Monoids of sequences over finite abelian groups defined via zero-sums with respect to a given set of weights and applications to factorizations of norms of algebraic integers

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MONOIDS OF SEQUENCES OVER FINITE ABELIAN GROUPS DEFINED VIA ZERO-SUMS WITH RESPECT TO A GIVEN SET OF WEIGHTS AND APPLICATIONS TO FACTORIZATIONS OF NORMS OF ALGEBRAIC INTEGERS

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Abstract. The investigation of the arithmetic of monoids of zero-sum sequences over finite abelian groups is a classical subject due to their crucial role in understanding the arithmetic of (transfer) Krull monoids. More recently, sequences that admit, for a given set of weights, a weighted zero-sum received increased attention. Yet, the focus was on zero-sum constants rather than the arithmetic of the monoids formed by these sequences. We begin a systematic study of the arithmetic of these monoids. We show that for a wide class of weights unions of sets of lengths are intervals and we obtain various results on the elasticity of these monoids. More detailed results are obtained for the special case of plus-minus weighted sequences. Moreover, we apply our results to obtain results on factorizations of norms of algebraic integers.

1. Introduction

The investigation of zero-sum sequences over (finite) abelian groups by now has a considerable tradition (see, e.g., [8], [12, Chapters 5 and 6], [18, Part II], [27, Chapter 9]). We recall that a collection of elements $g_1 \dots g_l$ of a finite abelian group $(G, +, 0_G)$ is said to have sum zero, if the sum of all these elements is the neutral element of the group, that is, $g_1 + \cdots + g_l = 0_G$. One can consider sets or sequences of elements, in the former case there are no repetitions of elements in the latter case there are repetitions of elements; to be precise, usually one does not take the ordering of the elements in the sequence into account, indeed formally sequences in this context are elements of the free abelian monoid over G. We refer to Section 2 for further details.

Besides the study of zero-sum constants such as the Erdős–Ginzburg–Ziv constant and the Davenport constant, considerable effort was put into the investigation of the arithmetic of the monoids of zero-sum sequences over abelian groups, mainly finite abelian groups. A main reason for this is that they are an important class of auxiliary monoids in factorization theory (see, e.g., [9, 10, 12]). Every Krull monoid, in particular the multiplicative monoid of every Dedekind domain, admits a transfer homomorphism to a monoid of zero-sum sequences. Another reason is that they are monoids that are easily described yet show rich phenomena regarding their arithmetic.

In recent years the investigation of zero-sum problems was extended by introducing 'weights'. Intuitively, this means that rather than considering simply the sum

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of the elements, one allows to assign weights to the elements, for example allowing weights from the set $\{1,3\}$ means that one considers sums of the form $\sum_{i=1}^{l} w_i g_i$ where $w_i \in \{1,3\}$, that is one can assign a different 'weight' to some elements by chosing w_i to equal 3 rather than 1.

Of course in a finite abelian group, 'weight' has to be understood figuratively. But we recall that an early example of a zero-sum problem originated from the question of the existence of points in an integral lattice whose barycenter is again a lattice point [20], and in that context the idea of assigning different weights to the points would make sense in a more literal sense.

It turns out that for certain applications a more general notion of 'weights' is relevant. The generalization becomes intuitive, when one interprets for an integer w the notion 'multiplication by w' as an endomorphism of the abelian group G. Then, the idea to allow any endomorphism of G as a 'weight,' rather than just those induced by multiplication by an integer becomes very natural. Generalizations of the notion of 'weight' even beyond that are possible and appear in the literature, but we will not consider them in the current paper (see [29] and [18, Chapter 16]).

This generalization, introducing weights, received considerable interest (see, e.g., [2, 3]). Those investigations were however focused on the investigation of zero-sum constants. In this paper, we start an investigation of the monoids of sequences over finite abelian groups that admit zero-sums with weights. For a precise definition see Section 2.

After collecting the main definitions, we study the basic algebraic properties of these monoids. Then, we investigate certain classical arithmetical invariants for these monoids in detail, namely elasticities and unions of sets of lengths. It turns out that these investigations bear similarity to those of the arithmetic of product-one sequences over non-abelian groups (see the recent papers [13, 24]). We end by showing that our results are not only a natural generalization of existing results but that they have actual applications, too. We give an arithmetic application in Section 7, namely we show that these monoids arise when investigating monoids of norms of algebraic integers; a closely related connection already appears in [19] and more implicitly in [23, Section 9.2].

2. Preliminaries

We recall some definitions and notations. For the most part, our notations are fairly common in factorization theory and we follow [12]. By $\mathbb{N}$ and $\mathbb{N}_0$ we denote the set of positive and non-negative integers, respectively. For real numbers a and b, we denote by $[a,b] = \{x \in \mathbb{Z} \colon a \le x \le b\}$ the interval of integers.

In general, we use additive notation for abelian groups. We denote by C_n a cyclic group of order n. For $(G,+,0)$, a finite abelian group, there are uniquely determined $1 < n_1 \mid \cdots \mid n_r$ such that $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$. One calls r the rank, denoted $\mathsf{r}(G)$ and n_r the exponent of G, denoted $\exp(G)$; the exponent of a group of order 1 is 1 and its rank is 0. By a basis of G we mean an independent generating family of elements $(e_1,\dots,e_s)$ of G; the family $(e_1,\dots,e_s)$ is called independent if $\sum_{i=1}^{s} a_i e_i = 0$ with $a_i \in \mathbb{Z}$ implies that $a_i e_i = 0$ for each $i \in [1,s]$. For a basis $(e_1,\dots,e_s)$ of G each element of G can be written in a unique way as $\sum_{i=1}^{s} a_i e_i$ with $a_i \in [0, \operatorname{ord}(e_i) - 1]$.

For subsets A, B of G let $A + B = \{a + b \colon a \in A,\, b \in B\}$ denote the sumset of A and B.

In this paper, by a *monoid* we always mean a commutative semigroup with identity that satisfies the cancellation law (that is, if a, b, c are elements of the monoid with $ab = ac$, then $b = c$ follows). In general, we use multiplicative notation for monoids. If we want to include the non-commutative case, we stress it explicitly and speak of not necessarily commutative monoids (they still have an identity and satisfy the cancellation law).

An element a of a monoid H is called *invertible* if there exists an element $a' \in H$ such $aa' = 1_H$ where 1_H denotes the identity of H. The set of invertible elements of H is denoted by $H^\times$; it is a subgroup of H. We call the monoid reduced if 1_H is the only invertible element of H. We call $H_{\rm red} = H/H^\times$ the reduced monoid associated to H. An element $a \in H$ is called irreducible or an atom (in H) if $a = bc$ with $b, c \in H$ implies that b or c are invertible. The set of irreducible elements of H is denoted by $\mathcal{A}(H)$. An element $p \in H \setminus H^\times$ is called prime if $p \mid bc$ with $b, c \in H$ implies that $p \mid b$ or $p \mid c$. The set of prime elements of H is denoted by $\mathcal{P}(H)$. It is not hard to see that $\mathcal{P}(H) \subseteq \mathcal{A}(H)$. In general, equality does not hold. Indeed, for an atomic monoid H equality holds if and only if H is factorial, that is, up to ordering and associates, each element has a unique factorization into irreducible elements.

For a set P a *sequence* over P is formally defined as an element of $\mathcal{F}(P)$ the free abelian monoid over P. Thus, for a sequence $S \in \mathcal{F}(P)$, there exist unique $v_p \in \mathbb{N}_0$, all but finitely many equal to 0, such that $S = \prod_{p \in P} p^{v_p}$. Then, one calls v_p the multiplicity of p in S or also its p-adic valuation of S; it is denoted by $\mathsf{v}_p(S)$. Alternatively, there exist up to ordering uniquely determined $p_1, \dots, p_\ell \in P$ (not necessarily distinct) such that $S = p_1 \dots p_\ell$. Thus, informally, a sequence is a collection of elements of P where repetitions are allowed and the ordering of elements is disregarded; one might also call these unordered sequences or multi-sets.

We call the identity element of the monoid of sequences, the empty sequence, and simply denote it by 1 unless there is a risk of confusion. Further, we denote by $|S| = \ell$ the length of S.

Formally, a subsequence of S is a sequence T that divides S in the monoid of sequences, that is $T = \prod_{i \in I} p_i$ for some $I \subseteq [1, \ell]$. Moreover, we denote by $T^{-1}S$ the sequence fulfilling $(T^{-1}S)T = S$, that is $T^{-1}S = \prod_{i \in [1,\ell] \setminus I} p_i$. This matches the intuitive idea of a subsequence of a sequence. If P' is a set and $f : P \to P'$ some map, then f can be extended to a homomorphism of monoids from $\mathcal{F}(P)$ to $\mathcal{F}(P')$, which we continue to denote by f. In particular, for a sequence S we denote by $-S$ the sequence where each term g in S is replaced by $-g$.

Often we consider sequences over a subset G_0 of an abelian group. In this case, for a sequence $S = g_1 \dots g_\ell \in \mathcal{F}(G_0)$ we denote by $\sigma(S) = \sum_{i=1}^{\ell} g_i \in G$ its sum, and the set $\Sigma(S) = \{\sigma(T) \colon 1 \neq T \mid S\}$ is called the set of (nonempty) subsums of S. A sequence whose sum is 0, the neutral element of the group, is called a zero-sum sequence. The sequence S is called zero-sum free if $0 \notin \Sigma(S)$. The set of all sequences over G_0 that are zero-sum sequences is denoted by $\mathcal{B}(G_0)$, and it is easy to see that $\mathcal{B}(G_0)$ is a submonoid of $\mathcal{F}(G_0)$. If G' is an abelian group and $f : G \to G'$ is a group homomorphism, then the image of $\mathcal{B}(G)$ under f is contained in $\mathcal{B}(G')$.

Next, we recall the notion of sequences that admit a weighted zero-sum. Traditionally, one had taken sets of integers as sets of weights. For $W \subseteq \mathbb{Z}$ a set of weights, an element of the form $\sum_{i=1}^{\ell} w_i g_i$ with $w_i \in W$ is called a W-weighted sum of S. However, for our current application another, more general, notion of weights is necessary, which also already appears in the literature see, e.g., [18].

For a subset $\Omega \subseteq \mathrm{End}(G)$ of endomorphisms of the finite abelian group G, an element $\sum_{i=1}^{\ell} \omega_i(g_i)$ with $\omega_i \in \Omega$ is called an Ω-weighted sum of S. We denote by $\sigma_\Omega(S)$ the set of all Ω-weighted sums of S. We say that S admits an Ω-weighted zero-sum if $0 \in \sigma_\Omega(S)$; we also call such a sequence an Ω-weighted zero-sum sequence. We denote the set of all Ω-weighted zero-sum sequences over G_0 by $\mathcal{B}_\Omega(G_0)$. Explicitly, a sequence $S = g_1 \dots g_l$ with $g_i \in G_0$ is in $\mathcal{B}_\Omega(G_0)$ if there exist $\omega_i \in \Omega$ such that $\omega_1(g_1) + \cdots + \omega_l(g_l) = 0$. If there is no risk of confusion we just write ωg instead of $\omega(g)$.

An element is called an Ω-weighted subsum of S if it is an Ω-weighted sum of a non-empty subsequence of S. We denote the set of all Ω-weighted subsums of S by $\Sigma_\Omega(S)$. The sequence S is called Ω-weighted zero-sum free if $0 \notin \Sigma_\Omega(S)$. In this context, we call Ω a set of weights.

To see the link between the two notions it suffices to recall that, for an integer w, multiplication by an integer w induces an endomorphism of the abelian group G. Thus, this can be considered as a generalization of the notion of weights. For the sake of completeness, we note that different integers can induce the same endomorphism, in that sense it is not a generalization in a very strict sense. However, this is essentially inconsequential in our context, and in any case it is common to only consider sets of integral weights that do not contain distinct integers that are congruent modulo the exponent of the group, in which case each integer does yield a distinct endomorphism. Thus, for all practical purposes, the latter generalizes the former notion of weights. We recall that there is an even more general notion of weights for sequences (where rather than endomorphisms of an abelian group, one considers homomophisms between two abelian groups), see [29].

The case $\Omega = \{\mathrm{id}_G\}$, corresponds to the problem without weights. It should be noted though that $\sigma_{\{\mathrm{id}_G\}}(S)$ is not $\sigma(S)$ but $\{\sigma(S)\}$. Especially when used as a subscript, we use the symbol $\pm$ to denote the set of weights $\{+\mathrm{id}_G, -\mathrm{id}_G\}$, and we use the terminology plus-minus weighted or $\pm$-weighted to refer to this set of weights; to emphasis this we usually write $+\mathrm{id}_G$ instead of id_G.

We recall some more concepts from factorization theory. A monoid H is called atomic if each non-invertible element of H can be written as a (finite) product of irreducible elements. The monoid of factorizations of H, denoted $\mathsf{Z}(H)$, is the monoid $\mathcal{F}(\mathcal{A}(H_{\mathrm{red}}))$. Informally, the elements of the monoid of factorizations correspond to factorizations of elements of H into irreducible elements where factorizations that differ only by the ordering of the terms or multiplication by units are considered as equal.

The homomorphism $\pi_H : \mathsf{Z}(H) \to H_{\mathrm{red}}$, which maps the formal product $a_1 \dots a_k$ to its value, is called the factorization homomorphism. It is surjective if and only if H is atomic. For $a \in H$, the set $\mathsf{Z}_H(a) = \pi_H^{-1}(aH^\times)$ is called the set of factorizations of a in H; if the monoid H is obvious from context we drop H from the notation.

