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Monodromy Group of Appell's System (F_4)

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Introduction

Appell's hypergeometric function $F_4(\alpha, \beta, \gamma, \gamma'; x, y)$ is defined by

$$F_4(\alpha, \beta, \gamma, \gamma', x, y)=\sum_{m,n=0}^{\infty}\frac{(\alpha, m+n)(\beta, m+n)}{(\gamma, m)(\gamma', n)(1, m)(1, n)}x^m y^n$$

where (a, k) denotes the factorial function;

$$(a, k)=\Gamma(a+k)/\Gamma(a) .$$

We assume $\gamma, \gamma'\neq 0, -1, -2, \cdots$ throughout this paper. This power series converges in the domain $\{(x, y)\in C^2; \sqrt{|x|}+\sqrt{|y|}<1\}$ and satisfies the following system of partial differential equations.

$$(F_4)\quad \begin{aligned} &x(1-x)r-y^2t-2xys+[\gamma-(\alpha+\beta+1)x]p-(\alpha+\beta+1)yq-\alpha\beta z=0\\ &y(1-y)t-x^2r-2xys+[\gamma'-(a+\beta+1)y]q-(\alpha+\beta+1)xp-\alpha\beta z=0 .\end{aligned}$$

Where z is the unknown function and

$$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y},\ r=\frac{\partial^2 z}{\partial x^2},\ s=\frac{\partial^2 z}{\partial x\partial y},\ t=\frac{\partial^2 z}{\partial y^2} .$$

This system (F_4) is equivalent to a completely integrable system of linear differential equations of rank 4 whose coefficients are rational functions with poles only on $L=L_1\cup L_2\cup L_3\cup C$ and A in P^2, where

$$L_1=\{x=0\} ,\quad L_2=\{y=0\} ,\quad L_3=\text{line at infinity}$$
$$C=\{(x-y)^2-2(x+y)+1=0\} ,\quad A=\{x+y=1\} .$$

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As we prove in §1, A is an apparent singularity, i.e. any solution of the system (F_4) does not branch at A. Hence we have the monodromy representation:

$$\rho\colon \pi_1(P^2-L,\ P_0) \longrightarrow GL(4,\ C)\ ,$$

where $\pi_1(P^2-L,\ P_0)$ denotes the fundamental group of P^2-L with base point P_0.

In this paper, we calculate the generators $\rho(\gamma_i)$, $i=1, 2, 3$ of the monodromy group, where γ_i, $i=1, 2, 3$ are the generators of $\pi_1(P^2-L,\ P_0)$, which is thoroughly investigated in the appendix.

The monodromy representations of the Appell's hypergeometric equations (F_i), $i=1, 2, 3$ are calculated explicitly by several authors. They made use of the Appell's integral representations whose integrands are power products of linear functions. For this kind of integral, the fundamental method of calculating the monodromy is known ([3]). System (F_4) has no such integral representation. We make use of Aomoto's integral representation whose integrand contains a power of a quadratic polynomial.

During the author was preparing this paper, he was informed that Prof. K. Takano was calculating the monodromy group of (F_4) by quite a different method.

Finally, the author wishes to express his grateful thanks to Prof. K. Aomoto for suggesting this problem and for his patient encouragement.

Throughout this paper we use the notation:

$$e(a)=\exp(2\pi\sqrt{-1}\ a)\ .$$

Contents

§1. Preliminaries.

The system (F_4) is equivalent to a completely integrable system of differential equations of the form:

$$(F_4) \qquad d\begin{pmatrix} z \\ p \\ q \\ s \end{pmatrix} = \omega \begin{pmatrix} z \\ p \\ q \\ s \end{pmatrix}$$

where $\omega=(\omega_{ij})$ is a 4×4 matrix of rational 1-forms which have poles only on $L\cup A$.

LEMMA 1. A is an apparent singularity.

PROOF. Let $D=\{(u, v)\in C^2;\ uv\neq 0\}$. $\pi\colon (u, v)\rightarrow(x, y)=(u^2, v^2)$ gives a covering $D\rightarrow P^2-L_1\cup L_2\cup L_3$. Let $(\tilde{F}_4)$ be the pull back of the system (F_4) to D:

$$(\tilde{F}_4)\quad \begin{aligned} &(1-u^2)r-v^2t-2uvs-\left[\gamma-\frac{1}{2}-\left(\alpha+\beta+\frac{1}{2}\right)u^2\right]\frac{1}{u}p \\ &\qquad -2\left(\alpha+\beta+\frac{1}{2}\right)vq-4\alpha\beta z=0 \\ &(1-v^2)t-u^2r-2uvs-\left[\gamma'-\frac{1}{2}-\left(\alpha+\beta+\frac{1}{2}\right)v^2\right]\frac{1}{v}q \\ &\qquad -2\left(\alpha+\beta+\frac{1}{2}\right)up-4\alpha\beta z=0 \end{aligned}$$

where z is the unknown function and $p=\partial z/\partial u$, $q=\partial z/\partial v$, $r=\partial^2 z/\partial u^2$, $s=\partial^2 z/\partial u\partial v$, $t=\partial^2 z/\partial v^2$.

An easy calculation shows that the characteristic variety $\tilde{L}$ of $(\tilde{F}_4)$ is given by

$$\tilde{L}=\bigcup_{\varepsilon_i=\pm 1}\{(u, v)\in D;\ \varepsilon_1 u+\varepsilon_2 v=1\}\ .$$

Hence, by the theorem of Bernstein-Sato ([5]), any solution of the system $(\tilde{F}_4)$ is holomorphic on $D-\tilde{L}$. Since $\pi(\tilde{L})=C$, the lemma is proved.

PROPOSITION 1. *Any solution of Lauricella's hypergeometric equation* (F_c) *(see* [4], [7]*) of* n *variables is holomorphic on* P^n-L *where*

$$L=\bigcup_{i=1}^{n}\{x_i=0\}\cup\{\text{line at infinity}\}\cup C$$

$$C=\bigcup_{\varepsilon_i=\pm 1}\left\{\sum_{i=1}^{n}\varepsilon_i\sqrt{x_i}=1\right\}\ .$$

The proof is analogous to that of Lemma 1.

§2. Integral representation of F_4.

Put

$$\lambda_1=\beta-\gamma',\ \lambda_2=\beta-\gamma,\ \lambda_3=-\alpha,\ \lambda_4=\gamma+\gamma'-\beta-2$$
$$\lambda_5=-(\lambda_1+\lambda_2+\lambda_3+2\lambda_4)\ .$$

We pose the following assumtion throughout this paper:

ASSUMPTION.

I) $\lambda_i,\ \lambda_j+\lambda_3\notin Z,\ 1\leqq i\leqq 5,\ 1\leqq j\neq 3\leqq 5.$

II) $\lambda_i+\lambda_j+\lambda_4,\ 2\lambda_i+\lambda_j+\lambda_4,\ \lambda_i+\lambda_j+2\lambda_4\notin Z$

where $i,\ j=1,\ 2,\ 5$ and $i\neq j$.

