MONOMIAL IDEALS, ALMOST COMPLETE INTERSECTIONS AND THE WEAK LEFSCHETZ PROPERTY

JUAN C. MIGLIORE*, ROSA M. MIRÓ-ROIG**, UWE NAGEL+

Abstract. Many algebras are expected to have the Weak Lefschetz property though this is often very difficult to establish. We illustrate the subtlety of the problem by studying monomial and some closely related ideals. Our results exemplify the intriguing dependence of the property on the characteristic of the ground field, and on arithmetic properties of the exponent vectors of the monomials.

Contents

1. Introduction 1
2. Tools for studying the WLP 3
3. A class of monomial ideals 5
4. Monomial almost complete intersections in any codimension 8
5. An almost monomial almost complete intersection 14
6. Monomial almost complete intersections in three variables 17
7. A proof of half of Conjecture 6.8 20
8. Final Comments 26
References 27

1. Introduction

Let A be a standard graded Artinian algebra over the field K. Then A is said to have the *Weak Lefschetz property (WLP)* if there is a linear form $L \in (A)_1$ such that, for all integers j, the multiplication map

$$\times L : (A)_{j-1} \to (A)_j$$

has maximal rank, i.e. it is injective or surjective. In this case, the linear form L is called a *Lefschetz element* of A. (We will often abuse notation and say that the corresponding ideal has the WLP.) The Lefschetz elements of A form a Zariski open, possibly empty, subset of $(A)_1$. Part of the great interest in the WLP stems from the fact that its presence puts severe constraints on the possible Hilbert functions (see [6]), which can appear in various disguises (see, e.g., [13]). Though many algebras are expected to have the WLP, establishing this property is often rather difficult. For example, it is open whether every

* Part of the work for this paper was done while the first author was sponsored by the National Security Agency under Grant Number H98230-07-1-0036.
** Part of the work for this paper was done while the second author was partially supported by MTM2007-61104.
+ Part of the work for this paper was done while the third author was sponsored by the National Security Agency under Grant Number H98230-07-1-0065.
The authors thank Fabrizio Zanello for useful and enjoyable conversations related to some of this material. They also thank David Cook II for useful comments.

complete intersection of height four over a field of characteristic zero has the WLP. (This is true if the height is at most 3 by [6].)

In some sense, this note presents a case study of the WLP for monomial ideals and almost complete intersections. Our results illustrate how subtle the WLP is. In particular, we investigate its dependence on the characteristic of the ground field K. The following example (Example 7.7) illustrates the surprising effect that the characteristic can have on the WLP. Consider the ideal $I = (x^{10}, y^{10}, z^{10}, x^3y^3z^3) \subset R = K[x, y, z]$. Our methods show that R/I fails to have the WLP in characteristics 2, 3 and 11, but possesses it in all other characteristics.

One starting point of this paper has been Example 3.1 in [4], where Brenner and Kaid show that, over an algebraically closed field of characteristic zero, any ideal of the form $(x^3, y^3, z^3, f(x, y, z))$, with $\deg f = 3$, fails to have the WLP if and only if $f \in (x^3, y^3, z^3, xyz)$. In particular, the latter ideal is the *only* such monomial ideal that fails to have the WLP. This paper continues the study of this question.

The example of Brenner and Kaid satisfies several interesting properties. In this paper we isolate several of these properties and examine the question of whether or not the WLP holds for such algebras, and we see to what extent we can generalize these properties and still get meaningful results. Some of our results hold over a field of arbitrary characteristic, while others show different ways in which the characteristic plays a central role in the WLP question. (Almost none are characteristic zero results.) Most of our results concern monomial ideals, although in Section 5 and Section 8 we show that even minor deviations from this property can have drastic effects on the WLP. Most of our results deal with almost complete intersections in three or more variables, but we also study ideals with more generators (generalizing that of Brenner and Kaid in a different way).

More specifically, we begin in Section 2 with some simplifying tools for studying the WLP. These are applied throughout the paper. We also recall the construction of basic double linkage.