For $z \in \mathsf{Z}(H)$, one calls $|z|$, which is defined as $\mathsf{Z}(H)$ is a free monoid, the length of the factorization; informally it is the number of irreducible elements in the factorization where multiplicities are taken into account. Moreover, one calls $\mathsf{L}_H(a) = \{|z| \colon z \in \mathsf{Z}_H(a)\}$ the set of length of a in H. Moreover, the system of set of lengths of H is defined as $\mathcal{L}(H) = \{\mathsf{L}_H(a) \colon a \in H\}$.

The monoid is called a bounded factorization monoid, BF-monoid for short, if $\mathsf{L}_H(a)$ is finite for each $a \in H$. Similarly, monoids for which even the sets of factorizations are finite are called finite factorization monoids. We note that H is factorial if and only if $|\mathsf{Z}_H(a)| = 1$ for each $a \in H$. One says that H is half-factorial if $|\mathsf{L}_H(a)| = 1$ for each $a \in H$.

Let H and $\mathcal{B}$ be monoids. A monoid homomorphism $\Theta : H \to \mathcal{B}$ is called a transfer homomorphism when it has the following two properties:

T1 $\mathcal{B} = \Theta(H)\mathcal{B}^\times$ and $\Theta^{-1}(\mathcal{B}^\times) = H^\times$.

T2 If $u \in H$ and $b, c \in \mathcal{B}$ with $\Theta(u) = bc$, then there exist $v, w \in H$ such that $u = vw$, $\Theta(v) \simeq b$ and $\Theta(w) \simeq c$.

The relevance of this notion is due to the fact that it preserves many arithmetical properties. In particular, if $\Theta : H \to \mathcal{B}$ is a transfer homomorphism, then $\mathsf{L}_H(a) = \mathsf{L}_{\mathcal{B}}(\Theta(a))$, and $\mathcal{L}(H) = \mathcal{L}(\mathcal{B})$. We refer, for example, to [9, Section 1.3]. More recently, this notion was generalized to not necessarily commutative monoids; we refer to [4] yet do not recall the details here.

There are various arithmetical invariants that are derived from sets of lengths. We recall some of them. For a more complete presentation see, e.g., [10, 12, 16].

Let $k \in \mathbb{N}$ and let H be an atomic monoid. To avoid complications in trivial corner cases we assume that $H \neq H^\times$. Then

$$\mathcal{U}_k(H) = \bigcup_{\mathsf{L} \in \mathcal{L}(H), k \in \mathsf{L}} \mathsf{L}$$

denotes the union of sets of length of H containing k. Furthermore, one sets $\rho_k(H) = \sup \mathcal{U}_k(H)$ and $\lambda_k(H) = \min \mathcal{U}_k(H)$.

The value $\rho_k(H)$ is sometimes called the k-th local elasticity of H. This terminology is derived from that of the elasticity of a monoid, denoted $\rho(H)$. If $A \subseteq \mathbb{N}$, we call $\rho(A) = \sup A / \min A \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$ the elasticity of A, and we set $\rho(\{0\}) = 1$. The elasticity of an element $a \in H$, denoted $\rho(a)$, is just $\rho(\mathsf{L}_H(a))$. Finally, the elasticity of H, denoted $\rho(H)$, is defined as $\sup\{\rho(a) \colon a \in H\} \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$. It is not difficult to see that $\rho(H) = \lim_{k \to \infty} \rho_k(H)/k$.

We end by recalling the definition of the set of (successive) distances of a monoid; we do not study it specifically in this paper, but need to invoke it in some arguments. For $A \subseteq \mathbb{Z}$, we denote by $\Delta(A)$ the set of (successive) distances of A, that is the set of all $d \in \mathbb{N}$ for which there exists some $\ell \in A$ such that $A \cap [\ell, \ell + d] = \{\ell, \ell + d\}$. Clearly, $\Delta(A) \subseteq \{d\}$ if and only if A is an arithmetical progression with difference d. A set A is called an interval if it is an arithmetical progression with difference 1. We set $\Delta(H) = \bigcup_{a \in H} \Delta(\mathsf{L}(a))$. The monoid is half-factorial if and only if $\rho(H) = 1$ if and only if $\Delta(H) = \emptyset$.

We briefly recall the notion of Krull monoids. For a submonoid H' of a monoid H, we say that $H' \subseteq H$ is saturated when $a \mid b$ in H if and only if $a \mid b$ in H'. A monoid H that is a saturated submonoid of a factorial monoid is called a Krull monoid. It is well-known that a Krull monoid admits a transfer homomorphism to a monoid of zero-sum sequences over its class group, more precisely to the monoid of zero-sum sequences over the subsets of classes containing prime divisors. A monoid is called transfer Krull if it admits a transfer homomorphism to a monoid of zero-sum sequences (see [10, Section 4]). Numerous monoids of arithmetical interest are Krull monoids or at least transfer Krull monoids, and they are no doubt the most intensely studied class of monoids in factorization theory (see [10, 16, 26]). In the current paper, they do not play that prominent a role, thus we refrain from giving further details.

3. The monoid of sequences that admit an Ω-weighted zero-sum

The purpose of this section is to introduce monoids of sequences over a finite abelian group G that admit an Ω-weighted zero-sum for a general set of weights $\Omega \subseteq \mathrm{End}(G)$ and to establish first results on their arithmetic. We establish common finiteness results. In Section 5 we will refine some of these results. In Section 6 more detailed results are obtained for the case of plus-minus weights, that is for the special case $\Omega = \{+\mathrm{id}_G, -\mathrm{id}_G\}$.

We denote, for G_0 a subset of a finite abelian group, by $\mathcal{B}_\Omega(G_0) = \{S \in \mathcal{F}(G_0) \colon 0 \in \sigma_\Omega(S)\}$ the set of sequences over G_0 that admit an Ω-weighted zero-sum over G_0. As remarked earlier such sequences are also called Ω-weighted zero-sum sequences, and one thus might refer to $\mathcal{B}_\Omega(G_0)$ as the set of Ω-weighted zero-sum sequences. To avoid confusion we stress that the sums are 'weighted' not the sequences themselves; the elements of $\mathcal{B}_\Omega(G_0)$ are just sequences over G_0, that is $\mathcal{B}_\Omega(G_0) \subseteq \mathcal{F}(G_0)$.

Noting that $\sigma_\Omega(S_1S_2) = \sigma_\Omega(S_1) + \sigma_\Omega(S_2)$, and $\sigma_\Omega(1_{\mathcal{F}(G_0)}) = \{0\}$, it follows that $\mathcal{B}_\Omega(G_0)$ is a submonoid of $\mathcal{F}(G_0)$. Since $\{\omega(\sigma(S)) \colon \omega \in \Omega\} \subseteq \sigma_\Omega(S)$, it follows that if $0 = \sigma(S)$, then $0 \in \sigma_\Omega(S)$, that is, $\mathcal{B}(G_0) \subseteq \mathcal{B}_\Omega(G_0)$.

Since $\mathcal{B}_\Omega(G_0)$ is a submonoid of the free monoid $\mathcal{F}(G_0)$, it follows that $\mathcal{B}_\Omega(G_0)$ is atomic and even a BF-monoid (see [12, Corollary 1.3.3]). We show that $\mathcal{B}_\Omega(G_0)$ is finitely generated, which yields various additional finiteness results for its arithmetical invariants. To show that the monoid is finitely generated, we need to study the set of irreducible elements $\mathcal{B}_\Omega(G_0)$.

By definition, a sequence $S \in \mathcal{B}_\Omega(G_0) \setminus \{1\}$ is irreducible in $\mathcal{B}_\Omega(G_0)$, in other words $S \in \mathcal{A}(\mathcal{B}_\Omega(G_0))$ if it is not possible to write $S = S_1S_2$ with $S_1, S_2 \in \mathcal{B}_\Omega(G_0) \setminus \{1\}$. We call such a sequence a minimal Ω-weighted zero-sum sequence. We stress that in contrast to the problem without weights this definition is not, in general, equivalent to saying that the Ω-weighted zero-sum sequence S has no proper and non-empty Ω-weighted zero-sum subsequence. In other words, it is possible that $S = S_1T$ with $S, S_1 \in \mathcal{B}_\Omega(G_0)$ and $T \in \mathcal{F}(G)$ yet $T \notin \mathcal{B}_\Omega(G_0)$. The point is that $\sigma_\Omega(S_1), \sigma_\Omega(T)$ are subsets of G and it is well possible that for subsets A, B of an abelian group one has $0 \in A+B = \{a+b \colon a \in A,\, b \in B\}$ and $0 \in A$ yet $0 \notin B$, while for elements $a, b \in G$ of course $0 = a + b$ and $0 = a$ implies $0 = b$. That is, $\mathcal{B}_\Omega(G_0)$ is not necessarily a saturated submonoid of $\mathcal{F}(G_0)$, and thus not necessarily Krull. Of course, in some specific cases it might still be Krull. We discuss this problem towards the end of this section.

The Davenport constants play an important role in the investigations of the arithmetic of monoids of zero-sum sequences. As mentioned earlier there are varied investigations on Davenport constants with weights. However, some care is needed as those constants do not correspond to the constants most relevant in the present context. To explain the situation we recall two definitions from Cziszter, Domokos and Geroldinger [5, Section 2.5].

Let H be a BF-monoid and let $|\cdot| : H \to (\mathbb{N}_0, +)$ be a homomorphism of monoids, which in this context is called a degree function, e.g., if H is a subset of a free monoid then the usual length function is a degree function. Then, for $k \in \mathbb{N}$, the k-th large Davenport constant of H (with respect to the given degree function) is defined as $\sup\{|a|\colon a \in \mathcal{M}_k(H)\}$ where $\mathcal{M}_k(H) = \{a \in H\colon \max \mathsf{L}(a) \le k\}$. For $k = 1$, the index is dropped and $\mathsf{D}(H) = \mathsf{D}_1(H)$ is called the Davenport constant of H.

Let H be a submonoid of a free monoid F and let $|\cdot|$ denote the usual length function on F, and for $k \in \mathbb{N}$ let $\mathcal{M}_k^*(H)$ be the set of all $f \in F$ such that f is not divisible (in F) by a product of k non-units in H. The k-th small Davenport constant of H, denoted $\mathsf{d}_k(H)$, is defined as $\sup\{|f|\colon f \in \mathcal{M}_k^*(H)\}$. Again, for $k = 1$ one just writes $\mathsf{d}(H)$ and calls it the small Davenport constant of H. From the definition it is follows that $1 + \mathsf{d}_k(H)$ is the smallest $l \in \mathbb{N} \cup \{\infty\}$ such that every $f \in F$ with length at least l is divisible (in F) by a product of k non-units in H.

In many common situations it is true that $1 + \mathsf{d}_k(H) \le \mathsf{D}_k(H)$ and not rarely even equality holds. Most notably this is the case for $H = \mathcal{B}(G)$, which allows to use the two definitions interchangeably. However, in general this is not true and it is even possible that $\mathsf{d}_k(H)$ exceeds $\mathsf{D}_k(H)$. In particular, the Davenport constant with weights that one usually finds in the literature and which is often denoted by $\mathsf{D}_\Omega(G)$, is in fact $1 + \mathsf{d}(\mathcal{B}_\Omega(G))$, yet not $\mathsf{D}(\mathcal{B}_\Omega(G))$. In light of this, to avoid any risk of confusion we systematically use the notation for monoids, and do not use the usual short-hand notation that drops $\mathcal{B}$.

We recall a well-known finiteness result for the Davenport constant, see for example [12, Theorem 3.4.2].

Proposition 3.1. *Let G be an abelian group and let $G_0 \subseteq G$ be a finite subset. Then $\mathsf{D}(\mathcal{B}(G_0))$ is finite.*

We proceed to show, for subsets of finite abelian groups, that $\mathsf{D}(\mathcal{B}_\Omega(G_0))$ is bounded above by $\mathsf{D}(\mathcal{B}(G))$.

Lemma 3.2. *Let G be a finite abelian group and let $G_0 \subseteq G$. Then $\mathsf{D}(\mathcal{B}_\Omega(G_0)) \le \mathsf{D}(\mathcal{B}(G))$. Moreover $\mathcal{A}(\mathcal{B}_\Omega(G_0)) \cap \mathcal{B}(G_0) \subseteq \mathcal{A}(\mathcal{B}(G))$.*

Proof. Let $g_1 \dots g_\ell$ be a sequence in $\mathcal{A}(\mathcal{B}_\Omega(G_0))$. Then there exists $\omega_i \in \Omega$ such that $\sum_{i=1}^{\ell} \omega_i g_i = 0$. We assert that $(\omega_1 g_1)\dots(\omega_\ell g_\ell) \in \mathcal{A}(\mathcal{B}(G))$. By construction the sum of the sequence is 0. It remains to show that it is a minimal zero-sum sequence. Assume that that there is some $\emptyset \ne I \subsetneq [1,\ell]$ such that $\prod_{i\in I}(\omega_i g_i)$ and $\prod_{i\in[1,\ell]\setminus I}(\omega_i g_i)$ are zero-sum sequences. Then $(\prod_{i\in I} g_i)$ and $(\prod_{i\in[1,\ell]\setminus I} g_i)$ are Ω-weighted zero-sum sequences. A contradiction.

