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Monodromy of Variations of Hodge Structure

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Abstract. We present a survey of the properties of the monodromy of local systems on quasi-projective varieties which underlie a variation of Hodge structure. In the last section, a less widely known version of a Noether–Lefschetz-type theorem is discussed.

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1. Introduction

The kth primitive cohomology group $H^k_{\mathrm{prim}}(X)$ of a smooth projective variety X is known to carry a weight k polarized rational Hodge structure. For a family of those over a smooth connected base manifold S, the cohomology groups glue together to a local system on S. Such a local system is completely determined by its *monodromy representation*, i.e. the induced representation of $\pi_1(S, s_0)$ on the kth cohomology group V of the fibre at s_0. For a *projective family*, i.e. one which is locally embeddable in $S \times \mathbb{P}^N$ (projection onto the first factor giving the family), the polarizations glue together so that the monodromy representation lands in $\mathrm{Aut}(V, Q)$, where Q is the polarization. The polarized Hodge structures on the primitive kth rational cohomology groups of the fibre glue to a *polarized variation of weight k rational Hodge structures*. All of these notions will be reviewed in Sections 2, 3 and 4. For the moment, think of the latter as a certain filtration on the complexification of V that varies holomorphically with s (and satisfies some additional properties to be made explicit later).

The main question that we are considering is 'what are the restrictions on a local system if it underlies a polarized variation of weight k Hodge structures?'

In examples coming from geometry, the base manifold S is often a smooth quasi-projective variety. In that case, it is customary to choose a smooth projective compactification $\overline{S}$ such that the complement $D = \overline{S} \setminus S$ is a divisor with normal crossings, i.e. locally D has an equation $z_1 z_2 \cdots z_a = 0$, where $z_1, \ldots, z_s$ are local

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coordinates in S. Then S looks like $U = (\Delta^*)^a \times \Delta^{s-a}$, where Δ is a unit disk in $\mathbb{C}$ and $\Delta^* = \Delta \setminus \{0\}$. The *local monodromy operators* are then given by the images in $\mathrm{Aut}(V, Q)$ under the monodromy representation of the a loops around the origin (in counter clockwise direction) that generate the fundamental group of this open set U. In fact, for most of what follows S could be Zariski-open in a compact analytic manifold and we can then choose a smooth compact analytic compactification $\overline{S}$ by adding a suitable normal crossing divisor. Any such S will be called a *compactifiable complex manifold*. Note that a given complex manifold may have several compactifications, which are not even bimeromorphic! For an example due to Serre, see [11, Ex. VI.3.2].

In the setting of variations of polarized rational Hodge structure over a compactifiable manifold S, there are serious restrictions on the monodromy which we recall in Section 5 such as the ‘Monodromy Theorem’, the ‘Theorem of the Fixed Part’ and a very restrictive finiteness result saying that there are at most finitely many conjugacy-classes of N-dimensional representations that yield local systems underlying a polarized variation of Hodge structure (of any weight) over a given S. In Section 6 we briefly discuss the proof from [2] that the locus where a given integral cycle of type (p, p) stays of type (p, p) is algebraic when S is algebraic. We also relate the monodromy group and the Mumford–Tate group of the Hodge structure at a very general stalk.

The classical Noether–Lefschetz theorem tells us that the only algebraic cycles on a generic surface in $\mathbb{P}^3$ of degree $\geqslant 4$ are the hypersurface sections. In Section 7, we generalize this to polarizable variations of Hodge structures with ‘big monodromy’. See Theorem 17. This result is folklore among experts, but we think that it is useful to dispose of an elementary proof.

2. Local Systems and Monodromy

Given a ring A, a *local system of A-modules* can be defined for any topological space S: it is a sheaf $\mathbb{V}$ of A_S-modules with the property that every point $x \in S$ has a neighbourhood U such that the natural map $\mathbb{V}(U) \to \mathbb{V}_x$ is an isomorphism. This is a functorial construction: any continuous map $f\colon T \to S$ induces a morphism of ringed spaces $(T, A_T) \to (S, A_S)$ and for any local system $\mathbb{V}$ of A-modules on S, the pull back $f^*(\mathbb{V})$ is a local system of A-modules on T.