PROPOSITION 2. (*K. Aomoto*) *For sufficiently small x, y, we have*

$$(2.1)\quad F_4(\alpha,\ \beta,\ \gamma,\ \gamma';\ x,\ y)=(-1)^{\beta-\gamma-2\gamma'}\frac{\sin\pi(\beta-\gamma)}{\pi}$$
$$\times\frac{\Gamma(\gamma)\cdot(\gamma')}{\Gamma(\gamma-\gamma'-\beta-1)}\int_\Delta\int u^{\beta-\gamma'}v^{\beta-\gamma}(1-xu-yv)^{-\alpha}(u+v-uv)^{\gamma+\gamma'-\beta-2}dudv\ ,$$

where $\Delta=\{(u,\ v)\in R^2;\ 0\leqq u\leqq 1,\ 0\geqq v,\ u+v-uv\geqq 0\}$.

PROOF. By the assumption posed above, we can regularize the right hand side in Hadamard's sense by taking a suitable cycle Δ_{reg} disjoint from the singularities (see [1]). Hence we have the estimates on Δ_{reg}:

$$|u^{\beta-\gamma'+m}v^{\beta-\gamma+n}|\leqq K_1^m\times K_2^n\qquad m,\ n=0,\ 1,\ 2,\ \cdots$$

For some positive constants K_1, K_2. Substitute the expression

$$(1-xu-yv)^{-\alpha}=\sum_{m,n=0}^{\infty}\frac{(\alpha,\ m+n)}{(1,\ m)(1,\ n)}x^m y^n u^m v^n$$

in the integrand. On account of the above estimates, we can interchange integration and summation so that

$$\iint_\Delta u^{\beta-\gamma'}v^{\beta-\gamma}(1-xu-yv)^{-\alpha}(u+v-uv)^{\gamma+\gamma'-\beta-2}dudv$$
$$=\sum_{m,n=0}^{\infty}\frac{(\alpha,\ m+n)}{(1,\ m)(1,\ n)}x^m y^n\iint_{\Delta_{\mathrm{reg}}}u^{\beta-\gamma'+m}v^{\beta-\gamma+n}(u+v-uv)^{\gamma+\gamma'-\beta-2}dudv\ .$$

On the other hand we have

$$\iint_{\Delta_{\mathrm{reg}}}u^{\beta-\gamma'+m}v^{\beta-\gamma+n}(u+v-uv)^{\gamma+\gamma'-\beta-2}dudv$$
$$=(-1)^{\gamma+2\gamma'-\beta}\frac{\pi\Gamma(\gamma+\gamma'-\beta-1)}{\sin\pi(\beta-\gamma)\Gamma(\gamma)\Gamma(\gamma')}\cdot\frac{(\beta,\ m+n)}{(\gamma,\ m)(\gamma',\ n)}$$

which completes the proof.

Next we shall transform the domain of integration Δ_{reg}. We shall use the following notations:

$U(u, v)=u^{\lambda_1}v^{\lambda_2}(1-xu-yv)^{\lambda_3}(u+v-uv)^{\lambda_4}$

$\omega=dU/U$

$S=\{(u, v)\in C^2: U(u, v)=0\}$

$X=C^2-S$, $Y=\{u=1\}\cap X$

$\bar{X}=P^2$, $\bar{S}=S\cup\{\text{line an infinity}\}$

$\bar{X}_R$(resp. $\bar{S}_R$)=real parts of $\bar{X}$(resp. $\bar{S}$)

$\nabla_\omega=d+\omega$: Gauss-Manin connection attached to ω.

S_ω: Local system defined by many-valuedness of U.

$S_{-\omega}$: Dual local system of S_ω.

$\theta: \pi_1(\bar{X}-\bar{S}, x_0)\to C^*$: The characteristic homomorphism of $S_{-\omega}$, where x_0 is a certain fixed point.

∂_θ: Boundary operation of homology groups of $\bar{X}-\bar{S}$ with coefficients in $S_{-\omega}$.

$H^*(X, \nabla_\omega)$: De Rham rational twisted cohomology group (see [2]).

$H_*(X, S_{-\omega})$: Homology group with coefficients in $S_{-\omega}$.

By the comparison theorem of Deligne-Grothendieck (see Theorem 6.2 in [6]), we have the duality:

$$H_i(X, S_{-\omega})\times H^i(X, \nabla_\omega)\longrightarrow C \qquad (2.2)$$
$$\langle c, \phi\rangle\longrightarrow\int_c U\cdot\phi$$

$$H_j(Y, S_{-\omega}|_Y)\times H^j(Y, \nabla_\omega|_Y)\longrightarrow C \qquad (2.3)$$
$$\langle d, \psi\rangle\longrightarrow\int_d U|_Y\cdot\psi$$

where $|_Y$ denotes the restriction to Y.

In the following, we shall take $(x, y)\in P^2-L$.

LEMMA 2. $\dim_C H_2(X, S_{-\omega})=4$, $\dim_C H_1(Y, S_{-\omega}|_Y)=1$.

PROOF. Since X has a finite simple covering, we have

$$\sum_{i=0}^{4}(-1)^i \dim_C H_i(X, S_{-\omega})=\chi(X) ,$$

where $\chi(X)$ denotes the Euler characteristic of X. Due to the vanishing theorem of Aomoto (Theorem 4.2 in [2]), we have

$$H^i(X, \nabla_\omega)=0,\ i\neq 2 .$$

One can easily check that $\chi(X)=\chi(P^2)-\chi(\bar{S})=4$. Latter claim is analogously proved.

The vanishing theorem of Aomoto gives the following commutative exact sequence (coefficients are omitted):

$$\begin{tikzcd}
0 \arrow[r] & H^1(Y) \arrow[r, "(\nabla_\omega)_*"] \arrow[d, no head, "\wr"] & H^2(X, Y) \arrow[r] \arrow[d, no head, "\wr"] & H^2(X) \arrow[r] \arrow[d, no head, "\wr"] & 0 \\
0 & H_1(Y) \arrow[l] & H_2(X, Y) \arrow[l, "(\partial_\theta)_*"'] & H_2(X) \arrow[l] & 0 \arrow[l]
\end{tikzcd} \tag{2.4}$$

where $(\nabla_\omega)_*$ and $(\partial_\theta)_*$ are the canonical homomorphisms of relative exact sequence. Let c_1, c_2, c_3, c_4 be a basis of $H_2(X, S_{-\omega})$. In view of the above exact sequences, we can take $c_1, \cdots, c_4$ and $c_5=\Delta_{\mathrm{reg}}$ as a basis of the relative homology group $H_2(X, Y, S_{-\omega})$.

Put

$$\begin{aligned}
&\phi_1=du\wedge dv,\ \phi_2=\frac{-udu\wedge dv}{1-xu-yv},\ \phi_3=\frac{-vdu\wedge du}{1-xu-yv} \\
&\phi_4=\frac{uvdu\wedge dv}{1-xu-yv},\ \phi_5=(\nabla_\omega)_*(\psi) ,
\end{aligned} \tag{2.5}$$

where ψ is a basis of $H^1(Y, \nabla_\omega|_Y)$.

$\phi_1, \cdots, \phi_4$ is a basis of $H^2(X, \nabla_\omega)$ provided that $(x, y)\notin L\cup A$.