Thus, for each $S \in \mathcal{A}(\mathcal{B}_\Omega(G_0))$ there exists some $S' \in \mathcal{A}(\mathcal{B}(G))$ of the same lengths. This implies that directly that $\mathsf{D}(\mathcal{B}_\Omega(G_0)) \le \mathsf{D}(\mathcal{B}(G))$.

The additional statement is readily seen by recalling that $\mathcal{B}(G_0) \subseteq \mathcal{B}_\Omega(G_0)$ and thus each factorization in $\mathcal{B}(G_0)$ yields one in $\mathcal{B}_\Omega(G_0)$. □

Theorem 3.3. *Let G be a finite abelian group. Then $1+\mathsf{d}(\mathcal{B}_\Omega(G)) \le \mathsf{D}(\mathcal{B}_\Omega(G)) \le \mathsf{D}(\mathcal{B}(G))$.*

Proof. By Lemma 3.2 we have $\mathsf{D}(\mathcal{B}_\Omega(G)) \le \mathsf{D}(\mathcal{B}(G))$. We now show that $1 + \mathsf{d}(\mathcal{B}_\Omega(G)) \le \mathsf{D}(\mathcal{B}_\Omega(G))$. Let S be a sequence of length $\ell = \mathsf{D}(\mathcal{B}_\Omega(G)))$. We show that it has a non-empty Ω-weighted zero-sum subsequence. We consider the sequence $(-\sigma(S))S$, which is in $\mathcal{B}(G)$ and thus also in $\mathcal{B}_\Omega(G)$ by the inclusion $\mathcal{B}(G) \subseteq \mathcal{B}_\Omega(G)$. Now, $|(-\sigma(S))S| > \mathsf{D}(\mathcal{B}_\Omega(G))$. Consequently, it is not a minimal Ω-weighted zero-sum sequence and there exit non-empty $S_1, S_2 \in \mathcal{B}_\Omega(G)$ such that

$(-\sigma(S))S = S_1S_2$. It follows that S_1 or S_2 is a subsequence of S, establishing that it has a non-empty Ω-weighted zero-sum subsequence.

Thus, we established that every sequence of length $\mathsf{D}(\mathcal{B}_\Omega(G))$ has a non-empty Ω-weighted zero-sum subsequence. Since by definition $1 + \mathsf{d}(\mathcal{B}_\Omega(G))$ is the smallest positive integer with this property, we have $1 + \mathsf{d}(\mathcal{B}_\Omega(G)) \le \mathsf{D}(\mathcal{B}_\Omega(G))$. □

We record the following direct corollary.

Corollary 3.4. *Let G be a finite abelian group and let $G_0 \subseteq G$. Let $\Omega \subseteq \operatorname{End}(G)$ be a set of weights. The monoid $\mathcal{B}_\Omega(G_0)$ is finitely generated.*

Proof. Since the length of elements of $\mathcal{A}(\mathcal{B}_\Omega(G_0))$ is bounded above by $\mathsf{D}(\mathcal{B}_\Omega(G_0))$, which is finite by Lemma 3.2, it follows that the set $\mathcal{A}(\mathcal{B}_\Omega(G_0))$ is finite, that is, the monoid is finitely generated. □

This result has immediate and strong consequences for the arithmetic of these monoids, which we discuss belows. However, first we establish another lower bound on the Davenport constant that we need later on.

Lemma 3.5. *Let $G = G_1 \oplus G_2$ be a finite abelian group. Let $\Omega \subseteq \operatorname{End}(G)$ be a set of endomorphisms that forms a group under composition and such that $\omega(G_i) \subseteq G_i$ for $i \in \{1, 2\}$. Then $\mathsf{D}(\mathcal{B}_\Omega(G)) \ge \mathsf{D}(\mathcal{B}_\Omega(G_1)) + \mathsf{D}(\mathcal{B}_\Omega(G_2)) - 1$.*

Proof. For $i \in \{1, 2\}$, let A_i be an element of $\mathcal{A}(\mathcal{B}_\Omega(G_i))$ of maximal length; furthermore let g_i be some fixed element of A_i and let $A_i = g_iF_i$. Since $0 \in \sigma_\Omega(A_i)$ for $i \in \{1, 2\}$, there is some $\omega_i \in \Omega$ such that $\omega_i g_i \in -\sigma_\Omega(F_i)$. We consider $A = (\omega_1 g_1 + \omega_2 g_2)F_1F_2$ and assert that it is contained in $\mathcal{A}(\mathcal{B}_\Omega(G))$. We start by showing that $0 \in \sigma_\Omega(A)$. Since Ω is a group, there is some $\epsilon \in \Omega$ such that $\epsilon \circ \omega_i = \omega_i$ for $i \in \{1, 2\}$.

Now, $\epsilon(\omega_1 g_1 + \omega_2 g_2) \in -(\sigma_\Omega(F_1) + \sigma_\Omega(F_2)) = -(\sigma_\Omega(F_1F_2))$, implies that $0 \in \sigma_\Omega(A)$. It remains to show that there is no decomposition $A = A'A''$ with non-empty A' and A'' such that $0 \in \sigma_\Omega(A')$ and $0 \in \sigma_\Omega(A'')$. Assume to the contrary that such a decomposition exists. Without loss we may assume that $\omega_1 g_1 + \omega_2 g_2$ occurs in A'. We write $A' = (\omega_1 g_1 + \omega_2 g_2)F_1'F_2'$ and $A'' = F_1''F_2''$ where $F_i = F_i'F_i''$ for $i \in \{1, 2\}$.

Since $0 \in \sigma_\Omega(A'')$ and $\sigma_\Omega(F_i'') \subseteq G_i$ for $i \in \{1, 2\}$, it follows that $0 \in \sigma_\Omega(F_i'')$. Moreover, there is some $\omega \in \Omega$ such that $\omega(\omega_1 g_1 + \omega_2 g_2) \in -\sigma_\Omega(F_1'F_2')$. It follows that $\omega(\omega_i g_i) \in -\sigma_\Omega(F_i')$ for $i \in \{1, 2\}$. Now, since $\omega \circ \omega_i \in \Omega$, this implies that $0 \in \sigma_\Omega(g_iF_i')$. Thus, $A_i = (g_iF_i')F_i''$ and $0 \in \sigma_\Omega(g_iF_i')$ and $0 \in \sigma_\Omega(F_i'')$. Since at least one of F_1'' and F_2'' is non-empty and g_1F1' and g_2F_2' are of course both non-empty, we get a contradiction to A_1 or A_2 being irreducible. □

We now turn to the arithmetic of monoids of weighted zero-sum sequences.

Theorem 3.6. *Let G be a finite abelian group and let $G_0 \subseteq G$. Let $\Omega \subseteq \operatorname{End}(G)$ be a set of weights. Let $H = \mathcal{B}_\Omega(G_0)$.*

1. *The set $\Delta(H)$ and the constant $\rho(H)$ are finite.*
2. *There is some $M \in \mathbb{N}_0$ such that each set of lengths L of H is an almost arithmetical multiprogression with bound M and difference $d \in \Delta(H) \cup \{0\}$, that is, $L = y + (L_1 \cup L^* \cup L_2) \subseteq y + \mathcal{D} + d\mathbb{Z}$ with $y \in \mathbb{N}_0$, $\{0, d\} \subseteq \mathcal{D} \subseteq [0, d]$, $L_1, -L_2 \subseteq [1, M]$, $\min L^* = 0$ and $L^* = [0, \max L^*] \cap \mathcal{D} + d\mathbb{Z}$.*

(3) *There is some $M' \in \mathbb{N}_0$ such that each $k \in \mathbb{N}_0$ the set $\mathcal{U}_k(H)$ is an almost arithmetical progression with bound M' and difference* $\min \Delta(H)$*, that is,* $\mathcal{U}_k(H) = y' + (U_1 \cup U^* \cup U_2) \subseteq y + d\mathbb{Z}$ *with* $y \in \mathbb{N}_0$*,* $U_1, -U_2 \subseteq [1, M']$*,* $\min U^* = 0$ *and* $U^* = [0, \max U^*] \cap d\mathbb{Z}$*.*

Proof. By Corollary 3.4 the monoid is finitely generated. The claim now follows from results on finitely generated monoids, specifically see [12, Theorem 3.1.4 and 4.4.11], and for the final part see [6, Theorem 3.6]. □

It is also known that various other arithmetic invariants of H are finite, including the catenary degree $\mathsf{c}(H)$ and the tame degree $\mathsf{t}(H)$, in particular the monoid is locally tame; moreover in the result above the set of distances $\Delta(H)$ can be replaced by $\Delta^*(H)$ and it is know that the elasticity is accepted. We refer to the references mentioned in the proof just above and [16, Section 3].

We recall that the second point of the result is referred to as Structure Theorem for Sets of Lengths, while the third is called Structure Theorem for Unions.

In Section 5 we refine the Structure Theorem for Unions for this class of monoids showing that for a wide class of weights the sets are indeed arithmetic progressions, with difference 1, i.e., intervals of integers.

We end this section by some more algebraic results on these monoids. We show that in general they are not Krull monoids, not even transfer Krull monoids. However, they are still C-monoids; we refer to [12, Section 2.9] for a definition. For the special case of plus-minus weighted zero-sum sequences it is possible to characterize completely when such a monoid is Krull and transfer Krulll. This is done in Proposition 3.8, which is due to Geroldinger and Zhong [17] including the main idea of the lemma preceding it.

Lemma 3.7. *Let G be an abelian group with* $\exp(G) \geq 3$*. Let Ω be a set of weights such that* $\{+\mathrm{id}_G, -\mathrm{id}_G\} \subseteq \Omega \subseteq \mathrm{Aut}(G)$*. Then $\mathcal{B}_\Omega(G)$ is not a transfer Krull monoid.*

Proof. Assume to the contrary that there is a transfer homomorphism $\theta \colon \mathcal{B}_\Omega(G) \to \mathcal{B}(G_0)$, where G_0 is a subset of any abelian group. Let $g \in G$ with $\mathrm{ord}(g) \geq 3$. We observe that

$$A_1 = g^2,\ A_2 = (2g)^2, \quad \text{and} \quad A_3 = g^2(2g)$$

are atoms of $\mathcal{B}_\Omega(G)$; indeed, that $A_1, A_2, A_3 \in \mathcal{B}_\Omega(G)$ follows from $\{+\mathrm{id}_G, -\mathrm{id}_G\} \subseteq \Omega$, that they are irreducible follows from $\omega(g)$ and $\omega(2g)$ being non-zero for each $\omega \in \Omega$.

Since $A_3^2 = A_1^2 A_2$, we have $\theta(A_3^2) = \theta(A_1^2 A_2)$ and it follows that

$$\theta(A_3)^2 = \theta(A_1)^2 \theta(A_2) \in \mathcal{B}(G_0) \subseteq \mathcal{F}(G_0)\,.$$

Therefore $\theta(A_1)^2$ divides $\theta(A_3)^2$ in $\mathcal{F}(G_0)$ whence $\theta(A_1)$ divides $\theta(A_3)$ in $\mathcal{F}(G_0)$. This implies that $\theta(A_1)$ divides $\theta(A_3)$ in $\mathcal{B}(G_0)$ and hence $\theta(A_1) = \theta(A_3)$ (because both elements are atoms). Thus we get $\theta(A_2) = 1$, a contradiction to the first condition in the definition of transfer homomorphism. □

Proposition 3.8. *Let G be an abelian group and $\mathcal{B}_\pm(G)$ the monoid of $\pm$-zero-sum sequences. Then the following statements are equivalent:*

(a) *G is an elementary 2-group.*
(b) *$\mathcal{B}_\pm(G)$ is a Krull monoid.*
(c) *$\mathcal{B}_\pm(G)$ is a transfer Krull monoid.*

Proof. (a) $\Rightarrow$ (b) If G is an elementary 2-group, then $\mathcal{B}_{\pm}(G) = \mathcal{B}(G)$ as $-g = g$ for each $g \in G$, and $\mathcal{B}(G)$ is a Krull monoid.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) Since the conditions on the set of weights in Lemma 3.7 hold, $\mathcal{B}_{\pm}(G)$ can only be a transfer Krull monoid when G contains no element of order at least 3, that is G is an elementary 2-group □

In other words, unless $\mathcal{B}_{\pm}(G) = \mathcal{B}(G)$, the monoid $\mathcal{B}_{\pm}(G)$ is not a Krull monoid. However, we now show that for finite abelian G and Ω a set of weights, the monoid $\mathcal{B}_{\Omega}(G)$ is a C-monoid. To this end we recall a result by Cziszter, Domokos and Geroldinger [5, Proposition 2.6.3] in a spcial case.

