Monoid Domain Constructions of Antimatter Domains

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Abstract. An integral domain without irreducible elements is called an antimatter domain. We give some monoid domain constructions of antimatter domains. Among other things, we show that if D is a GCD domain with quotient field K that is algebraically closed, real closed, or perfect of characteristic $p > 0$, then the monoid domain $D[X; \mathbb{Q}^+]$ is an antimatter GCD domain. We also show that a GCD domain D is antimatter if and only if $P^{-1} = D$ for each maximal t-ideal P of D.

Let D be an integral domain with quotient field K. By an *irreducible element* or *atom* of D we mean a nonunit $x \in D^\star = D - \{0\}$ such that $x = uv$, $u, v \in D$, implies u or v is a unit. The domain D is *atomic* if each nonzero nonunit of D is expressible as a finite product of atoms. However, it may happen that a domain does not have any atoms. Such domains, called *antimatter domains*, were introduced by Coykendall, Dobbs, and Mullins [**5**]. A somewhat obvious example of an antimatter domain is a valuation domain whose maximal ideal is not principal [**5**, Proposition 1]. Another example is a field which, ironically, is also an example of an atomic domain. It is patent that if D is an antimatter domain, or any integral domain for that matter, then $D[X]$ is not antimatter, as $X + r$ is an atom in $D[X]$ for all $r \in D$. On the other hand, the monoid domain $\mathbb{C}[X; \mathbb{Q}^+]$, where $\mathbb{Q}^+$ is the monoid of nonnegative rationals under addition, is an antimatter domain (Theorem 1). But $\mathbb{Q}[X; \mathbb{Q}^+]$ is not antimatter as $X - 2$ is irreducible. (If $X - 2$ properly factors in $\mathbb{Q}[X; \mathbb{Q}^+]$, then $X - 2$ properly factors in some $\mathbb{Q}[X^{1/n}]$ since $\mathbb{Q}^+$ is locally cyclic (that is, each finitely generated submonoid of $\mathbb{Q}^+$ is contained in a cyclic submonoid of $\mathbb{Q}^+$). But by Eisenstein's Criterion, $X - 2 = (X^{1/n})^n - 2$ is irreducible in $\mathbb{Q}[X^{1/n}]$).

The purpose of this paper is to explore the following question. For an integral domain D and torsionless cancellative monoid S (always written additively), when is the monoid domain $D[X; S]$ antimatter? Certainly, if $D[X; S]$ is antimatter, then D and S must be antimatter (a monoid S is antimatter if it has no atoms where atoms are defined in the obvious way). However, as both $\mathbb{Q}$ and $(\mathbb{Q}^+, +)$ are antimatter while $\mathbb{Q}[X; \mathbb{Q}^+]$ is not, the converse is false. In this note we show that if D is an antimatter GCD domain with quotient field K algebraically closed, real closed, or perfect of characteristic $p > 0$, (Theorems 1, 2, and 5), then $D[X; \mathbb{Q}^+]$ is an antimatter domain. Our standard references are [**6**], [**7**], and [**10**].

In the case where $D = K$ is an algebraically closed or real closed field, we can show that $D[X; S]$ is antimatter in slightly more generality than the case $S = (\mathbb{Q}^+, +)$. Let us call a monoid S *pure* if (1) S is (order-isomorphic to) a submonoid of $(\mathbb{Q}^+, +)$, (2) S is locally cyclic, and (3) for each $s \in S$, there is a natural number $n > 1$ (depending on s) with $s/n \in S$. We remark that in the presence of (2) and (3), condition (1) can be replaced by either S is totally ordered and each $s \geq 0$ or that S is reduced, cancellative, and torsionless. Examples of pure monoids include $(\mathbb{Q}^+, +)$ and $(\mathbb{Z}_T^+, +)$ where $\mathbb{Z}_T^+ = \{n/t | n \in \mathbb{Z}^+,\ t \in T\}$ with T a multiplicatively closed subset of $\mathbb{Z}^+ = \{0, 1, 2, \cdots\}$. We will consider a pure monoid S to actually be a submonoid of $(\mathbb{Q}^+, +)$. With this in mind, note that if $s_1, s_2 \in S$ with $s_1 < s_2$, then $s_2 - s_1 \in S$. Indeed, $\langle s_1, s_2 \rangle \subseteq \langle s \rangle$ for some $s \in S$; so $s_1 = ns$ and $s_2 = ms$ where necessarily $n < m$. Then $s_2 - s_1 = ms - ns = (m - n)s \in S$. Observe that for S pure and K any field, $K[X; S]$ is a nonatomic Bezout domain. For by [**6**, Theorem 13.6] a monoid domain $K[X; S]$ over a field K and monoid S is Bezout if and only if S is isomorphic to a submonoid of $(\mathbb{Q}, +)$. And if S is pure and $0 \neq s \in S$, then $s/n \in S$ for some $n > 1$, so $X^s = (X^{s/n})^n$ and hence $K[X; S]$ does not satisfy ACCP, or equivalently since $K[X; S]$ is Bezout, is not atomic.