In fact we have

$$\begin{aligned}
&\int_{c_5} U\phi_1=F_4,\ \int_{c_5} U\phi_2=\frac{\partial F_4}{\partial x},\ \int_{c_5} U\phi_3=\frac{\partial F_4}{\partial y} , \\
&\int_{c_5} U\phi_4=\frac{\partial^2 F_4}{\partial x\partial y} .
\end{aligned} \tag{2.6}$$

Lemma 3. *For any $(x, y)\notin L\cup A$, $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ form a basis of $H^2(X, Y, \nabla_\omega)$.*

Proof. We may take ϕ_5 to be rational with respect to (u, v, x, y). First we note that $\phi_1, \cdots, \phi_4$ can be seen as elements of $H^2(X, Y, \nabla_\omega)$ naturally and that they are linearly independent. Suppose we have, in $H^2(X, Y, \nabla_\omega)$,

$$\phi_5=\alpha_1\phi_1+\alpha_2\phi_2+\alpha_3\phi_3+\alpha_4\phi_4$$

where $\alpha_1, \cdots, \alpha_4$ are rational functions of x and y. The integration over the cycle $c_5=\Delta_{\mathrm{reg}}$ gives

$$\alpha_1 F_4+\alpha_2\frac{\partial F_4}{\partial x}+\alpha_3\frac{\partial F_4}{\partial y}+\alpha_4\frac{\partial^2 F_4}{\partial x\partial y}=\int_{c_5}U\phi_5\,. \tag{2.7}$$

In view of (2.4), the right hand side can also be regarded as the duality between $H^1(Y, \nabla_\omega|_Y)$ and $H_1(Y, S_{-\omega}|_Y)$:

$$\int_{c_5}U\phi_5=Cte\times y^{\lambda_3}\times\int_0^\infty u^{\lambda_2}\Big(\frac{x-1}{y}-u\Big)^{\lambda_3}du\,. \tag{2.8}$$

Restricting to $y=c$ with $c\neq 0, 4$, the left hand side of (2.7) does not branch at $x=1$ as we saw in §1. But the right hand side does. This gives a contradiction.

PROPOSITION 3. *There exists certain constants* K_j, $j=1, \cdots, 4$ *such that*

$$\int_{c_5}U\phi_i=\sum_{j=1}^{4}K_j\times\int_{c_j}U\phi_i,\ 1\leqq i\leqq 4\,. \tag{2.9}$$

PROOF. The column vector

$$Y_c={}^t\Big(\int_c U\phi_1, \cdots, \int_c U\phi_4\Big)$$
$$c\in H_2(X, Y, S_{-\omega})$$

satisfies the following system:

$$dY_c=\tilde{\omega}\cdot Y_c\,,$$

where $\tilde{\omega}$ denotes a rational matrix 1-form of x, y of the form:

$$\tilde{\omega}=\begin{array}{cc} 4 & 1 \\ \left(\begin{matrix}\omega & 0\\ 0 & \omega'\end{matrix}\right) & \begin{matrix}4\\ 1\,.\end{matrix}\end{array}$$

Hence we have

$${}^t\Big(\int_{c_5}U\phi_1, \cdots, \int_{c_5}U\phi_4\Big)={}^t\Big(\int_{c_5}U\phi_1, \cdots, \int_{c_5}U\phi_4\Big)\,.$$

Therefore, ${}^t\Big(\int_{c_5}U\phi_1, \cdots, \int_{c_5}U\phi_4\Big)$ is a linear combination of the fundamental system of solutions:

$$ {}^t\left(\int_{c_i} U\phi_1, \cdots, \int_{c_i} U\phi_4\right), \quad 1 \leqq i \leqq 4 . $$

§ 3. Construction of cycles attached to certain fibering of X.

In the following, we shall assume that $(x, y) \in P^2 - L$ is real.

Since $X_R - S_R$ is simply connected, we can define the regularization of cycles of the relative homology group:

$$ Z_2(X_R, S_R, C) \xrightarrow{\text{reg}} Z_2(\bar{X}-\bar{S}, S_{-\omega}) \tag{3.1} $$

(relative cycles) (twisted cycles)

and the map reg_*:

$$ H_2(X_R, S_R, C) \xrightarrow{\text{reg}_*} H_2(\bar{X}-\bar{S}, S_{-\omega}) \quad \text{(see [1])} . \tag{3.2} $$

DEFINITION 1. The image of the map reg_* will be called the group of real cycles and any other cycles will be called imaginary cycles.

PROPOSITION 4. *Let*

$$ \{(u, v) \in C^2;\ 1 - xu - yv = 0\} \cap \{(u, v) \in C^2;\ u + v - uv = 0\} $$

be real. Then we have the exact sequence

$$ H_2(\bar{X}_R, \bar{S}_R, C) \xrightarrow{\text{reg}_*} H_2(\bar{X}-\bar{S}, S_{-\omega}) \longrightarrow 0 . $$

To prove Proposition 4, we first construct a fibering $\bar{X}-\bar{S} \to B$. Let $\bar{B}$ be a projective line in P^2 which passes through the origin P_1 and the contact point P_2 of C and L_3. $\bar{B}$ does not pass through the other singular points of L (see Figure 1). We set $P_3 = (\bar{B} - P_2) \cap C$. We take real base point $P_0 = (x_0, y_0)$ in $\bar{B}$ between P_3 and P_1. We set

$$ B = \bar{B} - \{P_0, P_1\} . $$

For any $(x, y) \in B$, put (see Figure 2)

$$ F(x, y) = \{(u, v) \in \bar{X} - \bar{S},\ 1 - xu - yv = 0\} . $$

Finally, define the map $p: X - S \to B$ by $p(u, v) = (x, y)$ if $(u, v) \in F(x, y)$. We note that P_i, $i = 0, 1, 2, 3$ are real and the singlar fibres are located on P_2 and P_3.

We may suppose, by suitable coordinization, $P_0 = \infty$ and P_i, $i = 1, 2, 3$

are located as in the following Figure 3:

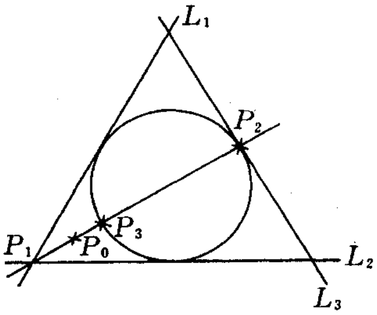


FIGURE 1

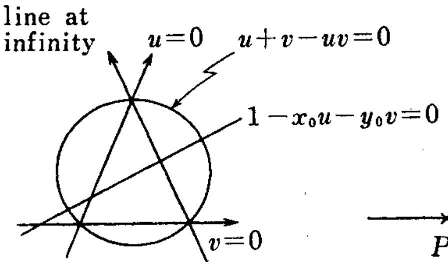


FIGURE 2

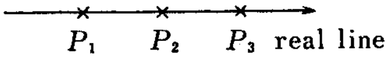


FIGURE 3

We set

$$\bar{F}(x, y)\cap\{v=0\}=q_1,\ \bar{F}(x, y)\cap\{u+v-uv=0\}=\{q_2, q_3\}$$
$$\bar{F}(x, y)\cap\{u=0\}=q_4,\ \bar{F}(x, y)\cap\{\text{line at infinity}\}=q_0\ ,$$

where $\bar{F}(x, y)$ denotes the closure of $F(x, y)$ in $\bar{X}$. Note that all of $\bar{F}(x, y)$ have one common point q_0.