If S is locally and globally arcwise connected the *monodromy representation* of $\mathbb{V}$ can be defined as follows. For any arc $\gamma\colon [0, 1] \to S$ the pull-back $\gamma^*\mathbb{V}$ is a local system of A-modules on $[0, 1]$. In particular, we obtain a canonical identification of $(\gamma^*\mathbb{V})_0 \simeq \mathbb{V}_{\gamma(0)}$ with $(\gamma^*\mathbb{V})_1 \simeq \mathbb{V}_{\gamma(1)}$. Choosing a base point $s_0 \in S$ this yields the monodromy representation

$$\rho_{\mathbb{V}}\colon \pi_1(S, s_0) \to \mathrm{Aut}_A(\mathbb{V}_{s_0}).$$

The image of ρ is called the *monodromy group* of the local system $\mathbb{V}$.

The construction of the monodromy representation from a local system has an inverse: let us start with a representation

$$\rho: \pi_1(S, s_0) \to \mathrm{Aut}_A(V)$$

for some A-module V. Let p: $(\tilde{S}, \tilde{s}_0) \to (S, s_0)$ be a universal covering space for (S, s_0). Then we define a sheaf of A_S-modules $\mathbb{V}$ on S as the sheaf associated to the presheaf which to $U \subset S$ open associates the set of all locally constant functions f: $p^{-1}(U) \to V$ such that $f(\gamma(y)) = \rho(\gamma) f(y)$ for all $y \in p^{-1}(U),\ \gamma \in \pi_1(S, s_0)$.

These two constructions define an equivalence between the category of local systems of A-modules on S and the category of modules over the group ring $A[\pi_1(S, s_0)]$. We refer to [3] for more details.

EXAMPLE 1. A *family* p: $X \to S$ *of compact differentiable manifolds* is nothing but a proper submersion between differentiable manifolds. The kth cohomology group $V = H^k(X_s)$ of the fibre X_s at s defines a local system of Abelian groups of finite type on S. If S is connected, fixing s_0 we obtain the monodromy group of the family as the associated representation of $\pi_1(S, s_0)$ on $H^k(X_s)$.

Monodromy representations need not be fully reducible, but in the sequel we mainly consider those. Irreducible representations give *indecomposable local systems*. If we have a representation on a $\mathbb{Q}$-vector space which stays irreducible under field-extensions we say that the representation is *absolutely irreducible*.

There is one particular type of such representations, namely representations with ‘big’ monodromy group in the following sense.

DEFINITION 2. Let $\mathbb{V}$ be a local system of $\mathbb{Q}$-vector spaces on S with monodromy representation

$$\rho: \pi_1(S, s_0) \to \mathrm{Gl}(V), \quad V := \mathbb{V}_{s_0}.$$

(1) The smallest algebraic subgroup of $\mathrm{Gl}(V)$ containing the monodromy group $\rho(\pi_1(S, s_0))$, i.e. the Zariski-closure of the monodromy group, is denoted $\rho(\pi_1(S, s_0))^{\mathrm{cl}}$.

(2) Suppose that in addition V carries a nondegenerate bilinear form Q which is either symmetric or anti-symmetric and which is preserved by the monodromy group. The latter is said to be *big* if the connected component H_{s_0} of $\rho(\pi_1(S, s_0))^{\mathrm{cl}}$ acts irreducibly on $V_{\mathbb{C}}$.

The proof of the following lemma is left to the reader:

LEMMA 3. *Suppose $\mathbb{V}$ is a local system on a connected complex manifold S with big monodromy group. Suppose that π: $S' \to S$ is a finite covering map. Then $\pi^*\mathbb{V}$ has also big monodromy.*

Indeed, the reader can check that the inclusion $\pi_1(S', s'_0) \to \pi_1(S, s_0)$ (where $\pi(s'_0) = s$) induces an isomorphism $H_{s'_0} \simeq H_{s_0}$. It follows that a local system with big monodromy group stays indecomposable after pull-back by any finite covering map.

To determine which monodromy groups are big we need a way to determine the Zariski-closure of subgroups of $\mathrm{Aut}(V, Q)$; we cite from [7]:

LEMMA 4 *Let V be a finite-dimensional complex vector space of dimension n equipped with a nondegenerate symmetric bilinear form Q which is either symmetric or anti-symmetric. Let $M \subset \mathrm{Aut}(V, Q)$ be an algebraic subgroup.*