Proposition 3.9. *Let H be a finitely generated and reduced monoid. Suppose that H is a submonoid of a free monoid $\mathcal{F}(P)$. The following statements are equivalent:*

(1) *H is a C-monoid defined in $\mathcal{F}(P)$ and for every $p \in P$ there is an $a \in H$ such that $\mathsf{v}_p(a) > 0$.*
(2) *For every $a \in \mathcal{F}(P)$ there is an $n_a \in \mathbb{N}$ such that $a^{n_a} \in H$.*

Theorem 3.10. *Let G be a finite abelian group and let $G_0 \subseteq G$. Let $\Omega \subseteq \mathrm{End}(G)$ be a set of weights. The monoid $\mathcal{B}_{\Omega}(G_0)$ is a C-monoid defind in $\mathcal{F}(G_0)$.*

Proof. By Proposition 3.9, it suffices to show that for $S \in \mathcal{F}(G_0)$ there is an $n \in \mathbb{N}$ such that $S^n \in \mathcal{B}_{\Omega}(G_0)$. Let $\omega \in \Omega$. Let n denote the least common multiple of $\{\mathrm{ord}(\omega(g)) \colon g \mid S\}$. Then $0 = \sigma(\omega(S^n)) \in \sigma_{\Omega}(S^n)$. □

4. Some general auxiliary results

We collect some results that are useful for our investigations but are not specific to monoids of (weighted) zero-sum sequences. For the most part they concern the sets $\mathcal{U}_k(H)$ and related notions. We refer to [9, Lemma 5.2] for further details.

Lemma 4.1. *Let H be an atomic monoid with $k, l \in \mathbb{N}_0$ then:*

(1) *$\mathcal{U}_k(H) = \{k\}$ for $k \in \{0,1\}$ and $k \in \mathcal{U}_k(H)$ for each $k \in \mathbb{N}$.*
(2) *For $k, l \in \mathbb{N}$ we have $l \in \mathcal{U}_k(H)$ if and only if $k \in \mathcal{U}_l(H)$.*
(3) *$\mathcal{U}_k(H) + \mathcal{U}_l(H) \subseteq \mathcal{U}_{k+l}(H)$.*
(4) *$\lambda_{k+l}(H) \le \lambda_k(H) + \lambda_l(H) \le k + l \le \rho_k(H) + \rho_l(H) \le \rho_{k+l}(H)$.*
(5) *$\rho_k(H) \le k\rho(H)$ and $k \le \lambda_k(H)\rho(H)$.*

We make some further general observations on these constants.

Lemma 4.2. *Let H be an atomic monoid. Let $k \in \mathbb{N}$.*

(1) *$\rho_{\lambda_k(H)}(H) \ge k$.*
(2) *If $\rho_k(H)$ is finite, then $\lambda_{\rho_k(H)}(H) \le k$.*

Proof. 1. Since by definition $\lambda_k(H) \in \mathcal{U}_k(H)$, it follows by Lemma 4.1.2 that $k \in \mathcal{U}_{\lambda_k(H)}(H)$. Since by definition $\rho_{\lambda_k(H)}(H) = \sup \mathcal{U}_{\lambda_k(H)}(H)$, it is thus plain that $\rho_{\lambda_k(H)}(H) \ge k$.

2. Since $\rho_k(H)$ is finite, we have $\rho_k(H) \in \mathcal{U}_k(H)$, and again by Lemma 4.1.2 $k \in \mathcal{U}_{\rho_k(H)}(H)$ and hence $\lambda_{\rho_k(H)}(H) \le k$. □

Lemma 4.3. *Let H be an atomic monoid. Let $k \in \mathbb{N}$. Suppose that $\mathcal{U}_i(H)$ is an interval for each $i \le k$. Then, we have $\lambda_k(H) = \min\{i \colon \rho_i(H) \ge k\}$.*

Proof. Let $j \in \mathbb{N}$ be minimal such that $\rho_j(H) \geq k$; note that since $\rho_k(H) \geq k$ by Lemma 4.2 such a j exists and $j \leq k$.

Since $j \leq k \leq \rho_j(H)$ and $\mathcal{U}_j(H)$ is an interval, it follows that $k \in \mathcal{U}_j(H)$. Thus, by Lemma 4.1.2 we have $j \in \mathcal{U}_k(H)$ and therefore $\lambda_k(H) \leq j$.

Since by Lemma 4.2, we have $\rho_{\lambda_k(H)}(H) \geq k$, it follows that $\min\{i \colon \rho_i(H) \geq k\} \leq \lambda_k(H)$. □

Lemma 4.4. *Let H be an atomic monoid. Let $k \in \mathbb{N}$. Suppose that $\mathcal{U}_i(H)$ is an interval for each $i \leq k$. Then, we have $\rho_k(H) = \sup\{i \colon \lambda_i(H) \leq k\}$.*

Proof. Let $j \in \mathbb{N}$ with $j \geq k$ such that $\lambda_j(H) \leq k$; note that since $\lambda_k(H) \leq k$ such a j exists.

Since $\lambda_j(H) \leq k \leq j$ and $\mathcal{U}_j(H)$ is an interval, it follows that $k \in \mathcal{U}_j(H)$. Thus, by Lemma 4.1.2 we have $j \in \mathcal{U}_k(H)$ and therefore $\rho_k(H) \geq j$. Thus, $\rho_k(H) \geq \sup\{i \colon \lambda_i(H) \leq k\}$.

If $\sup\{i \colon \lambda_i(H) \leq k\}$ is infinite, it follows that $\rho_k(H) = \infty$. Assume that $\sup\{i \colon \lambda_i(H) \leq k\}$ is finite and let $j > \sup\{i \colon \lambda_i(H) \leq k\}$. Assume that $\rho_k(H) \geq j$. Since $\mathcal{U}_k(H)$ is an interval, it follows that $j \in \mathcal{U}_k(H)$. Yet, this implies that $k \in \mathcal{U}_j(H)$ and thus $k \leq \lambda_j(H)$, a contradiction to $j > \sup\{i \colon \lambda_i(H) \leq k\}$. Thus $\rho_k(H) < j$ and the claim follows. □

The following lemma is a slight generalization of [24, Lemma 5.1].

Lemma 4.5. *Let P be a set and let $S_1, \cdots, S_k, T_1, \cdots, T_\ell \in \mathcal{F}(P)$ be non-empty sequences such that*

$$S_1 \dots S_k = T_1 \dots T_\ell.$$

If $k < \ell$, then there exists some $i_0 \in [1, k]$ and $j_1, j_2 \in [1, \ell]$ such that $p_1 p_2 \mid S_{i_0}$ and $p_1 \mid T_{j_1}$ and $p_2 \mid T_{j_2}$.

Proof. We assume that $k < \ell$ and proceed by induction on k. Set $k = 1$. Suppose for any two h, h' with $hh' \mid S_1$ there are no $j, j' \in [1, \ell]$ such that $h \mid T_j$ and $h' \mid T_{j'}$. It follows that $S_1 = S_k \mid T_j$ for some $j \in [1, \ell]$ say $S_k \mid T_\ell$. Now, we get

$$1_{\mathcal{F}(G)} = T_1 T_2 \dots (T_\ell S_k^{-1})$$

a contradiction since T_1 is non empty. Suppose now $k \geq 2$ and assume the claim is true for $k-1$ we have $S_1 \dots S_k = T_1 \dots T_\ell$; as before we obtain that $S_k \mid T_\ell$ say. We consider $S_1 \dots S_{k-1} = T_1 \cdots T_{\ell-1}(T_\ell S_k^{-1})$. Now, the claim follows by the induction hypotheses applied to $S_1 \dots S_{k-1} = T_1 \dots T_{\ell-1}$. □

In the lemma below, which is essentially in [1] see in particular Theorem 2.1, we adopt the convention that $a/0 = \infty$ for $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. Note that the condition that H is not factorial guarantees that $\mathcal{A}(H) \setminus \mathcal{P}(H) \neq \emptyset$. Of course, for a factorial monoid H one has $\rho(H) = 1$, thus nothing is lost by excluding this case.

Lemma 4.6. *Let H be an atomic monoid that is not factorial. Let $r : H \to (\mathbb{R}_{\geq 0}, +)$ be a monoid homomorphism.*

1. *Let $r_1 = \inf\{r(a) \colon a \in \mathcal{A}(H)\}$ and let $R_1 = \sup\{r(a) \colon a \in \mathcal{A}(H)\}$. Then $\rho(H) \leq R_1/r_1$.*
2. *Let $r_2 = \inf\{r(a) \colon a \in \mathcal{A}(H) \setminus \mathcal{P}(H)\}$ and let $R_2 = \sup\{r(a) \colon a \in \mathcal{A}(H) \setminus \mathcal{P}(H)\}$. Then $\rho(H) \leq R_2/r_2$.*

Proof. Since r is a monoid homomorphism, it follows that $r(u) = 0$ for each $u \in H^\times$. Without loss we can assume that the monoid is reduced.

1. Let $a_1, \dots, a_k, b_1, \dots, b_l \in \mathcal{A}(H)$ such that $a_1 \dots a_k = b_1 \dots b_l$. It suffices to show that $l/k \le R_1/r_1$. We note that

$$kR_1 \ge r(a_1) + \cdots + r(a_k) = r(a_1 \dots a_k) = r(b_1 \dots b_l) = r(b_1) + \cdots + r(b_l) \ge lr_1,$$

and the claim follows.

2. Let $a_1, \dots, a_k, b_1, \dots, b_l \in \mathcal{A}(H)$ such that $a_1 \dots a_k = b_1 \dots b_l$. Suppose first that none of $a_1, \dots, a_k, b_1, \dots, b_l$ is prime. Then, of course, we can conclude

$$kR_2 \ge r(a_1) + \cdots + r(a_k) = r(a_1 \dots a_k) = r(b_1 \dots b_l) = r(b_1) + \cdots + r(b_l) \ge lr_2.$$

Suppose that this is not the case, say, renumbering if necessary, $a_{(k-r)+1}, \dots, a_k$ are prime while $a_1, \dots, a_{k-r}$ are not prime. It follows, renumbering if necessary, that $a_{k-r+i} = b_{l-r+i}$ for each $1 \le i \le r$ and that $a_1 \dots a_{k-r} = b_1 \dots b_{l-r}$. If $k - r = 0$, our claim is trivially true and we assume that $k - r \ne 0$. Since now none of $a_1, \dots a_{k-r}, b_1, \dots, b_{l-r}$ are prime (note that if one of the b_j is prime also one of the a_i would be prime), we get as above

$$(k - r)R_2 \ge (l - r)r_2$$

and thus

$$\frac{k - r}{l - r} \le \frac{R_2}{r_2}.$$

Now for $k \ge l$ we have that $\frac{k-r}{l-r} \ge \frac{k}{l}$ and thus

$$\frac{k}{l} \le \frac{R_2}{r_2}.$$

This completes the proof as for $k < l$ the inequality holds trivially. $\square$

5. Results on $\mathcal{U}_k(H)$ for monoids of weighted zero-sum sequences

The purpose of this section is to obtain various results for $\mathcal{U}_k(H)$ for monoids of weighted zero-sum sequences that go beyond what was already established in Theorem 3.6. First, we establish that under some assumptions on the weights these sets are intervals, that is, arithmetic progressions with difference 1. We then proceed to study the maxima and minima of these sets, that is $\rho_k(H)$ and $\lambda_k(H)$, which in combination yields a complete description of these sets.

For the proof of our results, we use the results of Section 3 of [7] that are valid for $\mathcal{B}_\Omega(G)$; we summarize them in the following lemma.

Lemma 5.1. *Let H be an atomic monoid. Suppose that $\Delta(H) \ne \emptyset$ and $d = \min \Delta(H)$. Then, we have:*

(1) *$\Delta(\mathcal{U}_k(H)) \subseteq d\mathbb{N}$, and there exists $k^* \in \mathbb{N}$ such that $\min \Delta(\mathcal{U}_k(H)) = d$ for all $k \ge k^*$.*
(2) *$\sup \Delta(\mathcal{U}_k(H)) \le \sup \Delta(H)$ for all $k \in \mathbb{N}$.*
(3) *If $k \in \mathbb{N}$ and $\mathcal{U}_m(H) \cap \mathbb{N}_{\ge m}$ is an arithmetical progression with difference d for all $m \in [\lambda_k(H), k]$, then $\mathcal{U}_k(H) \cap [0, k]$ is an arithmetical progression with difference d.*
(4) *The following statements are equivalent:*
    (a) *$\mathcal{U}_k(H) \cap \mathbb{N}_{\ge k}$ is an arithmetical progression with difference d.*
    (b) *$\mathcal{U}_k(H)$ is an arithmetical progression with difference d for all $k \in \mathbb{N}$.*

We show that the sets $\mathcal{U}_k(\mathcal{B}_\Omega(G))$ are intervals if the set of weights $\Omega \subseteq \mathrm{End}(G)$ is a group with respect to composition of endomorphisms. We stress that Ω might be a group while not containing id_G, which makes some slight additional complication in the argument. Indeed, in some other results we assume in addition that $\mathrm{id}_G \in \Omega$, in other words we make the stronger assumption that Ω is a subgroup of $\mathrm{Aut}(G)$.

Theorem 5.2. *Let G be a finite abelian group. Let $\Omega \subseteq \mathrm{End}(G)$. If Ω is a group with respect to composition of endomorphisms, then $\mathcal{U}_k(\mathcal{B}_\Omega(G))$ is an interval for each $k \in \mathbb{N}$.*

Proof. By Lemma 5.1 it suffices to show that $\mathcal{U}_k(\mathcal{B}_\Omega(G)) \cap \mathbb{N}_{\ge k}$ is an interval for each $k \in \mathbb{N}$. This means we need to show that $[k, \rho_k(\mathcal{B}_\Omega(G))] \subseteq \mathcal{U}_k(\mathcal{B}_\Omega(G))$.