Next we construct a Lefschetz type cell decomposition of $\bar{X}-\bar{S}$ compatible with the above fibring $p: \bar{X}-\bar{S}\rightarrow B$. We take a cell decomposition of B as follows.

$$B=\varDelta_+\cup\varDelta_-\cup\bigcup_{i=0}^{3} I_i\cup P_2\cup P_3\ ,$$
$$\varDelta_+(\varDelta_-)=\text{upper half plane (lower half plane)}\ ,$$
$$I_i=(P_i, P_{i+1})\quad i=0, 1, 2, 3\ ,\quad \text{where}\quad P_4=P_0\ .$$

By choosing suitable coordinates of $\bar{X}-\bar{S}$, we may suppose q_0 is a point at infinity on every $\bar{F}(x, y)$ and $q_1, \cdots, q_4$ are as in the following Figure 4 for $(x, y)\in I_0$:

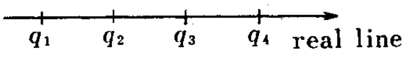


FIGURE 4

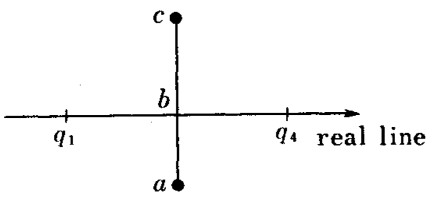


FIGURE 5

We take cell decomposition of $F(x, y)$ as follows:

$$F(x, y)=\delta_+\cup\delta_-\cup\bigcup_{i=0}^{4} l_i,\ (x, y)\in I_0$$

(3.3) δ_+(resp. δ_-) = upper half plane (resp. lower half plane)
$l_i=(q_i,\ q_{i+1})$, where $q_5=q_0$
$F(x,\ y)=\delta'_+\cup\delta'_-\cup l_{1b}\cup l_{b4}\cup l_{ab}\cup l_{bc}\cup o,\ (x,\ y)\in I_2$

(3.4) δ'_+(resp. δ'_-) = slitted upper half plane (resp. slitted lower half plane)
$l_{1b}=(q_1,\ o),\ l_{b4}=(o,\ q_1)$,
$l_{ab}=(q_2,\ o)$ or $(q_3,\ o)$ = slit in the lower half plane
$l_{bc}=(o,\ q_3)$ or $(o,\ q_2)$ = slit in the lower half plane
o: middle point of the segment $\overline{q_2q_3}$.

Notice that o is a real fixed point provided that $(x,\ y)\in I_2$. We can define the unique cell decomposition of $\bar{X}-\bar{S}$ compatible with (3.3) and (3.4).

PROOF OF PROPOSITION 4. Let $(x,\ y)=(x_0,\ y_0)=P_0$ be as above. We shall observe the boundary operation in detail. We have

1) $$0=\partial_\theta(I_i\times\delta_\pm)=I_i\times\partial_\theta\delta_\pm+\partial I_i\times\delta_\pm$$
2) $$0=\partial_\theta(\varDelta_\pm\times l_i)$$

in $H_2(X,\ S_{-\omega})$. Since $\partial I_1=P_1,\ \partial I_3=-P_3$, putting $i=1,\ 3$ in 1), every 2-cycle of type $P_1\times\delta_\pm$ is a linear combination of cycles of type $I_j\times l_k$, i.e. real cycles. We have only to show the following four imaginary cycles can be written by linear combinations of real cycles:

$$I_2\times\{l_{1b}-l_{ab}\}+I_3\times l_1,\ I_2\times\{l_{1b}+l_{bc}\}+I_3\times l_1\ ,$$
$$I_2\times\{l_{b4}-l_{bc}\}+I_3\times l_3,\ I_2\times\{l_{b4}+l_{ab}\}+I_3\times l_3\ .$$

Putting $i=1,\ 3$ in 2) and noting that the imaginary cycle appears only once in each formula, we obtain the desired expressions.

REMARK. For general S defined over R, Proposition 4 is not always true. For example, for

$$U(u,\ v)=(u^3+v^3-1)^2u^2\ ,$$

put $S=\{U(u,\ v)=0\}$, then we have

$$\dim_C H_2(\bar{X}_R,\ \bar{S}_R,\ C)=4\ ,\quad\text{and}\quad \dim_C H_2(\bar{X}-\bar{S},\ S_{-\omega})=6\ .$$

Now we specify a basis of real cycles $\{c_1,\ c_2,\ c_3,\ c_4\}$ for $P_0=(x_0,\ y_0)$ as in the following figure: (The other real cycles are also numbered.)

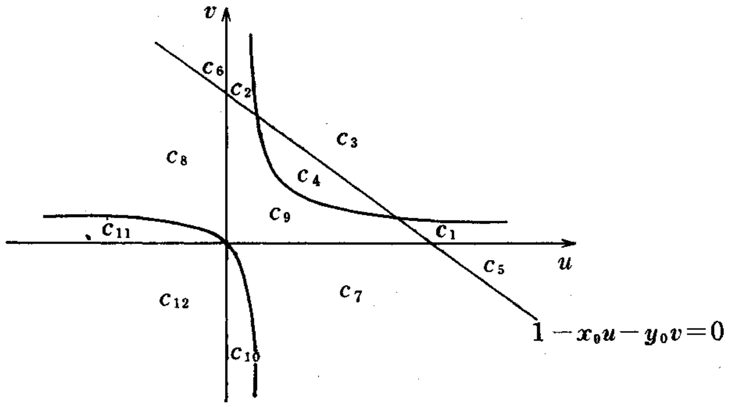


FIGURE 6

PROPOSITION 5. *c_1, c_2, c_3, c_4 span the homology group $H_2(X, S_{-\omega})$.*

PROOF. By Proposition 4, it suffices to show c_1, c_2, c_3, c_4 actually span the real cycles. Let b be the curve in $F(x, y)$, $(x, y) \in I_0$ as in the following figure.

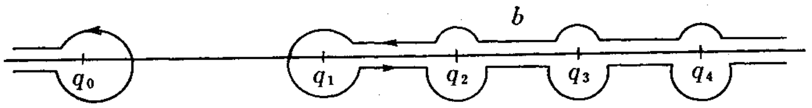


FIGURE 7

Since b is homotopically zero in $F(x, y)$, we have

$$0=I_0\times\{(1-e(\lambda_1+2\lambda_4+\lambda_3+\lambda_5))l_1+(1-e(\lambda_1+\lambda_4+\lambda_3+\lambda_5))l_2 \\ +(1-e(\lambda_1+\lambda_3+\lambda_5))l_3+(1-e(\lambda_3+\lambda_5))l_4\} .$$

Hence

$$c_5=\frac{1}{e(\lambda_3+\lambda_5)-1}\{(1-e(\lambda_1+2\lambda_4+\lambda_3+\lambda_5))c_1+(1-e(\lambda_1+\lambda_4+\lambda_3+\lambda_5))c_3 \\ +(1-e(\lambda_1+\lambda_3+\lambda_5))c_2\} . \tag{3.5}$$

By the same way, we have

$$c_6=\frac{1}{e(\lambda_1+\lambda_2+2\lambda_4)-1}\{(1-e(\lambda_1+2\lambda_4))c_1+(1-e(\lambda_1+\lambda_4))c_3 \\ +(1-e(\lambda_1))c_2\} . \tag{3.6}$$

Fibring by linear pencils through the origin and by the same argument as above, we can show that c_9, c_{12} are linear combinations of c_i, $i=1, \cdots, 4$. Fibring by linear pencils parallel to $\{v=0\}$ (resp. $\{u=0\}$), we can show that c_7, c_{10}(resp. c_8, c_{11}) are linear combinations of c_5, c_{12}(resp. c_6, c_{12}).