Theorem 1. *Let K be an algebraically closed field and S a pure monoid. Then $K[X; S]$ is an antimatter Bezout domain.*

Proof. We have already remarked that $K[X; S]$ is Bezout. Let f be a nonzero nonunit of $K[X; S]$; so $f = k_1 X^{s_1} + \cdots + k_n X^{s_n}$ where $0 \leq s_1 < \cdots < s_n$ and each $k_i \neq 0$. Now $f = X^{s_1}(k_1 + k_2 X^{s_2 - s_1} + \cdots + k_n X^{s_n - s_1})$ where as previously noted $s_i - s_1 \in S$. First, suppose that $s_1 > 0$. Choose $n_1 > 1$ with $s_1/n_1 \in S$. Then $X^{s_1} = (X^{s_1/n_1})^{n_1}$ and hence f is not irreducible. Next suppose that $s_1 = 0$, so $n > 1$. Choose $q \in S$ with $\langle s_1, \cdots, s_n \rangle \subseteq \langle q \rangle$. Then f factors into linear factors in $K[X^q]$ since K is algebraically closed. Now a typical linear factor of f in $K[X^q]$ has the form $\ell_0 + \ell_1 X^q$, $\ell_0, \ell_1 \in K$ with $\ell_1 \neq 0$. Choose $m > 1$ with $q/m \in S$. Then $\ell_0 + \ell_1 X^q = \ell_0 + \ell_1 (X^{q/m})^m$ and is not irreducible in $K[X^{q/m}]$. Thus f is not irreducible in $K[X; S]$. So $K[X; S]$ is an antimatter domain. □

Recall that a field K is *real closed* if K is formally real (that is, -1 is not a sum of squares) and K has no proper formally real algebraic extensions. Using Zorn's Lemma, every formally real field F is contained in a real closed field K that is algebraic over F. Also, if K is a real closed field, then $K(\sqrt{-1})$ is algebraically closed. If K is formally real, then $K(X)$ is again formally real for any set X of indeterminates. Thus $K(X)$ is contained in a real closed field. So there are plenty of real closed fields in addition to $\mathbb{R}$. For results on real closed fields, the reader is referred to [**9**, Section 5.1].

Theorem 2. *Let K be a real closed field and S a pure monoid. Then $K[X; S]$ is an antimatter Bezout domain.*

Proof. We have already remarked that $K[X; S]$ is Bezout. Let $f = k_1 X^{s_1} + \cdots + k_n X^{s_n}$, $s_1 < \cdots < s_n$, $k_i \neq 0$, be a nonzero nonunit of $K[X; S]$. As in the proof of Theorem 1, f is not irreducible if $s_1 > 0$. So suppose that $s_1 = 0$ and hence $n > 1$. Choose $q \in S$ with $\langle s_1, \cdots, s_n \rangle \subseteq \langle q \rangle$ and $m > 1$ with $q/m \in S$. Choose $m' > 1$ with $q/mm' \in S$. Then f as a polynomial in $K[X^{q/mm'}]$ has $\deg f \geq mm' > 2$. But over a real closed field an irreducible polynomial has degree one or two. Hence f is not irreducible in $K[X^{q/mm'}]$ and hence not irreducible in $K[X; S]$. □

We want to extend Theorems 1 and 2 to the case where D is a GCD domain. Thus it is of interest to know when a GCD domain is antimatter. In [**5**, Proposition

2.1] it was shown that a valuation domain (V, M) is antimatter if and only if $M^{-1} = V$, that is, M is not principal. We generalize this result. For a nonzero (fractional) ideal I of a domain D recall that $I_v = (I^{-1})^{-1}$ where $I^{-1} = [D{:}I]$ and $I_t = \bigcup\{J_v | 0 \neq J \subseteq I, J \text{ is finitely generated}\}$. An ideal I is called a *t-ideal* if $I = I_t$. A proper integral t-ideal is contained in a maximal proper integral t-ideal and a maximal t-ideal is prime.

