ follows from the standard Harnack estimate

$$\text{(4.9 d)} \qquad \sup_{S(a,\varepsilon r)} V \le \frac{1-\varepsilon}{(1+\varepsilon)^{2n-1}}\,\mu\big(V, S(a,r)\big)$$

when ε is small. As $\sup_{S(a,r)} V = \sup_{B(a,r)} V$, Formula (4.9 c) can also be rewritten $\nu(dd^cV, a) = \liminf_{z\to a} V(z)/\log|z-a|$. Since $\sup_{S(a,r)} V$ is a convex (increasing) function of $\log r$, we infer that

$$\text{(4.9 e)} \qquad V(z) \le \gamma \log|z-a| + \mathrm{O}(1)$$

with $\gamma = \nu(dd^cV, a)$, and $\nu(dd^cV, a)$ is the largest constant γ which satisfies this inequality. Thus $\nu(dd^cV, a) = \gamma$ is equivalent to V having a logarithmic pole of coefficient γ.

(4.10) Special case. Take in particular $V = \log|f|$ where f is a holomorphic function on X. The Lelong-Poincaré formula shows that $dd^c \log|f|$ is equal to the zero divisor $[Z_f] = \sum m_j[H_j]$, where H_j are the irreducible components of $f^{-1}(0)$ and m_j is the multiplicity of f on H_j. The trace $\frac{1}{2\pi}\Delta f$ is then the euclidean area measure of Z_f (with corresponding multiplicities m_j). By Formula (4.9 c), we see that the Lelong number $\nu([Z_f], a)$ is equal to the vanishing order $\mathrm{ord}_a(f)$, that is, the smallest integer m such that $D^\alpha f(a) \ne 0$ for some multiindex α with $|\alpha| = m$. In dimension $n = 1$, we have $\frac{1}{2\pi}\Delta f = \sum m_j \delta_{a_j}$. Then (4.9 *a*) is the usual Jensen formula

$$\mu\big(\log|f|, S(0,r)\big) - \log|f(0)| = \int_0^r \nu(t)\frac{dt}{t} = \sum m_j \log\frac{r}{|a_j|}$$

where $\nu(t)$ is the number of zeros a_j in the disk $D(0,t)$, counted with multiplicities m_j.

(4.11) Example. Take $\varphi(z) = \log\max|z_j|^{\lambda_j}$ where $\lambda_j > 0$. Then $B(r)$ is the polydisk of radii $(e^{r/\lambda_1}, \ldots, e^{r/\lambda_n})$. If some coordinate z_j is non zero, say z_1, we can write $\varphi(z)$ as $\lambda_1 \log|z_1|$ plus some function depending only on the $(n-1)$ variables $z_j/z_1^{\lambda_1/\lambda_j}$. Hence $(dd^c\varphi)^n = 0$ on $\mathbb{C}^n \setminus \{0\}$. It will be shown later that

$$(4.11\,\mathrm{a}) \qquad (dd^c\varphi)^n = \lambda_1 \ldots \lambda_n\, \delta_0.$$

We now determine the measures μ_r. At any point z where not all terms $|z_j|^{\lambda_j}$ are equal, the smallest one can be omitted without changing φ in a neighborhood of z. Thus φ depends only on $(n-1)$-variables and $(dd^c\varphi_{\geq r})^n = 0$, $\mu_r = 0$ near z. It follows that μ_r is supported by the distinguished boundary $|z_j| = e^{r/\lambda_j}$ of the polydisk $B(r)$. As φ is invariant by all rotations $z_j \longmapsto e^{\mathrm{i}\theta_j} z_j$, the measure μ_r is also invariant and we see that μ_r is a constant multiple of $d\theta_1 \ldots d\theta_n$. By formula (4.2) and (4.11 a) we get

$$(4.11\,\mathrm{b}) \qquad \mu_r = \lambda_1 \ldots \lambda_n\, (2\pi)^{-n} d\theta_1 \ldots d\theta_n.$$

In particular, the Lelong number $\nu(dd^cV, \varphi)$ is given by

$$\nu(dd^cV, \varphi) = \lim_{r\to-\infty} \frac{\lambda_1 \ldots \lambda_n}{r} \int_{\theta_j \in [0,2\pi]} V(e^{r/\lambda_1 + \mathrm{i}\theta_1}, \ldots, e^{r/\lambda_n + \mathrm{i}\theta_n})\, \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n}.$$

These numbers have been introduced and studied by Kiselman [Ki4]. We call them *directional Lelong numbers* with coefficients $(\lambda_1, \ldots, \lambda_n)$. For an arbitrary current T, we define

$$(4.11\,\mathrm{c}) \qquad \nu(T, x, \lambda) = \nu\big(T, \log\max|z_j - x_j|^{\lambda_j}\big).$$

The above formula for $\nu(dd^cV, \varphi)$ combined with the analogue of Harnack's inequality (4.9 d) for polydisks gives

$$\begin{aligned} \nu(dd^cV, x, \lambda) &= \lim_{r\to 0} \frac{\lambda_1 \ldots \lambda_n}{\log r} \int V(r^{1/\lambda_1} e^{\mathrm{i}\theta_1}, \ldots, r^{1/\lambda_n} e^{\mathrm{i}\theta_n})\, \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n} \\ (4.11\,\mathrm{d}) \qquad &= \lim_{r\to 0} \frac{\lambda_1 \ldots \lambda_n}{\log r} \sup_{\theta_1, \ldots, \theta_n} V(r^{1/\lambda_1} e^{\mathrm{i}\theta_1}, \ldots, r^{1/\lambda_n} e^{\mathrm{i}\theta_n}). \end{aligned}$$

5. Comparison Theorems for Lelong Numbers

Let T be a closed positive current of bidimension (p,p) on a Stein manifold X equipped with a semi-exhaustive plurisubharmonic weight φ. We first show that the Lelong numbers $\nu(T, \varphi)$ only depend on the asymptotic behaviour of φ near the polar set $S(-\infty)$. In a precise way:

(5.1) First comparison theorem. *Let $\varphi, \psi : X \longrightarrow [-\infty, +\infty[$ be continuous plurisubharmonic functions. We assume that φ, ψ are semi-exhaustive on* $\operatorname{Supp} T$ *and that*

$$\ell := \limsup \frac{\psi(x)}{\varphi(x)} < +\infty \qquad \text{as } \ x \in \operatorname{Supp} T \ \text{ and } \ \varphi(x) \to -\infty.$$

Then $\nu(T, \psi) \le \ell^p \nu(T, \varphi)$, and the equality holds if $\ell = \lim \psi/\varphi$.

Proof. Definition 3.4 shows immediately that $\nu(T, \lambda\varphi) = \lambda^p \nu(T, \varphi)$ for every scalar $\lambda > 0$. It is thus sufficient to verify the inequality $\nu(T, \psi) \le \nu(T, \varphi)$ under the hypothesis $\limsup \psi/\varphi < 1$. For all $c > 0$, consider the plurisubharmonic function

$$u_c = \max(\psi - c, \varphi).$$

Let R_φ and R_ψ be such that $B_\varphi(R_\varphi) \cap \operatorname{Supp} T$ and $B_\psi(R_\psi) \cap \operatorname{Supp} T$ be relatively compact in X. Let $r < R_\varphi$ and $a < r$ be fixed. For $c > 0$ large enough, we have $u_c = \varphi$ on $\varphi^{-1}([a, r])$ and Stokes' formula gives

$$\nu(T, \varphi, r) = \nu(T, u_c, r) \ge \nu(T, u_c).$$

The hypothesis $\limsup \psi/\varphi < 1$ implies on the other hand that there exists $t_0 < 0$ such that $u_c = \psi - c$ on $\{u_c < t_0\} \cap \operatorname{Supp} T$. We infer

$$\nu(T, u_c) = \nu(T, \psi - c) = \nu(T, \psi),$$

hence $\nu(T, \psi) \le \nu(T, \varphi)$. The equality case is obtained by reversing the roles of φ and ψ and observing that $\lim \varphi/\psi = 1/l$. □

Assume in particular that $z^k = (z_1^k, \ldots, z_n^k)$, $k = 1, 2$, are coordinate systems centered at a point $x \in X$ and let

$$\varphi_k(z) = \log |z^k| = \log\big(|z_1^k|^2 + \ldots + |z_n^k|^2\big)^{1/2}.$$

We have $\lim_{z \to x} \varphi_2(z)/\varphi_1(z) = 1$, hence $\nu(T, \varphi_1) = \nu(T, \varphi_2)$ by Th. 5.1.

(5.2) Corollary. *The usual Lelong numbers $\nu(T, x)$ are independent of the choice of local coordinates.* □

This result had been originally proved by [Siu] with a much more delicate proof. Another interesting consequence is:

(5.3) Corollary. *On an open subset of $\mathbb{C}^n$, the Lelong numbers and Kiselman numbers are related by*

$$\nu(T, x) = \nu\big(T, x, (1, \ldots, 1)\big).$$

Proof. By definition, the Lelong number $\nu(T,x)$ is associated with the weight $\varphi(z) = \log|z-x|$ and the Kiselman number $\nu\big(T,x,(1,\ldots,1)\big)$ to the weight $\psi(z) = \log\max|z_j - x_j|$. It is clear that $\lim_{z\to x}\psi(z)/\varphi(z) = 1$, whence the conclusion. □

Another consequence of Th. 5.1 is that $\nu(T,x,\lambda)$ is an increasing function of each variable λ_j. Moreover, if $\lambda_1 \le \ldots \le \lambda_n$, we get the inequalities

$$\lambda_1^p\nu(T,x) \le \nu(T,x,\lambda) \le \lambda_n^p\nu(T,x).$$

These inequalities will be improved in section 7 (see Cor. 7.14). For the moment, we just prove the following special case.

(5.4) Corollary. *For all $\lambda_1,\ldots,\lambda_n > 0$ we have*

$$\big(dd^c\log\max_{1\le j\le n}|z_j|^{\lambda_j}\big)^n = \Big(dd^c\log\sum_{1\le j\le n}|z_j|^{\lambda_j}\Big)^n = \lambda_1\ldots\lambda_n\,\delta_0.$$

Proof. In fact, our measures vanish on $\mathbb{C}^n\smallsetminus\{0\}$ by the arguments explained in example 4.11. Hence they are equal to $c\,\delta_0$ for some constant $c\ge 0$ which is simply the Lelong number of the bidimension (n,n)-current $T=[X]=1$ with the corresponding weight. The comparison theorem shows that the first equality holds and that

$$\Big(dd^c\log\sum_{1\le j\le n}|z_j|^{\lambda_j}\Big)^n = \ell^{-n}\Big(dd^c\log\sum_{1\le j\le n}|z_j|^{\ell\lambda_j}\Big)^n$$

for all $\ell>0$. By taking ℓ large and approximating $\ell\lambda_j$ with $2[\ell\lambda_j/2]$, we may assume that $\lambda_j = 2s_j$ is an even integer. Then formula (3.6) gives

$$\begin{aligned}\int_{\sum|z_j|^{2s_j}<r^2}\Big(dd^c\log\sum|z_j|^{2s_j}\Big)^n &= r^{-2n}\int_{\sum|z_j|^{2s_j}<r^2}\Big(dd^c\sum|z_j|^{2s_j}\Big)^n\\ &= s_1\ldots s_n\,r^{-2n}\int_{\sum|w_j|^2<r^2}2^n\Big(\frac{\mathrm{i}}{2\pi}d'd''|w|^2\Big)^n = \lambda_1\ldots\lambda_n\end{aligned}$$

by using the $s_1\ldots s_n$-sheeted change of variables $w_j = z_j^{s_j}$. □

Now, we assume that $T=[A]$ is the current of integration over an analytic set $A\subset X$ of pure dimension p (cf. P. Lelong[Le1]). The above comparison theorem will enable us to give a simple proof of P. Thie's main result [Th]: the Lelong number $\nu([A],x)$ can be interpreted as the multiplicity of the analytic set A at point x.

Let $x\in A$ be a given point and $\mathcal{I}_{A,x}$ the ideal of germs of holomorphic functions at x vanishing on A. Then, one can find local coordinates

$z = (z_1, \ldots, z_n)$ on X centered at x such that there exist distinguished Weierstrass polynomials $P_j \in \mathcal{I}_{A,x}$ in the variable z_j, $p < j \leq n$, of the type

$$(5.5) \qquad P_j(z) = z_j^{d_j} + \sum_{k=1}^{d_j} a_{j,k}(z_1, \ldots, z_{j-1}) z_j^{d_j-k}\ , \ \ a_{j,k} \in \mathcal{M}^k_{\mathbb{C}^{j-1},0}$$

where $\mathcal{M}_{X,x}$ is the maximal ideal of X at x.

Indeed, we will prove this property by induction on $\operatorname{codim} X = n - p$. We fix a coordinate system $(w_1, \ldots, w_n)$ by which we identify the germ (X, x) to $(\mathbb{C}^n, 0)$.

If $n - p \geq 1$, there exists a non zero element $f \in \mathcal{I}_{A,x}$. Let d be the smallest integer such that $f \in \mathcal{M}^d_{\mathbb{C}^n,0}$ and let $e_n \in \mathbb{C}^n$ be a non zero vector such that $\lim_{t\to 0} f(te_n)/t^d \neq 0$. Complete e_n into a basis $(\tilde{e}_1, \ldots, \tilde{e}_{n-1}, e_n)$ of $\mathbb{C}^n$ and denote by $(\tilde{z}_1, \ldots, \tilde{z}_{n-1}, z_n)$ the corresponding coordinates. The Weierstrass preparation theorem gives a factorization $f = gP$ where P is a distinguished polynomial of type (5.5) in the variable z_n and where g is an invertible holomorphic function at point x. If $n - p = 1$, the polynomial $P_n = P$ satisfies the requirements. Observe that a generic choice of e_n actually works, since e_n only has to avoid the algebraic hypersurface $f_d(z) = 0$ where f_d is the polynomial of lowest degree in the Taylor expansion of f at 0.

If $n - p \geq 2$, $\mathcal{O}_{A,x} = \mathcal{O}_{X,x}/\mathcal{I}_{A,x}$ is a $\mathcal{O}_{\mathbb{C}^{n-1},0} = \mathbb{C}\{\tilde{z}_1, \ldots, \tilde{z}_{n-1}\}$-module of finite type, i.e. the projection $\mathrm{pr} : (X, x) \approx (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ is a finite morphism of (A, x) onto a germ $(Z, 0) \subset (\mathbb{C}^{n-1}, 0)$ of dimension p. The induction hypothesis applied to $\mathcal{I}_{Z,0} = \mathcal{O}_{\mathbb{C}^{n-1},0} \cap \mathcal{I}_{A,x}$ implies the existence of a new basis $(e_1, \ldots, e_{n-1})$ of $\mathbb{C}^{n-1}$ and of Weierstrass polynomials $P_{p+1}, \ldots, P_{n-1} \in \mathcal{I}_{Z,0}$, of the type (5.5) with respect to the coordinates $(z_1, \ldots, z_{n-1})$ associated with $(e_1, \ldots, e_{n-1})$. The polynomials $P_{p+1}, \ldots, P_n$ show that the expected property also holds in codimension $n - p$.

For any polynomial $Q(w) = w^d + a_1 w^{d-1} + \ldots + a_d \in \mathbb{C}[w]$, the roots w of Q satisfy

$$(5.6) \qquad |w| \leq 2 \max_{1\leq k\leq d} |a_k|^{1/k},$$

otherwise $Q(w)w^{-d} = 1 + a_1 w^{-1} + \ldots + a_d w^{-d}$ would have a modulus larger than $1 - (2^{-1} + \ldots + 2^{-d}) = 2^{-d}$, a contradiction. Let us denote $z = (z', z'')$ with $z' = (z_1, \ldots, z_p)$ and $z'' = (z_{p+1}, \ldots, z_n)$. As $a_{j,k} \in \mathcal{M}^k_{\mathbb{C}^{j-1},0}$, we get

$$|a_{j,k}(z_1, \ldots, z_{j-1})| = \mathrm{O}\big((|z_1| + \ldots + |z_{j-1}|)^k\big) \quad \text{if} \ \ j > p,$$

and we deduce from (5.5), (5.6) that $|z_j| = O(|z_1| + \ldots + |z_{j-1}|)$ on (A, x). Therefore, we get:

(5.7) Lemma. *For a generic choice of coordinates $z' = (z_1, \ldots, z_p)$ and $z'' = (z_{p+1}, \ldots, z_n)$ on (X, x), the germ (A, x) is contained in a cone $|z''| \le C|z'|$.* □

We use this property to compute the Lelong number of $[A]$ at point x. When $z \in A$ tends to x, the functions

$$\varphi(z) = \log|z| = \log(|z'|^2 + |z''|^2)^{1/2}, \qquad \psi(z) = \log|z'|.$$

are equivalent. As φ, ψ are semi-exhaustive on A, Th. 5.1 implies

$$\nu([A], x) = \nu([A], \varphi) = \nu([A], \psi).$$

Let $B' \subset \mathbb{C}^p$ the ball of center 0 and radius r', $B'' \subset \mathbb{C}^{n-p}$ the ball of center 0 and radius $r'' = Cr'$. The inclusion of germ (A, x) in the cone $|z''| \le C|z'|$ shows that for r' small enough the projection

$$\mathrm{pr} : A \cap (B' \times B'') \longrightarrow B'$$

is proper. The fibers are finite by (5.5). Hence this projection is a ramified covering with finite sheet number m (see Fig. 3).