§ 4. Monodromy group with respect to $\{c_1, c_2, c_3, c_4\}$.

In view of Proposition 5, for the calculation of the monodromy group, it suffices to observe the variation:

$$\delta\colon \pi_1(P^2-L,\, P_0) \longrightarrow \operatorname{Aut} H_2(X,\, S_{-\infty})$$

with respect to the basis $\{c_1, c_2, c_3, c_4\}$.

We define $\gamma_i \in \pi_1(P^2-L, P_0)$ $i=1, 2, 3$ as follows (see Figure 8).

Put $\varepsilon_0 = x_0 = y_0$ and take a small positive number ε_1 such that $0<\varepsilon_1<\varepsilon_0^2$.

i) $\gamma_1=[\tau_1\cdot\sigma_1\cdot\tau_1^{-1}]$ ($[\gamma]$ denotes the homotopy class of the loop γ) where

τ_1: a segment $\varepsilon_1 \leqq x \leqq \varepsilon_0$ on $y=\varepsilon_0$ joining $x=\varepsilon_0$ to $x=\varepsilon_1$,

σ_1: a circle $x=\varepsilon_1 e^{i\theta}$ with $0\leqq\theta\leqq 2\pi$ on $y=\varepsilon_0$.

ii) $\gamma_2=[\tau_2\cdot\sigma_2\cdot\tau_2^{-1}]$
where

τ_2: a segment $\varepsilon_1 \leqq y \leqq \varepsilon_0$ on $x=\varepsilon_0$ joining $y=\varepsilon_0$ to $y=\varepsilon_1$,

σ_2: a circle $y=\varepsilon_1 e^{i\theta}$ with $0\leqq\theta\leqq 2\pi$ on $x=\varepsilon_0$.

iii) $\gamma_3=[\tau_3\cdot\sigma_3\cdot\tau_3^{-1}]$
where

τ_3: a segment $\varepsilon_0 \leqq x(=y) \leqq \dfrac{1}{4}-\varepsilon_1$ on $x=y$ joining $x(=y)=\varepsilon_0$ to $x(=y)=\dfrac{1}{4}-\varepsilon_1$,

σ_3: a circle $x-\dfrac{1}{4}\left(=y-\dfrac{1}{4}\right)=-\varepsilon_1 e^{i\theta}$ with $0\leqq\theta\leqq 2\pi$.

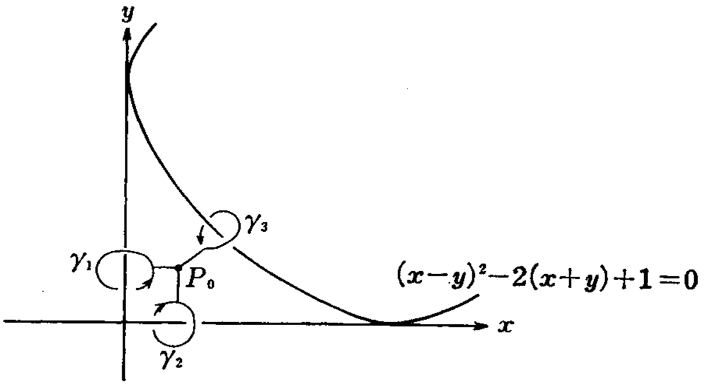


FIGURE 8

$$\gamma_1, \gamma_2, \gamma_3 \text{ actually generate } \pi_1(P^2-L, P_0)\,.$$

For the proof, see Appendix.

Now we calculate the monodromy group with respect to the basis $\{c_1, c_2, c_3, c_4\}$ by the method of F. Pham's generalized Lefschetz principle (see [3] and [8]).

I) Variation along γ_1.

$$\begin{aligned}
&c_1 \longrightarrow e(-(\lambda_1+\lambda_4))c_1\\
&c_2 \longrightarrow c_2\\
&c_3 \longrightarrow c_3+(e(\lambda_4)-e(\lambda_2+\lambda_3+\lambda_4+\lambda_5))c_1+(e(\lambda_2+\lambda_4)-e(\lambda_2+\lambda_3+\lambda_4+\lambda_5))c_5\\
&c_4 \longrightarrow c_4+(e(\lambda_2+\lambda_4+\lambda_5)-e(\lambda_4))c_1+(e(\lambda_2+\lambda_4+\lambda_5)-e(\lambda_2+\lambda_4))c_5\,.
\end{aligned}$$

Using the formula (3.5) in the proof of Proposition 5, we get

$$\begin{aligned}&(c_1, c_2, c_3, c_4)\\ &\longrightarrow (c_1, c_2, c_3, c_4)\end{aligned}\begin{bmatrix} e(-(\lambda_1+\lambda_4)) & 0 & 0 & h_1(\lambda_1, \lambda_2)\\ 0 & 1 & e(-(\lambda_1+\lambda_4))(1-e(\lambda_1)) & h_2(\lambda_1, \lambda_2)\\ 0 & 0 & e(-(\lambda_1+\lambda_4)) & h_3(\lambda_1, \lambda_2)\\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

where

$$\begin{aligned}
h_1(\lambda_1, \lambda_2) &= \frac{e(\lambda_4)(e(\lambda_2)-1)(e(-\lambda_3)-1)}{e(\lambda_1+\lambda_2+2\lambda_4)-1}\\
h_2(\lambda_1, \lambda_2) &= \frac{e(\lambda_2+\lambda_4)(e(\lambda_5)-1)(1-e(\lambda_1))}{e(\lambda_1+\lambda_2+2\lambda_4)-1}\\
h_3(\lambda_1, \lambda_2) &= \frac{e(\lambda_2+\lambda_4)(e(\lambda_5)-1)(1-e(\lambda_1+\lambda_4))}{e(\lambda_1+\lambda_2+2\lambda_4)-1}\,.
\end{aligned}$$

II) Variation along γ_2.

$$\begin{aligned}
&c_1 \longrightarrow c_1\\
&c_2 \longrightarrow e(-(\lambda_2+\lambda_4))c_2\\
&c_3 \longrightarrow c_3+(1-e(\lambda_1+\lambda_3+\lambda_5))c_2+(1-e(\lambda_3+\lambda_5))c_6\\
&c_4 \longrightarrow c_4+(e(\lambda_1+\lambda_5)-1)c_2+(e(\lambda_5)-1)c_6\,.
\end{aligned}$$

Using the formula (3.6) in the proof of Proposition 5, we get

$$\begin{aligned}&(c_1, c_2, c_3, c_4)\\ &\longrightarrow (c_1, c_2, c_3, c_4)\end{aligned}\begin{bmatrix} 1 & 0 & e(-\lambda_2)-1 & e(\lambda_4)h_2(\lambda_2, \lambda_1)\\ 0 & e(-(\lambda_2+\lambda_4)) & 0 & e(-\lambda_4)h_1(\lambda_2, \lambda_1)\\ 0 & 0 & e(-(\lambda_2+\lambda_4)) & h_3(\lambda_2, \lambda_1)\\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

III) Variation along γ_3.

$$(c_1, c_2, c_3, c_4) \longrightarrow (c_1, c_2, c_3, c_4)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -e(\lambda_3) & -e(\lambda_3+\lambda_4) & e(\lambda_3)(1+e(\lambda_4)) & -e(\lambda_3+\lambda_4) \end{bmatrix}.$$