Let $\ell \in [k, \rho_k(\mathcal{B}_\Omega(G))]$ be minimal such that $[\ell, \rho_k(\mathcal{B}_\Omega(G))] \subseteq \mathcal{U}_k(\mathcal{B}_\Omega(G))$. This is well-defined as of course for $\ell = \rho_k(\mathcal{B}_\Omega(G))$ we have $[\ell, \rho_k(\mathcal{B}_\Omega(G))] \subseteq \mathcal{U}_k(\mathcal{B}_\Omega(G))$. We want to show that $\ell = k$. Assume to the contrary $\ell > k$.

We consider the set of all $B \in \mathcal{B}_\Omega(G)$ with $\{k, j\} \subseteq \mathsf{L}(B)$ for some $j \ge \ell$. Let B_0 be such an element such that $|B_0|$ is minimal among all these elements. Now, let $B_0 = U_1 \dots U_k = V_1 \dots V_j$.

By Lemma 4.5 we may assume that there are

$$g_1 g_2 \mid U_1 \text{ such that } g_1 \mid V_{j-1} \text{ and } g_2 \mid V_j.$$

Let $\omega_i \in \Omega$ such that $\Sigma_{i=1}^{|U_1|} \omega_i g_i = 0$. Let $g_0 = \omega_1 g_1 + \omega_2 g_2$. Put $U_1' = g_0 (g_1 g_2)^{-1} U_1$ and $V_{j-1}' = g_0 V_{j-1} V_j (g_1 g_2)^{-1}$. Since $-g_0 = -(\omega_1 g_1 + \omega_2 g_2) \in \sigma_\Omega((g_1 g_2)^{-1} U_1)$ and since Ω is a group $\omega(-g_0) \in \sigma_\Omega((g_1 g_2)^{-1} U_1)$ for some (in fact, for each) $\omega \in \Omega$, it follows that $0 \in \sigma_\Omega(U_1')$. We assert that $0 \in \sigma_\Omega(V_{j-1}')$ holds as well. To see this note that $0 \in \sigma_\Omega(V_{j-1})$ implies that $-\omega_1' g_1 \in \sigma_\Omega(g_1^{-1} V_{j-1})$ for some $\omega_1' \in \Omega$. Since Ω is a group, it follows that $-\omega_1' g_1 \in \sigma_\Omega(g_1^{-1} V_{j-1})$ for every $\omega_1' \in \Omega$ in particular $-\omega_1 g_1 \in \sigma_\Omega(g_1^{-1} V_{j-1})$. In the same way we obtain $-\omega_2 g_2 \in \sigma_\Omega(g_2^{-1} V_j)$. Thus, $-g_0 = -(\omega_1 g_1 + \omega_2 g_2) \in \sigma_\Omega(V_{j-1} V_j (g_1 g_2)^{-1})$ and since Ω is a group we have $\omega(-g_0) \in \sigma_\Omega(V_{j-1} V_j (g_1 g_2)^{-1})$ for some (in fact, for each) $\omega \in \Omega$. Thus $0 \in \sigma_\Omega(V_{j-1}')$.

Thus $U_1', V_{j-1}' \in \mathcal{B}_\Omega(G)$. Indeed, U_1' is in $\mathcal{A}(\mathcal{B}_\Omega(G))$, as a factorization of U_1' would directly yield a factorization of U_1. For V_{j-1}' this is however not clear at this point. Let

$$B_0' = U_1' U_2 \dots U_k = V_1 \dots V_{j-2} V_{j-1}'$$

It is clear that $B_0' \in \mathcal{B}_\Omega(G)$ and $|B_0'| < |B_0|$.

Since U_1' is an atom, it follows that $k \in \mathsf{L}(B_0')$. By our assumption on B_0 it thus follows that $\mathsf{L}(B_0')$ does not contain any element greater than or equal to l, that is $\max \mathsf{L}(B_0') < \ell$.

Since $B_0' = V_1 \dots V_{j-2} V_{j-1}'$, it follows that $j - 2 + \mathsf{L}_{\mathcal{B}_\Omega(G)}(V_{j-1}') \subseteq \mathsf{L}_{\mathcal{B}_\Omega(G)}(B_0')$. Since $\max \mathsf{L}(B_0') < \ell$, it follows that $j - 2 + \max \mathsf{L}_{\mathcal{B}_\Omega(G)}(V_{j-1}') < \ell$.

As V_{j-1}' is not the empty sequence, $\max \mathsf{L}_{\mathcal{B}_\Omega(G)}(V_{j-1}') \ge 1$. Finally $j \ge \ell$, yields the following chain of inequalities: $\ell - 2 + 1 \le j - 2 + \max \mathsf{L}_{\mathcal{B}_\Omega(G)}(V_{j-1}') < \ell$. Thus, $j - 2 + \max \mathsf{L}_{\mathcal{B}_\Omega(G)}(V_{j-1}') = \ell - 1$, and $\ell - 1 \in \mathsf{L}(B_0')$. Since $k \in \mathsf{L}(B_0')$, too. It follows that $\ell - 1 \in \mathcal{U}_k(\mathcal{B}_\Omega(G))$. Thus $[\ell - 1, \rho_k(\mathcal{B}_\Omega(G))] \subseteq \mathcal{U}_k(\mathcal{B}_\Omega(G))$. A contradiction to the definition of ℓ. □

We proceed to establish a result that is useful for investigations of elasticities and related problems.

Lemma 5.3. *Let G be a finite abelian group. Let $\Omega \subseteq \mathrm{End}(G)$ and let $j \in [2, \mathsf{D}(\mathcal{B}_\Omega(G))]$.*

1. *If Ω is a semigroup with respect to composition, then there exists some $A \in \mathcal{A}(\mathcal{B}_\Omega(G))$ with $|A| = j$.*
2. *If $\Omega \subseteq \mathrm{Aut}(G)$ is a subgroup, then there exists some $B \in \mathcal{B}_\Omega(G)$ such that $\{2, j\} \subseteq \mathsf{L}(B)$.*

Proof. 1. Let $C \in \mathcal{A}(\mathcal{B}_\Omega(G))$ with length $l = \mathsf{D}(\mathcal{B}_\Omega(G))$. Suppose that $C = \prod_{i=1}^{l} g_i$ and $\sum_{i=1}^{l} \omega_i g_i = 0$ with $\omega_i \in \Omega$. Let $s = \sum_{i=1}^{l-j+1} \omega_i g_i$ and $A = s \prod_{i=l-j+2}^{l} g_i$. Then, $A \in \mathcal{B}_\Omega(G)$, since for $\omega \in \Omega$ we have $\omega(\sum_{i=1}^{l} \omega_i(g_i)) = 0$ and thus $\omega(s) + \sum_{i=l-j+2}^{l} (\omega \circ \omega_i)(g_i) = 0$. Moreover, it follows that $A \in \mathcal{A}(\mathcal{B}_\Omega(G))$, since a non-trivial factorization of A would directly yield a non-trivial factorization of C; note that we need again that Ω is a semigroup. Since $|A| = j$, this proves the first assertion.

2. Let $A \in \mathcal{A}(\mathcal{B}_\Omega(G))$ with $|A| = j$. Note that $0 \nmid A$. It is easy to see that $-A \in \mathcal{A}(\mathcal{B}_\Omega(G))$. We consider $B = (-A)A$. By definition $2 \in \mathsf{L}(B)$. For each $g \in G$, one has $(-g)g \in \mathcal{B}_\Omega(G)$; note that since $\omega \in \Omega$ is an endomorphisms of G one always has $\omega(-g) = -\omega(g)$. Since Ω contains only monomorphisms, it follows that $(-g)g \in \mathcal{A}(\mathcal{B}_\Omega(G))$. Thus $\max \mathsf{L}(B) = |A|$, and the claim is established. □

Lemma 5.4. *Let G be a finite abelian group. Let $\Omega \subseteq \mathrm{End}(G)$. Let $k \in \mathbb{N}$. Then $\rho_{2k}(\mathcal{B}_\Omega(G)) \ge k\,\mathsf{D}(\mathcal{B}_\Omega(G))$ and $\rho_{2k+1}(\mathcal{B}_\Omega(G)) \ge 1 + k\,\mathsf{D}(\mathcal{B}_\Omega(G))$.*

Proof. Let $A \in \mathcal{A}(\mathcal{B}_\Omega(G))$ with maximal length. We know that $-A \in \mathcal{A}(\mathcal{B}_\Omega(G))$ and consider $B = (-A)^k A^k$. By definition, $2k \in \mathsf{L}(B)$. Since for each $g \in G$, one has $(-g)g \in \mathcal{B}_\Omega(G)$ (note that since $\omega \in \Omega$ is an endomorphisms of G one always has $\omega(-g) = -\omega(g)$). It follows that $\max \mathsf{L}(B) \ge k|A|$ and the claim is established. The second claim is an immediate consequence of the former, e.g., we can consider $0B$. □

We now want to use Lemma 4.6 to establish that $\rho_{2k}(\mathcal{B}_\Omega(G)) = k\,\mathsf{D}(\mathcal{B}_\Omega(G))$ in various cases. To this end we need to have a lower bound of 2 on the length of atoms that are not prime. Somewhat surprisingly it turns out that this is not true for all sets of weights.

Example 5.5. Let $G = C_2 \oplus C_6$ let $e_1, e_2 \in G$ be independent with $\mathrm{ord}(e_1) = 2$ and $\mathrm{ord}(e_2) = 6$. Let $\Omega = \{+2\,\mathrm{id}_G, +\mathrm{id}_G, -\mathrm{id}_G\}$. Then e_1 is in $\mathcal{A}(\mathcal{B}_\Omega(G))$ as $2\,\mathrm{id}_G(e_1) = 0$. Yet, e_1 is not in $\mathcal{P}(\mathcal{B}_\Omega(G))$ as $e_1 \nmid e_1(e_1 + e_2)e_2$ while $e_1 \mid (e_1(e_1 + e_2)e_2)^2$.

However under some conditions on the set of weights this holds true.

Lemma 5.6. *Let G be a finite abelian group. Let $\Omega \subseteq \mathrm{End}(G)$. Let $A \in \mathcal{A}(\mathcal{B}_\Omega(G)) \setminus \mathcal{P}(\mathcal{B}_\Omega(G))$.*

1. *If Ω only contains monomorphismes, then $|A| \ge 2$.*
2. *If Ω is a commutative semigroup with respect to composition, then $|A| \ge 2$.*

Proof. Assume that Ω only contains monomorphismes. It follows directly that the only elements in $\mathcal{B}_\Omega(G)$ of length less than 2 are the empty sequence and the sequence 0. The former is not in $\mathcal{A}(\mathcal{B}_\Omega(G))$ while the latter is in $\mathcal{P}(\mathcal{B}_\Omega(G))$ and the first claim follows.

Now assume that Ω is closed under composition. Suppose that there is an atom of length 1, say, for $a \in G$, we have $a \in \mathcal{B}_\Omega(G)$. This means that there is some $\omega' \in \Omega$ such that $\omega'(a) = 0$. We need to show that $a \in \mathcal{P}(\mathcal{B}_\Omega(G))$. Let $C, D \in \mathcal{B}_\Omega(G)$ such that $a \mid CD$ (the divisibility holds in $\mathcal{B}_\Omega(G)$). We need to assert that $a \mid C$ or $a \mid D$ (in $\mathcal{B}_\Omega(G)$). Without loss, assume that C contains a, in other words a divides C in $\mathcal{F}(G)$. We need to show that a divides C in $\mathcal{B}_\Omega(G)$. Let $C = af_1 \dots f_r$. We know that there are $\omega_0, \omega_1, \dots, \omega_r$ such that $\omega_0(a) + \omega_1(f_1) + \dots + \omega_r(f_r) = 0$. We apply ω' to this expression and obtain, $(\omega'\omega_0)(a) + (\omega'\omega_1)(f_1) + \dots + (\omega'\omega_r)(f_r) = 0$. Now $(\omega'\omega_0)(a) = (\omega_0\omega')(a) = \omega_0(0) = 0$. Thus $(\omega'\omega_1)(f_1) + \dots + (\omega'\omega_r)(f_r) = 0$, and since $\omega'\omega_i \in \Omega$ for each $i \in [1, r]$, it follows that $f_1 \dots f_r \in \mathcal{B}_\Omega(G)$, establishing the claim. □

The results established so far allow to determine for various sets of weights Ω the constants $\rho_k(\mathcal{B}_\Omega(G))$ for even k. The case of odd k is more complex and we address it for the particular case of plus-minus weights in the following section.

Theorem 5.7. *Let G be a finite abelian group. Let $\Omega \subseteq \mathrm{End}(G)$. If Ω only contains monomorphismes or if Ω is a commutative semigroup, with respect to composition, then $\rho_{2k}(\mathcal{B}_\Omega(G)) = k\,\mathsf{D}(\mathcal{B}_\Omega(G))$ for each $k \in \mathbb{N}$. Moreover,*

$$1 + k\,\mathsf{D}(\mathcal{B}_\Omega(G)) \le \rho_{2k+1}(\mathcal{B}_\Omega(G)) \le k\,\mathsf{D}(\mathcal{B}_\Omega(G)) + \left\lfloor \frac{\mathsf{D}(\mathcal{B}_\Omega(G))}{2} \right\rfloor.$$

In particular, in this case $\rho(\mathcal{B}_\Omega(G)) = \mathsf{D}(\mathcal{B}_\Omega(G))/2$.