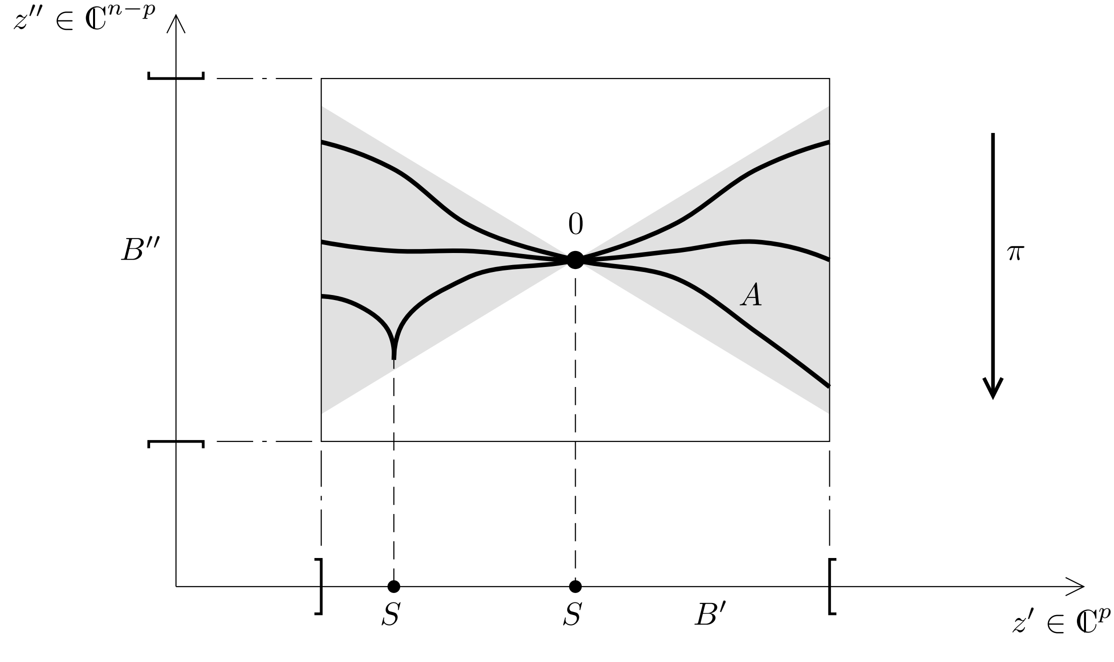

Fig. 3. Ramified covering π and ramification locus S

Let us apply formula (3.6) to ψ: for every $t < r'$ we get

$$\begin{aligned}\nu([A],\psi,\log t) &= t^{-2p}\int_{\{\psi<\log t\}}[A]\wedge\Big(\frac{1}{2}dd^c e^{2\psi}\Big)^p\\ &= t^{-2p}\int_{A\cap\{|z'|<t\}}\Big(\frac{1}{2}\mathrm{pr}^\star dd^c|z'|^2\Big)^p\\ &= m\,t^{-2p}\int_{\mathbb{C}^p\cap\{|z'|<t\}}\Big(\frac{1}{2}dd^c|z'|^2\Big)^p = m,\end{aligned}$$

hence $\nu(T,\psi)=m$. Here, we used the fact that pr is actually a covering with m sheets over the complement of the ramification locus $S\subset B'$, which is of zero Lebesgue measure. We thus obtain a new proof of Thie's result [Th] that $\nu([A],x)$ is equal to the multiplicity of A at x :

(5.8) Theorem. *Let A be an analytic set of dimension p in a complex manifold of dimension n. For every point $x\in A$, there exist local coordinates*

$$z=(z',z''),\quad z'=(z_1,\ldots,z_p),\quad z''=(z_{p+1},\ldots,z_n)$$

centered at x and balls $B'\subset\mathbb{C}^p$, $B''\subset\mathbb{C}^{n-p}$ of radii r',r'' in these coordinates, such that $A\cap(B'\times B'')$ is contained in the cone $|z''|\le(r''/r')|z'|$. The multiplicity of A at x is defined as the number m of sheets of any such ramified covering map $A\cap(B'\times B'')\longrightarrow B'$. Then $\nu([A],x)=m$.

There is another interesting version of the comparison theorem which compares the Lelong numbers of two currents obtained as intersection products (in that case, we take the same weight for both).

(5.9) Second comparison theorem. *Let $u_1,\ldots,u_q$ and $v_1,\ldots,v_q$ be plurisubharmonic functions such that each q-tuple satisfies the hypotheses of Th. 2.5 with respect to T. Suppose moreover that $u_j=-\infty$ on $\operatorname{Supp}T\cap\varphi^{-1}(-\infty)$ and that*

$$\ell_j:=\limsup\frac{v_j(z)}{u_j(z)}<+\infty\qquad\text{when}\quad z\in\operatorname{Supp}T\smallsetminus u_j^{-1}(-\infty),\quad\varphi(z)\to-\infty.$$

Then

$$\nu(dd^cv_1\wedge\ldots\wedge dd^cv_q\wedge T,\varphi)\le\ell_1\ldots\ell_q\,\nu(dd^cu_1\wedge\ldots\wedge dd^cu_q\wedge T,\varphi).$$

Proof. By homogeneity in each factor v_j, it is enough to prove the inequality with constants $\ell_j=1$ under the hypothesis $\limsup v_j/u_j<1$. We set

$$w_{j,c}=\max\{v_j-c,u_j\}.$$

Our assumption implies that $w_{j,c}$ coincides with v_j-c on a neighborhood $\operatorname{Supp}T\cap\{\varphi<r_0\}$ of $\operatorname{Supp}T\cap\{\varphi<-\infty\}$, thus

$$\nu(dd^c v_1 \wedge \ldots \wedge dd^c v_q \wedge T, \varphi) = \nu(dd^c w_{1,c} \wedge \ldots \wedge dd^c w_{q,c} \wedge T, \varphi)$$

for every c. Now, fix $r < R_\varphi$. Proposition 2.9 shows that the current $dd^c w_{1,c} \wedge \ldots \wedge dd^c w_{q,c} \wedge T$ converges weakly to $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ when c tends to $+\infty$. By Prop. 3.12 we get

$$\limsup_{c\to+\infty} \nu(dd^c w_{1,c} \wedge \ldots \wedge dd^c w_{q,c} \wedge T, \varphi) \leq \nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T, \varphi). \ \square$$

(5.10) Corollary. *If $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ is well defined, then at every point $x \in X$ we have*

$$\nu\big(dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T, x\big) \geq \nu(dd^c u_1, x) \ldots \nu(dd^c u_q, x)\, \nu(T, x).$$

Proof. Apply (5.9) with $\varphi(z) = v_1(z) = \ldots = v_q(z) = \log|z - x|$ and observe that $\ell_j := \limsup v_j/u_j = 1/\nu(dd^c u_j, x)$ (there is nothing to prove if $\nu(dd^c u_j, x) = 0$). $\square$

Finally, we present an interesting stability property of Lelong numbers due to [Siu]: almost all slices of a closed positive current T along linear subspaces passing through a given point have the same Lelong number as T. Before giving a proof of this, we need a useful formula known as *Crofton's formula*.

(5.11) Lemma. *Let α be a closed positive (p,p)-form on $\mathbb{C}^n \setminus \{0\}$ which is invariant under the unitary group $U(n)$. Then α has the form*

$$\alpha = \big(dd^c \chi(\log|z|)\big)^p$$

where χ is a convex increasing function. Moreover α is invariant by homotheties if and only if χ is an affine function, i.e. $\alpha = \lambda\,(dd^c \log|z|)^p$.

Proof. A radial convolution $\alpha_\varepsilon(z) = \int_{\mathbb{R}} \rho(t/\varepsilon)\, \alpha(e^t z)\, dt$ produces a smooth form with the same properties as α and $\lim_{\varepsilon\to 0} \alpha_\varepsilon = \alpha$. Hence we can suppose that α is smooth on $\mathbb{C}^n \setminus \{0\}$. At a point $z = (0, \ldots, 0, z_n)$, the (p,p)-form $\alpha(z) \in \bigwedge^{p,p}(\mathbb{C}^n)^\star$ must be invariant by $U(n-1)$ acting on the first $(n-1)$ coordinates. We claim that the subspace of $U(n-1)$-invariants in $\bigwedge^{p,p}(\mathbb{C}^n)^\star$ is generated by $(dd^c|z|^2)^p$ and $(dd^c|z|^2)^{p-1} \wedge \mathrm{i}dz_n \wedge d\overline{z}_n$. In fact, a form $\beta = \sum \beta_{I,J} dz_I \wedge d\overline{z}_J$ is invariant by $U(1)^{n-1} \subset U(n-1)$ if and only if $\beta_{I,J} = 0$ for $I \neq J$, and invariant by the permutation group $\mathcal{S}_{n-1} \subset U(n-1)$ if and only if all coefficients $\beta_{I,I}$ (resp. $\beta_{Jn,Jn}$) with $I, J \subset \{1, \ldots, n-1\}$ are equal. Hence

$$\beta = \lambda \sum_{|I|=p} dz_I \wedge d\overline{z}_I + \mu\Big(\sum_{|J|=p-1} dz_J \wedge d\overline{z}_J\Big) \wedge dz_n \wedge d\overline{z}_n.$$

This proves our claim. As $d|z|^2 \wedge d^c|z|^2 = \frac{\mathrm{i}}{\pi}|z_n|^2 dz_n \wedge d\overline{z}_n$ at $(0,\ldots,0,z_n)$, we conclude that

$$\alpha(z) = f(z)(dd^c|z|^2)^p + g(z)(dd^c|z|^2)^{p-1} \wedge d|z|^2 \wedge d^c|z|^2$$

for some smooth functions f, g on $\mathbb{C}^n \setminus \{0\}$. The $U(n)$-invariance of α shows that f and g are radial functions. We may rewrite the last formula as

$$\alpha(z) = u(\log|z|)(dd^c \log|z|)^p + v(\log|z|)(dd^c \log|z|)^{p-1} \wedge d\log|z| \wedge d^c \log|z|.$$

Here $(dd^c \log|z|)^p$ is a positive (p,p)-form coming from $\mathbb{P}^{n-1}$, hence it has zero contraction in the radial direction, while the contraction of the form $(dd^c \log|z|)^{p-1} \wedge d\log|z| \wedge d^c \log|z|$ by the radial vector field is $(dd^c \log|z|)^{p-1}$. This shows easily that $\alpha(z) \geq 0$ if and only if $u, v \geq 0$. Next, the closedness condition $d\alpha = 0$ gives $u' - v = 0$. Thus u is increasing and we define a convex increasing function χ by $\chi' = u^{1/p}$. Then $v = u' = p\chi'^{p-1}\chi''$ and

$$\alpha(z) = \big(dd^c \chi(\log|z|)\big)^p.$$

If α is invariant by homotheties, the functions u and v must be constant, thus $v = 0$ and $\alpha = \lambda(dd^c \log|z|)^p$. □

(5.12) Corollary (Crofton's formula)**.** *Let dv be the unique $U(n)$-invariant measure of mass 1 on the Grassmannian $G(p,n)$ of p-dimensional subspaces in $\mathbb{C}^n$. Then*

$$\int_{S \in G(p,n)} [S]\, dv(S) = (dd^c \log|z|)^{n-p}.$$

Proof. The left hand integral is a closed positive bidegree $(n-p, n-p)$ current which is invariant by $U(n)$ and by homotheties. By lemma 5.11, this current must coincide with the form $\lambda(dd^c \log|z|)^{n-p}$ for some $\lambda \geq 0$. The coefficient λ is the Lelong number at 0. As $\nu([S], 0) = 1$ for every S, we get $\lambda = \int_{G(p,n)} dv(S) = 1$. □

We now recall the basic facts of slicing theory (see Federer [Fe] and Harvey [Ha]). Let $\sigma : M \to M'$ be a submersion of smooth differentiable manifolds and let Θ be a *locally flat* current on M, that is a current which can be written locally as $\Theta = U + dV$ where U, V have locally integrable coefficients. It can be shown that every current Θ such that both Θ and $d\Theta$ have measure coefficients is locally flat; in particular, closed positive currents are locally flats. Then, for almost every $x' \in M'$, there is a well defined slice $\Theta_{x'}$, which is the current on the fiber $\sigma^{-1}(x')$ defined by

$$\Theta_{x'} = U_{\restriction \sigma^{-1}(x')} + dV_{\restriction \sigma^{-1}(x')}.$$

The restrictions of U, V to the fibers exist for almost all x' by the Fubini theorem. It is easy to show by a regularization $\Theta_\varepsilon = \Theta \star \rho_\varepsilon$ that the slices of a closed positive current are again closed and positive: in fact $U_{\varepsilon,x'}$ and $V_{\varepsilon,x'}$ converge to $U_{x'}$ and $V_{x'}$ in L^1_{loc}, thus $\Theta_{\varepsilon,x'}$ converges weakly to $\Theta_{x'}$ for almost every x'. This kind of slicing can be referred to as *parallel slicing* (if we think of σ as being a projection map). The kind of slicing we need (where the slices are taken over linear subspaces passing through a given point) is of a slightly different nature and is called *concurrent slicing*.

The possibility of concurrent slicing is proved as follows. Let T be a closed positive current of bidimension (p,p) in the ball $B(0,R)\subset\mathbb{C}^n$. Let

$$Y=\big\{(x,S)\in\mathbb{C}^n\times G(q,n)\,;\,x\in S\big\}$$

be the total space of the tautological rank q vector bundle over the Grassmannian $G(q,n)$, equipped with the obvious projections

$$\sigma:Y\longrightarrow G(q,n),\qquad \pi:Y\longrightarrow\mathbb{C}^n.$$

We set $Y_R=\pi^{-1}(B(0,R))$ and $Y^\star_R=\pi^{-1}(B(0,R)\smallsetminus\{0\})$. The restriction π_0 of π to $Y^\star_R$ is a submersion onto $B(0,R)\smallsetminus\{0\}$, so we have a well defined pull-back $\pi_0^\star T$ over $Y^\star_R$. We would like to extend it as a pull-back $\pi^\star T$ over Y_R, so as to define slices $T_{\restriction S}=(\pi^\star T)_{\restriction\sigma^{-1}(S)}$; of course, these slices can be non zero only if the dimension of S is at least equal to the degree of T, i.e. if $q\ge n-p$. We first claim that $\pi_0^\star T$ has locally finite mass near the zero section $\pi^{-1}(0)$ of σ. In fact let ω_G be a unitary invariant Kähler metric over $G(q,n)$ and let $\beta=dd^c|z|^2$ in $\mathbb{C}^n$. Then we get a Kähler metric on Y defined by $\omega_Y=\sigma^\star\omega_G+\pi^\star\beta$. If $N=(q-1)(n-q)$ is the dimension of the fibers of π, the projection formula $\pi_\star(u\wedge\pi^\star v)=(\pi_\star u)\wedge v$ gives

$$\pi_\star\omega_Y^{N+p}=\sum_{1\le k\le p}\binom{N+p}{k}\beta^k\wedge\pi_\star(\sigma^\star\omega_G^{N+p-k}).$$

Here $\pi_\star(\sigma^\star\omega_G^{N+p-k})$ is a unitary and homothety invariant $(p-k,p-k)$ closed positive form on $\mathbb{C}^n\smallsetminus\{0\}$, so $\pi_\star(\sigma^\star\omega_G^{N+p-k})$ is proportional to $(dd^c\log|z|)^{n-k}$. With some constants $\lambda_k>0$, we thus get

$$\begin{aligned}\int_{Y^\star_r}\pi_0^\star T\wedge\omega_Y^{N+p}&=\sum_{0\le k\le p}\lambda_k\int_{B(0,r)\smallsetminus\{0\}}T\wedge\beta^k\wedge(dd^c\log|z|)^{k-p}\\&=\sum_{0\le k\le p}\lambda_k2^{-(p-k)}r^{-2(p-k)}\int_{B(0,r)\smallsetminus\{0\}}T\wedge\beta^p<+\infty.\end{aligned}$$

The Skoda-El Mir theorem 0.5 shows that the trivial extension $\widetilde{\pi}_0^\star T$ of $\pi_0^\star T$ is a closed positive current on Y_R. Of course, the zero section $\pi^{-1}(0)$ might also carry some extra mass of the desired current $\pi^\star T$. Since $\pi^{-1}(0)$ has codimension q, this extra mass cannot exist when $q>n-p=\operatorname{codim}\pi^\star T$ and we simply set $\pi^\star T=\widetilde{\pi}_0^\star T$. On the other hand, if $q=n-p$, we set

$$\pi^\star T:=\widetilde{\pi}_0^\star T+\nu(T,0)\,[\pi^{-1}(0)].\tag{5.13}$$

We can now apply parallel slicing with respect to $\sigma : Y_R \to G(q,n)$, which is a submersion: for almost all $S \in G(q,n)$, there is a well defined slice $T_{\restriction S} = (\pi^\star T)_{\restriction \sigma^{-1}(S)}$. These slices coincide with the usual restrictions of T to S if T is smooth.