(1) *If Q is anti-symmetric we suppose that M contains the transvections T_δ: $v \mapsto v + Q(v, \delta)\delta$, where δ runs over an M-orbit R which spans V. Then $M = \mathrm{Aut}(V, Q)$ $(= \mathrm{Sp}(V))$.*
(2) *If Q is symmetric, suppose that M contains the reflections R_δ: $v \mapsto v - Q(v, \delta)\delta$ in 'roots' δ, i.e. with $Q(\delta, \delta) = 2$ which form an R-orbit spanning V. Then either M is finite or $M = \mathrm{Aut}(V, Q)$ $(= \mathrm{O}(V))$.*

SPECIAL CASES. (1) Let $V = V_{\mathbb{Z}} \otimes \mathbb{C}$, where $V_{\mathbb{Z}}$ is a free finite rank $\mathbb{Z}$-module equipped with a nondegenerate anti-symmetric bilinear form. The Zariski-closure inside $\mathrm{Sp}(V)$ of the group $\mathrm{Sp}(V_{\mathbb{Z}})$ of symplectic automorphisms of the lattice V_Z is the full group $\mathrm{Sp}(V)$. This follows from the fact that $\mathrm{Sp}(V_{\mathbb{Z}})$ contains all symplectic transvections T_v, $v \in V_{\mathbb{Z}}$ and for given nonzero $\delta \in V_{\mathbb{Z}}$, the elements $T_v\delta$, $v \in V_{\mathbb{Z}}$ span already V. It follows that the Zariski-closure of any subgroup of finite index in $\mathrm{Sp}(V_{\mathbb{Z}})$ is also the full symplectic group.

(2) Let $V = V_{\mathbb{Z}} \otimes \mathbb{C}$, where V_Z is a free finite $\mathbb{Z}$-module equipped with a nondegenerate symmetric bilinear form Q. If Q is definite, the orthogonal group preserving the lattice $V_{\mathbb{Z}}$ is of course finite and equals its Zariski-closure. Hence it is never big. In general it will contain reflections R_δ in all roots $\delta \in V_{\mathbb{Z}}$. Assuming that these roots contain at least one orbit which spans the lattice, we conclude in the indefinite case that the Zariski closure of $\mathrm{Aut}(V_{\mathbb{Z}}, Q)$ is the full orthogonal group.

These special cases can be used in the following geometric context.

EXAMPLE 5. Consider a smooth subvariety X of $\mathbb{P}^N(\mathbb{C})$ of dimension $n+1$ spanning $\mathbb{P}^N(\mathbb{C})$. Its *dual variety* $X^\vee$ consists of the hyperplanes $H \subset \mathbb{P}^N(\mathbb{C})$ such that $X \cap H$ is singular. It is an irreducible hypersurface in the dual projective space $\mathbb{P}^N(\mathbb{C})^\vee$. Let S denote its complement. Over S we have the tautological family of hyperplane sections $\{Y_s\}$ of X. We have an orthogonal splitting (with respect to cup product)

$$H^n(Y_s; \mathbb{Q}) = H^n_{\mathrm{var}}(Y_s; \mathbb{Q}) \oplus H^n_{\mathrm{fixed}}(Y_s; \mathbb{Q}),$$

where $H^n_{\mathrm{fixed}}(Y_s; \mathbb{Q})$ is the part of $H^n(Y_s; \mathbb{Q})$ which is invariant under the monodromy group of the family. By the invariant cycle theorem ([4, Th. 4.1.1])

$$H^n_{\mathrm{fixed}}(Y_s; \mathbb{Q}) = \mathrm{Im}[H^n(X, \mathbb{Q}) \to H^n(Y_s; \mathbb{Q})].$$

A *Lefschetz pencil of hyperplane sections* of X is a family $\{Y_t\}_{t\in\ell}$ where ℓ is a line in $\mathbb{P}^N(\mathbb{C})^\vee$ which intersects $X^\vee$ transversely. By a theorem of Van Kampen [12], the map $\pi_1(S\cap\ell, s)\to\pi_1(S,s)$ is surjective for a Lefschetz pencil. Hence the monodromy groups for the families over S and over $\ell\cap S$ coincide.

We take as a special set of generators for $\pi_1(S\cap\ell, s)$ the classes of loops based at s surrounding each of the intersection points of ℓ and $X^\vee$ in counterclockwise direction. The Picard–Lefschetz formula computes the action of such a generator and the absolute irreducibility of the action of the monodromy group on $H^n_{\mathrm{var}}(Y_s;\mathbb{Q})$ follows from the fact that all the associated vanishing cycles are conjugate. See [13, Sect. 3] for details. In fact, Lemma 4 implies that the variations on the variable part of the cohomology groups of the families under consideration have big monodromy groups, unless the monodromy group is finite.