Proof. The lower bounds are established in Lemma 5.4. The upper bounds follow by Lemmas 4.6 and 5.6. The final claim is a direct consequence of the bounds and the fact that $\rho(H) = \sup_{k \in \mathbb{N}} \rho_k(H)/k$. □

The following result is known for monoids of zero-sum sequences without weights (see [9, Coroallry 5.4]). The structure of our proof is very similar to the version without weights.

Theorem 5.8. *Let G be a finite abelian group. Let $\Omega \subseteq \mathrm{Aut}(G)$ be a subgroup. Let D denote the Davenport constant of $\mathcal{B}_\Omega(G)$ and suppose that $\mathsf{D} \ge 2$. Then, for $k \in \mathbb{N}_0$, let $l \in \mathbb{N}_0$ and $j \in [0, \mathsf{D} - 1]$ such that $k = l\,\mathsf{D} + j$, we have*

$$\lambda_k(\mathcal{B}_\Omega(G)) = \begin{cases} 2l & \text{if } j = 0 \\ 2l+1 & \text{if } j \in [1, \rho_{2l+1}(\mathcal{B}_\Omega(G)) - l\,\mathsf{D}] \\ 2l+2 & \text{if } j \in [\rho_{2l+1}(\mathcal{B}_\Omega(G)) - l\,\mathsf{D} + 1, \mathsf{D} - 1] \end{cases}$$

Proof. For $|G| = 2$, the monoid $\mathcal{B}_\Omega(G)$ is half-factorial. Thus $\mathcal{U}_k(\mathcal{B}_\Omega(G)) = \{k\}$ for each k and the claim is basically trivial; note that by assumption $\mathsf{D} = 2$ (and not 1).

If $l = 0$, then for $j \in [0, 1]$ we have $\mathcal{U}_j(\mathcal{B}_\Omega(G)) = \{j\}$ and the claim is established; for $j \in [2, \mathsf{D} - 1]$ we know by Lemma 5.3 that there is a set of length that contains $\{2, j\}$, which shows that $\lambda_j(\mathcal{B}_\Omega(G)) \le 2$; as it cannot be strictly less than 2, the claim is established.

We can thus suppose that $l \ge 1$. If $j = 0$, the claim is a consequence of Theorem 5.7, because there is a set of lengths L with $\{2l, l\,\mathsf{D}\} \subseteq L$, this is because $\rho_{2l}(\mathcal{B}_\Omega(G)) = l\,\mathsf{D}(\mathcal{B}_\Omega(G))$, and there cannot exist an L' with $\{l', l\,\mathsf{D}\} \subseteq L'$ for some $l' < 2l$ as $l\,\mathsf{D}/l'$ would exceed the elasticity $D/2$ of the monoid $\mathcal{B}_\Omega(G)$.

Suppose $j \geq 1$. Since $k \leq \lambda_k(\mathcal{B}_\Omega(G))\rho(\mathcal{B}_\Omega(G))$ by Lemma 4.1 it follows that for $k = l\mathsf{D} + j$ one has

$$2l + \frac{2j}{\mathsf{D}} = \frac{l\mathsf{D} + j}{\mathsf{D}/2} \leq \lambda_{l\mathsf{D}+j}(\mathcal{B}_\Omega(G))$$

in particular $\lambda_{l\mathsf{D}+j}(\mathcal{B}_\Omega(G)) > 2l$ and thus $\lambda_{l\mathsf{D}+j}(\mathcal{B}_\Omega(G)) \geq 2l + 1$. In the other direction we have by Lemma 4.1 that

$$\lambda_{l\mathsf{D}+j}(\mathcal{B}_\Omega(G)) \leq \lambda_{l\mathsf{D}}(\mathcal{B}_\Omega(G)) + \lambda_j(\mathcal{B}_\Omega(G)) \leq 2l + \lambda_j(\mathcal{B}_\Omega(G)) \leq 2l + 2$$

where we used that $\lambda_{l\mathsf{D}}(\mathcal{B}_\Omega(G)) = 2l$ and $\lambda_j(\mathcal{B}_\Omega(G)) \leq 2$ as established already. For $j = 1$ we get that $2l < \lambda_{l\mathsf{D}+1}(\mathcal{B}_\Omega(G)) \leq 2l + \lambda_1(\mathcal{B}_\Omega(G)) = 2l + 1$, and thus $\lambda_{l\mathsf{D}+1}(\mathcal{B}_\Omega(G)) = 2l + 1$.

We assume that $j \geq 2$. If $j \in [2, \rho_{2l+1}(\mathcal{B}_\Omega(G)) - l\,\mathsf{D}]$, then $j + l\,\mathsf{D} \leq \rho_{2l+1}(\mathcal{B}_\Omega(G))$. Since $\mathcal{U}_{2l+1}(\mathcal{B}_\Omega(G))$ is an interval by Theorem 5.2, this implies that $j + l\,\mathsf{D} \in \mathcal{U}_{2l+1}(\mathcal{B}_\Omega(G))$ and thus $\lambda_{l\mathsf{D}+j}(\mathcal{B}_\Omega(G)) \leq 2l + 1$, which shows that $\lambda_{l\mathsf{D}+j}(\mathcal{B}_\Omega(G)) = 2l + 1$.

If $j > \rho_{2l+1}(\mathcal{B}_\Omega(G)) - l\,\mathsf{D}$, then $j + l\,\mathsf{D} > \rho_{2l+1}(\mathcal{B}_\Omega(G))$. This implies that $j + l\,\mathsf{D} \notin \mathcal{U}_{2l+1}(\mathcal{B}_\Omega(G))$ and thus $\lambda_{l\mathsf{D}+j}(\mathcal{B}_\Omega(G)) > 2l + 1$, which shows that $\lambda_{l\mathsf{D}+j}(\mathcal{B}_\Omega(G)) = 2l + 2$. □

6. Results for plus-minus weighted sequences

The purpose of this section is to establish further results on $\mathcal{B}_\Omega(G)$ in the specific case that the set of weights is equal to $\{+\operatorname{id}_G, -\operatorname{id}_G\}$, which we refer to as plus-minus weighted zero-sum sequences; we denote this set of weights by using the subscript $\pm$, that is, $\mathcal{B}_\pm(G)$ denotes $\mathcal{B}_{\{+\operatorname{id}_G, -\operatorname{id}_G\}}(G)$. Since $\{+\operatorname{id}_G, -\operatorname{id}_G\}$ is a commutative subgroup of $\operatorname{Aut}(G)$, the results of the preceding section are applicable, and we know that for G a finite abelian group:

- $\mathcal{U}_k(\mathcal{B}_\pm(G))$ is an interval for each $k \in \mathbb{N}$ (see Theorem 5.2).
- $\rho_{2k}(\mathcal{B}_\pm(G)) = k\,\mathsf{D}(\mathcal{B}_\pm(G))$ for each $k \in \mathbb{N}$ (see Theorem 5.7).

We will investigate on the one hand the actual value of $\mathsf{D}(\mathcal{B}_\Omega(G))$, and on the other hand investigate the value of $\rho_k(\mathcal{B}_\pm(G))$ for k odd. It turns out that the results depend on the parity of the order of the group.

We start by investigating the set of atoms of $\mathcal{B}_\pm(G)$. We remarked in Section 3 that $\mathcal{A}(\mathcal{B}_\pm(G)) \cap \mathcal{B}(G) \subseteq \mathcal{A}(\mathcal{B}(G))$. Conversely, while it is clear that $\mathcal{A}(\mathcal{B}(G)) \subseteq \mathcal{B}_\pm(G)$, the elements of $\mathcal{A}(\mathcal{B}(G))$, which are irreducible in the monoid $\mathcal{B}(G)$, might well not be irreducible in the larger monoid $\mathcal{B}_\pm(G)$. For example, in $C_4 = \langle e \rangle$, the sequence e^4 is a minimal zero-sum sequence over C_4, that is, $e^4 \in \mathcal{A}(\mathcal{B}(C_4))$. However, in $\mathcal{B}_\pm(C_4)$ it admits the factorization $e^2 \cdot e^2$. We show that that for groups of odd order this never happens.

Theorem 6.1. *Let G be an abelian group such that $|G|$ is odd. Then, $\mathcal{A}(\mathcal{B}(G)) \subseteq \mathcal{A}(\mathcal{B}_\pm(G))$.*

Proof. Let $A \in \mathcal{A}(\mathcal{B}(G))$. Since $\mathcal{B}(G) \subseteq \mathcal{B}_\pm(G)$, it follows that $A \in \mathcal{B}_\pm(G)$. Assume for a contradiction $A = A_1 \cdot A_2$ with non-empty A_1 and A_2 such that $0 \in \sigma_\pm(A_1)$ and $0 \in \sigma_\pm(A_2)$. We can now decompose A_1 and A_2 according to the choice of weights that lead to sum zero; this decomposition might not be unique. Let $A_1 = A_1^+ A_1^-$ such that $0 = \sigma(A_1^+) - \sigma(A_1^-)$, and likewise for A_2. Hence $\sigma(A_1^+) = \sigma(A_1^-)$ and

$\sigma(A_2^+) = \sigma(A_2^-)$. We have $A = A_1^+ A_1^- A_2^+ A_2^-$. We introduce some shorthand-notation $\sigma(A_1^-) = s_1^-$, $\sigma(A_1^+) = s_1^+$, $\sigma(A_2^-) = s_2^-$, and $\sigma(A_2^+) = s_2^+$. We noted that $s_1^+ = s_1^-$ and $s_2^+ = s_2^-$. Since $\sigma(A) = 0$ it follows that $s_1^+ + s_1^- + s_2^+ + s_2^- = 0$.

Consequently, we have $s_1^+ + s_1^+ + s_2^+ + s_2^+ = 0$, that is $2s_1^+ + 2s_2^+ = 0$. This means that $2(s_1^+ + s_2^+) = 0$ and since the order of G is odd, we have $s_1^+ + s_2^+ = 0$. Therefore, $A_1^+ A_2^+$ is a zero-sum subsequence of A (without weights). Since A is a minimal zero-sum sequence, this is only possible when $A_1^+ A_2^+$ is empty or equal to A. The same holds true for $A_1^+ A_2^-$, $A_1^- A_2^+$ and $A_1^- A_2^-$.

Now, exactly one of $A_1^+ A_2^+$ and $A_1^- A_2^-$ equals A and the other is empty. By symmetry we may assume that $A_1^+ A_2^+ = A$ and $A_1^- A_2^-$ is empty. Yet, then A_1^+ and A_2^+ are zero-sum sequences without weight and $A = A_1^+ A_2^+$, a contradiction. □

In particular, the result above shows that in case the order of G is odd, the inclusion $\mathcal{A}(\mathcal{B}_{\pm}(G)) \cap \mathcal{B}(G) \subseteq \mathcal{A}(\mathcal{B}(G))$ is an equality. However, this does not imply that $\mathcal{A}(\mathcal{B}_{\pm}(G)) = \mathcal{A}(\mathcal{B}(G))$ as in general there exist elements in $\mathcal{A}(\mathcal{B}_{\pm}(G))$ that are not in $\mathcal{B}(G)$. For example, for $C_3 = \langle e \rangle$, we have that $e^2 \in \mathcal{A}(\mathcal{B}_{\pm}(C_3))$ yet $e^2 \notin \mathcal{B}(C_3)$.

As an immediate corollary of this result we get that for groups of odd order the Davenport constant of $\mathcal{B}_{\pm}(G)$ is equal to the classical Davenport constant.

Corollary 6.2. *Let G be an abelian group of odd order. Then $\mathsf{D}(\mathcal{B}_{\pm}(G)) = \mathsf{D}(\mathcal{B}(G))$.*

Proof. Directly from Theorem 6.1 and Lemma 3.2. □

While the value of $\mathsf{D}(\mathcal{B}(G))$ is not known in general, there are known results that can be use to obtain explicit results for $\mathsf{D}(\mathcal{B}_{\pm}(G))$ for groups of odd order. In particular, for $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$ one has $\mathsf{D}(\mathcal{B}(G)) \geq 1 + \sum_{i=1}^{r}(n_i - 1)$ and equality is known to hold if G has rank at most two, i.e., $r \leq 2$, or if G is a p-group, i.e., n_r is a prime-power. Equality also holds in some other cases yet not in general, we refer to [8, Section 3] for an overview.

The situation regarding $\mathsf{D}(\mathcal{B}_{\pm}(G))$ for groups of even order is more complicated. We treat the case of cyclic groups of even order completely and then obtain a general lower bound. To do this we recall a result on the structure of long minimal zero-sum sequences and related concepts. Our presentation follows [9, Section 7]; the results are originally due to Savchev and Chen [25] and Yuan [28].

Definition 6.3. Let G be a finite abelian group.

(1) A sequence $S \in \mathcal{F}(G)$ is called *smooth*, or more precisely g-smooth, if there is some $g \in G$ such that $S = (n_1 g) \cdots (n_\ell g)$, where $1 = n_1 \leq \cdots \leq n_\ell$, $n = n_1 + \cdots + n_\ell < \operatorname{ord}(g)$ and $\Sigma(S) = \{g, 2g, \cdots, ng\}$.

(2) A sequence $S \in \mathcal{F}(G)$ is called a *splittable minimal zero-sum sequence* if $S \in \mathcal{A}(\mathcal{B}(G))$, and $S = (g_1 + g_2)T$ for some $g_1, g_2 \in G$ and $T \in \mathcal{F}(G)$ such that $g_1 g_2 T \in \mathcal{A}(\mathcal{B}(G))$.