(5.14) Theorem ([Siu]). *For almost all $S \in G(q,n)$ with $q \geq n-p$, the slice $T_{\restriction S}$ satisfies $\nu(T_{\restriction S}, 0) = \nu(T, 0)$.*

Proof. If $q = n-p$, the slice $T_{\restriction S}$ consists of some positive measure with support in $S \smallsetminus \{0\}$ plus a Dirac measure $\nu(T,0)\,\delta_0$ coming from the slice of $\nu(T,0)\,[\pi^{-1}(0)]$. The equality $\nu(T_{\restriction S}, 0) = \nu(T,0)$ thus follows directly from (5.13).

In the general case $q > n-p$, it is clearly sufficient to prove the following two properties:

(a) $\displaystyle \nu(T,0,r) = \int_{S\in G(q,n)} \nu(T_{\restriction S}, 0, r)\, dv(S)$ for all $r \in]0,R[$;

(b) $\nu(T_{\restriction S}, 0) \geq \nu(T,0)$ for almost all S.

In fact, (a) implies that $\nu(T,0)$ is the average of all Lelong numbers $\nu(T_{\restriction S},0)$ and the conjunction with (b) implies that these numbers must be equal to $\nu(T,0)$ for almost all S. In order to prove (a) and (b), we can suppose without loss of generality that T is smooth on $B(0,R) \smallsetminus \{0\}$. Otherwise, we perform a small convolution with respect to the action of $\mathrm{Gl}_n(\mathbb{C})$ on $\mathbb{C}^n$:

$$T_\varepsilon = \int_{g \in \mathrm{Gl}_n(\mathbb{C})} \rho_\varepsilon(g)\, g^\star T\, dv(g)$$

where (ρ_ε) is a regularizing family with support in an ε-neighborhood of the unit element of $\mathrm{Gl}_n(\mathbb{C})$. Then T_ε is smooth in $B(0,(1-\varepsilon)R) \smallsetminus \{0\}$ and converges weakly to T. Moreover, we have $\nu(T_\varepsilon, 0) = \nu(T,0)$ by (5.2) and $\nu(T_{\restriction S}, 0) \geq \limsup_{\varepsilon\to 0} \nu(T_{\varepsilon,\restriction S}, 0)$ by (3.12), thus (a), (b) are preserved in the limit. If T is smooth on $B(0,R) \smallsetminus \{0\}$, the slice $T_{\restriction S}$ is defined for all S and is simply the restriction of T to $S \smallsetminus \{0\}$ (carrying no mass at the origin).

(a) Here we may even assume that T is smooth at 0 by performing an ordinary convolution. As $T_{\restriction S}$ has bidegree $(n-p, n-p)$, we have

$$\nu(T_{\restriction S}, 0, r) = \int_{S\cap B(0,r)} T \wedge \alpha_S^{q-(n-p)} = \int_{B(0,r)} T \wedge [S] \wedge \alpha_S^{p+q-n}$$

where $\alpha_S = dd^c \log|w|$ and $w = (w_1, \ldots, w_q)$ are orthonormal coordinates on S. We simply have to check that

$$\int_{S\in G(q,n)} [S] \wedge \alpha_S^{p+q-n}\, dv(S) = (dd^c \log|z|)^p.$$

However, both sides are unitary and homothety invariant (p,p)-forms with Lelong number 1 at the origin, so they must coincide by Lemma 5.11.

(b) We prove the inequality when $S = \mathbb{C}^q \times \{0\}$. By the comparison theorem 5.1, for every $r > 0$ and $\varepsilon > 0$ we have

$$(5.15) \qquad \int_{B(0,r)} T \wedge \gamma_\varepsilon^p \geq \nu(T,0) \quad \text{where}$$

$$\gamma_\varepsilon = \frac{1}{2} dd^c \log(\varepsilon|z_1|^2 + \ldots + \varepsilon|z_q|^2 + |z_{q+1}|^2 + \ldots + |z_n|^2).$$

We claim that the current γ_ε^p converges weakly to

$$[S] \wedge \alpha_S^{p+q-n} = [S] \wedge \Big(\frac{1}{2} dd^c \log(|z_1|^2 + \ldots + |z_q|^2)\Big)^{p+q-n}$$

as ε tends to 0. In fact, the Lelong number of γ_ε^p at 0 is 1, hence by homogeneity

$$\int_{B(0,r)} \gamma_\varepsilon^p \wedge (dd^c|z|^2)^{n-p} = (2r^2)^p$$

for all $\varepsilon, r > 0$. Therefore the family (γ_ε^p) is relatively compact in the weak topology. Since $\gamma_0 = \lim \gamma_\varepsilon$ is smooth on $\mathbb{C}^n \smallsetminus S$ and depends only on $n-q$ variables ($n - q \leq p$), we have $\lim \gamma_\varepsilon^p = \gamma_0^p = 0$ on $\mathbb{C}^n \smallsetminus S$. This shows that every weak limit of (γ_ε^p) has support in S. Each of these is the direct image by inclusion of a unitary and homothety invariant $(p+q-n, p+q-n)$-form on S with Lelong number equal to 1 at 0. Therefore we must have

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon^p = (i_S)_\star(\alpha_S^{p+q-n}) = [S] \wedge \alpha_S^{p+q-n},$$

and our claim is proved (of course, this can also be checked by direct elementary calculations). By taking the limsup in (5.15) we obtain

$$\nu(T_{\restriction S}, 0, r+0) = \int_{\overline{B}(0,r)} T \wedge [S] \wedge \alpha_S^{p+q-n} \geq \nu(T,0)$$

(the singularity of T at 0 does not create any difficulty because we can modify T by a dd^c-exact form near 0 to make it smooth everywhere). Property (b) follows when r tends to 0. □

6. Siu's Semicontinuity Theorem

Let X, Y be complex manifolds of dimension n, m such that X is Stein. Let $\varphi : X \times Y \longrightarrow [-\infty, +\infty[$ be a continuous plurisubharmonic function. We assume that φ is *semi-exhaustive* with respect to $\operatorname{Supp} T$, i.e. that for every compact subset $L \subset Y$ there exists $R = R(L) < 0$ such that

$$(6.1) \qquad \{(x,y) \in \operatorname{Supp} T \times L\,;\ \varphi(x,y) \le R\} \subset\subset X \times Y.$$

Let T be a closed positive current of bidimension (p,p) on X. For every point $y \in Y$, the function $\varphi_y(x) := \varphi(x,y)$ is semi-exhaustive on $\operatorname{Supp} T$; one can therefore associate with y a generalized Lelong number $\nu(T,\varphi_y)$. Proposition 3.13 implies that the map $y \mapsto \nu(T,\varphi_y)$ is upper semi-continuous, hence the upperlevel sets

$$(6.2) \qquad E_c = E_c(T,\varphi) = \{y \in Y; \nu(T,\varphi_y) \ge c\}\ ,\ \ c > 0$$

are closed. Under mild additional hypotheses, we are going to show (following [De4]) that the sets E_c are in fact analytic subsets of Y.

(6.3) Definition. *We say that a function $f(x,y)$ is locally Hölder continuous with respect to y on $X \times Y$ if every point of $X \times Y$ has a neighborhood Ω on which*

$$|f(x,y_1) - f(x,y_2)| \le M|y_1 - y_2|^\gamma$$

for all $(x,y_1) \in \Omega$, $(x,y_2) \in \Omega$, with some constants $M > 0$, $\gamma \in\,]0,1]$, and suitable coordinates on Y.

(6.4) Theorem ([De4])**.** *Let T be a closed positive current on X and let*

$$\varphi : X \times Y \longrightarrow [-\infty,+\infty[$$

be a continuous plurisubharmonic function. Assume that φ is semi-exhaustive on $\operatorname{Supp} T$ and that $e^{\varphi(x,y)}$ is locally Hölder continuous with respect to y on $X \times Y$. Then the upperlevel sets

$$E_c(T,\varphi) = \{y \in Y; \nu(T,\varphi_y) \ge c\}$$

are analytic subsets of Y.

This theorem can be rephrased by saying that $y \longmapsto \nu(T,\varphi_y)$ is upper semi-continuous with respect to the analytic Zariski topology. As a special case, we get the following important result of [Siu]:

(6.5) Corollary. *If T is a closed positive current of bidimension (p,p) on a complex manifold X, the upperlevel sets $E_c(T) = \{x \in X\,;\ \nu(T,x) \ge c\}$ of the usual Lelong numbers are analytic subsets of dimension $\le p$.*

Proof. The result is local, so we may assume that $X \subset \mathbb{C}^n$ is an open subset. Theorem 6.4 with $Y = X$ and $\varphi(x,y) = \log|x-y|$ shows that $E_c(T)$ is analytic. Moreover, Prop. 3.11 implies $\dim E_c(T) \le p$. □

(6.6) Generalization. Theorem 6.4 can be applied more generally to weight functions of the type

$$\varphi(x,y) = \max_j \ \log\Big(\sum_k |F_{j,k}(x,y)|^{\lambda_{j,k}}\Big)$$

where $F_{j,k}$ are holomorphic functions on $X \times Y$ and where $\gamma_{j,k}$ are positive real constants; in this case e^{φ} is Hölder continuous of exponent $\gamma = \min\{\lambda_{j,k}, 1\}$ and φ is semi-exhaustive with respect to the whole space X as soon as the projection $\mathrm{pr}_2 : \bigcap F_{j,k}^{-1}(0) \longrightarrow Y$ is proper and finite.

For example, when $\varphi(x,y) = \log\max|x_j - y_j|_j^{\lambda}$ on an open subset X of $\mathbb{C}^n$, we see that the upperlevel sets for Kiselman's numbers $\nu(T,x,\lambda)$ are analytic in X (a result first proved in [Ki4]). More generally, set $\psi_\lambda(z) = \log\max|z_j|^{\lambda_j}$ and $\varphi(x,y,g) = \psi_\lambda\big(g(x-y)\big)$ where $x, y \in \mathbb{C}^n$ and $g \in \mathrm{Gl}(\mathbb{C}^n)$. Then $\nu(T,\varphi_{y,g})$ is the Kiselman number of T at y when the coordinates have been rotated by g. It is clear that φ is plurisubharmonic in (x,y,g) and semi-exhaustive with respect to x, and that e^{φ} is locally Hölder continuous with respect to (y,g). Thus the upperlevel sets

$$E_c = \{(y,g) \in X \times \mathrm{Gl}(\mathbb{C}^n)\,;\, \nu(T,\varphi_{y,g}) \geq c\}$$

are analytic in $X \times \mathrm{Gl}(\mathbb{C}^n)$. However this result is not meaningful on a manifold, because it is not invariant under coordinate changes. One can obtain an invariant version as follows. Let X be a manifold and let $J^k\mathcal{O}_X$ be the bundle of k-jets of holomorphic functions on X. We consider the bundle S_k over X whose fiber $S_{k,y}$ is the set of n-tuples of k-jets $u = (u_1, \ldots, u_n) \in (J^k\mathcal{O}_{X,y})^n$ such that $u_j(y) = 0$ and $du_1 \wedge \ldots \wedge du_n(y) \neq 0$. Let (z_j) be local coordinates on an open set $\Omega \subset X$. Modulo $O(|z-y|^{k+1})$, we can write

$$u_j(z) = \sum_{1\leq|\alpha|\leq k} a_{j,\alpha}(z-y)^{\alpha}$$

with $\det(a_{j,(0,\ldots,1_k,\ldots,0)}) \neq 0$. The numbers $((y_j),(a_{j,\alpha}))$ define a coordinate system on the total space of $S_{k\restriction\Omega}$. For $(x,(y,u)) \in X \times S_k$, we introduce the function

$$\varphi(x,y,u) = \log\max|u_j(x)|^{\lambda_j} = \log\max\Big|\sum_{1\leq|\alpha|\leq k} a_{j,\alpha}(x-y)^{\alpha}\Big|^{\lambda_j}$$

which has all properties required by Th. 6.4 on a neighborhood of the diagonal $x = y$, i.e. a neighborhood of $X \times_X S_k$ in $X \times S_k$. For k large, we claim that Kiselman's directional Lelong numbers

$$\nu(T,y,u,\lambda) := \nu(T,\varphi_{y,u})$$

with respect to the coordinate system (u_j) at y do not depend on the selection of the jet representives and are therefore canonically defined on S_k. In fact, a change of u_j by $O(|z-y|^{k+1})$ adds $O(|z-y|^{(k+1)\lambda_j})$ to e^{φ}, and we have $e^{\varphi} \geq O(|z-y|^{\max\lambda_j})$. Hence by (5.1) it is enough to take $k+1 \geq \max\lambda_j/\min\lambda_j$. Theorem 6.4 then shows that the upperlevel sets $E_c(T,\varphi)$ are analytic in S_k. $\square$

Now we give the detailed proof of Th. 6.4. As the result is local on Y, we may assume without loss of generality that Y is a ball in $\mathbb{C}^m$. After addition of a constant to φ, we may also assume that there exists a compact subset $K \subset X$ such that

$$\{(x,y) \in X \times Y; \varphi(x,y) \le 0\} \subset K \times Y.$$

By Th. 5.1, the Lelong numbers depend only on the asymptotic behaviour of φ near the (compact) polar set $\varphi^{-1}(-\infty) \cap (\mathrm{SuppT} \times Y)$. We can add a smooth strictly plurisubharmonic function on $X \times Y$ to make φ strictly plurisuharmonic. Then Richberg's approximation theorem for continuous plurisubharmonic functions shows that there exists a smooth plurisubharmonic function $\widetilde{\varphi}$ such that $\varphi \le \widetilde{\varphi} \le \varphi + 1$. We may therefore assume that φ is smooth on $(X \times Y) \setminus \varphi^{-1}(-\infty)$.

• **First step**: *construction of a local plurisubharmonic potential.*

Our goal is to generalize the usual construction of plurisubharmonic potentials associated with a closed positive current (cf. P. Lelong[Le2] and H. Skoda[Sk1]). We replace here the usual kernel $|z-\zeta|^{-2p}$ arising from the hermitian metric of $\mathbb{C}^n$ by a kernel depending on the weight φ. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be an increasing function such that $\chi(t) = t$ for $t \le -1$ and $\chi(t) = 0$ for $t \ge 0$. We consider the half-plane $H = \{z \in \mathbb{C}\,;\, \mathrm{Re} z < -1\}$ and associate with T the potential function V on $Y \times H$ defined by

$$V(y,z) = -\int_{\mathrm{Re} z}^{0} \nu(T, \varphi_y, t)\chi'(t)\, dt. \tag{6.7}$$

For every $t > \mathrm{Re}\, z$, Stokes' formula gives

$$\nu(T, \varphi_y, t) = \int_{\varphi(x,y)<t} T(x) \wedge (dd^c_x \widetilde{\varphi}(x,y,z))^p$$

with $\widetilde{\varphi}(x,y,z) := \max\{\varphi(x,y), \mathrm{Re} z\}$. The Fubini theorem applied to (6.7) gives

$$\begin{aligned} V(y,z) &= -\int_{\substack{x\in X, \varphi(x,y)<t \\ \mathrm{Re}\, z<t<0}} T(x) \wedge (dd^c_x \widetilde{\varphi}(x,y,z))^p\ \chi'(t) dt \\ &= \int_{x \in X} T(x) \wedge \chi(\widetilde{\varphi}(x,y,z))\,(dd^c_x \widetilde{\varphi}(x,y,z))^p. \end{aligned}$$

For all $(n-1, n-1)$-form h of class C^∞ with compact support in $Y \times H$, we get

$$\begin{aligned} \langle dd^c V, h\rangle &= \langle V, dd^c h\rangle \\ &= \int_{X\times Y\times H} T(x) \wedge \chi(\widetilde{\varphi}(x,y,z))(dd^c \widetilde{\varphi}(x,y,z))^p \wedge dd^c h(y,z). \end{aligned}$$

Observe that the replacement of dd^c_x by the total differentiation $dd^c = dd^c_{x,y,z}$ does not modify the integrand, because the terms in dx, $d\overline{x}$ must have total bidegree (n,n). The current $T(x)\wedge\chi(\widetilde\varphi(x,y,z))h(y,z)$ has compact support in $X\times Y\times H$. An integration by parts can thus be performed to obtain

$$\langle dd^cV,h\rangle=\int_{X\times Y\times H}T(x)\wedge dd^c(\chi\circ\widetilde\varphi(x,y,z))\wedge(dd^c\widetilde\varphi(x,y,z))^p.h(y,z).$$

On the corona $\{-1\le\varphi(x,y)\le 0\}$ we have $\widetilde\varphi(x,y,z)=\varphi(x,y)$, whereas for $\varphi(x,y)<-1$ we get $\widetilde\varphi<-1$ and $\chi\circ\widetilde\varphi=\widetilde\varphi$. As $\widetilde\varphi$ is plurisubharmonic, we see that $dd^cV(y,z)$ is the sum of the positive $(1,1)$-form

$$(y,z)\longmapsto\int_{\{x\in X;\varphi(x,y)<-1\}}T(x)\wedge(dd^c_{x,y,z}\widetilde\varphi(x,y,z))^{p+1}$$

and of the $(1,1)$-form independent of z

$$y\longmapsto\int_{\{x\in X;-1\le\varphi(x,y)\le 0\}}T\wedge dd^c_{x,y}(\chi\circ\varphi)\wedge(dd^c_{x,y}\varphi)^p.$$

As φ is smooth outside $\varphi^{-1}(-\infty)$, this last form has locally bounded coefficients. Hence $dd^cV(y,z)$ is ≥ 0 except perhaps for locally bounded terms. In addition, V is continuous on $Y\times H$ because $T\wedge(dd^c\widetilde\varphi_{y,z})^p$ is weakly continuous in the variables (y,z) by Cor. 1.6. We therefore obtain the following result.