For Lefschetz pencils of hypersurfaces of $\mathbb{P}^{n+1}$ or complete intersections of codimension p in $\mathbb{P}^{n+p}$ we can apply the above to $V=H^n_{\mathrm{var}}(Y_s;\mathbb{C})$, Q the intersection pairing (if n is odd) or $(-1)^{n/2}$ times the intersection pairing (if n is even). By the criterion, $\rho(\pi_1(S,s))^{\mathrm{cl}}$ is the full symplectic group if n is odd and either finite or the full orthogonal group if n is even. If n is even and $\rho(\pi_1(S,s))$ is finite, Q must be definite. Observing that the variable cohomology in this case is just the primitive cohomology, the Hodge Riemann bilinear relations tell us however that the signature is (a,b) with a, resp. b the sum of the Hodge numbers $\dim H^{p,q}_{\mathrm{prim}}$ with p even, resp. odd. Since even-dimensional hypersurfaces always have $H^{n,n}_{\mathrm{prim}}\neq 0$, the form Q can only be definite if all Hodge numbers $h^{p,q}$ with $p\neq q$ are zero. It is not too hard to see [5] that this only happens for a quadric hypersurface, a cubic surface or an even-dimensional intersection of two quadrics. So apart from these three classes, the primitive cohomology of a complete intersection of projective space gives a local system with big monodromy group.

3. Hodge Structures

Let us recall some definitions concerning Hodge structures. A *Hodge structure of weight k* is a finitely generated Abelian group V together with a direct sum decomposition

$$V_{\mathbb{C}}=\bigoplus_{p+q=k}V^{p,q},\quad \text{with } V^{p,q}=\overline{V^{q,p}}\quad \textit{the Hodge decomposition}$$

on the complexification $V_{\mathbb{C}}=V\otimes\mathbb{C}$. The decreasing *Hodge filtration* on $V_{\mathbb{C}}$ is defined by

$$F^pV_{\mathbb{C}}=\bigoplus_{r\geqslant p}V^{r,k-r}.$$

From this we see that

$$V^{p,k-p}\simeq \mathrm{Gr}^p_F V_{\mathbb{C}}=F^p/F^{p+1}$$

under the projection $F^p\to\mathrm{Gr}^p_F$.

For a Hodge structure V we define its *Weil operator* as the automorphism C of $V_{\mathbb{C}}$ defined by $Cv = i^{p-q}v$ for $v \in V^{p,q}$. A *polarization* of a Hodge structure V of weight k is a bilinear form $Q\colon V \otimes V \to \mathbb{Z}$ which is $(-1)^k$-symmetric and such that for its $\mathbb{C}$-bilinear extension to $V_{\mathbb{C}}$

(1) The orthogonal complement of F^p is F^{k-p+1};
(2) The Hermitian form $(u, v) \mapsto Q(Cu, \overline{v})$ is positive definite.

The first condition is equivalent with $Q(V^{p,q}, V^{r,s}) = 0$ for $(p, q) \neq (s, r)$ and that Q gives a perfect pairing between $V^{p,q}$ and $V^{q,p}$.

In a similar way one can define $\mathbb{Q}$-Hodge structures and $\mathbb{R}$-Hodge structures. The notion of morphism of Hodge structures is the obvious one: the complexification of the given group homomorphism should respect the Hodge decompositions. It is sufficient to require that the Hodge filtrations be respected.

The category of Hodge structures has tensor products and internal Homs. The category of polarized $\mathbb{Q}$-Hodge structures is abelian and semisimple.

Let $\mathbb{Z}(k)$ denote the subgroup $(2\pi i)^k\mathbb{Z} \subset \mathbb{C} = \mathbb{Z}(k)^{-k,-k}$ which is a Hodge structure of weight $-2k$. Then a polarization Q of a Hodge structure V of weight k induces a morphism of Hodge structures $V \otimes V \to \mathbb{Z}(-k)$.

The cohomology groups of compact Kähler manifolds are Hodge structures. Their primitive cohomology groups are polarized $\mathbb{R}$-Hodge structures. For projective manifolds the primitive cohomology groups are even polarized $\mathbb{Q}$-Hodge structures. Algebraic cycles of codimension p define (p, p)-classes in cohomology group and the Hodge conjecture predicts that the rational (p, p)-classes are carried by algebraic cycles. This motivates the following definition.

