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Monoid of Multisets and Subsets

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Summary. The monoid of functions yielding elements of a group is introduced. The monoid of multisets over a set is constructed as such monoid where the target group is the group of natural numbers with addition. Moreover, the generalization of group operation onto the operation on subsets is present. That generalization is used to introduce the group 2^G of subsets of a group G.

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The articles [21], [22], [15], [3], [17], [10], [5], [14], [11], [7], [16], [20], [9], [8], [19], [6], [13], [1], [18], [23], [24], [12], [2], and [4] provide the notation and terminology for this paper.

1. Updating

We adopt the following convention: x, y are arbitrary, X, Y, Z are sets, and n is a natural number. We now define two new constructions. Let D be a non-empty set, and let d be an element of D. Then $\{d\}$ is a non-empty subset of D. Let D be a non-empty set, and let X_1, X_2 be subsets of D. Then $X_1 \cup X_2$ is a subset of D. Let D be a non-empty set, and let X_1 be a subset of D, and let X_2 be a non-empty subset of D. Then $X_1 \cup X_2$ is a non-empty subset of D. Let D_1, D_2, D be non-empty sets. A binary function from D_1, D_2 into D is a function from $[: D_1, D_2 :]$ into D.

Let f be a function, and let x_1, x_2, y be arbitrary. The functor $f(x_1, x_2)(y)$ is defined by:

(Def.1) $f(x_1, x_2)(y) = f(\langle x_1, x_2 \rangle)(y)$.

The following proposition is true

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(1) For all functions f, g and for arbitrary x_1, x_2, x such that $\langle x_1, x_2\rangle \in \operatorname{dom} f$ and $g = f(x_1, x_2)$ and $x \in \operatorname{dom} g$ holds $f(x_1, x_2)(x) = g(x)$.

Let A, D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D^A, and let x_1 be an element of D_1, and let x_2 be an element of D_2, and let x be an element of A. Then $f(x_1, x_2)(x)$ is an element of D. Let A be a set, and let D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D, and let g_1 be a function from A into D_1, and let g_2 be a function from A into D_2. Then $f^\circ(g_1, g_2)$ is an element of D^A. Let A be a non-empty set, and let n be a natural number, and let x be an element of A. Then $n \longmapsto x$ is a finite sequence of elements of A. We introduce the functor $n \dot{\longmapsto} x$ as a synonym of $n \longmapsto x$. Let D be a non-empty set, and let A be a set, and let d be an element of D. Then $A \longmapsto d$ is an element of D^A. Let A be a set, and let D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D, and let d be an element of D_1, and let g be a function from A into D_2. Then $f^\circ(d, g)$ is an element of D^A. Let A be a set, and let D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D, and let g be a function from A into D_1, and let d be an element of D_2. Then $f^\circ(g, d)$ is an element of D^A.

We now state the proposition

(2) For all functions f, g and for every set X holds $(f \restriction X) \cdot g = f \cdot (X \restriction g)$.

The scheme *NonUniqFuncDEx* concerns a set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the following condition is met:

- for every x such that $x \in \mathcal{A}$ there exists an element y of $\mathcal{B}$ such that $\mathcal{P}[x, y]$.

2. Monoid of functions into a semigroup

Let D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D, and let A be a set. The functor f_A° yields a binary function from ${D_1}^A$, ${D_2}^A$ into D^A and is defined by:

(Def.2) for every element f_1 of ${D_1}^A$ and for every element f_2 of ${D_2}^A$ holds $(f_A^\circ)(f_1, f_2) = f^\circ(f_1, f_2)$.

Next we state the proposition

(3) For all non-empty sets D_1, D_2, D and for every binary function f from D_1, D_2 into D and for every set A and for every function f_1 from A into D_1 and for every function f_2 from A into D_2 and for every x such that $x \in A$ holds $(f_A^\circ)(f_1, f_2)(x) = f(f_1(x), f_2(x))$.