THEOREM. *The monodromy group of the system (F_4) with respect to the fundamental system of solutions ${}^t\left(\int_{d_i} U\phi_1, \cdots, \int_{d_i} U\phi_4\right)$, $1 \leqq i \leqq 4$, is generated by*

$$\rho(\gamma_1)=\begin{bmatrix} e(-(\lambda_1+\lambda_4)) & 0 & 0 & f_1(\lambda_1, \lambda_2) \\ 0 & 1 & e(-\lambda_1)-1 & f_2(\lambda_1, \lambda_2) \\ 0 & 0 & e(-(\lambda_1+\lambda_4)) & f_3(\lambda_1, \lambda_2) \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\rho(\gamma_2)=\begin{bmatrix} 1 & 0 & e(-\lambda_2)-1 & f_2(\lambda_2, \lambda_1) \\ 0 & e(-(\lambda_2+\lambda_4)) & 0 & f_1(\lambda_2, \lambda_1) \\ 0 & 0 & e(-\lambda_2+\lambda_4)) & f_3(\lambda_2, \lambda_1) \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\rho(\gamma_3)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -e(\lambda_3) & -e(\lambda_3) & e(\lambda_3)(1+e(\lambda_4)) & -e(\lambda_3+\lambda_4) \end{bmatrix},$$

where

$$f_1(\lambda_1, \lambda_2)=\frac{e(\lambda_4)(e(\lambda_2)-1)(e(-\lambda_3)-1)}{e(\lambda_1+\lambda_2+2\lambda_4)-1}$$

$$f_2(\lambda_1, \lambda_2)=\frac{e(\lambda_2+2\lambda_4)(e(\lambda_5)-1)(1-e(\lambda_1))}{e(\lambda_1+\lambda_2+2\lambda_4)-1}$$

$$f_3(\lambda_1, \lambda_2)=\frac{e(\lambda_2+\lambda_4)(e(\lambda_5)-1)(1-e(\lambda_1+\lambda_4))}{e(\lambda_1+\lambda_2+2\lambda_4)-1}$$

and

$$d_i=c_i,\ i \neq 2,\ d_2=e(-\lambda_4)c_2\ .$$

Appendix

In this appendix, we shall show that γ_1, γ_2, γ_3 generate $\pi_1(P^2-L, P_0)$

and satisfy the following generating relations:

$$\gamma_1\gamma_2=\gamma_2\gamma_1,\ (\gamma_1\gamma_3)^2=(\gamma_3\gamma_1)^2,\ (\gamma_2\gamma_3)^2=(\gamma_3\gamma_2)^2\ ,$$

which define an Artin group of rank three of infinite type. We shall use the pencil section method of Zariski-Van Kampen ([9]). In the following, the elements of the fundamental group will be identified with their representative loops.

We take a linear pencil Λ_ξ passing through the point P_0 such that $\{x+y=0\}\cap\Lambda_\xi=\xi$. By suitable coordinization of $\{x+y=0\}$, we may suppose

$\Lambda_{-\infty}$, Λ_{α_3} are tangents of C,
$\Lambda_{\alpha_{-2}}$, Λ_{α_2} pass through $L_2\cap C$ and $L_1\cap C$ respectively,
$\Lambda_{\alpha_{-1}}$, Λ_{α_1} are parallel to L_2 and L_1 respectively,
Λ_{α_0} passes through the origin (see Figure 8),

and

$$-\infty<\alpha_{-2}<\alpha_{-1}<\alpha_0<\alpha_1<\alpha_2<\alpha_3,\ (\alpha_{-1}<0<\alpha_0)\ .$$

Let $\pi: C^2-L_1\cup L_2\cup C\cup\Lambda_0\rightarrow C-\{\alpha_i,\ i=-2,\cdots 3\}\cup\{0\}$ be the map defined by $\pi(x)=\xi$ for $x\in\Lambda_\xi$ (ξ is uniquely determined by x). Then we have a fibre bundle

$$(C^2-L_1\cup L_2\cup C\cup\Lambda_0,\ \pi,\ C-\{\alpha_i,\ i=-2,\cdots,3\}\cup\{0\})\ .$$

We take special generators of $\pi_1(C-\{\alpha_i,\ i=-2,\cdots 3\},\ 0)$ as follows (see Figure 9):

$$\begin{aligned}
&g_0=\tau_0\cdot\sigma_0\cdot\tau_0^{-1},\ g_1=(\tau_0\cdot\sigma_0^-\cdot\tau_1)\cdot\sigma_1\cdot(\tau_0\cdot\sigma_0^-\cdot\tau_1)^{-1}\ ,\\
&g_2=(\tau_0\cdot\sigma_0^-\cdot\tau_1\cdot\tau_1^-\cdot\tau_2)\cdot\sigma_2\cdot(\tau_0\cdot\sigma_0^-\cdot\tau_1\cdot\sigma_1^-\cdot\tau_2)^{-1}\ ,\\
&g_3=(\tau_0\cdot\sigma_0^-\cdot\tau_1\cdot\sigma_1^-\cdot\tau_2\cdot\sigma_2^-\cdot\tau_3)\cdot\sigma_3\cdot(\tau_0\cdot\sigma_0^-\cdot\tau_1\cdot\sigma_1^-\cdot\tau_2\cdot\sigma_2^-\cdot\tau_3)^{-1}\ ,\\
&g_{-1}=\tau_{-1}\cdot\sigma_{-1}\cdot\tau_{-1}^{-1},\ g_{-2}=(\tau_{-1}\cdot\sigma_{-1}^{+}\cdot\tau_{-2})\cdot\sigma_{-2}\cdot(\tau_{-1}\cdot\sigma_{-1}^{+}\cdot\tau_{-2})^{-1}\ ,
\end{aligned}$$

where

τ_{-1} is the segment from 0 to $\alpha_{-1}-\varepsilon$,
τ_{-2} is the segment from $\alpha_{-1}+\varepsilon$ to $\alpha_{-2}-\varepsilon$,

σ_i^+(resp. σ_i^-), $i=-2,\cdots,3$, is the upper half circle (resp. lower half circle) around α_i with radius ε which is oriented counterclockwise, and

$$\sigma_i=\sigma_i^-\cdot\sigma_i^+,\ i\geqq 0,\ \sigma_i=\sigma_i^+\cdot\sigma_i^-,\ i<0\ .$$