(6.8) Proposition. *There exists a positive plurisubharmonic function $\rho\in C^\infty(Y)$ such that $\rho(y)+V(y,z)$ is plurisubharmonic on $Y\times H$.*

If we let $\mathrm{Re}\,z$ tend to $-\infty$, we see that the function

$$U_0(y)=\rho(y)+V(y,-\infty)=\rho(y)-\int_{-\infty}^0\nu(T,\varphi_y,t)\chi'(t)dt$$

is locally plurisubharmonic or identically $-\infty$ on Y. Moreover, it is clear that $U_0(y)=-\infty$ at every point y such that $\nu(T,\varphi_y)>0$. If Y is connected and $U_0\not\equiv-\infty$, we already conclude that the density set $\bigcup_{c>0}E_c$ is pluripolar in Y.

• **Second step**: *application of Kiselman's minimum principle.*

We refer to Kiselman [Ki1] for a proof (in a somewhat more general situation) of the following basic minimum principle for plurisubharmonic functions.

(6.9) Kiselman's minimum principle. *Let M be a complex manifold, let $\omega\subset\mathbb{R}^n$ be a convex open subset and Ω be the "tube domain" $\Omega=\omega+\mathrm{i}\mathbb{R}^n$. For every plurisubharmonic function $v(\zeta,z)$ on $M\times\Omega$ that does not depend on* $\mathrm{Im}\,z$*, the function*

$$u(\zeta) = \inf_{z\in\Omega} v(\zeta, z)$$

is plurisubharmonic or locally $\equiv -\infty$ *on* M. □

Let $a \ge 0$ be arbitrary. The function

$$Y \times H \ni (y,z) \longmapsto \rho(y) + V(y,z) - a\mathrm{Re}z$$

is plurisubharmonic and independent of $\mathrm{Im}\, z$. By 6.9, the Legendre transform

$$U_a(y) = \inf_{r<-1} \big\{\rho(y) + V(y,r) - ar\big\}$$

is locally plurisubharmonic or $\equiv -\infty$ on Y.

(6.10) Lemma. *Let* $y_0 \in Y$ *be a given point.*

(a) *If* $a > \nu(T, \varphi_{y_0})$, *then* U_a *is bounded below on a neighborhood of* y_0.

(b) *If* $a < \nu(T, \varphi_{y_0})$, *then* $U_a(y_0) = -\infty$.

Proof. By definition of V (cf. (6.7)) we have

$$V(y,r) \le -\nu(T,\varphi_y,r)\int_r^0 \chi'(t)dt = r\nu(T,\varphi_y,r) \le r\nu(T,\varphi_y). \tag{6.11}$$

Then clearly $U_a(y_0) = -\infty$ if $a < \nu(T,\varphi_{y_0})$. On the other hand, if $\nu(T,\varphi_{y_0}) < a$, there exists $t_0 < 0$ such that $\nu(T,\varphi_{y_0},t_0) < a$. Fix $r_0 < t_0$. The semi-continuity property (3.13) shows that there exists a neighborhood ω of y_0 such that $\sup_{y\in\omega} \nu(T,\varphi_y,r_0) < a$. For all $y \in \omega$, we get

$$V(y,r) \ge -C - a\int_r^{r_0} \chi'(t)dt = -C + a(r - r_0),$$

and this implies $U_a(y) \ge -C - ar_0$. □

(6.12) Theorem. *If* Y *is connected and if* $E_c \ne Y$, *then* E_c *is a closed complete pluripolar subset of* Y, *i.e. there exists a continuous plurisubharmonic function* $w : Y \longrightarrow [-\infty, +\infty[$ *such that* $E_c = w^{-1}(-\infty)$.

Proof. We first observe that the family (U_a) is increasing in a, that $U_a = -\infty$ on E_c for all $a < c$ and that $\sup_{a<c} U_a(y) > -\infty$ if $y \in Y \setminus E_c$ (apply Lemma 6.10). For any integer $k \ge 1$, let $w_k \in C^\infty(Y)$ be a plurisubharmonic regularization of $U_{c-1/k}$ such that $w_k \ge U_{c-1/k}$ on Y and $w_k \le -2^k$ on $E_c \cap Y_k$ where $Y_k = \{y \in Y\,;\, d(y,\partial Y) \ge 1/k\}$. Then Lemma 6.10 (a) shows that the family (w_k) is uniformly bounded below on every compact subset of $Y \setminus E_c$. We can also choose w_k uniformly bounded above on every compact subset of Y because $U_{c-1/k} \le U_c$. The function

$$w = \sum_{k=1}^{+\infty} 2^{-k} w_k$$

satifies our requirements. □

• **Third step**: *estimation of the singularities of the potentials* U_a.

(6.13) Lemma. *Let* $y_0 \in Y$ *be a given point,* L *a compact neighborhood of* y_0, $K \subset X$ *a compact subset and* r_0 *a real number* < -1 *such that*

$$\{(x,y) \in X \times L; \varphi(x,y) \le r_0\} \subset K \times L.$$

Assume that $e^{\varphi(x,y)}$ *is locally Hölder continuous in* y *and that*

$$|f(x,y_1) - f(x,y_2)| \le M|y_1 - y_2|^\gamma$$

for all $(x,y_1,y_2) \in K \times L \times L$*. Then, for all* $\varepsilon \in\,]0,1[$*, there exists a real number* $\eta(\varepsilon) > 0$ *such that all* $y \in Y$ *with* $|y - y_0| < \eta(\varepsilon)$ *satisfy*

$$U_a(y) \le \rho(y) + \big((1-\varepsilon)^p \nu(T,\varphi_{y_0}) - a\big)\Big(\gamma \log|y - y_0| + \log\frac{2eM}{\varepsilon}\Big).$$

Proof. First, we try to estimate $\nu(T,\varphi_y,r)$ when $y \in L$ is near y_0. Set

$$\begin{cases} \psi(x) = (1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 & \text{if} \quad \varphi_{y_0}(x) \le r - 1 \\ \psi(x) = \max\big(\varphi_y(x), (1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2\big) & \text{if} \quad r - 1 \le \varphi_{y_0}(x) \le r \\ \psi(x) = \varphi_y(x) & \text{if} \quad r \le \varphi_{y_0}(x) \le r_0 \end{cases}$$

and verify that this definition is coherent when $|y - y_0|$ is small enough. By hypothesis

$$|e^{\varphi_y(x)} - e^{\varphi_{y_0}(x)}| \le M|y - y_0|^\gamma.$$

This inequality implies

$$\begin{aligned} \varphi_y(x) &\le \varphi_{y_0}(x) + \log\big(1 + M|y - y_0|^\gamma e^{-\varphi_{y_0}(x)}\big) \\ \varphi_y(x) &\ge \varphi_{y_0}(x) + \log\big(1 - M|y - y_0|^\gamma e^{-\varphi_{y_0}(x)}\big). \end{aligned}$$

In particular, for $\varphi_{y_0}(x) = r$, we have $(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 = r - \varepsilon/2$, thus

$$\varphi_y(x) \ge r + \log(1 - M|y - y_0|^\gamma e^{-r}).$$

Similarly, for $\varphi_{y_0}(x) = r - 1$, we have $(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 = r - 1 + \varepsilon/2$, thus

$$\varphi_y(x) \le r - 1 + \log(1 + M|y - y_0|^\gamma e^{1-r}).$$

The definition of ψ is thus coherent as soon as $M|y - y_0|^\gamma e^{1-r} \le \varepsilon/2$, i.e.

$$\gamma \log |y - y_0| + \log \frac{2eM}{\varepsilon} \le r.$$

In this case ψ coincides with φ_y on a neighborhood of $\{\psi = r\}$, and with

$$(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2$$

on a neighborhood of the polar set $\psi^{-1}(-\infty)$. By Stokes' formula applied to $\nu(T, \psi, r)$, we infer

$$\nu(T, \varphi_y, r) = \nu(T, \psi, r) \ge \nu(T, \psi) = (1-\varepsilon)^p \nu(T, \varphi_{y_0}).$$

From (6.11) we get $V(y, r) \le r\nu(T, \varphi_y, r)$, hence

$$U_a(y) \le \rho(y) + V(y, r) - ar \le \rho(y) + r\big(\nu(T, \varphi_y, r) - a\big),$$

$$U_a(y) \le \rho(y) + r\big((1-\varepsilon)^p \nu(T, \varphi_{y_0}) - a\big). \tag{6.14}$$

Suppose $\gamma \log |y - y_0| + \log(2eM/\varepsilon) \le r_0$, i.e. $|y - y_0| \le (\varepsilon/2eM)^{1/\gamma} e^{r_0/\gamma}$; one can then choose $r = \gamma \log |y - y_0| + \log(2eM/\varepsilon)$, and by (6.14) this yields the inequality asserted in Lemma 6.13. □

• **Fourth step**: *application of Hörmander's L^2 estimates.*

The end of the proof rests upon the following crucial result, known as the Hörmander-Bombieri-Skoda theorem (cf. [Hö], [Bo] and [Sk1,Sk2]).

(6.15) Theorem. *Let u be a plurisubharmonic function on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every point $z_0 \in \Omega$ such that e^{-u} is integrable in a neighborhood of z_0, there exists a holomorphic function F on Ω such that $F(z_0) = 1$ and*

$$\int_\Omega \frac{|F(z)|^2 e^{-u(z)}}{(1 + |z|^2)^{n+\varepsilon}} d\lambda(z) < +\infty.$$

(6.16) Corollary. *Let u be a plurisubharmonic function on a complex manifold Y. The set of points in a neighborhood of which e^{-u} is not integrable is an analytic subset of Y.*

Proof. The result is local, so we may assume that Y is a ball in $\mathbb{C}^n$. Then the set of non integrability points of e^{-u} is the intersection of all hypersurfaces $F^{-1}(0)$ defined by the holomorphic functions F such that $\int_Y |F|^2 e^{-u} d\lambda < +\infty$. Indeed F must vanish at any non integrability point, and on the other hand Th. 6.15 shows that one can choose $F(z_0) = 1$ at any integrability point z_0. □

End of proof of Th. 6.4. The main idea in what follows is due to Kiselman [Ki2]. For $a, b > 0$, we let $Z_{a,b}$ be the set of points in a neighborhood of which $\exp(-U_a/b)$ is not integrable. Then $Z_{a,b}$ is analytic, and as the family (U_a) is increasing in a, we have $Z_{a',b'} \supset Z_{a'',b''}$ if $a' \le a''$, $b' \le b''$.

Let $y_0 \in Y$ be a given point. If $y_0 \notin E_c$, then $\nu(T, \varphi_{y_0}) < c$ by definition of E_c. Choose a such that $\nu(T, \varphi_{y_0}) < a < c$. Lemma 6.10 (a) implies that U_a is bounded below in a neighborhood of y_0, thus $\exp(-U_a/b)$ is integrable and $y_0 \notin Z_{a,b}$ for all $b > 0$.

On the other hand, if $y_0 \in E_c$ and if $a < c$, then Lemma 6.13 implies for all $\varepsilon > 0$ that

$$U_a(y) \le (1-\varepsilon)(c-a)\gamma \log|y - y_0| + C(\varepsilon)$$

on a neighborhood of y_0. Hence $\exp(-U_a/b)$ is non integrable at y_0 as soon as $b < (c-a)\gamma/2m$, where $m = \dim Y$. We obtain therefore

$$E_c = \bigcap_{\substack{a<c \\ b<(c-a)\gamma/2m}} Z_{a,b}.$$

This proves that E_c is an analytic subset of Y. □

Finally, we use corollary 6.5 to derive an important decomposition formula for currents, which is again due to [Siu]. We first begin by two simple observations.

(6.17) Lemma. *If T is a closed positive current of bidimension (p,p) and A is an irreducible analytic set in X, we set*

$$m_A = \inf\{x \in A\,;\, \nu(T,x)\}.$$

Then $\nu(T,x) = m_A$ for all $x \in A \setminus \bigcup A'_j$, where (A'_j) is a countable family of proper analytic subsets of A. We say that m_A is the generic Lelong number of T along A.

Proof. By definition of m_A and $E_c(T)$, we have $\nu(T,x) \ge m_A$ for every $x \in A$ and

$$\nu(T,x) = m_A \quad \text{on} \;\; A \setminus \bigcup_{c\in\mathbb{Q},\, c>m_A} A \cap E_c(T).$$

However, for $c > m_A$, the intersection $A \cap E_c(T)$ is a proper analytic subset of A. □

(6.18) Proposition. *Let T be a closed positive current of bidimension (p,p) and let A be an irreducible p-dimensional analytic subset of X. Then $\mathbb{1}_A T = m_A[A]$, in particular $T - m_A[A]$ is positive.*

Proof. As the question is local and as a closed positive current of bidimension (p, p) cannot carry any mass on a $(p-1)$-dimensional analytic subset, it is enough to work in a neighborhood of a regular point $x_0 \in A$. Hence, by choosing suitable coordinates, we can suppose that X is an open set in $\mathbb{C}^n$ and that A is the intersection of X with a p-dimensional linear subspace. Then, for every point $a \in A$, the inequality $\nu(T, a) \geq m_A$ implies

$$\sigma_T\big(B(a,r)\big) \geq m_A\,\pi^p r^{2p}/p! = m_A \sigma_{[A]}\big(B(a,r)\big)$$

for all r such that $B(a,r) \subset X$. Now, set $\Theta = T - m_A[A]$ and $\beta = dd^c|z|^2$. Our inequality says that $\int \mathbb{1}_{B(a,r)}\Theta \wedge \beta^p \geq 0$. If we integrate this with respect to some positive continuous function f with compact support in A, we get $\int_X g_r \Theta \wedge \beta^p \geq 0$ where

$$g_r(z) = \int_A \mathbb{1}_{B(a,r)}(z)\, f(a)\, d\lambda(a) = \int_{a \in A \cap B(z,r)} f(a)\, d\lambda(a).$$

Here g_r is continuous on $\mathbb{C}^n$, and as r tends to 0 the function $g_r(z)/(\pi^p r^{2p}/p!)$ converges to f on A and to 0 on $X \smallsetminus A$, with a global uniform bound. Hence we obtain $\int \mathbb{1}_A f\, \Theta \wedge \beta^p \geq 0$. Since this inequality is true for all continuous functions $f \geq 0$ with compact support in A, we conclude that the measure $\mathbb{1}_A \Theta \wedge \beta^p$ is positive. By a linear change of coordinates, we see that

$$\mathbb{1}_A \Theta \wedge \Big(dd^c \sum_{1 \leq j \leq n} \lambda_j |u_j|^2\Big)^n \geq 0$$

for every basis $(u_1, \ldots, u_n)$ of linear forms and for all coefficients $\lambda_j > 0$. Take $\lambda_1 = \ldots = \lambda_p = 1$ and let the other λ_j tend to 0. Then we get $\mathbb{1}_A \Theta \wedge \mathrm{i} du_1 \wedge d\overline{u}_1 \wedge \ldots \wedge du_p \wedge d\overline{u}_p \geq 0$. This implies $\mathbb{1}_A \Theta \geq 0$, or equivalently $\mathbb{1}_A T \geq m_A[A]$. By a result of Skoda [Sk3], we know that $\mathbb{1}_A T$ is a closed positive current, thus $\mathbb{1}_A T = \lambda[A]$ with $\lambda \geq 0$. We have just seen that $\lambda \geq m_A$. On the other hand, $T \geq \mathbb{1}_A T = \lambda[A]$ clearly implies $m_A \geq \lambda$. □

(6.19) Siu's decomposition formula. *If T is a closed positive current of bidimension (p, p), there is a unique decomposition of T as a (possibly finite) weakly convergent series*

$$T = \sum_{j \geq 1} \lambda_j [A_j] + R, \qquad \lambda_j > 0,$$

where $[A_j]$ is the current of integration over an irreducible p-dimensional analytic set $A_j \subset X$ and where R is a closed positive current with the property that $\dim E_c(R) < p$ for every $c > 0$.

Uniqueness. If T has such a decomposition, the p-dimensional components of $E_c(T)$ are $(A_j)_{\lambda_j \geq c}$, for $\nu(T, x) = \sum \lambda_j \nu([A_j], x) + \nu(R, x)$ is non zero only on $\bigcup A_j \cup \bigcup E_c(R)$, and is equal to λ_j generically on A_j $\big($more precisely,

$\nu(T,x) = \lambda_j$ at every regular point of A_j which does not belong to any intersection $A_j \cup A_k$, $k \neq j$ or to $\bigcup E_c(R)\big)$. In particular A_j and λ_j are unique.