DEFINITION 6. Let V be a rational Hodge structure of even weight $2p$. Any $v \in V$ of pure type (p, p) is called a *Hodge element*.

Given any rational Hodge structure V, define for $u \in \mathbb{C}^*$ the operator $C(u) \in \mathrm{Aut}(V_{\mathbb{C}})$ as multiplication by $u^{-p}\bar{u}^{-q}$ on $V^{p,q}$. Note that $C = C(-i)$. This defines a rational representation of the algebraic group $\mathbb{C}^*$ on $V_{\mathbb{C}}$ with image $S(V)$. The *special Mumford–Tate group* $\mathrm{MT}'(V)$ of any rational Hodge structure $V = V_{\mathbb{Q}}$ is the smallest algebraic subgroup of $\mathrm{Gl}(V)$ whose set of real points contains $S(V)$. The *Mumford–Tate group* $\mathrm{MT}(V)$ is defined as the smallest algebraic subgroup of $\mathrm{Gl}(V) \times \mathbb{G}_m$ defined over $\mathbb{Q}$ whose set of real points contains the image of the map $C \times N\colon \mathbb{C}^* \to \mathrm{Gl}(V_{\mathbb{R}}) \times \mathbb{G}_m(\mathbb{R})$ where $N(z) = z\overline{z}$.

The Mumford–Tate group is useful to describe Hodge elements inside twisted tensor spaces related to V defined by

$$T^{m,n}V(p) = V^{\otimes m} \otimes (V^\vee)^{\otimes n} \otimes \mathbb{Q}(p), \quad (m-n)k - 2p = 0.$$

Indeed, this is a Hodge structure of weight $(m-n)k - 2p = 0$ by assumption. On the other hand this is a representation for $\mathrm{MT}(V) \subset \mathrm{GL}(V) \times \mathbb{G}_m$ as follows: $\mathrm{GL}(V)$ acts in the obvious way on $V^{\otimes m} \otimes (V^\vee)^{\otimes n}$ and trivially on $\mathbb{Q}(p)$, whereas $\mathbb{G}_m$ acts trivially on $V^{\otimes m} \otimes (V^\vee)^{\otimes n}$ and by the character $z \mapsto z^p$ on $\mathbb{Q}(p)$.

THEOREM 7. *The Mumford–Tate group is exactly the* (*largest*) *algebraic subgroup of* $\mathrm{Gl}(V_{\mathbb{C}}) \times \mathbb{C}^*$ *which fixes all Hodge elements inside* $T^{m,n}V(p)$ *for all* (m, n, p) *such that* $(m-n)k - 2p = 0$. *Conversely, an element of* $T^{m,n}V(p)$ *with* (m, n, p) *such that* $(m-n)k - 2p = 0$ *is a Hodge element if and only if it is fixed by* $\mathrm{MT}(V)$. *Every subspace of* $T^{m,n}V(p)$ *invariant under* $\mathrm{MT}(V)$ *is a Hodge substructure.*

See [9, Prop. 3.4].

4. Variations of Hodge Structure

Let S be a complex manifold. A *variation of Hodge structure of weight* k on S consists of the following data:

(1) a local system $\mathbb{V}_{\mathbb{Z}}$ of finitely generated Abelian groups on S;
(2) a finite decreasing filtration $\{\mathcal{F}^p\}$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_S$ by holomorphic subbundles (the *Hodge filtration*).

These data must satisfy the following conditions:

(1) for each $s \in S$ the filtration $\{\mathcal{F}^p(s)\}$ of $\mathbb{V}(s) \simeq \mathbb{V}_{\mathbb{Z},s} \otimes_{\mathbb{Z}} \mathbb{C}$ defines on the finitely generated Abelian group $\mathbb{V}_{\mathbb{Z},s}$ a Hodge structure of weight k;
(2) the connection $\nabla\colon \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega^1_S$ whose sheaf of horizontal sections is $\mathbb{V}_{\mathbb{C}}$ satisfies the *Griffiths' transversality condition* $\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_S$.

EXAMPLES. (1) Let V be a Hodge structure of weight k and $s_0 \in S$ a base point. Suppose that one has a representation $\rho\colon \pi_1(S, s_0) \to \mathrm{Aut}(V)$. Then the local system $\mathbb{V}(\rho)$ associated to ρ underlies a locally constant variation of Hodge structure. In this case the Hodge bundles $\mathcal{F}^p$ are even locally constant, so that $\nabla(\mathcal{F}^p) \subset \mathcal{F}^p \otimes \Omega^1_S$. This property characterizes the local systems of Hodge structures among the variations of Hodge structure. In case ρ is the trivial representation, we denote the corresponding variation by V_S.