In Section 3 we consider the class of monomial ideals in $K[x_1, \dots, x_r]$ of the form

$$(x_1^k, x_2^k, \dots, x_r^k) + (\text{all squarefree monomials of degree } d).$$

Note that the example of Brenner and Kaid is of this form. Our main result in this section (Theorem 3.3) says that when $d = 2$ we always have the WLP, but if $d = 3$ and $k \geq 2$ then we have two cases: if K has characteristic 2 then we never have the WLP, but if the characteristic is not 2 then we have the WLP if and only if k is even.

In Section 4, we consider almost complete intersections of the form $(x_1^r, \dots, x_r^r, x_1 \cdots x_r)$ with $r \geq 3$ (note that the result of Brenner and Kaid dealt with the case $r = 3$ in characteristic zero). Our main result for these algebras is that they always fail to have the WLP, regardless of the characteristic. The proof is surprisingly difficult.

In Section 5 we explicitly illustrate the fact that even a minuscule change in the ideal can affect the WLP. Specifically, we consider the ideals of the form

$$(x_1^r, \dots, x_r^r, x_1 \cdots x_{r-1} \cdot (x_1 + x_r)).$$

We show that this has the same Hilbert function as the corresponding ideal in the previous section, but the WLP behavior is very different. For example, the two ideals

$$(x_1^4, \dots, x_4^4, x_1x_2x_3x_4) \quad \text{and} \quad (x_1^4, \dots, x_4^4, x_1x_2x_3(x_1 + x_4))$$

have the same Hilbert function, but the former never has the WLP while the latter has the WLP if and only if the characteristic of K is not two or five.

In Section 6 we turn to monomial almost complete intersections in three variables, generalizing the Brenner-Kaid example in a different direction. To facilitate this study, we assume that the algebra is also level (as is the case for Brenner and Kaid's example). We give a number of results in this section, which depend on the exponent vectors of the monomials. We end with a conjectured classification of the level Artinian monomial ideals in three variables that fail to have the WLP (Conjecture 6.8). The work in Sections 6 and 7 proves most of this conjecture. We end the paper in Section 8 with some suggestive computations and natural questions coming from our work.

2. Tools for studying the WLP

In this section we establish various general results that help to study the WLP and that are used throughout the remainder of this paper. Throughout this paper we set $R = K[x_1, \dots, x_r]$, where K is a field. Sometimes we will have specific values of r (usually 3) and sometimes we will have further restrictions on the field K.

Our first results singles out the crucial maps to be studied if we consider the WLP of a level algebra. Recall that an Artinian algebra is called *level* if its socle is concentrated in one degree.

Proposition 2.1. *Let R/I be an Artinian standard graded algebra and let L be a general linear form. Consider the homomorphisms $\phi_d : (R/I)_d \to (R/I)_{d+1}$ defined by multiplication by L, for $d \geq 0$.*

(a) *If ϕ_{d_0} is surjective for some d_0 then ϕ_d is surjective for all $d \geq d_0$.*
(b) *If R/I is level and ϕ_{d_0} is injective for some $d_0 \geq 0$ then ϕ_d is injective for all $d \leq d_0$.*
(c) *In particular, if R/I is level and $\dim(R/I)_{d_0} = \dim(R/I)_{d_0+1}$ for some d_0 then R/I has the WLP if and only if ϕ_{d_0} is injective (and hence is an isomorphism).*

Proof. Consider the exact sequence

$$0 \to \frac{[I : L]}{I} \to R/I \xrightarrow{\times L} (R/I)(1) \to (R/(I, L))(1) \to 0$$

where $\times L$ in degree d is just ϕ_d. This shows that the cokernel of ϕ_d is just $(R/(I, L))_{d+1}$ for any d. If ϕ_{d_0} is surjective, then $(R/(I, L))_{d_0+1} = 0$, and the same necessarily holds for all subsequent twists since R/I is a standard graded algebra. Then (a) follows immediately.

For (b), recall that the K-dual of the finite length module R/I is a shift of the canonical module of R/I, which we will denote simply by M. Since R/I is level, M is generated in the first degree. But now if we consider the graded homomorphism of M to itself induced by multiplication by L, a similar analysis (recalling that M is generated in the first degree) gives that once this multiplication is surjective in some degree, it is surjective thereafter. The result on R/I follows by duality.