For simplicity we adopt the following convention: A will denote a set, D will denote a non-empty set, a will denote an element of D, o, o' will denote binary operations on D, and f, g, h will denote functions from A into D. The following propositions are true:

(4) If o is commutative, then $o^\circ(f, g) = o^\circ(g, f)$.

(5) If o is associative, then $o^\circ(o^\circ(f, g), h) = o^\circ(f, o^\circ(g, h))$.

(6) If a is a unity w.r.t. o, then $o^\circ(a, f) = f$ and $o^\circ(f, a) = f$.

(7) If o is idempotent, then $o^\circ(f, f) = f$.

(8) If o is commutative, then o°_A is commutative.

(9) If o is associative, then o°_A is associative.

(10) If a is a unity w.r.t. o, then $A \longmapsto a$ is a unity w.r.t. o°_A.

(11) If o has a unity, then $\mathbf{1}_{o^\circ_A} = A \longmapsto \mathbf{1}_o$ and o°_A has a unity.

(12) If o is idempotent, then o°_A is idempotent.

(13) If o is invertible, then o°_A is invertible.

(14) If o is cancelable, then o°_A is cancelable.

(15) If o has uniquely decomposable unity, then o°_A has uniquely decomposable unity.

(16) If o absorbs o', then o°_A absorbs o'°_A.

(17) For all non-empty sets D_1, D_2, D, E_1, E_2, E and for every binary function o_1 from D_1, D_2 into D and for every binary function o_2 from E_1, E_2 into E such that $o_1 \leq o_2$ holds ${o_1}^\circ_A \leq {o_2}^\circ_A$.

Let G be a half group structure, and let A be a set. The functor G^A yielding a half group structure is defined by:

(Def.3) (i) $G^A = \langle(\text{the carrier of } G)^A, (\text{the operation of } G)^\circ_A, A \longmapsto \mathbf{1}_{\text{the operation of } G}$ **qua** an element of (the carrier of $G)^A$ **qua** a non-empty set$\rangle$ if G is unital,

(ii) $G^A = \langle(\text{the carrier of } G)^A, (\text{the operation of } G)^\circ_A\rangle$, otherwise.

In the sequel G denotes a half group structure. We now state two propositions:

(18) The carrier of $G^X =$ (the carrier of $G)^X$ and the operation of $G^X =$ (the operation of $G)^\circ_X$.

(19) x is an element of G^X if and only if x is a function from X into the carrier of G.

Let G be a half group structure, and let A be a set. Then G^A is a constituted functions half group structure.

We now state two propositions:

(20) For every element f of G^X holds $\operatorname{dom} f = X$ and $\operatorname{rng} f \subseteq$ the carrier of G.

(21) For all elements f, g of G^X if for every x such that $x \in X$ holds $f(x) = g(x)$, then $f = g$.

Let G be a half group structure, and let A be a non-empty set, and let f be an element of G^A. Then $\operatorname{rng} f$ is a non-empty subset of G. Let a be an element of A. Then $f(a)$ is an element of G.

We now state the proposition

(22) For all elements f_1, f_2 of G^D and for every element a of D holds $(f_1 \cdot f_2)(a) = f_1(a) \cdot f_2(a)$.

Let G be a unital half group structure, and let A be a set. Then G^A is a well unital constituted functions strict monoid structure.

One can prove the following propositions:

(23) For every G being a unital half group structure holds the unity of $G^X = X \longmapsto \mathbf{1}_{\text{the operation of } G}$.

(24) Let G be a half group structure. Let A be a set. Then

(i) if G is commutative, then G^A is commutative,

(ii) if G is associative, then G^A is associative,

(iii) if G is idempotent, then G^A is idempotent,

(iv) if G is invertible, then G^A is invertible,

(v) if G is cancelable, then G^A is cancelable,

(vi) if G has uniquely decomposable unity, then G^A has uniquely decomposable unity.

(25) For every subsystem H of G holds H^X is a subsystem of G^X.