(2) Let $f\colon X \to S$ be a proper and smooth morphism of complex algebraic manifolds. We have seen that the cohomology groups $H^k(X_s)$ of the fibres X_s fit together into a local system. This local system, by the fundamental results of Griffiths underlies a variation of Hodge structure on S such that the Hodge structure at s is just the Hodge structure we have on $H^k(X_s)$. This case will be referred to as *the geometric case*.

The notion of a morphism of variations of Hodge structure is defined in the obvious way.

A *polarization* of a variation of Hodge structure $\mathbb{V}$ of weight k on X is a morphism of variations $Q\colon \mathbb{V} \otimes \mathbb{V} \to \mathbb{Z}(-k)_S$ which induces on each fibre a polarization of the corresponding Hodge structure of weight k.

EXAMPLE 8. Let $\mathbb{V}$ be a polarized variation of Hodge structure on a connected complex manifold which is purely of type (p, p). Then $\mathbb{V}$ has a finite monodromy group.

Indeed, the Hodge structure plays no role here; one just needs the fact that the isometry group of a lattice is finite.

5. Properties of Variations of Hodge Structure

There is another way of describing variations of Hodge structure using period domains as introduced by Griffiths which we briefly recall. Fixing a rational vector space V, a 'weight' k, a nondegenerate $(-1)^k$-symmetric rational bilinear form Q on V, and a vector $(h^0, \dots, h^k) \in \mathbb{Z}_{\geqslant 0}^{k+1}$ such that $h^i = h^{k-i}$ and $\sum h^i = \dim V$, the possible Hodge structures of weight k polarized by Q having Hodge numbers $h^{p,q} = h^p$ are parametrized by the homogenous space $D = \mathrm{Aut}(V_{\mathbb{R}}, Q)/H$, where H is the subgroup of $\mathrm{Aut}(V_{\mathbb{R}}, Q)$ stabilizing any fixed Hodge structure of the given type. The space D turns out to be a complex manifold. Given any variation of Hodge structure of the same type over a base S, the map which to $s \in S$ assigns the point in D corresponding to the Hodge structure on V given by x yields a multiply valued holomorphic map $S \to D$, which becomes uni-valued after dividing out by the action of the monodromy group Γ. In this way one arrives at the *period map* $p\colon S \to D/\Gamma$. Period maps have the property that they are *locally liftable*, i.e. locally in a contractible open subset $U \subset S$, the map lifts to $U \to D$. Moreover such lifts are *horizontal*. This is a translation of the Griffiths transversality relation. Conversely, locally liftable holomorphic maps $S \to D/\Gamma$ with horizontal local lifts are period maps for variations of Hodge structures on S (of the given type).

As shown by Griffiths [10, Theorem (10.1)] period domains have invariant metrics with special curvature properties which force period maps with source a polydisk (equipped with the Poincaré metric) to be distance decreasing.

First of all this imposes a strong restriction on local monodromy transformations as shown by Borel, see [14, Lemma 4.5, Thm 6.1].

THEOREM 9 (Monodromy theorem). *Let $\mathbb{V}$ be a polarized variation of Hodge structure on the punctured disc Δ^*. Then the monodromy operator T is quasi-unipotent. More precisely: if $\ell = \max(\{p - q \mid V_t^{p,q} \neq 0\})$ and $T = T_s T_u$ is the Jordan decomposition if T with T_u unipotent, then $(T_u - I)^{\ell+1} = 0$.*

Secondly [10, Theorem (9.5)], given a compactifiable base, say $S \subset \overline{S}$ compactified to a smooth $\overline{S}$ by a normal crossing divisor, this implies that we may extend the period map across points $s' \in \overline{S}$ all of whose local monodromy-operators have finite order. This yields a compactifiable $S' \subset \overline{S}$ whose image under the period map is a closed analytic subset of D/Γ. See [10, Theorem (9.6)].

As a third implication we have Deligne's finiteness result from [8]:

THEOREM 10. *Fix a connected compactifiable S and an integer N. There are at most finitely many conjugacy classes of rational representations of $\pi_1(S)$ of dimension N giving local systems that occur as a direct factor of a polarized variation of Hodge structure on S (of any weight).*

A result whose proof requires much more Hodge theory is the following.