Part (c) follows immediately from (a) and (b). $\square$

If the field is infinite and the K-algebra satisfies the WLP for some linear form, then it does for a general linear form. For monomial ideals there is no need to consider a general linear form.

Proposition 2.2. *Let $I \subset R$ be an Artinian monomial ideal and assume that the field K is infinite. Then R/I has the WLP if and only if $x_1 + \cdots + x_r$ is a Lefschetz element for R/I.*

Proof. Set $A = R/I$ and let $L = a_1x_1 + \cdots + a_rx_r$ be a general linear form in R. Thus, we may assume that each coefficient a_i is not zero and, in particular, $a_r = 1$. Let $J \subset S := K[x_1, \ldots, x_{r-1}]$ be the ideal that is generated by elements that are obtained from the minimal generators of I after substituting $a_1x_1 + \cdots + a_{r-1}x_{r-1}$ for x_r. Then $A/LA \cong S/J$.

Each minimal generator of J is of the form $x_1^{j_1} \cdots x_{r-1}^{j_{r-1}}(a_1x_1+\cdots+a_{r-1}x_{r-1})^{j_r}$. Replacing it by $(a_1x_1)^{j_1} \cdots (a_{r-1}x_{r-1})^{j_{r-1}}(-a_1x_1 + \cdots - a_{r-1}x_{r-1})^{j_r}$ does not change the ideal J because $a_1 \cdots a_{r-1} \neq 0$. Using the isomorphism $K[y_1, \ldots, y_{r-1}] \to S$, $y_i \mapsto a_ix_i$, we see that A/LA and $A/(x_1 + \cdots + x_r)A$ have the same Hilbert function. Since we can decide whether L is a Lefschetz element for A by solely looking at the Hilbert function of A/LA, the claim follows. □

If A is an Artinian K-algebra with the WLP and E is an extension field of K, then also $A \otimes_K E$ has the WLP. However, the converse is not clear. We pose this as a problem.

Problem 2.3. Is it true that A has the WLP if and only if $A \otimes_K E$ has the WLP?

Proposition 2.2 shows that the answer is affirmative in the case of monomial ideals.

Corollary 2.4. *Let E be an extension field of the infinite field K. If $I \subset R$ is an Artinian monomial ideal, then R/I has the WLP if and only if $(R/I) \otimes_K E$ does.*

The following result applies if we can hope that the multiplication by a linear form is surjective.

Proposition 2.5. *Let $I \subset R = K[x_1, \ldots, x_r]$, where K is a field and $A = R/I$ is Artinian. Let d be any degree such that $h_A(d-1) \geq h_A(d) > 0$. Let L be a linear form, let $\bar{R} = R/(L)$ and let $\bar{I}$ be the image of I in $\bar{R}$. Denote by $\bar{A}$ the quotient $\bar{R}/\bar{I}$. Consider the minimal free $\bar{R}$-resolution of $\bar{A}$:*

$$0 \to \bigoplus_{i=1}^{p_{r-1}} \bar{R}(-b_i) \to \cdots \to \bigoplus_{j=1}^{p_1} \bar{R}(-a_j) \to \bar{R} \to \bar{A} \to 0$$

where $a_1 \leq \cdots \leq a_{p_1}$ and $b_1 \leq \cdots \leq b_{p_{r-1}}$. Then the following are equivalent:

(a) *the multiplication by L from A_{d-1} to A_d fails to be surjective;*
(b) *$\bar{A}_d \neq 0$;*
(c) *$b_{p_{r-1}} \geq d + r - 1$;*
(d) *Let $G_1, \ldots, G_{r-1}$ be a regular sequence in $\bar{I}$ of degrees $c_1, \ldots, c_{r-1}$ respectively that extends to a minimal generating set for $\bar{I}$. Then there exists a form $F \in \bar{R}$ of degree $\leq c_1 + \cdots + c_{r-1} - (d + r - 1)$, non-zero modulo $(G_1, \ldots, G_{r-1})$, such that $F \cdot \bar{I} \subset (G_1, \ldots, G_{r-1})$.*

Proof. From the exact sequence

$$\cdots \to A_{d-1} \xrightarrow{\times L} A_d \to (R/(I, L))_d \to 0$$

it follows that the multiplication fails to be surjective if and only if $\bar{A}_d = (R/(I, L))_d \neq 0$. The latter holds if and only if

$$d \leq \text{socle degree of } \bar{A} = b_{p_{r-1}} - (r-1),$$

from which the equivalence of (a), (b) and (c) follows.