Theorem 6.4. *Let G be cyclic of order $n \geq 3$.*

(1) *If $S \in \mathcal{F}(G)$ is zero-sum free and $|S| \geq (n+1)/2$, then S is g-smooth for some $g \in G$ with $\operatorname{ord}(g) = n$.*

(2) *Let $A \in \mathcal{A}(\mathcal{B}(G))$ be of length $|A| \geq \lfloor n/2 \rfloor + 2$. Then $A = (n_1 g) \cdots (n_\ell g)$ for some $g \in G$ with $\operatorname{ord}(g) = n$, $1 = n_1 \leq \cdots \leq n_\ell$, $n_1 + \cdots + n_\ell = n$. Moreover, if A is not splittable, then $A = g^n$.*

Lemma 6.5. *Let $g \in G$ and $k, \ell, n_1, \cdots, n_\ell \in \mathbb{N}$ such that $\ell \ge k/2$ and $n = n_1 + \cdots + n_\ell < k \le \mathrm{ord}(g)$. If $1 \le n_1 \le \cdots \le n_\ell$ and $S = (n_1 g) \dots (n_\ell g)$, then $\Sigma(S) = \{g, 2g, \cdots ng\}$, and S is g-smooth.*

We establish a lower bound for $\mathsf{D}(\mathcal{B}_\pm(C_n))$ for even n.

Lemma 6.6. *Let $n \ge 2$ be even. Then*

$$\mathsf{D}(\mathcal{B}_\pm(C_n)) \ge 1 + \frac{n}{2}.$$

Proof. Let $n = 2m$. Let $C_n = \langle e \rangle$. For $n = 2$, we note that e^2 is an element of $\mathcal{A}(\mathcal{B}_\pm(C_n))$, which establishes the claim in this case.

Now we assume that $n \ge 4$. We show that $A = e^m(me)$ is an element of $\mathcal{A}(\mathcal{B}_\pm(C_n))$.

Suppose that $A = A_1 A_2$ with $0 \in \sigma_\pm(A_i)$ for $i \in \{1, 2\}$. Without loss we can assume that $me \mid A_1$. Let $A_1 = (me)e^k$ with $k \in [0, m]$.

Since $0 \in \sigma_\pm(A_i)$ it follows that there exist $\epsilon_1, \dots, e_k \in \{+\mathrm{id}_G, -\mathrm{id}_G\}$ such that $me + \sum_{i=1}^k \epsilon_i e = 0$ (we can assume that the weight of me is $+\mathrm{id}_G$). Yet, this means that $\sum_{i=1}^k \epsilon_i e = me$. Since $\sum_{i=1}^k \epsilon_i e$ is equal to de where d is the difference between the numbers of weights $+\mathrm{id}_G$ and $-\mathrm{id}_G$, we get that $|d|$ is at most k. Thus, $de = me$ is only possible for $k = m$. Consequently, $A_1 = A$ and A is indeed in $\mathcal{A}(\mathcal{B}_\pm(C_n))$. □

Theorem 6.7. *Let n be even. Then, we have*

$$\mathsf{D}(B_\pm(C_n)) = 1 + \frac{n}{2}.$$

Proof. The claim is easily established for $n = 2$; we assume $n \ge 4$. By Lemma 6.6 we know that $\mathsf{D}(\mathcal{B}_\pm(C_n)) \ge 1 + n/2$. It remains to show that $\mathsf{D}(\mathcal{B}_\pm(C_n)) \le 1 + n/2$. Let $S = g_1 \cdots g_\ell \in \mathcal{A}(\mathcal{B}_\pm(C_n))$ and assume for a contradiction that $|S| \ge n/2 + 2$. Since $0 \in \sigma_\pm(S)$ there exist $\epsilon_i \in \{+\mathrm{id}_G, -\mathrm{id}_G\}$ such that $(\epsilon_1 g_1) + \cdots + (\epsilon_\ell g_\ell) = 0$.

Hence $A = (\epsilon_1 g_1) \dots (\epsilon_\ell g_\ell) \in \mathcal{B}(C_n)$. In fact this zero-sum sequence A must be minimal, since a decomposition of A in $\mathcal{B}(C_n)$ would imply a decomposition of S in $\mathcal{B}_\pm(C_n)$.

By Theorem 6.4 there exists $e \in C_n$ such that $C_n = \langle e \rangle$ and moreover we can write $A = (a_1 e) \cdots (a_\ell e)$ with $a_i \in [1, n]$ and $\sum_{i=1}^\ell a_i = n$. We assume that $1 \le a_1 \le \cdots \le a_\ell$.

We show that $a_1 = a_2 = a_3 = a_4 = 1$. Assume for a contradiction that $a_4 \ge 2$. Then $\sum_{i=1}^\ell a_i \ge 3 + 2 \cdot (\ell - 3) \ge 3 + 2 \cdot (n/2 - 1) > n$, a contradiction.

Hence $A = e^4 (a_5 e) \dots (a_\ell e)$. We consider $T = e^{-2} A$. Now, we write $T = (b_1 e)(b_2 e) \cdots (b_{\ell'} e)$ where $b_1 \le \cdots \le b_{\ell'}$, $\sum_{i=1}^{\ell'} b_i = n - 2$ and $b_1 = b_2 = 1$. Notice that $\ell' = \ell - 2$ and $\ell' \ge n/2$. We now note that Lemma 6.5 can be applied to T. We have that $\Sigma(T) = \{e, 2e, \dots, (\mathrm{ord}(e) - 2)e\}$. Let $T_1 \mid T$ such that $\sigma(T_1) = ((n-2)/2)e$. Set $T = T_1 T_2$. Hence $\sigma(T_1) = \sigma(T_2)$.

Therefore we can decompose $A = e^2 \cdot T_1 T_2$, and $0 \in \sigma_\pm(e^2)$ and $0 \in \sigma_\pm(T_1 T_2)$. Therefore $A \notin \mathcal{A}(\mathcal{B}_\pm(G))$. Since $\{+\mathrm{id}_G, -\mathrm{id}_G\}$ is a group under composition of endomorphisms, we can conclude $S \notin \mathcal{A}(\mathcal{B}_\pm(G)$, a contradiction. This establishes $\mathsf{D}(\mathcal{B}_\pm(C_n)) \le 1 + n/2$. □

Corollary 6.8. *Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$ and let $t \in [0, r]$ be maximal such that $2 \nmid n_i$. Then*

$$\mathsf{D}(\mathcal{B}_{\pm}(G)) \geq 1 + \sum_{i=1}^{t}(n_i - 1) + \sum_{i=t+1}^{r} \frac{n_i}{2}.$$

Proof. By Lemma 3.5 we know that

$$\mathsf{D}(\mathcal{B}_{\pm}(G)) \geq 1 + \sum_{i=1}^{r}(\mathsf{D}(\mathcal{B}_{\pm}(C_{n_i})) - 1).$$

The claim now follows using the fact that $\mathsf{D}(\mathcal{B}_{\pm}(C_{n_i}))$ is equal to n_i for odd n_i (see Corollary 6.2) and equal to $1 + n_i/2$ for even n_i (see Theorem 6.7). □

With n_i and r, t as above, we denote $\mathsf{D}^*(\mathcal{B}_{\pm}(G)) = 1+\sum_{i=1}^{t}(n_i-1)+\sum_{i=t+1}^{r} n_i/2$. It would be interesting to have further results on the question of equality in the inequality $\mathsf{D}(\mathcal{B}_{\pm}(G)) \geq \mathsf{D}^*(\mathcal{B}_{\pm}(G))$, for example for groups of rank two of even order or for 2-groups.

Finally, we point out that the result above show that $\mathsf{D}(\mathcal{B}_{\pm}(G))$ and $1+\mathsf{d}(\mathcal{B}_{\pm}(G))$ are quite different. We recall that the later is bounded above by $1 + \langle \log_2 |G| \rangle$; we refer to [2] and [21] for further results on this constant.

We conclude the section by some results on sets of lengths and elasticities in case G is a group of odd order.

Proposition 6.9. *Let G be a group of odd order. For each $B \in \mathcal{B}(G)$ we have $\mathsf{Z}_{\mathcal{B}(G)}(B) \subseteq \mathsf{Z}_{\mathcal{B}_{\pm}(G)}(B)$ and, in particular, $\mathsf{L}_{\mathcal{B}(G)}(B) \subseteq \mathsf{L}_{\mathcal{B}_{\pm}(G)}(B)$.*

Proof. Let $B \in \mathcal{B}(G)$. Let $z \in \mathsf{Z}_{\mathcal{B}(G)}(B)$. This means $z = A_1 \dots A_k$ with $A_i \in \mathcal{A}(G)$. Yet, since the order of G is odd, by Theorem 6.1, we have $\mathcal{A}(G) \subseteq \mathcal{A}(\mathcal{B}_{\pm}(G)$ and thus $A_i \in \mathcal{A}(\mathcal{B}_{\pm}(G))$ for each $1 \leq i \leq k$. That is, $z \in \mathsf{Z}_{\mathcal{B}_{\pm}(G)}(B)$. The claim on the set of lengths is immediate. □

The preceding result allows to obtain results on elasticities.

Corollary 6.10. *Let G be a group of odd order. For $k \in \mathbb{N}$, we have $\mathcal{U}_k(\mathcal{B}(G)) \subseteq \mathcal{U}_k(\mathcal{B}_{\pm}(G))$, and in particular $\rho_k(\mathcal{B}(G)) \leq \rho_k(\mathcal{B}_{\pm}(G))$, and $\lambda_k(\mathcal{B}(G)) \geq \lambda_k(\mathcal{B}_{\pm}(G))$.*

Proof. This is immediate from Proposition 6.9 and the definitions. □

In Theorem 5.7 we already determined $\rho_{2k}(\mathcal{B}_{\pm}(G))$ and remarked that the problem for odd index is more complicated. We now show how we can use results obtained for the problem without weights.

Proposition 6.11. *Let G be a group of odd order. Let $\mathsf{D} = \mathsf{D}(\mathcal{B}(G))$. Let $k \in \mathbb{N}_0$.*

1. *We have $\rho_{2k}(\mathcal{B}_{\pm}(G)) = \rho_{2k}(\mathcal{B}(G)) = k\,\mathsf{D}$ and $\rho(\mathcal{B}_{\pm}(G)) = \rho(\mathcal{B}(G)) = \mathsf{D}/2$.*
2. *If $\rho_{2k+1}(\mathcal{B}(G)) = k\,\mathsf{D} + \lfloor \mathsf{D}/2 \rfloor$, then*

$$\rho_{2k+1}(\mathcal{B}_{\pm}(G)) = \rho_{2k+1}(\mathcal{B}(G)) = k\,\mathsf{D} + \left\lfloor \frac{\mathsf{D}}{2} \right\rfloor.$$

Proof. By Theorem 6.1 we have $\mathsf{D} = \mathsf{D}(\mathcal{B}_{\pm}(G))$. The first part is now immediate from Theorem 5.7. For the second part, again by Theorem 5.7 we have $\rho_{2k+1}(\mathcal{B}_{\pm}(G)) \leq k\,\mathsf{D} + \lfloor \mathsf{D}/2 \rfloor$. Since by Corollary 6.10, we have $\rho_{2k+1}(\mathcal{B}(G)) \leq \rho_{2k+1}(\mathcal{B}_{\pm}(G))$ the claim follows. □

For a detailed investigation of the question when $\rho_{2k+1}(\mathcal{B}(G)) = k\,\mathsf{D} + \lfloor \mathsf{D}/2 \rfloor$ holds, we refer to [11]. Various results there, by the result above, give directly the analogous result for $\mathcal{B}_{\pm}(G)$. There is no point in essentially just copying all these results. By way of example we state one result.

Corollary 6.12. *Let $r \geq 2$ and let n be a power of an odd prime. Then for each $k \in \mathbb{N}$ we have*

$$\rho_{2k+1}(\mathcal{B}_{\pm}(C_n^r)) = k(1 + r(n-1)) + \frac{r(n-1)}{2}.$$

Proof. This is a direct consequence of Proposition 6.11 and [11, Corollary 4.3] that establishes $\rho_{2k+1}(\mathcal{B}(C_n^r)) = k(1 + r(n-1)) + (r(n-1))/2$ (note that we plugged in the value of $\mathsf{D}(C_n^r)$ and evaluated the floor-function). □

7. The arithmetic of monoids of norms of rings of algebraic integers

In the present section we relate the arithmetic of certain multiplicative submonoids of the natural numbers, defined via norms of rings of algebraic integers, to the arithmetic of monoids of weighted zero-sum sequences over finite abelian groups. A relation between problems on norms of algebraic integers and weighted zero-sum sequences was investigated in [19]; our application is closely related but distinct.

To fix ideas and notations, we recall some standard notions and results from algebraic number theory (see, e.g., [22, 23]).

Let K denote a number field of degree d, and assume that the extension $K/\mathbb{Q}$ is a Galois extension; let $\mathrm{Gal}(K/\mathbb{Q})$ denote its Galois group. Let $\mathcal{O}_K$ denote the ring of algebraic integers of K. Let $\mathrm{Cl}(\mathcal{O}_K)$ denote its ideal class group.