THEOREM 11. *Let $\mathbb{V}$ be a variation of $\mathbb{Q}$-Hodge structure of weight k on a compactifiable complex manifold S. Then $H^0(S, \mathbb{V})$ has a $\mathbb{Q}$-Hodge structure of weight k in such a way that for each $s \in S$ the restriction map $H^0(S, \mathbb{V}) \hookrightarrow \mathbb{V}_s$ is a morphism of $\mathbb{Q}$-Hodge structures.*

As $H^0(S, \mathbb{V})$ is the fixed part of $\mathbb{V}_s$ under the action of $\pi_1(S, s)$, this theorem is mostly called the *Theorem of the fixed part*. It was first proved by Griffiths [10] in the case where S is compact, then by Deligne [4] in the algebraic case where $f \colon X \to S$ is a proper and smooth morphism and $\mathbb{V} = R^k f_* \mathbb{Q}_X$, and finally by Schmid [14] in general.

This theorem has several interesting consequences. A first obvious consequence is that any variation of Hodge structure on a simply-connected compactifiable complex manifold is in fact constant.

Secondly we deduce that a global horizontal section of $\mathbb{V}_{\mathbb{C}}$ which at one point is of type (p, q), i.e. lies in $\mathbb{V}_s^{p,q}$ for some $s \in S$, is everywhere of type (p, q). Applying this to $\mathrm{Hom}(\mathbb{V}, \mathbb{V}')$ where $\mathbb{V}$ and $\mathbb{V}'$ are variations of Hodge structures of the same weight k, and $(p, q) = (0, 0)$ we get the

COROLLARY 12 (Rigidity theorem). *Every morphism of Hodge structures $\mathbb{V}_s \to \mathbb{V}'_s$ which intertwines the actions of $\pi_1(S, s)$ extends to a morphism of variations of Hodge structure $\mathbb{V} \to \mathbb{V}'$.*

Finally we conclude:

COROLLARY 13. *The category of polarizable variations of $\mathbb{Q}$-Hodge structures on a fixed compactifiable base is semisimple.*

Proof. Suppose that $\mathbb{V}'$ is a subvariation of $\mathbb{V}$ and suppose that $\mathbb{V}$ is polarized. Take $s \in S$. The Hodge structure $\mathbb{V}_s$ is polarized, and this polarization induces an orthogonal projector in $\mathrm{End}(\mathbb{V}_s)$ with image equal to $\mathbb{V}'_s$, which commutes with the action of the monodromy group. Hence, it extends to a projector in $\mathrm{End}_{\mathrm{VHS}}(\mathbb{V})$ with image $\mathbb{V}'$. □

6. The Locus of a Hodge Element

Let us consider a variation of Hodge structure over any smooth connected complex base S of even weight $k = 2p$. We consider a section v of $\mathbb{V}_{\mathbb{Z}}$ on the universal cover of S and we let Y_v be the locus of all $s \in S$ where some determination $v(s')$, $s' \mapsto s$ is of type (p, p). This locus is an analytic subvariety of S since the condition to belong to the Hodge bundle $\mathcal{F}^p$ is analytic and a local section v of $\mathbb{V}_Z$ is a Hodge element in $\mathbb{V}_s$ precisely when $v(s) \in \mathcal{F}^p$. In case $Y_v \neq S$ we call v *special*. The union of all Y_v, with v special forms a thin subset of S. We call $s \in S$ *very general* with respect to $\mathbb{V}$ if it lies in the complement of this set. The very general points of S with respect to $\mathbb{V}$ form a dense subset. Now, if $s \in S$ is very general, by definition any Hodge element in $\mathbb{V}_s$ extends to give a multivalued horizontal section of $\mathbb{V}$ everywhere of type (p, p).

We can now show how the monodromy group is related to the Mumford–Tate group of the Hodge structure at a very general $s \in S$ using the characterisation (Proposition 7) of $\mathrm{MT}(\mathbb{V}_s)$ as the largest rationally defined algebraic subgroup of $\mathrm{Gl}(\mathbb{V}_s) \times \mathbb{C}^*$ fixing the Hodge elements in $\mathbb{V}_s^{m,n}(p)$, for all triples (m, n, p) with $(m-n)k-2p = 0$. So we look at $s \in S$ which is very general for all local systems $\mathbb{V}(m, n)(p)$ with $(m-n)k-2p = 0$. Then there is a local system $\mathbb{H}(m, n, p)$ on S whose stalk at s is $\mathrm{Hodge}(\mathbb{V}_s^{m,n}(p))$. Using this we deduce:

PROPOSITION 14. *Let S be a smooth complex variety. For very general $s \in S$ a finite index subgroup of the monodromy group is contained in the Mumford–Tate group of the Hodge structure on $\mathbb{V}_s$.*

Proof. The Hodge structure on $\mathbb{H}(m, n, p)_s$ is polarizable and so there is a positive definite quadratic form on this space invariant under monodromy. Hence, the monodromy acts on $\mathbb{H}(m, n, p)$ through a finite group. The Mumford–Tate group being algebraic, the Noetherian property then implies that finitely many triples (m, n, p) determine the Mumford–Tate group and so a finite index subgroup π' of the fundamental group has its image in the Mumford–Tate group. □

In case S is quasi-projective, the validity of the Hodge conjecture would imply that analytic sets Y_v in fact are *algebraic*. Surprisingly there is an independent proof; it is a consequence of the following result due to Cattani, Deligne and Kaplan [2]:

THEOREM 15. *Fix a natural number m. Suppose that we have a polarized variation $\mathbb{V}$ of even weight $k = 2p$ on a compactifiable $S \subset \overline{S}$ whose compactifying divisor has normal crossings. Define*

$$S^{(m)} = \{s \in S \mid (\mathbb{V}_\mathbb{Z})_s \text{ contains a Hodge element } v \text{ such that } Q(v, v) \leqslant m\}.$$

Then the closure of $S^{(m)}$ in $\overline{S}$ is a finite union of closed analytic subspaces.

As we noted before, the fact that $S^{(m)}$ a finite disjoint union of analytic subspaces of S is not hard; the difficult point is the assertion about the behaviour near the boundary.

The following result is due to André [1], who stated it for variations of mixed Hodge structure:

THEOREM 16. *Let S be a compactifiable complex manifold and let $\mathbb{V}$ be a polarizable variation of Hodge structure on S. Let $\Sigma \subset S$ denote the subset of points which are not very general with respect to $\mathbb{V}$. Then*

(1) *For all $s \in S \setminus \Sigma$ the connected component H_s of the algebraic monodromy group is a normal subgroup of the derived group M^{der} of the generic Mumford–Tate group $M = \mathrm{MT}(\mathbb{V}_s)$;*
(2) *If $\mathrm{MT}(\mathbb{V}_s)$ is Abelian for some $s \in S$ then $H_s = M^{\mathrm{der}}$ for every $s \in S \setminus \Sigma$.*

7. Noether–Lefschetz Properties

In this section S is any connected complex manifold and $\mathbb{V}$ is a polarizable variation of Hodge structure on S and we assume that the monodromy representation is big. We recall that if $q\colon S' \to S$ is a finite unramified cover, $q^*\mathbb{V}$ still has a big monodromy group.

The fact that $\mathbb{V}$ underlies a variation of Hodge structure imposes severe restrictions of Noether–Lefschetz type:

THEOREM 17. *Let there be given a (rational) polarized weight k variation of Hodge structure $\mathbb{V}$ over S with big monodromy. If $s \in S$ is very general with respect to* $\mathrm{Hom}(\mathbb{V}, \mathbb{V})$*, then $\mathbb{V}_s$ has no nontrivial rational Hodge substructures.*

Proof. Any projector $p\colon \mathbb{V}_s \to \mathbb{V}_s$ onto a Hodge substructure extends to a multivalued flat section of $\mathrm{Hom}(\mathbb{V}, \mathbb{V})$ everywhere of type $(0, 0)$. This flat section generates a sub Hodge structure of type $(0, 0)$ which by Example 8. has finite monodromy. Replacing S by a finite unramified cover, we may assume that the flat section is uni-valued, i.e. invariant under the monodromy. This means that the projector p intertwines every element from the monodromy group and thus defines a subsystem of $\mathbb{V}$. Since the latter is irreducible, this subsystem is either zero or all of $\mathbb{V}$. □

Remark. The proof from [6] asserting the truth of the theorem for certain variations related to K3-surfaces can be applied to our setting. The crucial result to use here is Proposition 14. Clearly this gives a much less elementary proof.

Using the above proposition and Example 5, we deduce:

COROLLARY 18. *Except for quadric hypersurfaces, cubic surfaces and even-dimensional intersections of two quadrics, the primitive cohomology of a generic smooth complete intersection in complex projective space does not contain nontrivial Hodge substructures.*

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