(26) For every G being a unital half group structure and for every subsystem H of G such that $\mathbf{1}_{\text{the operation of } G} \in$ the carrier of H holds H^X is a monoidal subsystem of G^X.

Let G be a unital associative commutative cancelable half group structure with uniquely decomposable unity, and let A be a set. Then G^A is a commutative cancelable constituted functions strict monoid with uniquely decomposable unity.

3. Monoid of multisets over a set

Let A be a set. The functor $A^{\otimes}_{\omega}$ yields a commutative cancelable constituted functions strict monoid with uniquely decomposable unity and is defined by:

(Def.4) $A^{\otimes}_{\omega} = \langle \mathbb{N}, +, 0 \rangle^A$.

Next we state the proposition

(27) The carrier of $X^{\otimes}_{\omega} = \mathbb{N}^X$ and the operation of $X^{\otimes}_{\omega} = (+_{\mathbb{N}})^{\circ}_{X}$ and the unity of $X^{\otimes}_{\omega} = X \longmapsto 0$.

Let A be a set. A multiset over A is an element of $A^{\otimes}_{\omega}$.

Next we state two propositions:

(28) x is a multiset over X if and only if x is a function from X into $\mathbb{N}$.

(29) For every multiset m over X holds $\operatorname{dom} m = X$ and $\operatorname{rng} m \subseteq \mathbb{N}$.

Let A be a non-empty set, and let m be a multiset over A. Then $\operatorname{rng} m$ is a non-empty subset of $\mathbb{N}$. Let a be an element of A. Then $m(a)$ is a natural number.

Next we state two propositions:

(30) For all multisets m_1, m_2 over D and for every element a of D holds $(m_1 \otimes m_2)(a) = m_1(a) + m_2(a)$.

(31) $\chi_{Y,X}$ is a multiset over X.

Let us consider Y, X. Then $\chi_{Y,X}$ is a multiset over X. Let us consider X, and let n be a natural number. Then $X \longmapsto n$ is a multiset over X. Let A be a non-empty set, and let a be an element of A. The functor χa yields a multiset over A and is defined as follows:

(Def.5) $\chi a = \chi_{\{a\},A}$.

One can prove the following proposition

(32) For every non-empty set A and for all elements a, b of A holds $(\chi a)(a) = 1$ and also if $b \neq a$, then $(\chi a)(b) = 0$.

For simplicity we follow a convention: A denotes a non-empty set, a denotes an element of A, p, q denote finite sequences of elements of A, and m_1, m_2 denote multisets over A. Next we state the proposition

(33) If for every a holds $m_1(a) = m_2(a)$, then $m_1 = m_2$.

Let A be a set. The functor $A^{\otimes}$ yields a strict monoidal subsystem of $A^{\otimes}_{\omega}$ and is defined as follows:

(Def.6) for every multiset f over A holds $f \in$ the carrier of $A^{\otimes}$ if and only if $f^{-1}(\mathbb{N} \setminus \{0\})$ is finite.

The following three propositions are true:

(34) χa is an element of $A^{\otimes}$.

(35) $\mathrm{dom}(\{x\} \upharpoonright (p \frown \langle x \rangle)) = \mathrm{dom}(\{x\} \upharpoonright p) \cup \{\mathrm{len}\, p + 1\}$.

(36) If $x \neq y$, then $\mathrm{dom}(\{x\} \upharpoonright (p \frown \langle y \rangle)) = \mathrm{dom}(\{x\} \upharpoonright p)$.

Let A be a non-empty set, and let p be a finite sequence of elements of A. The functor $|p|$ yields a multiset over A and is defined as follows:

(Def.7) for every element a of A holds $|p|(a) = \mathrm{card}\, \mathrm{dom}(\{a\} \upharpoonright p)$.

We now state several propositions:

(37) $|\varepsilon_A|(a) = 0$.

(38) $|\varepsilon_A| = A \longmapsto 0$.

(39) $|\langle a \rangle| = \chi a$.

(40) $|p \frown \langle a \rangle| = |p| \otimes \chi a$.