Existence. Let $(A_j)_{j\geq 1}$ be the countable collection of p-dimensional components occurring in one of the sets $E_c(T)$, $c \in \mathbb{Q}_+^\star$, and let $\lambda_j > 0$ be the generic Lelong number of T along A_j. Then Prop. 6.18 shows by induction on N that $R_N = T - \sum_{1\leq j\leq N} \lambda_j[A_j]$ is positive. As R_N is a decreasing sequence, there must be a limit $R = \lim_{N\to+\infty} R_N$ in the weak topology. Thus we have the asserted decomposition. By construction, R has zero generic Lelong number along A_j, so $\dim E_c(R) < p$ for every $c > 0$. □

It is very important to note that some components of lower dimension can actually occur in $E_c(R)$, but they cannot be subtracted because R has bidimension (p,p). A typical case is the case of a bidimension $(n-1,n-1)$ current $T = dd^c u$ with $u = \log(|F_j|^{\gamma_1} + \ldots |F_N|^{\gamma_N})$ and $F_j \in \mathcal{O}(X)$. In general $\bigcup E_c(T) = \bigcap F_j^{-1}(0)$ has dimension $< n-1$. In that case, an important formula due to King plays the role of (6.19). We state it in a somewhat more general form than its original version [Kg].

(6.20) King's formula. *Let $F_1,\ldots,F_N$ be holomorphic functions on a complex manifold X, such that the zero variety $Z = \bigcap F_j^{-1}(0)$ has codimension $\geq p$, and set $u = \log \sum |F_j|^{\gamma_j}$ with arbitrary coefficients $\gamma_j > 0$. Let $(Z_k)_{k\geq 1}$ be the irreducible components of Z of codimension p exactly. Then there exist multiplicities $\lambda_k > 0$ such that*

$$(dd^c u)^p = \sum_{k\geq 1} \lambda_k [Z_k] + R,$$

where R is a closed positive current such that $\mathbb{1}_Z R = 0$ and $\operatorname{codim} E_c(R) > p$ for every $c > 0$. Moreover the multiplicities λ_k are integers if $\gamma_1,\ldots,\gamma_N$ are integers, and $\lambda_k = \gamma_1 \ldots \gamma_p$ if $\gamma_1 \leq \ldots \leq \gamma_N$ and some partial Jacobian determinant of $(F_1,\ldots,F_p)$ of order p does not vanish identically along Z_k.

Proof. Observe that $(dd^c u)^p$ is well defined thanks to Cor. 2.11. The comparison theorem 5.9 applied with $\varphi(z) = \log|z-x|$, $v_1 = \ldots = v_p = u$, $u_1 = \ldots = u_p = \varphi$ and $T = 1$ shows that the Lelong number of $(dd^c u)^p$ is equal to 0 at every point of $X \setminus Z$. Hence $E_c((dd^c u)^p)$ is contained in Z and its $(n-p)$-dimensional components are members of the family (Z_k). The asserted decomposition follows from Siu's formula 6.19. We must have $\mathbb{1}_{Z_k} R = 0$ for all irreducible components of Z: when $\operatorname{codim} Z_k > p$ this is automatically true, and when $\operatorname{codim} Z_k = p$ this follows from (6.18) and the fact that $\operatorname{codim} E_c(R) > p$. If $\det(\partial F_j/\partial z_k)_{1\leq j,k\leq p} \neq 0$ at some point $x_0 \in Z_k$, then $(Z,x_0) = (Z_k,x_0)$ is a smooth germ defined by the equations

$F_1 = \ldots = F_p = 0$. If we denote $v = \log \sum_{j \le p} |F_j|^{\gamma_j}$ with $\gamma_1 \le \ldots \le \gamma_N$, then $u \sim v$ near Z_k and Th. 5.9 implies $\nu((dd^c u)^p, x) = \nu((dd^c v)^p, x)$ for all $x \in Z_k$ near x_0. On the other hand, if $G := (F_1, \ldots, F_p) : X \to \mathbb{C}^p$, Cor. 5.4 gives

$$(dd^c v)^p = G^\star \Big(dd^c \log \sum_{1 \le j \le p} |z_j|^{\gamma_j} \Big)^p = \gamma_1 \ldots \gamma_p \, G^\star \delta_0 = \gamma_1 \ldots \gamma_p \, [Z_k]$$

near x_0. This implies that the generic Lelong number of $(dd^c u)^p$ along Z_k is $\lambda_k = \gamma_1 \ldots \gamma_p$. The integrality of λ_k when $\gamma_1, \ldots, \gamma_N$ are integers will be proved in the next section. □

7. Transformation of Lelong Numbers by Direct Images

Let $F : X \to Y$ be a holomorphic map between complex manifolds of respective dimensions $\dim X = n$, $\dim Y = m$, and let T be a closed positive current of bidimension (p, p) on X. If $F_{\restriction \operatorname{Supp} T}$ is proper, the direct image $F_\star T$ is defined by

$$\langle F_\star T, \alpha \rangle = \langle T, F^\star \alpha \rangle \tag{7.1}$$

for every test form α of bidegree (p, p) on Y. This makes sense because $\operatorname{Supp} T \cap F^{-1}(\operatorname{Supp} \alpha)$ is compact. It is easily seen that $F_\star T$ is a closed positive current of bidimension (p, p) on Y.

(7.2) Example. Let $T = [A]$ where A is a p-dimensional irreducible analytic set in X such that $F_{\restriction A}$ is proper. We know by Remmert's theorem [Re1,2] that $F(A)$ is an analytic set in Y. Two cases may occur. Either $F_{\restriction A}$ is generically finite and F induces an étale covering $A \smallsetminus F^{-1}(Z) \longrightarrow F(A) \smallsetminus Z$ for some nowhere dense analytic subset $Z \subset F(A)$, or $F_{\restriction A}$ has generic fibers of positive dimension and $\dim F(A) < \dim A$. In the first case, let $s < +\infty$ be the covering degree. Then for every test form α of bidegree (p, p) on Y we get

$$\langle F_\star [A], \alpha \rangle = \int_A F^\star \alpha = \int_{A \smallsetminus F^{-1}(Z)} F^\star \alpha = s \int_{F(A) \smallsetminus Z} \alpha = s \, \langle [F(A)], \alpha \rangle$$

because Z and $F^{-1}(Z)$ are negligible sets. Hence $F_\star [A] = s [F(A)]$. On the other hand, if $\dim F(A) < \dim A = p$, the restriction of α to $F(A)_{\mathrm{reg}}$ is zero, and therefore so is this the restriction of $F^\star \alpha$ to A_{reg}. Hence $F_\star [A] = 0$. □

Now, let ψ be a continuous plurisubharmonic function on Y which is semi-exhaustive on $F(\operatorname{Supp} T)$ (this set certainly contains $\operatorname{Supp} F_\star T$). Since $F_{\restriction \operatorname{Supp} T}$ is proper, it follows that $\psi \circ F$ is semi-exhaustive on $\operatorname{Supp} T$, for

$$\operatorname{Supp} T \cap \{\psi \circ F < R\} = F^{-1}\big(F(\operatorname{Supp} T) \cap \{\psi < R\}\big).$$

(7.3) Proposition. *If $F(\operatorname{Supp} T) \cap \{\psi < R\} \subset\subset Y$, we have*

$$\nu(F_\star T, \psi, r) = \nu(T, \psi \circ F, r) \qquad \text{for all } r < R,$$

in particular $\nu(F_\star T, \psi) = \nu(T, \psi \circ F)$.

Here, we do not necessarily assume that X or Y are Stein; we thus replace ψ with $\psi_{\geq s} = \max\{\psi, s\}$, $s < r$, in the definition of $\nu(F_\star T, \psi, r)$ and $\nu(T, \psi \circ F, r)$.

Proof. The first equality can be written

$$\int_Y F_\star T \wedge \mathbb{1}_{\{\psi<r\}}(dd^c\psi_{\geq s})^p = \int_X T \wedge (\mathbb{1}_{\{\psi<r\}} \circ F)(dd^c\psi_{\geq s} \circ F)^p.$$

This follows almost immediately from the adjunction formula (7.1) when ψ is smooth and when we write $\mathbb{1}_{\{\psi<R\}} = \lim \uparrow g_k$ for some sequence of smooth functions g_k. In general, we write $\psi_{\geq s}$ as a decreasing limit of smooth plurisubharmonic functions and we apply our monotone continuity theorems (if Y is not Stein, Richberg's theorem shows that we can obtain a decreasing sequence of almost plurisubharmonic approximations such that the negative part of dd^c converges uniformly to 0; this is good enough to apply the monotone continuity theorem; note that the integration is made on compact subsets, thanks to the semi-exhaustivity assumption on ψ). □

It follows from this that understanding the transformation of Lelong numbers under direct images is equivalent to understanding the effect of F on the weight. We are mostly interested in computing the ordinary Lelong numbers $\nu(F_\star T, y)$ associated with the weight $\psi(w) = \log|w - y|$ in some local coordinates $(w_1, \ldots, w_m)$ on Y near y. Then Prop. 7.3 gives

$$\nu(F_\star T, y) = \nu(T, \log|F - y|) \qquad \text{with} \tag{7.4}$$

$$\log|F(z) - y| = \frac{1}{2}\log\sum|F_j(z) - y_j|^2, \qquad F_j = w_j \circ F.$$

We are going to show that $\nu(T, \log|F - y|)$ is bounded below by a linear combination of the Lelong numbers of T at points x in the fiber $F^{-1}(y)$, with suitable multiplicities attached to F at these points. These multiplicities can be seen as generalizations of the notion of multiplicity of an analytic map introduced by W. Stoll [St].

(7.5) Definition. *Let $x \in X$ and $y = F(x)$. Suppose that the codimension of the fiber $F^{-1}(y)$ at x is $\geq p$. Then we set $\mu_p(F, x) = \nu\big((dd^c\log|F - y|)^p, x\big)$.*

Observe that $(dd^c \log |F-y|)^p$ is well defined thanks to Cor. 2.10. The second comparison theorem 5.9 immediately shows that $\mu_p(F,x)$ is independent of the choice of local coordinates on Y (and also on X, since Lelong nombers do not depend on coordinates). By definition, $\mu_p(F,x)$ is the mass carried by $\{x\}$ of the measure

$$(dd^c \log |F(z)-y|)^p \wedge (dd^c \log |z-x|)^{n-p}.$$

We are going to give of more geometric interpretation of this multiplicity, from which it will follow that $\mu_p(F,x)$ is always a positive integer (in particular, the proof of (6.20) will be complete).

(7.6) Example. For $p = n = \dim X$, the assumption $\operatorname{codim}_x F^{-1}(y) \geq p$ means that the germ of map $F : (X,x) \longrightarrow (Y,y)$ is finite. Let U_x be a neighborhood of x such that $\overline{U}_x \cap F^{-1}(y) = \{x\}$, let W_y be a neighborhood of y disjoint from $F(\partial U_x)$ and let $V_x = U_x \cap F^{-1}(W_y)$. Then $F : V_x \to W_y$ is proper and finite, and we have $F_\star[V_x] = s\,[F(V_x)]$ where s is the local covering degree of $F : V_x \to F(V_x)$ at x. Therefore

$$\begin{aligned}\mu_n(F,x) &= \int_{\{x\}} \big(dd^c \log |F-y|\big)^n = \nu\big([V_x], \log |F-y|\big) = \nu\big(F_\star[V_x], y\big) \\ &= s\,\nu\big(F(V_x), y\big).\end{aligned}$$

In the particular case when $\dim Y = \dim X$, we have $(F(V_x),y) = (Y,y)$, so $\mu_n(F,x) = s$. In general, it is a well known fact that the ideal generated by $(F_1 - y_1, \ldots, F_m - y_m)$ in $\mathcal{O}_{X,x}$ has the same integral closure as the ideal generated by n generic linear combinations of the generators, that is, for a generic choice of coordinates $w' = (w_1, \ldots, w_n)$, $w'' = (w_{n+1}, \ldots, w_m)$ on (Y,y), we have $|F(z)-y| \leq C|w' \circ F(z)|$ (this is a simple consequence of Lemma 5.7 applied to $A = F(V_x)$). Hence for $p = n$, the comparison theorem 5.1 gives

$$\mu_n(F,x) = \mu_n(w' \circ F, x) = \text{local covering degree of } w' \circ F \ \text{ at } x,$$

for a generic choice of coordinates (w', w'') on (Y,y). □

(7.7) Geometric interpretation of $\mu_p(F,x)$. A formal application of Crofton's formula 5.12 shows, after a translation, that there is a small ball $B(x,r_0)$ on which

$$\begin{aligned}&(dd^c \log |F(z)-y|)^p \wedge (dd^c \log |z-x|)^{n-p} = \\ &\qquad \int_{S \in G(p,n)} (dd^c \log |F(z)-y|)^p \wedge [x+S]\, dv(S).\end{aligned} \tag{7.7 a}$$

For a rigorous proof of (7.7 a), we replace $\log |F(z)-y|$ by the smooth function $\frac{1}{2}\log(|F(z)-y|^2 + \varepsilon^2)$ and let ε tend to 0 on both sides. By (2.3) (resp. by (2.10)), the wedge product $(dd^c \log|F(z)-y|)^p \wedge [x+S]$ is well defined on a small ball $B(x,r_0)$ as soon as $x+S$ does not intersect $F^{-1}(y)\cap \partial B(x,r_0)$ (resp. intersects $F^{-1}(y)\cap B(x,r_0)$ at finitely many points); thanks to the assumption $\mathrm{codim}(F^{-1}(y),x)\ge p$, Sard's theorem shows that this is the case for all S outside a negligible closed subset E in $G(p,n)$ (resp. by Bertini, an analytic subset A in $G(p,n)$ with $A\subset E$). Fatou's lemma then implies that the inequality $\ge$ holds in (7.7 a). To get equality, we observe that we have bounded convergence on all complements $G(p,n)\smallsetminus V(E)$ of neighborhoods $V(E)$ of E. However the mass of $\int_{V(E)}[x+S]\,dv(S)$ in $B(x,r_0)$ is proportional to $v(V(E))$ and therefore tends to 0 when $V(E)$ is small; this is sufficient to complete the proof, since Prop. 2.6 (b) gives

$$\int_{z\in\overline{B}(x,r_0)} \big(dd^c\log(|F(z)-y|^2+\varepsilon^2)\big)^p \wedge \int_{S\in V(E)} [x+S]\,dv(S) \le C\,v(V(E))$$

with a constant C independent of ε. By evaluating (7.7 a) on $\{x\}$, we get

$$\text{(7.7 b)}\qquad \mu_p(F,x)=\int_{S\in G(p,n)\smallsetminus A} \nu\big((dd^c\log|F_{\restriction x+S}-z|)^p,x\big)\,dv(S).$$

Let us choose a linear parametrization $g_S:\mathbb{C}^p\to S$ depending analytically on local coordinates of S in $G(p,n)$. Then Theorem 6.4 with $T=[\mathbb{C}^p]$ and $\varphi(z,S)=\log|F\circ g_S(z)-y|$ shows that

$$\nu\big((dd^c\log|F_{\restriction x+S}-z|)^p,x\big)=\nu\big([\mathbb{C}^p],\log|F\circ g_S(z)-y|\big)$$

is Zariski upper semicontinuous in S on $G(p,n)\smallsetminus A$. However, (7.6) shows that these numbers are integers, so $S\mapsto \nu\big((dd^c\log|F_{\restriction x+S}-z|)^p,x\big)$ must be constant on a Zariski open subset in $G(p,n)$. By (7.7 b), we obtain

$$\text{(7.7 c)}\qquad \mu_p(F,x)=\mu_p(F_{\restriction x+S},x)=\text{local degree of } w'\circ F_{\restriction x+S}\ \text{ at } x$$

for generic subspaces $S\in G(p,n)$ and generic coordinates $w'=(w_1,\ldots,w_p)$, $w''=(w_{p+1},\ldots,w_m)$ on (Y,y). □

(7.8) Example. Let $F:\mathbb{C}^n\longrightarrow\mathbb{C}^n$ be defined by

$$F(z_1,\ldots,z_n)=(z_1^{s_1},\ldots,z_n^{s_n}),\qquad s_1\le\ldots\le s_n.$$

We claim that $\mu_p(F,0)=s_1\ldots s_p$. In fact, for a generic p-dimensional subspace $S\subset\mathbb{C}^n$ such that $z_1,\ldots,z_p$ are coordinates on S and $z_{p+1},\ldots,z_n$ are linear forms in $z_1,\ldots,z_p$, and for generic coordinates $w'=(w_1,\ldots,w_p)$, $w''=(w_{p+1},\ldots,w_n)$ on $\mathbb{C}^n$, we can rearrange w' by linear combinations so that $w_j\circ F_{\restriction S}$ is a linear combination of $(z_j^{s_j},\ldots,z_n^{s_n})$ and has non zero coefficient in $z_j^{s_j}$ as a polynomial in $(z_j,\ldots,z_p)$. It is then an exercise to show that $w'\circ F_{\restriction S}$ has covering degree $s_1\ldots s_p$ at 0 [compute inductively the roots $z_n, z_{n-1},\ldots,z_j$ of $w_j\circ F_{\restriction S}(z)=a_j$ and use (5.6) to show that the s_j values of z_j lie near 0 when $(a_1,\ldots,a_p)$ are small]. □

We are now ready to prove the main result of this section, which describes the behaviour of Lelong numbers under proper morphisms. A similar weaker result was already proved in [De2] ($\mu_p(F,x)$ was not optimal).