THEOREM 3. (1) *Suppose that D is an integral domain in which every irreducible element is prime (e.g., a GCD domain). If $P^{-1} = D$ for each maximal t-ideal P of D, then D is antimatter.*
(2) *If D is an antimatter GCD domain, then $P^{-1} = D$ for each maximal t-ideal of D.*

PROOF. (1) Suppose that D has an irreducible element p. By hypothesis, p is prime. Hence (p) is a maximal t-ideal [**8**, Proposition 1.3]. But then $(p)^{-1} = D$, a contradiction.
(2) Suppose that D is an antimatter GCD domain. Let P be a maximal t-ideal of D. Let $x/y \in P^{-1}$ where $x, y \in D^*$. Since D is a GCD domain, we can assume that $[x, y] = 1$. Suppose that $x/y \notin D$, so y is a nonunit. Now $(x/y)P \subseteq D$ gives $xP \subseteq (y)$. For $0 \neq p \in P$, $y | xp$. But then $[x, y] = 1$ gives $y | p$. Hence $P \subseteq (y) \neq D$ and thus $P = (y)$ since P is a maximal t-ideal. But then y is prime and hence irreducible, a contradiction. Hence $P^{-1} = D$. □

Thus a GCD (and hence a Bezout domain) domain is antimatter if and only if $P^{-1} = D$ for each maximal t-ideal P of D. However, we will later give an example (Example 1) of an antimatter pre-Schreier domain with a maximal ideal M satisfying $M^{-1} \neq D$ (and hence M is a maximal t-ideal).

Recall that a saturated multiplicatively closed subset S of D is a *splitting set* if for each $x \in D^*$, $x = as$ for some $a \in D$ and $s \in S$ such that $aD \cap tD = atD$ for all $t \in S$.

LEMMA 1. *Let D be an integral domain and S a splitting set of D. Then D is antimatter if and only if S contains no atoms and D_S is antimatter.*

PROOF. ($\Rightarrow$) Suppose that D is antimatter. Then certainly S contains no atoms. By [**1**, Corollary 1.4(d)], each atom of D_S is an associate in D_S of an atom of D. Since D is antimatter, so is D_S. ($\Leftarrow$) Suppose that x is an atom of D. Then since x is an atom either $x \in S$ or $xD \cap tD = xtD$ for all $s \in S$. Since S contains no atoms, the second case must hold. But then by [**1**, Corollary 1.4(c)], x is an atom of D_S, a contradiction. □

THEOREM 4. *Let D be an antimatter GCD domain with quotient field K that is either algebraically closed or real closed. Then $D[X; \mathbb{Q}^+]$ is an antimatter GCD domain.*

PROOF. By [**6**, Theorem 14.5], $D[X; \mathbb{Q}^+]$ is a GCD domain. Since D is a GCD domain each nonzero element f of $D[X; \mathbb{Q}^+]$ has the form $f = r \sum_{i=1}^{n} a_i X^{q_i}$ where $[a_1, \cdots, a_n] = 1$. Moreover,

$$(\sum_{i=1}^{n} a_i X^{q_i}) D[X; \mathbb{Q}^+] \cap tD[X; \mathbb{Q}^+] = t(\sum_{i=1}^{n} a_i X^{q_i}) D[X; \mathbb{Q}^+]$$

for all $t \in D^*$. Hence D^* is a splitting set in $D[X;\mathbb{Q}^+]$. Now $D[X;\mathbb{Q}^+]_{D^*} = K[X;\mathbb{Q}^+]$ is an antimatter domain by either Theorem 1 or Theorem 2, respectively. Since D^* contains no atoms, $D[X;\mathbb{Q}^+]$ is antimatter by Lemma 1. □

Note that the ring of algebraic integers is an antimatter Bezout domain with algebraically closed quotient field. Other examples can be obtained via [**10**, Theorem 102]. We next give a characteristic $p > 0$ result.