(41) $|p \frown q| = |p| \otimes |q|$.

(42) $|n \dot{\longmapsto} a|(a) = n$ and for every element b of A such that $b \neq a$ holds $|n \dot{\longmapsto} a|(b) = 0$.

Next we state two propositions:

(43) $|p|$ is an element of $A^{\otimes}$.

(44) If x is an element of $A^{\otimes}$, then there exists p such that $x = |p|$.

4. Monoid of subsets of a semigroup

In the sequel a, b will be elements of D. Let D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D. The functor $^\circ f$ yields a binary function from 2^{D_1}, 2^{D_2} into 2^D and is defined by:

(Def.8) for every element x of $[: 2^{D_1}, 2^{D_2} :]$ holds $(^\circ f)(x) = f^\circ [: x_{\mathbf{1}}, x_{\mathbf{2}} :]$.

One can prove the following propositions:

(45) For all non-empty sets D_1, D_2, D and for every binary function f from D_1, D_2 into D and for every subset X_1 of D_1 and for every subset X_2 of D_2 holds $(^\circ f)(X_1, X_2) = f^\circ [: X_1, X_2 :]$.

(46) For all non-empty sets D_1, D_2, D and for every binary function f from D_1, D_2 into D and for every subset X_1 of D_1 and for every subset X_2 of D_2 and for arbitrary x_1, x_2 such that $x_1 \in X_1$ and $x_2 \in X_2$ holds $f(x_1, x_2) \in (^\circ f)(X_1, X_2)$.

(47) For all non-empty sets D_1, D_2, D and for every binary function f from D_1, D_2 into D and for every subset X_1 of D_1 and for every subset X_2 of D_2 holds $(^\circ f)(X_1, X_2) = \{f(a, b) : a \in X_1 \wedge b \in X_2\}$, where a ranges over elements of D_1, and b ranges over elements of D_2.

(48) If o is commutative, then $o^\circ [: X, Y :] = o^\circ [: Y, X :]$.

(49) If o is associative, then $o^\circ [: o^\circ [: X, Y :], Z :] = o^\circ [: X, o^\circ [: Y, Z :] :]$.

(50) If o is commutative, then $^\circ o$ is commutative.

(51) If o is associative, then $^\circ o$ is associative.

(52) If a is a unity w.r.t. o, then $o^\circ [: \{a\}, X :] = D \cap X$ and $o^\circ [: X, \{a\} :] = D \cap X$.

(53) If a is a unity w.r.t. o, then $\{a\}$ is a unity w.r.t. $^\circ o$ and $^\circ o$ has a unity and $\mathbf{1}_{^\circ o} = \{a\}$.

(54) If o has a unity, then $^\circ o$ has a unity and $\{\mathbf{1}_o\}$ is a unity w.r.t. $^\circ o$ and $\mathbf{1}_{^\circ o} = \{\mathbf{1}_o\}$.

(55) If o has uniquely decomposable unity, then $^\circ o$ has uniquely decomposable unity.

Let G be a half group structure. The functor 2^G yields a half group structure and is defined by:

(Def.9) (i) $2^G = \langle 2^{\text{the carrier of } G}, ^\circ(\text{the operation of } G), \{\mathbf{1}_{\text{the operation of } G}\}\rangle$ if G is unital,

(ii) $2^G = \langle 2^{\text{the carrier of } G}, ^\circ(\text{the operation of } G)\rangle$, otherwise.

Let G be a unital half group structure. Then 2^G is a well unital strict monoid structure.

One can prove the following three propositions:

(56) The carrier of $2^G = 2^{\text{the carrier of } G}$ and the operation of $2^G = {}^\circ(\text{the operation of } G)$.

(57) For every G being a unital half group structure holds the unity of $2^G = \{\mathbf{1}_{\text{the operation of } G}\}$.

(58) For every G being a half group structure holds if G is commutative, then 2^G is commutative and also if G is associative, then 2^G is associative and also if G has uniquely decomposable unity, then 2^G has uniquely decomposable unity.

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