(7.9) Theorem. *Let T be a closed positive current of bidimension (p,p) on X and let $F : X \longrightarrow Y$ be an analytic map such that the restriction $F_{\restriction \mathrm{Supp}\, T}$ is proper. Let $I(y)$ be the set of points $x \in \mathrm{Supp}\, T \cap F^{-1}(y)$ such that x is equal to its connected component in* $\mathrm{Supp}\, T \cap F^{-1}(y)$ *and* $\mathrm{codim}(F^{-1}(y),x) \geq p$. *Then we have*

$$\nu(F_\star T, y) \geq \sum_{x\in I(y)} \mu_p(F,x)\,\nu(T,x).$$

In particular, we have $\nu(F_\star T,y) \geq \sum_{x\in I(y)} \nu(T,x)$. This inequality no longer holds if the summation is extended to all points $x \in \mathrm{Supp}\, T \cap F^{-1}(Y)$ and if this set contains positive dimensional connected components: for example, if $F : X \longrightarrow Y$ is the blow-up of Y at a point and E is the exceptional divisor, then $T = [E]$ has direct image $F_\star[E] = 0$ thanks to (7.2).

Proof. We proceed in three steps.

Step 1. Reduction to the case of a single point x in the fiber. It is sufficient to prove the inequality when the summation is taken over an arbitrary finite subset $\{x_1,\ldots,x_N\}$ of $I(y)$. As x_j is equal to its connected component in $\mathrm{Supp}\, T \cap F^{-1}(y)$, it has a fondamental system of relative open-closed neighborhoods, hence there are disjoint neighborhoods U_j of x_j such that ∂U_j does not intersect $\mathrm{Supp}\, T \cap F^{-1}(y)$. Then the image $F(\partial U_j \cap \mathrm{Supp}\, T)$ is a closed set which does not contain y. Let W be a neighborhood of y disjoint from all sets $F(\partial U_j \cap \mathrm{Supp}\, T)$, and let $V_j = U_j \cap F^{-1}(W)$. It is clear that V_j is a neighborhood of x_j and that $F_{\restriction V_j} : V_j \to W$ has a proper restriction to $\mathrm{Supp}\, T \cap V_j$. Moreover, we obviously have $F_\star T \geq \sum_j (F_{\restriction V_j})_\star T$ on W. Therefore, it is enough to check the inequality $\nu(F_\star T,y) \geq \mu_p(F,x)\,\nu(T,x)$ for a single point $x \in I(y)$, in the case when $X \subset \mathbb{C}^n$, $Y \subset \mathbb{C}^m$ are open subsets and $x = y = 0$.

Step 2. Reduction to the case when F is finite. By (7.4), we have

$$\begin{aligned}\nu(F_\star T,0) &= \inf_{V\ni 0} \int_V T \wedge (dd^c \log|F|)^p\\ &= \inf_{V\ni 0}\lim_{\varepsilon\to 0} \int_V T \wedge \big(dd^c \log(|F| + \varepsilon|z|^N)\big)^p,\end{aligned}$$

and the integrals are well defined as soon as ∂V does not intersect the set $\mathrm{Supp}\, T \cap F^{-1}(0)$ (may be after replacing $\log|F|$ by $\max\{\log|F|, s\}$ with

$s \ll 0$). For every V and ε, the last integral is larger than $\nu(G_\star T, 0)$ where G is the finite morphism defined by

$$G : X \longrightarrow Y \times \mathbb{C}^n, \qquad (z_1, \ldots, z_n) \longmapsto (F_1(z), \ldots, F_m(z), z_1^N, \ldots, z_n^N).$$

We claim that for N large enough we have $\mu_p(F, 0) = \mu_p(G, 0)$. In fact, $x \in I(y)$ implies by definition $\operatorname{codim}(F^{-1}(0), 0) \geq p$. Hence, if $S = \{u_1 = \ldots = u_{n-p} = 0\}$ is a generic p-dimensional subspace of $\mathbb{C}^n$, the germ of variety $F^{-1}(0) \cap S$ defined by $(F_1, \ldots, F_m, u_1, \ldots, u_{n-p})$ is $\{0\}$. Hilbert's Nullstellensatz implies that some powers of $z_1, \ldots, z_n$ are in the ideal (F_j, u_k). Therefore $|F(z)| + |u(z)| \geq C|z|^a$ near 0 for some integer a independent of S (to see this, take coefficients of the u_k's as additional variables); in particular $|F(z)| \geq C|z|^a$ for $z \in S$ near 0. The comparison theorem 5.1 then shows that $\mu_p(F, 0) = \mu_p(G, 0)$ for $N \geq a$. If we are able to prove that $\nu(G_\star T, 0) \geq \mu_p(G, 0)\nu(T, 0)$ in case G is finite, the obvious inequality $\nu(F_\star T, 0) \geq \nu(G_\star T, 0)$ concludes the proof.

Step 3. Proof of the inequality $\nu(F_\star T, y) \geq \mu_p(F, x)\, \nu(T, x)$ when F is finite and $F^{-1}(y) = x$. Then $\varphi(z) = \log|F(z) - y|$ has a single isolated pole at x and we have $\mu_p(F, x) = \nu((dd^c\varphi)^p, x)$. It is therefore sufficient to apply the following Proposition. □

(7.10) Proposition. *Let φ be a semi-exhaustive continuous plurisubharmonic function on X with a single isolated pole at x. Then*

$$\nu(T, \varphi) \geq \nu(T, x)\, \nu((dd^c\varphi)^p, x).$$

Proof. Since the question is local, we can suppose that X is the ball $B(0, r_0)$ in $\mathbb{C}^n$ and $x = 0$. Set $X' = B(0, r_1)$ with $r_1 < r_0$ and $\Phi(z, g) = \varphi \circ g(z)$ for $g \in \mathrm{Gl}_n(\mathbb{C})$. Then there is a small neighborhood Ω of the unitary group $U(n) \subset \mathrm{Gl}_n(\mathbb{C})$ such that Φ is plurisubharmonic on $X' \times \Omega$ and semi-exhaustive with respect to X'. Theorem 6.4 implies that the map $g \mapsto \nu(T, \varphi \circ g)$ is Zariski upper semi-continuous on Ω. In particular, we must have $\nu(T, \varphi \circ g) \leq \nu(T, \varphi)$ for all $g \in \Omega \setminus A$ in the complement of a complex analytic set A. Since $\mathrm{Gl}_n(\mathbb{C})$ is the complexification of $U(n)$, the intersection $U(n) \cap A$ must be a nowhere dense real analytic subset of $U(n)$. Therefore, if dv is the Haar measure of mass 1 on $U(n)$, we have

$$\begin{aligned} \nu(T, \varphi) &\geq \int_{g \in U(n)} \nu(T, \varphi \circ g)\, dv(g) \\ &= \lim_{r \to 0} \int_{g \in U(n)} dv(g) \int_{B(0,r)} T \wedge (dd^c\varphi \circ g)^p. \end{aligned} \tag{7.11}$$

Since $\int_{g \in U(n)} (dd^c\varphi \circ g)^p dv(g)$ is a unitary invariant (p, p)-form on B, Lemma 5.11 implies

$$\int_{g\in U(n)} (dd^c\varphi\circ g)^p dv(g) = \big(dd^c\chi(\log|z|)\big)^p$$

where χ is a convex increasing function. The Lelong number at 0 of the left hand side is equal to $\nu((dd^c\varphi)^p,0)$, and must be equal to the Lelong number of the right hand side, which is $\lim_{t\to-\infty}\chi'(t)^p$ (to see this, use either Formula (3.5) or Th. 5.9). Thanks to the last equality, Formulas (7.11) and (3.5) imply

$$\begin{aligned}\nu(T,\varphi) &\geq \lim_{r\to 0}\int_{B(0,r)} T\wedge\big(dd^c\chi(\log|z|)\big)^p\\ &= \lim_{r\to 0}\chi'(\log r-0)^p\nu(T,0,r) \geq \nu((dd^c\varphi)^p,0)\,\nu(T,0). \qquad \square\end{aligned}$$

Another interesting question is to know whether it is possible to get inequalities in the opposite direction, i.e. to find upper bounds for $\nu(F_\star T,y)$ in terms of the Lelong numbers $\nu(T,x)$. The example $T=[\Gamma]$ with the curve $\Gamma: t\mapsto(t^a,t^{a+1},t)$ in $\mathbb{C}^3$ and $F:\mathbb{C}^3\to\mathbb{C}^2$, $(z_1,z_2,z_3)\mapsto(z_1,z_2)$, for which $\nu(T,0)=1$ and $\nu(F_\star T,0)=a$, shows that this may be possible only when F is finite. In this case, we have:

(7.12) Theorem. *Let $F:X\to Y$ be a proper and finite analytic map and let T be a closed positive current of bidimension (p,p) on X. Then*

(a) $$\nu(F_\star T,y)\leq \sum_{x\in\operatorname{Supp}T\cap F^{-1}(y)} \overline{\mu}_p(F,x)\,\nu(T,x)$$

where $\overline{\mu}_p(F,x)$ is the multiplicity defined as follows: if $H:(X,x)\to(\mathbb{C}^n,0)$ is a germ of finite map, we set

(b) $$\sigma(H,x)=\inf\big\{\alpha>0\,;\,\exists C>0,\ |H(z)|\geq C|z-x|^\alpha \text{ near } x\big\},$$

(c) $$\overline{\mu}_p(F,x)=\inf_G\frac{\sigma(G\circ F,x)^p}{\mu_p(G,0)},$$

where G runs over all germs of maps $(Y,y)\longrightarrow(\mathbb{C}^n,0)$ such that $G\circ F$ is finite.

Proof. If $F^{-1}(y)=\{x_1,\ldots,x_N\}$, there is a neighborhood W of y and disjoint neighborhoods V_j of x_j such that $F^{-1}(W)=\bigcup V_j$. Then $F_\star T=\sum(F_{\restriction V_j})_\star T$ on W, so it is enough to consider the case when $F^{-1}(y)$ consists of a single point x. Therefore, we assume that $F:V\to W$ is proper and finite, where V, W are neighborhoods of 0 in $\mathbb{C}^n$, $\mathbb{C}^m$ and $F^{-1}(0)=\{0\}$. Let $G:(\mathbb{C}^m,0)\longrightarrow(\mathbb{C}^n,0)$ be a germ of map such that $G\circ F$ is finite. Hilbert's Nullstellensatz shows that there exists $\alpha>0$ and $C>0$ such that $|G\circ F(z)|\geq C|z|^\alpha$ near 0. Then the comparison theorem 5.1 implies

$$\nu(G_\star F_\star T,0) = \nu(T, \log|G\circ F|) \le \alpha^p \nu(T,\log|z|) = \alpha^p\nu(T,0).$$

On the other hand, Th. 7.9 applied to $\Theta = F_\star T$ on W gives

$$\nu(G_\star F_\star T,0) \ge \mu_p(G,0)\,\nu(F_\star T,0).$$

Therefore

$$\nu(F_\star T,0) \le \frac{\alpha^p}{\mu_p(G,0)}\nu(T,0).$$

The infimum of all possible values of α is by definition $\sigma(G\circ F,0)$, thus by taking the infimum over G we obtain

$$\nu(F_\star T,0) \le \overline{\mu}_p(F,0)\,\nu(T,0). \qquad \square$$

(7.13) Example. Let $F(z_1,\ldots,z_n) = (z_1^{s_1},\ldots,z_n^{s_n})$, $s_1 \le \ldots \le s_n$ as in 7.8. Then we have

$$\mu_p(F,0) = s_1\ldots s_p, \qquad \overline{\mu}_p(F,0) = s_{n-p+1}\ldots s_n.$$

To see this, let s be the lowest common multiple of $s_1,\ldots,s_n$ and let $G(z_1,\ldots,z_n) = (z_1^{s/s_1},\ldots,z_n^{s/s_n})$. Clearly $\mu_p(G,0) = (s/s_{n-p+1})\ldots(s/s_n)$ and $\sigma(G\circ F,0) = s$, so we get by definition $\overline{\mu}_p(F,0) \le s_{n-p+1}\ldots s_n$. Finally, if $T = [A]$ is the current of integration over the p-dimensional subspace $A = \{z_1 = \ldots = z_{n-p} = 0\}$, then $F_\star[A] = s_{n-p+1}\ldots s_n\,[A]$ because $F_{\restriction A}$ has covering degree $s_{n-p+1}\ldots s_n$. Theorem 7.12 shows that we must have $s_{n-p+1}\ldots s_n \le \overline{\mu}_p(F,0)$, QED. If $\lambda_1 \le \ldots \le \lambda_n$ are positive real numbers and s_j is taken to be the integer part of $k\lambda_j$ as k tends to $+\infty$, Theorems 7.9 and 7.12 imply in the limit the following:

(7.14) Corollary. *For $0 < \lambda_1 \le \ldots \le \lambda_n$, Kiselman's directional Lelong numbers satisfy the inequalities*

$$\lambda_1\ldots\lambda_p\,\nu(T,x) \le \nu(T,x,\lambda) \le \lambda_{n-p+1}\ldots\lambda_n\,\nu(T,x). \qquad \square$$

(7.15) *Remark.* It would be interesting to have a direct geometric interpretation of $\overline{\mu}_p(F,x)$. In fact, we do not even know whether $\overline{\mu}_p(F,x)$ is always an integer.

8. A Schwarz Lemma. Application to Number Theory

In this section, we show how Jensen's formula and Lelong numbers can be used to prove a fairly general Schwarz lemma relating growth and zeros of entire functions in $\mathbb{C}^n$. In order to simplify notations, we denote by $|F|_r$ the supremum of the modulus of a function F on the ball of center 0 and radius r. Then, following [De1], we present some applications with a more arithmetical flavour.

(8.1) Schwarz lemma. *Let $P_1, \ldots, P_N \in \mathbb{C}[z_1, \ldots, z_n]$ be polynomials of degree δ, such that their homogeneous parts of degree δ do not vanish simultaneously except at* 0. *Then there is a constant $C \geq 2$ such that for all entire functions $F \in \mathcal{O}(\mathbb{C}^n)$ and all $R \geq r \geq 1$ we have*

$$\log |F|_r \leq \log |F|_R - \delta^{1-n} \nu([Z_F], \log |P|) \log \frac{R}{Cr}$$

where Z_F is the zero divisor of F and $P = (P_1, \ldots, P_N) : \mathbb{C}^n \longrightarrow \mathbb{C}^N$. Moreover

$$\nu([Z_F], \log |P|) \geq \sum_{w \in P^{-1}(0)} \mathrm{ord}(F, w)\, \mu_{n-1}(P, w)$$

where $\mathrm{ord}(F, w)$ *denotes the vanishing order of F at w and $\mu_{n-1}(P, w)$ is the $(n-1)$-multiplicity of P at w, as defined in* (7.5) *and* (7.7).