THEOREM 5. (1) *Let K be a perfect field of characteristic $p > 0$. Let S be a cardinal sum of copies of $\mathbb{Q}^+$. Then $K[X;S]$ is an antimatter GCD domain.*
(2) *Suppose that D is an antimatter GCD domain with quotient field K where K is a perfect field of characteristic $p > 0$. Then $D[X;\mathbb{Q}^+]$ is an antimatter GCD domain.*

PROOF. (1) Let $f = \sum_{i=1}^n k_i X^{s_i}$ be a nonzero nonunit of $K[X;S]$. Since K is perfect, each $\sqrt[p]{k_i} \in K$. Then $f = \sum_{i=1}^n k_i X^{s_i} = (\sum_{i=1}^n \sqrt[p]{k_i} X^{s_i/p})^p$ is not irreducible.
(2) By (1) $K[X;\mathbb{Q}^+] = D[X;\mathbb{Q}^+]_{D^*}$ is an antimatter domain. Then as in the proof of Theorem 4, $D[X;\mathbb{Q}^+]$ is an antimatter GCD domain. □

We next give the promised example showing that Theorem 3(2) can not be extended to pre-Schreier domains. We first recall some definitions and results. A nonzero element x of D is *primal* if whenever $x|yz$, $y,z \in D$, then $x = x_1x_2$ where $x_1|y$ and $x_2|z$. Call a primal element x *completely primal* if each factor of x is primal. Finally, D is *pre-Schreier* if each nonzero element of D is (completely) primal and an integrally closed pre-Schreier domain is called a *Schreier domain*. Schreier domains were introduced by P. M. Cohn [**3**] and the last author [**12**] introduced pre-Schreier domains. It is easy to see [**3**] that a GCD domain is Schreier. In [**3**, Theorem 5.3] (respectively, [**12**, p. 1901]) it was shown that an atom in a Schreier domain (respectively, pre-Schreier domain) is prime. So by Theorem 3(1) a pre-Schreier domain D is antimatter if $P^{-1} = D$ for each maximal t-ideal P of D. There do exist examples of Schreier domains that are not GCD domains [**2**, Example 2.10] and there do exist examples of antimatter domains (in which vacuously every irreducible element is prime) but which are not pre-Schreier [**2**, Proposition 3.10]. We next give an example of an antimatter pre-Schreier domain having a maximal ideal M that is a (maximal) t-ideal with $M^{-1} \neq D$.

EXAMPLE 1. *Let $D = \mathbb{Q} + (\{X^s | s \in \mathbb{Q}^+ - \{0\}\})\mathbb{R}[X;\mathbb{Q}^+]$. Then D is an antimatter pre-Schreier domain having $P = (\{X^s | s \in \mathbb{Q}^+ - \{0\}\})\mathbb{R}[X;\mathbb{Q}^+]$ as a maximal ideal with $(P^{-1})^{-1} = P = P^2$ and hence P is a maximal t-ideal with $P^{-1} \neq D$.*

Clearly P is a maximal ideal of D. For $f \in P$, $f = X^\alpha g$ where $\alpha > 0$. Then $f = (X^{\alpha/2})^2 g$; so f is not an atom and this also shows that $P = P^2$. If $f \in D - P$ is a nonunit, then $f = s(1+g)$ where $s \in \mathbb{Q}^*$ and $g \in P$. Now $1+g$ is a nonunit of the antimatter domain $\mathbb{R}[X;\mathbb{Q}^+]$ so we can write $1+g = (1+p_1)(1+p_2)$ where $p_1, p_2 \in P$ and $1+p_1, 1+p_2$ are nonunits of D. Hence D is antimatter. We show that $P^{-1} = \mathbb{R}[X;\mathbb{Q}^+]$. Certainly $\mathbb{R}[X;\mathbb{Q}^+] \subseteq P^{-1}$. Also, $P^{-1} = [D{:}P] \subseteq [\mathbb{R}[X;\mathbb{Q}^+]{:}P] = \mathbb{R}[X;\mathbb{Q}^+] \subseteq P^{-1}$ where the second equality follows since P is a noninvertible maximal ideal in the Bezout domain $\mathbb{R}[X;\mathbb{Q}^+]$. So $P^{-1} = \mathbb{R}[X;\mathbb{Q}^+]$. Now $P\mathbb{R}[X;\mathbb{Q}^+] = P$, so $P \subseteq (\mathbb{R}[X;\mathbb{Q}^+])^{-1} \subsetneq D$; that is, $P \subseteq P_v \neq D$. Since P is maximal, we have $P = P_v$. We next show that D is pre-Schreier. Let $T = D - P$.