For an ideal I of $\mathcal{O}_K$ let $\mathsf{N}(I) = |\mathcal{O}_K/I|$ denote the (numerical) norm of I. Moreover, for $a \in \mathcal{O}_K^*$ let $\mathsf{N}(a)$ denote the absolute norm of a, that is the norm of the principal ideal $a\mathcal{O}_K$, which is also equal to the absolute value of the norm of a in the extension $K/\mathbb{Q}$.

Let $\mathrm{Id}(\mathcal{O}_K)$ denote the set of all ideals of $\mathcal{O}_K$. Furthermore, let $\mathcal{H}(\mathcal{O}_K)$ denote the principal ideal of $\mathcal{O}_K$, and let $\mathrm{spec}(\mathcal{O}_K)$ denote the prime ideals of $\mathcal{O}_K$. It is well-known that every (non-zero) ideal has a unique representation as a product of (non-zero) prime ideals, in other words $\mathrm{Id}(\mathcal{O}_K) \setminus \{0\}$ is equal to $\mathcal{F}(\mathrm{spec}(\mathcal{O}_K) \setminus \{0\})$.

The norms mentioned above are multiplicative, that is, $\mathsf{N} : (\mathrm{Id}(\mathcal{O}_K) \setminus \{0\}, \cdot) \to (\mathbb{N}, \cdot)$ and $\mathsf{N} : (\mathcal{O}_K^*, \cdot) \to (\mathbb{N}, \cdot)$ are homomorphism of monoids. The set $\mathsf{N}(\mathcal{O}_K^*)$, that is the set of absolute norms of elements of $\mathcal{O}_K^*$, is thus a submonoid of $(\mathbb{N}, \cdot)$. We want to study its arithmetic.

In case the extension $K/\mathbb{Q}$ is Galois, for every prime number $p \in \mathbb{N}$ there are $e, f, g \in \mathbb{N}$, that depend on p, and distinct non-zero prime ideals $P_1, \dots, P_g \in \mathrm{spec}(\mathcal{O}_K) \setminus \{0\}$ such that $p\mathcal{O}_K = P_1^e \dots P_g^e$. Moreover, $d = efg$ and $\mathsf{N}(P_i) = p^f$. We recall that the only elements of $\mathrm{spec}(\mathcal{O}_K) \setminus \{0\}$ whose norm is a power of p are $P_1, \dots, P_g$.

Indeed, for every $P' \in \mathrm{spec}(\mathcal{O}_K) \setminus \{0\}$ there is a (unique) prime number p' such that P' occurs in the decomposition of $p'\mathcal{O}_K$; the norm of P' is then a power of p'.

Finally, we recall that the Galois group Γ acts transitively on the set $\{P_1, \dots, P_g\}$. For $P \in \mathrm{spec}(\mathcal{O}_K) \setminus \{0\}$ we denote the orbit of P under this action by $\mathrm{Orb}(P)$. We denote by $\mathrm{OrbSpec}(\mathcal{O}_K)$ the sets of orbits of the action of Γ on $\mathrm{spec}(\mathcal{O}_K) \setminus \{0\}$. Moreover for a prime number p we denote by $\mathrm{Orb}(p)$ the orbit of P for some prime ideal whose norm is a p-th power, which is well-defined by the facts that we just recalled. Also recall that the Galois group acts on the ideal class group by $\gamma[P] = [\gamma(P)]$.

Theorem 7.1. *Let K be a Galois number field with Galois group Γ and class group G. There is a transfer homomorphism from $\mathsf{N}(\mathcal{O}_K^*)$, the monoid of absolute norms of non-zero algebraic integers of K, to $\mathcal{B}_\Gamma(G)$, the monoid of Γ-weighted zero-sum sequences over the class group of K.*

Proof. Let $\tilde{\mathsf{cl}} : \mathrm{OrbSpec}(\mathcal{O}_K) \to \mathrm{Cl}(\mathcal{O}_K)$ we some *surjective* map such that for each $P \in \mathrm{spec}(\mathcal{O}_K) \setminus \{0\}$ we have $\tilde{\mathsf{cl}}(\mathrm{Orb}(P)) \in \{[\gamma(P)] : \gamma \in \Gamma\}$. Note that such a map actually exists; the only thing to be concerned about is surjectivity, yet this can be assured easily since each class contains infinitely many prime ideals. Moreover, let $\mathsf{cl} : \mathrm{spec}(\mathcal{O}_K) \setminus \{0\} \to \mathrm{Cl}(\mathcal{O}_K)$ denote the map given by $\mathsf{cl}(P) = \tilde{\mathsf{cl}}(\mathrm{Orb}(P))$. The map cl is fixed throughout the argument.

We now proceed to define the transfer homomorphism. For $n \in \mathsf{N}(\mathcal{O}_K^*)$ let $a_n \in \mathcal{O}_K^*$ such that $\mathsf{N}(a_n) = n$. There are uniquely determined $\mathsf{v}_P(a_n)$ such that $(a_n) = \prod_{P \in \mathrm{spec}(\mathcal{O}_K) \setminus \{0\}} P^{\mathsf{v}_P(a_n)}$. We set $\Theta(n) = \prod_{P \in \mathrm{spec}(\mathcal{O}_K) \setminus \{0\}} \mathsf{cl}(P)^{\mathsf{v}_P(a_n)} \in \mathcal{F}(\mathrm{Cl}(\mathcal{O}_K))$; note that we have to show that this is well-defined, that is, the definition only depends on n yet not the choice of a_n. Assuming that the map is well-defined, it is easy to see that it is a homomorphism. Concretely, if $n, m \in \mathsf{N}(\mathcal{O}_K^*)$ and $a_n, a_m \in \mathcal{O}_K$ such that $\mathsf{N}(a_n) = n$ and $\mathsf{N}(a_m) = m$, then $a_n a_m$ is an element of $\mathcal{O}_K$ of norm nm, and as $\mathsf{v}_P(a_n a_m) = \mathsf{v}_P(a_n) + \mathsf{v}_P(a_m)$ for each P. We see that $\Theta(nm) = \Theta(n)\Theta(m)$.

We now proceed to show that the map is well-defined. Let again $a_n \in \mathcal{O}_K^*$ such that $\mathsf{N}(a_n) = n$. Let $n = \prod_{i=1}^k p_i^{v_i}$ denote the factorization of n into prime powers (we assume that the p_i are distinct and that the v_i are non-zero).

Let $a_n \mathcal{O}_K = P_1 \dots P_L$ denote the factorization into prime ideals. Since $\mathsf{N}(a_n) = \mathsf{N}(P_1) \dots \mathsf{N}(P_L)$ and each $\mathsf{N}(P_i)$ is a prime-power it follows that we can write $\{1, \dots, L\} = \biguplus_{i=1}^k I_i$ where $j \in I_i$ if $\mathsf{N}(P_j)$ is a p_i-power. Furthermore it follows that for each p_i there is an integer f_i such that $\mathsf{N}(P_j) = p_i^{f_i}$ for each $j \in I_i$. Consequently, $|I_i| = v_i / f_i$ for each $i \in \{1, \dots, k\}$; note that these quantities depend on n only. Moreover, note that for $j, j' \in I_i$ the orbits of P_j and P_j' are the same and thus $\mathsf{cl}(P_j) = \mathsf{cl}(P_j')$. Thus, in any case, the contribution to $\Theta(n)$ does not depend on the choice of a_n.

We assert that $\Theta(n)$ is a sequence with a Γ-weighted zero-sum. To see this it suffices to note that $\sum_{P \in \mathrm{spec}(\mathcal{O}_K) \setminus \{0\}} [P] \mathsf{v}_P(a_n) = 0$, since (a_n) is a principal ideal, and for each P we have that $[P]$ is an element of $\{[\gamma(P)] : \gamma \in \Gamma\} = \{\gamma([P]) : \gamma \in \Gamma\}$.

Thus, we have a homomorphism $\Theta : \mathsf{N}(\mathcal{O}_K^*) \to \mathcal{B}_\Gamma(\mathrm{Cl}(\mathcal{O}_K))$. It remains to show that it is a transfer homomorphism.

First, we assert that the map is surjective. Let $S \in \mathcal{B}_\Gamma(\mathrm{Cl}(\mathcal{O}_K))$, say, $S = g_1 \dots g_r$. Let $\gamma_i \in \Gamma$ such that $\sum_{i=1}^r \gamma_i g_i = 0$. For each i, let $P_i \in \mathrm{spec}(\mathcal{O}_K) \setminus \{0\}$ such that $\mathsf{cl}(P_i) = g_i$ and more precisely $[P_i]$ is equal to $\gamma_i g_i$. We consider the ideal $I = P_1 \dots P_r$. Since $\sum_{i=1}^r [P_i] = \sum_{i=1}^r \gamma_i g_i = 0$, it follows that I is a principal ideal. Choosing for a some generating element of this principal ideal, it follows that $\Theta(\mathsf{N}(a)) = S$.

It is plain that only invertible elements are mapped to invertible elements. It remains to show that if $\Theta(n) = S_1 S_2$ with $S_1, S_2 \in \mathcal{B}_\Gamma(\mathrm{Cl}(\mathcal{O}_K))$ then there exit

$n_1, n_2 \in \mathsf{N}(\mathcal{O}_K)$ such that $n = n_1 n_2$ and $\Theta(n_1) = S_1$ and $\Theta(n_2) = S_2$. Let $S_1 = g_1^{(1)} \dots g_{k_1}^{(1)}$ and $S_2 = g_1^{(2)} \dots g_{k_2}^{(2)}$

Let $a_n \in \mathcal{O}_K$ such that $\mathsf{N}(a_n) = n$. By definition of Θ we know that $(a_n) = P_1^{(1)} \dots P_{k_1}^{(1)} P_1^{(2)} \dots P_{k_2}^{(2)}$ with non-zero prime ideals $P_i^{(j)}$ such that $\mathsf{cl}(P_i^{(j)}) = g_i^{(j)}$ for all i, j. Now, let $\gamma_i^{(j)} \in \Gamma$ such that $\sum_{i=1}^{k_j} \gamma_i^{(j)} g_i^{(j)} = 0$ for $j \in \{1, 2\}$.

Let $P_i^{(j),s}$ the prime ideal in the orbit of $P_i^{(j)}$ such that $[P_i^{(j),s}] = \gamma_i^{(j)} g_i^{(j)}$. Since the class of the ideal $\prod_{i=1}^{k_j} P_i^{(j),s}$ is the trivial class, this ideal is a principal ideal, say, it is equal to (b^j) with $b^j \in \mathcal{O}_K$. Let $n_j = \mathsf{N}(b^{(j)})$. Since $\mathsf{N}(P_i^{(j),s}) = \mathsf{N}(P_i^{(j)})$, it follows that $n_1 n_2 = n$. Moreover, since $\mathsf{cl}(P_i^{(j),s}) = \mathsf{cl}(P_i^{(j)})$, it follows that $\Theta(n_j) = S_j$. This establishes the claim. □

We point out some consequences of the preceding result.

Corollary 7.2. *Let K be a Galois number field with class group G. Let $H = \mathsf{N}(\mathcal{O}_K^*)$ be the monoid of absolute norms of non-zero algebraic integers of K.*

1. *The set $\Delta(H)$ and the constant $\rho(H)$ are finite.*
2. *For each $k \in \mathbb{N}$ the set $\mathcal{U}_k(H)$ is an interval.*
3. *There is some $M \in \mathbb{N}_0$ such that each set of lengths L of H is an almost arithmetical multiprogression with bound M and difference $d \in \Delta(H) \cup \{0\}$, that is, $L = y + (L_1 \cup L^* \cup L_2) \subseteq y + \mathcal{D} + d\mathbb{Z}$ with $y \in \mathbb{N}_0$, $\{0, d\} \subseteq \mathcal{D} \subseteq [0, d]$, $L_1, -L_2 \subseteq [1, M]$, $\min L^* = 0$ and $L^* = [0, \max L^*] \cap \mathcal{D} + d\mathbb{Z}$.*

Proof. By Theorem 7.1 we know that there is a tranfer homomorphism from H to $\mathcal{B}_\Gamma(G)$ where Γ denotes the Galois group of K. By Theorem 3.6 and Theorem 5.2 we know that $\mathcal{B}_\Gamma(G)$ has the claimed properties. Since all the properties depend on length of factorizations only, and transfer homomorphisms preserve sets of lengths (see Section 2 after we recall the definition of transfer homomorphism), the claim follows. □

In the case of quadratic number fields, we can apply our results on plus-minus weighted sequences.

Corollary 7.3. *Let K be a quadratic number field with odd class number. Then $\rho(\mathsf{N}(\mathcal{O}_K^*)) = \rho(\mathcal{O}_K^*)$ and $\rho_{2k}(\mathsf{N}(\mathcal{O}_K^*)) = \rho_{2k}(\mathcal{O}_K^*)$ for each $k \in \mathbb{N}$.*

Proof. As in the preceding corollary it suffices to establish the claim for $\mathcal{B}_\Gamma(G)$ where Γ denotes the Galois group of K. Since K is a quadratic number field it follows that $|\Gamma| = 2$. Moreover, if $\Gamma = \{\mathrm{id}, \gamma\}$, then $P\gamma(P)$ is a principal ideal for each prime ideal P of $\mathcal{O}_K$. Thus, $[P] + [\gamma(P)] = 0$ for each P, which implies that γ acts like $-\mathrm{id}_G$ on G. That is, in this case $\mathcal{B}_\Gamma(G) = \mathcal{B}_\pm(G)$. The claim now follows by Proposition 6.11. □

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