Proof. Our assumptions imply that P is a proper and finite map. The last inequality is then just a formal consequence of formula (7.4) and Th. 7.9 applied to $T = [Z_F]$. Let Q_j be the homogeneous part of degree δ in P_j. For $z_0 \in B(0, r)$, we introduce the weight functions

$$\varphi(z) = \log |P(z)|, \qquad \psi(z) = \log |Q(z - z_0)|.$$

Since $Q^{-1}(0) = \{0\}$ by hypothesis, the homogeneity of Q shows that there are constants $C_1, C_2 > 0$ such that

$$C_1 |z|^\delta \leq |Q(z)| \leq C_2 |z|^\delta \quad \text{on} \quad \mathbb{C}^n. \tag{8.2}$$

The homogeneity also implies $(dd^c\psi)^n = \delta^n\, \delta_{z_0}$. We apply the Lelong Jensen formula 4.5 to the measures $\mu_{\psi,s}$ associated with ψ and to $V = \log |F|$. This gives

$$\mu_{\psi,s}(\log |F|) - \delta^n \log |F(z_0)| = \int_{-\infty}^{s} dt \int_{\{\psi < t\}} [Z_F] \wedge (dd^c\psi)^{n-1}. \tag{8.3}$$

By (4.2), $\mu_{\psi,s}$ has total mass δ^n and has support in

$$\{\psi(z) = s\} = \{Q(z - z_0) = e^s\} \subset B\big(0, r + (e^s/C_1)^{1/\delta}\big).$$

Note that the inequality in the Schwarz lemma is obvious if $R \le Cr$, so we can assume $R \ge Cr \ge 2r$. We take $s = \delta \log(R/2) + \log C_1$; then

$$\{\psi(z) = s\} \subset B(0, r + R/2) \subset B(0, R).$$

In particular, we get $\mu_{\psi,s}(\log|F|) \le \delta^n \log|F|_R$ and formula (8.3) gives

$$\log|F|_R - \log|F(z_0)| \ge \delta^{-n} \int_{s_0}^{s} dt \int_{\{\psi<t\}} [Z_F] \wedge (dd^c\psi)^{n-1} \tag{8.4}$$

for any real number $s_0 < s$. The proof will be complete if we are able to compare the integral in (8.4) to the corresponding integral with φ in place of ψ. The argument for this is quite similar to the proof of the comparison theorem, if we observe that $\psi \sim \varphi$ at infinity. We introduce the auxiliary function

$$w = \begin{cases} \max\{\psi, (1-\varepsilon)\varphi + \varepsilon t - \varepsilon\} & \text{on } \{\psi \ge t-2\}, \\ (1-\varepsilon)\varphi + \varepsilon t - \varepsilon & \text{on } \{\psi \le t-2\}, \end{cases}$$

with a constant ε to be determined later, such that $(1-\varepsilon)\varphi + \varepsilon t - \varepsilon > \psi$ near $\{\psi = t-2\}$ and $(1-\varepsilon)\varphi + \varepsilon t - \varepsilon < \psi$ near $\{\psi = t\}$. Then Stokes' theorem implies

$$\int_{\{\psi<t\}} [Z_F] \wedge (dd^c\psi)^{n-1} = \int_{\{\psi<t\}} [Z_F] \wedge (dd^c w)^{n-1}$$

$$\ge (1-\varepsilon)^{n-1} \int_{\{\psi<t-2\}} [Z_F] \wedge (dd^c\varphi)^{n-1} \ge (1-\varepsilon)^{n-1} \nu([Z_F], \log|P|). \tag{8.5}$$

By (8.2) and our hypothesis $|z_0| < r$, the condition $\psi(z) = t$ implies

$$|Q(z-z_0)| = e^t \implies e^{t/\delta}/C_1^{1/\delta} \le |z - z_0| \le e^{t/\delta}/C_2^{1/\delta},$$

$$|P(z) - Q(z-z_0)| \le C_3(1+|z_0|)(1+|z|+|z_0|)^{\delta-1} \le C_4 r(r+e^{t/\delta})^{\delta-1},$$

$$\Big|\frac{P(z)}{Q(z-z_0)} - 1\Big| \le C_4 r e^{-t/\delta}(re^{-t/\delta}+1)^{\delta-1} \le 2^{\delta-1} C_4 r e^{-t/\delta},$$

provided that $t \ge \delta \log r$. Hence for $\psi(z) = t \ge s_0 \ge \delta \log(2^\delta C_4 r)$, we get

$$|\varphi(z) - \psi(z)| = \Big|\log\frac{|P(z)|}{|Q(z-z_0)|}\Big| \le C_5 r e^{-t/\delta}.$$

Now, we have

$$\big[(1-\varepsilon)\varphi + \varepsilon t - \varepsilon\big] - \psi = (1-\varepsilon)(\varphi - \psi) + \varepsilon(t - 1 - \psi),$$

so this difference is $< C_5 r e^{-t/\delta} - \varepsilon$ on $\{\psi = t\}$ and $> -C_5 r e^{(2-t)/\delta} + \varepsilon$ on $\{\psi = t-2\}$. Hence it is sufficient to take $\varepsilon = C_5 r e^{(2-t)/\delta}$. This number has to be < 1, so we take $t \ge s_0 \ge 2 + \delta\log(C_5 r)$. Moreover, (8.5) actually holds only if $P^{-1}(0) \subset \{\psi < t-2\}$, so by (8.2) it is enough to take $t \ge s_0 \ge 2 + \log(C_2(r+C_6)^\delta)$ where C_6 is such that $P^{-1}(0) \subset \overline{B}(0, C_6)$. Finally, we see that we can choose

$$s = \delta \log R - C_7, \qquad s_0 = \delta \log r + C_8,$$

and inequalities (8.4), (8.5) together imply

$$\log |F|_R - \log |F(z_0)| \geq \delta^{-n}\Big(\int_{s_0}^{s} (1 - C_5 r e^{(2-t)/\delta})^{n-1}\, dt\Big)\nu([Z_F], \log |P|).$$

The integral is bounded below by

$$\int_{C_8}^{\delta \log(R/r) - C_7} (1 - C_9 e^{-t/\delta})\, dt \geq \delta \log(R/Cr).$$

This concludes the proof, by taking the infimum when z_0 runs over $B(0, r)$.

□

(8.6) Corollary. *Let S be a finite subset of $\mathbb{C}^n$ and let δ be the minimal degree of algebraic hypersurfaces containing S. Then there is a constant $C \geq 2$ such that for all $F \in \mathcal{O}(\mathbb{C}^n)$ and all $R \geq r \geq 1$ we have*

$$\log |F|_r \leq \log |F|_R - \operatorname{ord}(F, S)\frac{\delta + n(n-1)/2}{n!} \log \frac{R}{Cr}$$

where $\operatorname{ord}(F, S) = \min_{w \in S} \operatorname{ord}(F, w)$.

Proof. In view of the Schwarz Lemma 8.1, we only have to select suitable polynomials $P_1, \ldots, P_N$. The vector space $\mathbb{C}[z_1, \ldots, z_n]_{<\delta}$ of polynomials of degree $< \delta$ in $\mathbb{C}^n$ has dimension

$$m(\delta) = \binom{\delta + n - 1}{n} = \frac{\delta(\delta + 1)\ldots(\delta + n - 1)}{n!}.$$

By definition of δ, the linear forms

$$\mathbb{C}[z_1, \ldots, z_n]_{<\delta} \longrightarrow \mathbb{C}, \qquad P \longmapsto P(w), \quad w \in S$$

vanish simultaneously only when $P = 0$. Hence we can find $m = m(\delta)$ points $w_1, \ldots, w_m \in S$ such that the linear forms $P \mapsto P(w_j)$ define a basis of $\mathbb{C}[z_1, \ldots, z_n]^\star_{<\delta}$. This means that there is a unique polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]_{<\delta}$ which takes given values $P(w_j)$ for $1 \leq j \leq m$. In particular, for every multiindex α, $|\alpha| = \delta$, there is a unique polynomial $R_\alpha \in \mathbb{C}[z_1, \ldots, z_n]_{<\delta}$ such that $R_\alpha(w_j) = w_j^\alpha$. Then the polynomials $P_\alpha(z) = z^\alpha - R_\alpha(z)$ have degree δ, vanish at all points w_j and their homogeneous parts of maximum degree $Q_\alpha(z) = z^\alpha$ do not vanish simultaneously except at 0. We simply use the fact that $\mu_{n-1}(P, w_j) \geq 1$ to get

$$\nu([Z_F], \log |P|) \geq \sum_{w \in P^{-1}(0)} \operatorname{ord}(F, w) \geq m(\delta) \operatorname{ord}(F, S).$$

Theorem 8.1 then gives the desired inequality, because $m(\delta)$ is a polynomial with positive coefficients and leading terms $(\delta^n + n(n-1)/2\, \delta^{n-1} + \ldots)/n!$

□

Let S be a finite subset of $\mathbb{C}^n$. According to Waldschmidt [Wa1], we introduce for every integer $t > 0$ a number $\omega_t(S)$ equal to the minimal degree of polynomials $P \in \mathbb{C}[z_1, \ldots, z_n]$ which vanish at order $\geq t$ at every point of S. The obvious subadditivity property

$$\omega_{t_1+t_2}(S) \leq \omega_{t_1}(S) + \omega_{t_2}(S)$$

easily shows that

$$\Omega(S) := \inf_{t>0} \frac{\omega_t(S)}{t} = \lim_{t\to+\infty} \frac{\omega_t(S)}{t}.$$

We call $\omega_1(S)$ the *degree* of S (minimal degree of algebraic hypersurfaces containing S) and $\Omega(S)$ the *singular degree* of S. If we apply Cor. 8.6 to a polynomial F vanishing at order t on S and fix $r = 1$, we get

$$\log |F|_R \geq t\frac{\delta + n(n-1)/2}{n!} \log \frac{R}{C} + \log |F|_1$$

with $\delta = \omega_1(S)$, in particular

$$\deg F \geq t\frac{\omega_1(S) + n(n-1)/2}{n!}.$$

The minimum of $\deg F$ over all such F is by definition $\omega_t(S)$. If we divide by t and take the infimum over t, we get the interesting inequality

$$\text{(8.7)} \qquad \frac{\omega_t(S)}{t} \geq \Omega(S) \geq \frac{\omega_1(S) + n(n-1)/2}{n!}.$$

(8.8) *Remark.* The constant $\frac{\omega_1(S)+n(n-1)/2}{n!}$ in (8.6) and (8.7) is optimal for $n = 1, 2$ but not for $n \geq 3$. It can be shown by means of Hörmander's L^2 estimates (see [Wa2]) that for every $\varepsilon > 0$ the Schwarz Lemma holds with coefficient $\Omega(S) - \varepsilon$ in Cor. 8.6 :

$$\log |F|_r \leq \log |F|_R - \mathrm{ord}(F, S)(\Omega(S) - \varepsilon) \log \frac{R}{C_\varepsilon r},$$

and that $\Omega(S) \geq (\omega_u(S)+1)/(u+n-1)$ for every $u \geq 1$ (this last inequality is due to Esnault-Viehweg [E-V], who used deep tools of algebraic geometry; Azhari [Az] reproved it recently by means of Hörmander's L^2 estimates). Rather simple examples (see [De1]) lead to the conjecture

$$\Omega(S) \geq \frac{\omega_u(S) + n - 1}{u + n - 1} \quad \text{for every} \quad u \geq 1.$$

The special case $u = 1$ of the conjecture is due to Chudnovsky [Ch].

Finally, let us mention that Cor. 8.6 contains Bombieri's theorem [Bo] on algebraic values of meromorphic maps satisfying algebraic differential equations. Recall that an entire function $F \in \mathcal{O}(\mathbb{C}^n)$ is said to be of order $\leq \rho$

if for every $\varepsilon > 0$ there is a constant C_ε such that $|F(z)| \le C_\varepsilon \exp(|z|^{\rho+\varepsilon})$. A meromorphic function is said to be of order $\le \rho$ if it can be written G/H where G, H are entire functions of order $\le \rho$.

(8.9) Theorem (Bombieri). *Let $F_1, \ldots, F_N$ be meromorphic functions on $\mathbb{C}^n$, such that $F_1, \ldots, F_d$, $n < d \le N$, are algebraically independent over $\mathbb{Q}$ and have finite orders $\rho_1, \ldots, \rho_d$. Let K be a number field of degree $[K : \mathbb{Q}]$. Suppose that the ring $K[f_1, \ldots, f_N]$ is stable under all derivations $d/dz_1, \ldots, d/dz_n$. Then the set S of points $z \in \mathbb{C}^n$, distinct from the poles of the F_j's, such that $(F_1(z), \ldots, F_N(z)) \in K^N$ is contained in an algebraic hypersurface whose degree δ satisfies*

$$\frac{\delta + n(n-1)/2}{n!} \le \frac{\rho_1 + \ldots + \rho_d}{d-n}[K : \mathbb{Q}].$$

Proof. If the set S is not contained in any algebraic hypersurface of degree $< \delta$, the linear algebra argument used in the proof of Cor. 8.6 shows that we can find $m = m(\delta)$ points $w_1, \ldots, w_m \in S$ which are not located on any algebraic hypersurface of degree $< \delta$. Let $H_1, \ldots, H_d$ be the denominators of $F_1, \ldots, F_d$. The standard arithmetical methods of transcendental number theory allow us to construct a sequence of entire functions in the following way: we set

$$G = P(F_1, \ldots, F_d)(H_1 \ldots H_d)^s$$

where P is a polynomial of degree $\le s$ in each variable with integer coefficients. The polynomials P are chosen so that G vanishes at a very high order at each point w_j. This amounts to solving a linear system whose unknowns are the coefficients of P and whose coefficients are polynomials in the derivatives of the F_j's (hence lying in the number field K). Careful estimates of size and denominators and a use of the Dirichlet-Siegel box principle leads to the following lemma (cf. for example Waldschmidt [Wa2]).

(8.10) Lemma. *For every $\varepsilon > 0$, there exist constants $C_1, C_2 > 0$, $r \ge 1$ and an infinite sequence G_t, $t \in T \subset \mathbb{N}$ (depending on m and on the choice of the points w_j), such that*

(a) *G_t vanishes at order $\ge t$ at all points $w_1, \ldots, w_m$;*

(b) $|G_t|_r \ge (C_1 t)^{-t\,[K:\mathbb{Q}]}$;

(c) $|G_t|_{R(t)} \le C_2^t$ *where* $R(t) = (t^{d-n}/\log t)^{1/(\rho_1+\ldots+\rho_d+\varepsilon)}$.

An application of Cor. 8.6 to $F = G_t$ and $R = R(t)$ gives the desired bound for the degree δ as t tends to $+\infty$ and ε tends to 0. If δ_0 is the largest integer which satisfies the inequality of Th. 8.9, we get a contradiction if we take $\delta = \delta_0 + 1$. This shows that S must be contained in an algebraic hypersurface of degree $\delta \le \delta_0$. □

9. Global Intersection Class and Self-intersection

Let X be a compact complex n-dimensional manifold. With every closed current Θ of degree k (or of bidegree (p,q) with $p+q=k$), we can associate a De Rham cohomology class $\{\Theta\} \in H^k_{DR}(X,\mathbb{R})$. In fact, it is well known that De Rham cohomology can be computed either by the complex of smooth differential forms or by the complex of currents: both complexes of sheaves are fine resolutions of the locally constant sheaf $\mathbb{R}$. Moreover, the assignment $\Theta \mapsto \{\Theta\}$ is continuous with respect to the weak topology of currents; this can be easily seen e.g. by Poincaré duality. In particular, with every analytic cycle $A = \sum \lambda_j A_j$ of pure dimension p in X is associated a De Rham cohomology class

$$\{A\} = \Big\{ \sum \lambda_j [A_j] \Big\} \in H^{2n-2p}_{DR}(X,\mathbb{R}). \tag{9.1}$$

When the coefficients λ_j are integers, the class $\{A\}$ lies in the image of $H^{2n-2p}(X,\mathbb{Z})$: this is for instance a consequence of the fact that every analytic set can be triangulated.

The wedge product of smooth differential forms defines a ring structure on De Rham cohomology. Given two currents Θ_1, Θ_2 on X, there is a well defined *intersection class* $\{\Theta_1\} \cdot \{\Theta_2\}$ in the cohomology ring, even when $\Theta_1 \wedge \Theta_2$ is not defined pointwise as a current. Especially, when $\deg \Theta_1 + \deg \Theta_2 = \dim_{\mathbb{R}} X$, the top degree class $\{\Theta_1\} \cdot \{\Theta_2\}$ can be considered as a number after integration over X. These simple observations show in fact that wedge products of closed positive currents cannot be defined in a reasonable way without further assumptions: if X is the blow-up of some other manifold at one point and $E \simeq \mathbb{P}^{n-1}$ is the exceptional divisor, then $\mathcal{O}(E)_{\restriction E} \simeq \mathcal{O}(-1)$ and so $\{E\}^n = \int_E c_1(\mathcal{O}(E))^{n-1} = (-1)^{n-1}$; thus $\{E\}^2 < 0$ if X is a surface! The same example shows that, in general, a closed positive $(1,1)$-current T cannot be approximated in the weak topology by smooth closed positive currents: a necessary condition for this is that $\{T\}^p \cdot \{Y\} \ge 0$ for every p-dimensional subvariety $Y \subset X$. However, a result proved in [De7] which we shall now recall shows that T can be approximated by closed real currents with small negative part controlled by the curvature of X. This result allows us to compute self-intersections by taking weak limits of products in which the original currents have been replaced by their regularizations. This technique will be applied here to get a fairly general self-intersection inequality for closed positive currents of bidegree $(1,1)$.

We say that a bidimension (p,p) current T is *almost positive* if there exists a smooth form v of bidegree $(n-p,n-p)$ such that $T+v \geq 0$. Similarly, a function φ on X is said to be *almost psh* if φ is locally equal to the sum of a psh (plurisubharmonic) function and of a smooth function; then the $(1,1)$-current $dd^c\varphi$ is almost positive; conversely, if a locally integrable function φ is such that $dd^c\varphi$ is almost positive, then φ is equal a.e. to an almost psh function. If T is closed and almost positive, the Lelong numbers $\nu(T,x)$ are well defined, since the negative part always contributes to zero. We refer to [De7] for a proof of the following basic approximation theorem:

(9.2) Theorem. *Let T be a closed almost positive $(1,1)$-current and let α be a smooth real $(1,1)$-form in the the same dd^c-cohomology class as T, i.e. $T = \alpha + dd^c\psi$ where ψ is an almost psh function. Let γ be a continuous real $(1,1)$-form such that $T \geq \gamma$. Suppose that $\mathcal{O}_{TX}(1)$ is equipped with a smooth hermitian metric such that the curvature form satisfies*

$$c\big(\mathcal{O}_{TX}(1)\big) + \pi^\star u \geq 0$$

with $\pi : P(T^\star X) \to X$ and with some nonnegative smooth $(1,1)$-form u on X. Fix a hermitian metric ω on X. Then for every $c > 0$, there is a sequence of closed almost positive $(1,1)$-currents $T_{c,k} = \alpha + dd^c\psi_{c,k}$ such that $\psi_{c,k}$ is smooth on $X \smallsetminus E_c(T)$ and decreases to ψ as k tends to $+\infty$ (in particular, $T_{c,k}$ is smooth on $X \smallsetminus E_c(T)$ and converges weakly to T on X), and

$$T_{c,k} \geq \gamma - \lambda_{c,k} u - \varepsilon_k \omega$$

where

(i) $\lambda_{c,k}(x)$ *is a decreasing sequence of continuous functions on X such that* $\lim_{k\to+\infty} \lambda_{c,k}(x) = \min\big(\nu(T,x), c\big)$ *at every point,*

(ii) $\lim_{k\to+\infty} \varepsilon_k = 0$,

(iii) $\nu(T_{c,k},x) = \big(\nu(T,x) - c\big)_+$ *at every point $x \in X$.* □

Here $\mathcal{O}_{TX}(1)$ is the canonical line bundle associated with TX over the hyperplane bundle $P(T^\star X)$. Observe that the theorem gives in particular approximants $T_{c,k}$ which are smooth everywhere on X if c is taken such that $c > \max_{x\in X} \nu(T,x)$. The equality in (iii) means that the procedure kills all Lelong numbers that are $\leq c$ and shifts all others downwards by c. Hence Th. 9.2 is an analogue over manifolds of Kiselman's procedure [Ki1,2] for killing Lelong numbers of a plurisubharmonic functions on an open subset of $\mathbb{C}^n$.