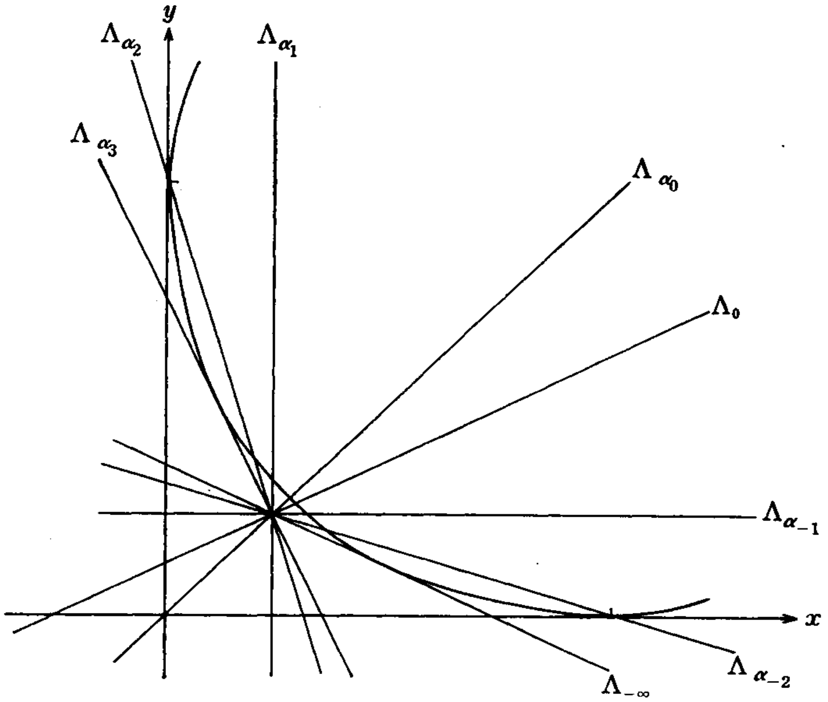


FIGURE 9

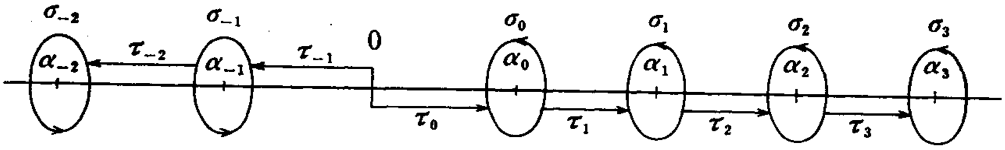


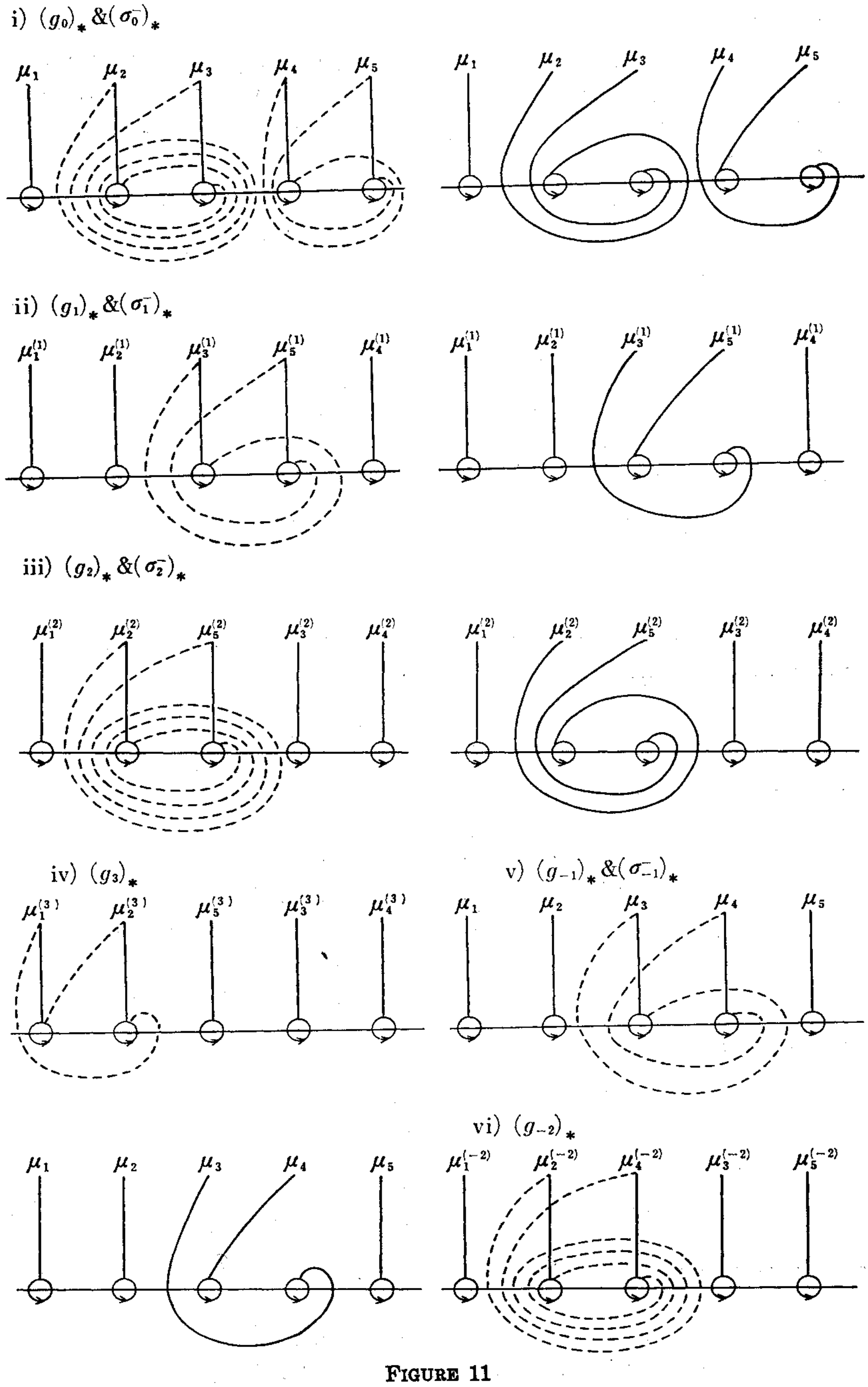
FIGURE 10

We take ξ_0 sufficiently small and positive, and take the generators $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ of $\pi_1(\Lambda_{\xi_0}-\Lambda_{\xi_0}\cap L, P_0)$ as indicated in Figure 10 (we may suppose, by suitable coordinization of Λ_ξ, $P_0=\sqrt{-1}\infty$ in Λ_ξ for any ξ),

μ_1: a loop enclosing counterclockwise only $\theta_1=\Lambda_{\xi_0}\cap C$
μ_2: a loop enclosing counterclockwise only $\theta_2=\Lambda_{\xi_0}\cap C$ with $\theta_1<\theta_2$
μ_3: a loop enclosing counterclockwise only $\theta_3=\Lambda_{\xi_0}\cap L_3$
μ_4: a loop enclosing counterclockwise only $\theta_4=\Lambda_{\xi_0}\cap L_2$
μ_5: a loop enclosing counterclockwise only $\theta_5=\Lambda_{\xi_0}\cap L_1$.