So $f \in T$ has the form $f = q(1+p)$ where $q \in \mathbb{Q}^*$ and $p \in P$. We show that elements of T are completely primal. Since T is saturated, it is enough to show that elements of the form $1+p$, $p \in P$, are primal. Suppose that $1+p|ab$ where $a, b \in D$. Then $1+p|ab$ in the Bezout (and hence Schreier) domain $\mathbb{R}[X;\mathbb{Q}^+]$*. So we can write $1+p = (1+q_1)(1+q_2)$, $q_1, q_2 \in P$ where $1+q_1|a$ and $1+q_2|b$ in $\mathbb{R}[X;\mathbb{Q}^+]$. Note that actually $1+q_1|a$ and $1+q|b$ in D. So $1+p$ is primal. By Nagata's Theorem for pre-Schreier domains (if S a saturated multiplicative set consisting of completely primal elements and D_S pre-Schreier, then D is pre-Schreier; see* [**3**] *for the Schreier case whose proof does not use integral closure), it is enough to show that D_T is pre-Schreier. Now $D_T = \mathbb{Q}+P\mathbb{R}[X;\mathbb{Q}^+]_T \subseteq \mathbb{R}[X;\mathbb{Q}^+]_T = \mathbb{R}[X;\mathbb{Q}^+]_{P\mathbb{R}[X;Q^+]}$ where $\mathbb{R}[X;\mathbb{Q}^+]_{P\mathbb{R}[X;\mathbb{Q}^+]}$ is a valuation domain. Since $P\mathbb{R}[X;\mathbb{Q}^+]_{P\mathbb{R}[X;\mathbb{Q}^+]}$ is not a principal ideal of $\mathbb{R}[X;\mathbb{Q}^+]_{P\mathbb{R}[X;\mathbb{Q}^+]}$, D_T is a Schreier domain* [**11**, Theorem 3.2]*. It is interesting to note that D is an ascending union of rings of the form $\mathbb{Q} + X^{\frac{1}{n!}}\mathbb{R}[X^{\frac{1}{n!}}]$, each of which is atomic but not pre-Schreier.*

We end with the following two results.

THEOREM 6. (1) *Let D be an integral domain with quotient field $K \neq D$, L be a field extension of K, $R = D + XL[X]$, and $T = \{f \in R | f(0) = 1\}$. Then D is antimatter if and only if R_T is antimatter.*
(2) *Let D be an antimatter Schreier domain and S a multiplicative set of D containing at least one nonunit. Let T be the saturated multiplicative set of $R = D + XD_S[X]$ generated by the prime elements of R. Then R_T is antimatter.*

PROOF. (1) Note that every nonzero element of R can be written as $kX^n(1 + Xf(X))$ where $n \geq 0$, $f(X) \in L[X]$, and $k \in K^*$ with $k \in D$ if $n = 0$. Thus in D_T each nonzero nonunit is an associate of kX^n with k and n as above. For $n \geq 2$, kX^n is clearly not an atom. For $n = 1$, $D \neq K$ gives that kX is not an atom since $kX = r(kX/r)$ for all nonunits $r \in D^*$. (It is essential that $D \neq K$ as $K + XL[X]$ is atomic.) And for $n = 0$, $k \in D^*$ properly factors in D if and only if it properly factors in R_T. It follows that D is antimatter if and only if R_T is antimatter.
(2) As remarked in [**4**], R is a Schreier domain. Hence R_T is also a Schreier domain. But in a Schreier domain atoms are the same thing as primes. Let $a(X)$ be a nonzero principal prime of R. Since D is antimatter, $a(X) \notin D$. Also, $a(0) \neq 0$ since X is not an atom because $X = s(X/s)$ where $s \in S$ is a nonunit. Thus $a(X)R \cap D = (0)$, so $R_{a(X)R} \supset K[X]$ and hence is a DVR. Also, since each such $a(X)$ extends to a prime of $K[X]$, no nonzero element of R is divisible by infinitely many nonassociate primes of R. Thus by [**1**, Proposition 1.6], T is a splitting set. Now there are no nonzero principal primes in R_T because if there were one, then by [**1**, Corollary 1.4], there would be a corresponding nonzero principal prime in $R - T$. But this is a contradiction since T is generated by all such primes. □

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