(9.3) Corollary. *Let Θ be a closed almost positive current of bidimension (p,p) and let $\alpha_1,\ldots,\alpha_q$ be closed almost positive $(1,1)$-currents such that*

$\alpha_1 \wedge \ldots \wedge \alpha_q \wedge \Theta$ *is well defined by application of criteria* 2.3 *or* 2.5*, when* α_j *is written locally as* $\alpha_j = dd^c u_j$*. Then*

$$\{\alpha_1 \wedge \ldots \wedge \alpha_q \wedge \Theta\} = \{\alpha_1\} \cdots \{\alpha_q\} \cdot \{\Theta\}.$$

Proof. Theorem 9.2 and the monotone continuity theorems of § 2 show that

$$\alpha_1 \wedge \ldots \wedge \alpha_q \wedge \Theta = \lim_{k \to +\infty} \alpha_1^k \wedge \ldots \wedge \alpha_q^k \wedge \Theta$$

where $\alpha_j^k \in \{\alpha_j\}$ is smooth. Since the result is by definition true for smooth forms, we conclude by the weak continuity of cohomology class assignment. □

Now, let X be a compact Kähler manifold equipped with a Kähler metric ω. The *degree* of a closed positive current Θ with respect to ω is by definition

$$\deg_\omega \Theta = \int_X \Theta \wedge \omega^p, \qquad \operatorname{bidim} \Theta = (p,p). \tag{9.4}$$

In particular, the degree of a p-dimensional analytic set $A \subset X$ is its volume $\int_A \omega^p$ with respect to ω. We are interested in the following problem.

(9.5) Problem. *Let* T *be a closed positive* $(1,1)$*-current on* X*. Is it possible to derive a bound for the codimension* p *components in the Lelong upperlevel sets* $E_c(T)$ *in terms of the cohomology class* $\{T\} \in H^2_{DR}(X, \mathbb{R})$ *?*

We introduce the sequence $0 = b_1 \le \ldots \le b_n \le b_{n+1}$ of "*jumping values*" b_p such that the dimension of $E_c(T)$ drops by one unit when c gets larger than b_p, namely $\operatorname{codim} E_c(T) = p$ with at least some component of codimension p when $c \in \,]b_p, b_{p+1}]$. Let $(Z_{p,k})_{k \ge 1}$ be the p-codimensional components occurring in any of the sets $E_c(T)$ for $c \in \,]b_p, b_{p+1}]$, and let

$$\nu_{p,k} = \min_{x \in Z_{p,k}} \nu(T,x) \in \,]b_p, b_{p+1}]$$

be the generic Lelong number of T along $Z_{p,k}$. Then we have the following self-intersection inequality.

(9.6) Theorem. *Suppose that* X *is Kähler and that* $\mathcal{O}_{TX}(1)$ *has a hermitian metric such that* $c\big(\mathcal{O}_{TX}(1)\big) + \pi^\star u \ge 0$*, where* u *is a smooth closed semi-positive* $(1,1)$*-form. For each* $p = 1, \ldots, n$*, the De Rham cohomology class* $(\{T\}+b_1\{u\}) \cdots (\{T\}+b_p\{u\})$ *can be represented by a closed positive current* Θ_p *of bidegree* (p,p) *such that*

$$\Theta_p \ge \sum_{k \ge 1} (\nu_{p,k} - b_1) \ldots (\nu_{p,k} - b_p)\, [Z_{p,k}] + (T_{\mathrm{abc}} + b_1 u) \wedge \ldots \wedge (T_{\mathrm{abc}} + b_p u)$$

where $T_{\rm abc} \ge 0$ is the absolutely continuous part in the Lebesgue decomposition of the coefficients of T into absolutely continuous and singular measures.

By neglecting the second term in the right hand side and taking the wedge product with ω^{n-p}, we get the following interesting consequence:

(9.7) Corollary. *If ω is a Kähler metric on X and if $\{u\}$ is a semipositive cohomology class such that $c_1\big(\mathcal{O}_{TX}(1)\big) + \pi^\star\{u\}$ is semipositive, the degrees of the components $Z_{p,k}$ with respect to ω satisfy the estimate*

$$\sum_{k=1}^{+\infty}(\nu_{p,k}-b_1)\ldots(\nu_{p,k}-b_p)\int_X [Z_{p,k}]\wedge\omega^{n-p} \le \big(\{T\}+b_1\{u\}\big)\cdots\big(\{T\}+b_p\{u\}\big)\cdot\{\omega\}^{n-p}.$$

By a semipositive cohomology class of type $(1,1)$, we mean a class in the closure of the Kähler cone of X. As a special case, if D is an effective divisor and $T=[D]$, we get a bound for the degrees of the p-codimensional singular strata of D in terms of a polynomial of degree p in the cohomology class $\{D\}$; the multiplicities $(\nu_{p,k}-b_1)\ldots(\nu_{p,k}-b_p)$ are then positive integers. The case when X is $\mathbb{P}^n$ or a homogeneous manifold is especially simple: then TX is generated by sections and we can take $u=0$; the bound is thus simply $\{D\}^p\cdot\{\omega\}^{n-p}$; the same is true more generally as soon as $\mathcal{O}_{TX}(1)$ is semipositive. The main idea of the proof is to kill the Lelong numbers of T up to the level b_j; then the singularities of the resulting current T_j occur only in codimension j and it becomes possible to define the wedge product $T_1\wedge\ldots\wedge T_p$ by means of Th. 2.5. Here are the details:

Proof of 9.6. We argue by induction on p. For $p=1$, Siu's decomposition formula shows that

$$T=\sum \nu_{1,k}[Z_{1,k}]+R,$$

and we have $R\ge T_{\rm abc}$ since the other part has singular measures as coefficients. The result is thus true with $\Theta_1=T$. Now, suppose that Θ_{p-1} has been constructed. For $c>b_p$, the current $T_{c,k}=\alpha+dd^c\psi_{c,k}$ produced by Th. 9.2 is such that $\operatorname{codim} L(\psi_{c,k}) = \operatorname{codim} E_c(T)\ge p$. Hence Cor. 2.10 shows that

$$\Theta_{p,c,k}=\Theta_{p-1}\wedge(T_{c,k}+c\,u+\varepsilon_k\omega)$$

is well defined. If ε_k tends to zero slowly enough, $T_{c,k}+c\,u+\varepsilon_k\omega$ is positive by (9.2 i), so $\Theta_{p,c,k}\ge 0$. Moreover, the cohomology class of $\Theta_{p,c,k}$ is $\{\Theta_{p-1}\}\cdot(\{T\}+c\{u\}+\varepsilon_k\{\omega\})$, converging to $\{\Theta_{p-1}\}\cdot(\{T\}+c\{u\})$.

Since the mass $\int_X \Theta_{p,c,k} \wedge \omega^{n-p}$ remains uniformly bounded, the family $(\Theta_{p,c,k})_{c\in]b_p,b_p+1],k\geq 1}$ is relatively compact in the weak topology. We define

$$\Theta_p = \lim_{c\to b_p+0}\lim_{k\to+\infty} \Theta_{p,c,k},$$

possibly after extracting some weakly convergent subsequence. Then $\{\Theta_p\} = \{\Theta_{p-1}\}\cdot(\{T\}+b_p\{u\})$, and so $\{\Theta_p\} = (\{T\}+b_1\{u\})\cdots(\{T\}+b_p\{u\})$. Moreover, we have

$$\begin{aligned}\nu(\Theta_p,x) &\geq \limsup_{c\to b_p+0}\limsup_{k\to+\infty}\ \nu\big(\Theta_{p-1}\wedge(T_{c,k}+c\,u+\varepsilon_k\omega),x\big)\\ &\geq \nu(\Theta_{p-1},x)\times\limsup_{c\to b_p+0}\limsup_{k\to+\infty}\ \nu(T_{c,k},x)\\ &\geq \nu(\Theta_{p-1},x)\big(\nu(T,x)-b_p\big)_+\end{aligned}$$

by application of (3.12), (5.10) and (9.2 iii). Hence by induction we get

$$\nu(\Theta_p,x)\geq\big(\nu(T,x)-b_1\big)_+\ldots\big(\nu(T,x)-b_p\big)_+,$$

in particular, the generic Lelong number of Θ_p along $Z_{p,k}$ is at least equal to $(\nu_{p,k}-b_1)\ldots(\nu_{p,k}-b_p)$. This already implies

$$\Theta_p\geq\sum_{k\geq 1}(\nu_{p,k}-b_1)\ldots(\nu_{p,k}-b_p)\,[Z_{p,k}].$$

Since the right hand side is Lebesgue singular, the desired inequality will be proved if we show in addition that

$$\Theta_{p,\mathrm{abc}}\geq(T_{\mathrm{abc}}+b_1u)\wedge\ldots\wedge(T_{\mathrm{abc}}+b_pu),$$

or inductively, that $\Theta_{p,\mathrm{abc}}\geq\Theta_{p-1,\mathrm{abc}}\wedge(T_{\mathrm{abc}}+b_pu)$. In order to do this, we simply have to make sure that $\lim_{k\to+\infty}T_{c,k,\mathrm{abc}}=T_{\mathrm{abc}}$ almost everywhere and use induction again. But our arguments are not affected if we replace $\psi_{c,k}$ by $\psi'_{c,k}=\max\{\psi,\psi_{c,k}-A_k\}$ with A_k converging quickly to $+\infty$. It is then easy to show that a suitable choice of A_k gives $\lim(dd^c\psi'_{c,k})_{\mathrm{abc}}=(dd^c\psi)_{\mathrm{abc}}$ almost everywhere (see Lemma 10.12 in [De6]). □

References

[Az] A. Azhari. — *Sur la conjecture de Chudnovsky-Demailly et les singularités des hypersurfaces algébriques*, Ann. Inst. Fourier, **40** (1990), 106–117.

[B-T1] E. Bedford and B.A. Taylor. — *The Dirichlet problem for a complex Monge-Ampère equation*, Invent. Math., **37** (1976), 1–44.

[B-T2] E. Bedford and B.A. Taylor. — *A new capacity for plurisubharmonic functions*, Acta Math., **149** (1982), 1–41.

[Bo] E. Bombieri. — *Algebraic values of meromorphic maps*, Invent. Math., **10** (1970), 267–287 and *Addendum*, Invent. Math. **11** (1970), 163–166.

[C-L-N] S.S. Chern, H.I. Levine and L. Nirenberg. — *Intrinsic norms on a complex manifold*, Global Analysis (papers in honor of K.Kodaira), p.119-139, Univ. of Tokyo Press, Tokyo, 1969.

[Chu] G.V. Chudnovsky. — *Singular points on complex hypersurfaces and multidimensional Schwarz lemma*, Sém. Delange-Pisot-Poitou, 21e année, 1979/80, Progress in Math. n° 12, ed. by Marie-José Bertin, Birkhäuser, 1981, 29–69.

[De1] J.-P. Demailly. — *Formules de Jensen en plusieurs variables et applications arithmétiques*, Bull. Soc. Math. France, **110** (1982), 75–102.

[De2] J.-P. Demailly. — *Sur les nombres de Lelong associés à l'image directe d'un courant positif fermé*, Ann. Inst. Fourier (Grenoble), **32** (1982), 37–66.

[De3] J.-P. Demailly. — *Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France (N.S.), **19** (1985), 1–124.

[De4] J.-P. Demailly. — *Nombres de Lelong généralisés, théorèmes d'intégralité et d'analyticité*, Acta Math., **159** (1987), 153–169.

[De5] J.-P. Demailly. — *Singular hermitian metrics on positive line bundles*, Proceedings of the Bayreuth conference "Complex algebraic varieties", April 2-6, 1990, edited by K. Hulek, T. Peternell, M. Schneider, F. Schreyer, Lecture Notes in Math. n° 1507, Springer-Verlag, Berlin, 1992.

[De6] J.-P. Demailly. — *A numerical criterion for very ample line bundles*, Preprint n° 153, Institut Fourier, Univ. Grenoble I, December 1990, submitted to J. Differential Geom.

[De7] J.-P. Demailly. — *Regularization of closed positive currents and Intersection Theory*, Preprint Univ. Grenoble I, Institut Fourier, May 1991.

[Dr] R.N. Draper. — *Intersection theory in analytic geometry*, Math. Annalen, **180** (1969), 175–204.

[E-M] H. El Mir. — *Sur le prolongement des courants positifs fermés*, Acta Math., **153** (1984), 1–45.

[E-V] H. Esnault and E. Viehweg. — *Sur une minoration du degré d'hypersurfaces s'annulant en certains points*, Math. Ann., **263** (1983), 75-86.

[Fe] H. Federer. — *Geometric measure theory*, Springer Verlag, Band 153, Berlin, Heidelberg, New-York, 1969.

[Ha] R. Harvey. — *Holomorphic chains and their boundaries*, Proc. of Symposia in Pure Math., Several Complex Variables, held in Williamstown 1975, vol. XXX, part I, ed. by R.O. Wells, 1977, 309–382.

[Hö] L. Hörmander. — *An introduction to Complex Analysis in several variables*, 3rd edition, North-Holland Math. libr., vol.7, Amsterdam, London, 1990.

[Kg] J.R. King. — *A residue formula for complex subvarieties*, Proc. Carolina Conf. on Holomorphic mappings and minimal surfaces, University of North Carolina, Chapel Hill, 1970, 43–56.

[Ki1] C.O. Kiselman. — *The partial Legendre transformation for plurisubharmonic functions*, Invent. Math., **49** (1978), 137–148.

[Ki2] C.O. Kiselman. — *Densité des fonctions plurisousharmoniques*, Bull. Soc. Math. France, **107** (1979), 295–304.

[Ki3] C.O. Kiselman. — *Sur la définition de l'opérateur de Monge-Ampère complexe*, Analyse Complexe, Proceedings of Journées Fermat (SMF), Toulouse, May 1983, Lecture Notes in Math., vol. 1094, Springer-Verlag, Berlin (1984), 139–150.

[Ki4] C.O. Kiselman. — *Un nombre de Lelong raffiné*, Preprint Uppsala 1986, Sém. d'Analyse Complexe et Géométrie 1985-87, Fac. des Sciences de Tunis et Fac. des Sciences et Techniques de Monastir, Maroc, 1987.

[Le1] P. Lelong. — *Intégration sur un ensemble analytique complexe*, Bull. Soc. Math. France, **85** (1957), 239–262.

[Le2] P. Lelong. — *Fonctionnelles analytiques et fonctions entières (n variables)*, Sém. de Math. Supérieures, 6e session, été 1967, Presses Univ. Montréal, 1968.

[Le3] P. Lelong. — *Plurisubharmonic functions and positive differential forms*, Gordon and Breach, New-York, and Dunod, Paris, 1969.

[Re1] R. Remmert. — *Projectionen analytischer Mengen*, Math. Annalen, **130** (1956), 410–441.

[Re2] R. Remmert. — *Holomorphe und meromorphe Abbildungen komplexer Räume*, Math. Annalen, **133** (1957), 328–370.

[Sib] N. Sibony. — *Quelques problèmes de prolongement de courants en analyse complexe*, Duke Math. J., **52** (1985), 157–197.

[Siu] Y.T. Siu. — *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math., **27** (1974), 53–156.

[Sk1] H. Skoda. — *Sous-ensembles analytiques d'ordre fini ou infini dans $\mathbb{C}^n$*, Bull. Soc. Math. France, **100** (1972), 353–408.

[Sk2] H. Skoda. — *Estimations L^2 pour l'opérateur $\overline{\partial}$ et applications arithmétiques*, Séminaire P. Lelong (Analyse), année 1975/76, Lecture Notes in Math. no 538, Springer-Verlag, Berlin (1977), 314–323.

[Sk3] H. Skoda. — *Prolongement des courants positifs fermés de masse finie*, Invent. Math., **66** (1982), 361–376.

[St] W. Stoll. — *The multiplicity of a holomorphic map*, Invent. Math., **2** (1966), 15–58.

[Th] P. Thie. — *The Lelong number of a point of a complex analytic set*, Math. Annalen, **172** (1967), 269–312.

[Wa1] M. Waldschmidt. — *Propriétés arithmétiques des fonctions de plusieurs variables (II)*, Séminaire P. Lelong (Analyse), année 1975/76, Lecture Notes in Math. no 538, Springer-Verlag, Berlin (1977), 108–135.

[Wa2] M. Waldschmidt. — *Nombres transcendants et groupes algébriques*, Astérisque n° 69-70, 1979.

(May 6, 1991, revised on May 18, 1992)