Each g_i defines an automorphism $(g_i)_*$ of $\pi_1(\Lambda_\xi-\Lambda_\xi\cap L, P_0)$. By the theorem of Van Kampen ([9]), relations

1) $$\mu_1\cdot\mu_2\cdot\mu_3\cdot\mu_4\cdot\mu_5=1$$
2) $$(g_i)_*(\mu_j)=\mu_j,\ j=1, \cdots, 5\ ,$$



i) $(g_0)_* \& (\sigma_0^-)_*$

ii) $(g_1)_* \& (\sigma_1^-)_*$

iii) $(g_2)_* \& (\sigma_2^-)_*$

iv) $(g_3)_*$

v) $(g_{-1})_* \& (\sigma_{-1}^-)_*$

vi) $(g_{-2})_*$

FIGURE 11

are the generating relations of $\pi_1(P_2-L, P_0)$. For the computation of 2), we take the generators $\mu_j^{(k)}k=-2, \cdots, 3$ of $\pi_1(\Lambda_\xi-\Lambda_\xi\cap L, P_0)$ for $\xi\in\tau_k$, with $\mu_j^{(0)}=\mu_j^{(-1)}=\mu_j$ as indicated in Figure 10. We shall compute

A) connection relation, namely, the relation of μ_j and $\mu_j^{(k)}$,

B) local monodromy relation, namely,

$$(g_\iota)_*(\mu_j^{(k)})=\mu_j^{(k)} .$$

i) *local monodromy relations*

$$(g_0)_*(\mu_2)=(\mu_2\mu_3)^2\mu_2(\mu_2\mu_3)^{-2}$$
$$(g_0)_*(\mu_3)=(\mu_2\mu_3)^2\mu_3(\mu_2\mu_3)^{-2}$$
$$(g_0)_*(\mu_4)=(\mu_4\mu_5)\mu_5(\mu_4\mu_5)^{-1} ,$$

i.e.,

$$(\mu_2\mu_3)^2=(\mu_3\mu_2)^2,\ \mu_4\mu_5=\mu_5\mu_4 .$$

connection relations

$$\mu_1^{(1)}=\mu_1,\ \mu_2^{(1)}=\mu_3^{-1}\mu_2\mu_3,\ \mu_3^{(1)}=(\mu_2\mu_3)^{-1}\mu_3(\mu_2\mu_3),\ \mu_4^{(1)}=\mu_5^{-1}\mu_4\mu_5,\ \mu_5^{(1)}=\mu_5.$$

Using local monodromy relations, we have

$$\mu_1^{(1)}=\mu_1,\ \mu_2^{(1)}=\mu_3^{-1}\mu_2\mu_3,\ \mu_3^{(1)}=\mu_2\mu_3\mu_2^{-1},\ \mu_4^{(1)}=\mu_4,\ \mu_5^{(1)}=\mu_5 .$$

ii) *local monodromy relations*

$$(g_1)_*(\mu_3^{(1)})=(\mu_3^{(1)}\mu_5^{(1)})\mu_3^{(1)}(\mu_3^{(1)}\mu_5^{(1)})^{-1}$$
$$(g_1)_*(\mu_5^{(1)})=(\mu_3^{(1)}\mu_5^{(1)})\mu_5^{(1)}(\mu_3^{(1)}\mu_5^{(1)})^{-1} ,$$

i.e.,

$$\mu_3^{(1)}\mu_5^{(1)}=\mu_5^{(1)}\mu_3^{(1)} .$$

Using the relations in i), we get

$$\mu_2\mu_3\mu_2^{-1}\mu_5=\mu_5\mu_2\mu_3\mu_2^{-1} .$$

connection relations

$$\mu_1^{(2)}=\mu_1,\ \mu_2^{(2)}=\mu_3^{-1}\mu_2\mu_3,\ \mu_3^{(2)}=\mu_2\mu_3\mu_2^{-1},\ \mu_4^{(2)}=\mu_4,\ \mu_5^{(2)}=\mu_5 .$$

iii) *local monodromy relations*

$$(g_2)_*(\mu_2^{(2)})=(\mu_2^{(2)}\mu_5^{(2)})^2\mu_2^{(2)}(\mu_2^{(2)}\mu_5^{(2)})^{-2}$$
$$(g_2)_*(\mu_5^{(2)})=(\mu_2^{(2)}\mu_5^{(2)})^2\mu_5^{(2)}(\mu_2^{(2)}\mu_5^{(2)})^{-2} .$$

Using ii), we have

$$(\mu_3^{-1}\mu_2\mu_3\mu_5)^2=(\mu_5\mu_3^{-1}\mu_2\mu_3)^2 .$$

connection relations

$$\mu_1^{(3)}=\mu_1,\ \mu_2^{(3)}=(\mu_5\mu_3)^{-1}\mu_2(\mu_3\mu_5),\ \mu_3^{(3)}=\mu_2\mu_3\mu_2^{-1},\ \mu_4^{(3)}=\mu_4 ,$$
$$\mu_5^{(3)}=\mu_3^{-1}\mu_2\mu_3\mu_5(\mu_3^{-1}\mu_2\mu_3)^{-1} .$$

iv) *local monodromy relations*

$$(g_3)_*(\mu_1^{(3)})=\mu_1^{(3)}\mu_2^{(3)}(\mu_1^{(3)})^{-1}$$
$$(g_3)_*(\mu_2^{(3)})=\mu_1^{(3)} .$$

Using iii), we have

$$\mu_3\mu_5\mu_1=\mu_2\mu_3\mu_5 .$$

v) *local monodromy relations*

$$(g_{-1})_*(\mu_3)=(\mu_3\mu_4)\mu_3(\mu_3\mu_4)^{-1}$$
$$(g_{-1})_*(\mu_4)=(\mu_3\mu_4)\mu_4(\mu_3\mu_4)^{-1} ,$$

i.e.,

$$\mu_3\mu_4=\mu_4\mu_3 .$$

connection relations

$$\mu_i^{(-2)}=\mu_i ,\quad i=1, \cdots, 5 .$$

vi) *local monodromy relations*

$$(g_{-2})_*(\mu_2^{(-2)})=(\mu_2^{(-2)}\mu_4^{(-2)})^2\mu_2^{(-2)}(\mu_2^{(-2)}\mu_4^{(-2)})^{-2}$$
$$(g_{-2})_*(\mu_4^{(-2)})=(\mu_2^{(-2)}\mu_4^{(-2)})^2\mu_4^{(-2)}(\mu_2^{(-2)}\mu_4^{(-2)})^{-2} .$$

Using v), we have

$$(\mu_2\mu_4)^2=(\mu_4\mu_2)^2 .$$

Now we shall simplify the above relations by using $\mu_1\mu_2\mu_3\mu_4\mu_5=1$. We can easily see that iii) is equivalant to

iii′) $$(\mu_1\mu_5)^2=(\mu_5\mu_1)^2 .$$

Using i) and iv), we have

$$\mu_1\mu_3\mu_5\mu_1\mu_4=1 .$$

Hence

*) $$\mu_3=(\mu_5\mu_1\mu_4\mu_1)^{-1}\ ,$$

**) $$\mu_2=(\mu_4\mu_1)^{-1}\mu_1\mu_4\mu_1\ .$$

Substituting these expressions to i), ii), iv), v) and vi), we get

i′) $$\mu_4\mu_5=\mu_5\mu_4\ ,\quad (\mu_4\mu_1)^2=(\mu_1\mu_4)^2\ .$$

Hence

**′) $$\mu_2=\mu_4\mu_1\mu_4^{-1}\ .$$

ii), iv), v) and vi) are deduced from i′), iii′) provided *) and **′). Therefore we get the generating relations of $\pi_1(P^2-L,\ P_0)$ as follows:

generators: μ_1, μ_4, μ_5,

generating relations:

$$\mu_4\mu_5=\mu_5\mu_4,\ (\mu_4\mu_1)^2=(\mu_1\mu_4)^2,\ (\mu_5\mu_1)^2=(\mu_1\mu_5)^2\ .$$

Finally we note

$$\gamma_1=\mu_5,\ \gamma_2=\mu_4^{(1)}=\mu_4,\ \gamma_3=\mu_1\ .$$

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