To show the equivalence of (c) and (d) we invoke liaison theory. Let

$$J = (G_1, \ldots, G_{r-1}) : \bar{I}.$$

A free resolution for J can be obtained from that of $\bar{I}$ and (G, G') by a standard mapping cone argument (see for instance [9]), as follows. We have the following commutative diagram (where the second one is the Koszul resolution for $(G_1, \ldots, G_{r-1})$):

$$\begin{array}{ccccccccccc} 0 & \to & \bigoplus_{i=1}^{p_{r-1}} \bar{R}(-b_i) & \to & \cdots & \to & \bigoplus_{j=1}^{p_1} \bar{R}(-a_j) & \to & \bar{I} & \to & 0 \\ & & \uparrow & & & & \uparrow & & \uparrow & & \\ 0 & \to & \bar{R}(-c_1 - \cdots - c_{r-1}) & \to & \cdots & \to & \bigoplus_{k=1}^{r-1} \bar{R}(-c_k) & \to & (G, G') & \to & 0 \end{array}$$

where the rightmost vertical arrow is an inclusion. This yields a free resolution for J (after splitting $\bigoplus_{k=1}^{r-1} \bar{R}(-c_k)$ and re-numbering the a_j, and setting $c := c_1 + \cdots + c_{r-1}$):

$$0 \to \bigoplus_{j=1}^{p_1-(r-1)} \bar{R}(a_j - c) \to \cdots \to \bigoplus_{i=1}^{p_{r-1}} \bar{R}(b_i - c) \oplus \bigoplus_{k=1}^{r-1} \bar{R}(-c_k) \to J \to 0.$$

Clearly $b_{p_{r-1}} \geq d + r - 1$ if and only if J has a minimal generator, F, of degree $\leq c - (d + r - 1)$. The result then follows from the definition of J as an ideal quotient. $\square$

We conclude this section by recalling a concept from liaison theory, which we do not state in the greatest generality.

Let $J \subset I \subset R = K[x_1, \ldots, x_r]$ be homogeneous ideals such that $\operatorname{codim} J = \operatorname{codim} I - 1$. Let $\ell \in R$ be a linear form such that $J : \ell = J$. Then the ideal $I' := \ell \cdot I + J$ is called a *basic double link* of I. The name stems from the fact that I' can be Gorenstein linked to I in two steps if I is unmixed and R/J is Cohen-Macaulay and generically Gorenstein ([8], Proposition 5.10). However, here we only need the relation among the Hilbert functions.

Lemma 2.6. *For each integer j,*

$$\dim_K(R/I')_j = \dim_K(R/I)_{j-1} + \dim_K(R/J)_j - \dim_K(R/J)_{j-1}.$$

Proof. This follows from the exact sequence (see [8], Lemma 4.8)

$$0 \to J(-1) \to J \oplus I(-1) \to I' \to 0.$$

$\square$

3. A class of monomial ideals

We now begin our study of a certain class of Artinian monomial ideals. Let $I_{r,k,d}$ be the monomial ideal defined by

$$(x_1^k, x_2^k, \ldots, x_r^k) + (\text{all squarefree monomials of degree } d). \tag{3.1}$$

Our first observation follows immediately be determining the socle of $R/I_{r,k,d}$. It shows that we may apply Proposition 2.1.

Proposition 3.1. *The inverse system for $I_{r,k,d}$ is generated by the module generated by all monomials of the form*

$$x_{i_1}^{k-1} \cdots x_{i_{d-1}}^{k-1}.$$

Corollary 3.2. *The algebra $R/I_{r,k,d}$ is level of socle degree $(k-1)(d-1)$ and socle type $\binom{r}{d-1}$.*

Concerning the WLP we have:

Theorem 3.3. *Consider the ring $R/I_{r,k,d}$.*

(a) *If $d = 2$, then it has the WLP.*
(b) *Let $d = 3$ and $k \geq 2$. Then:*
  (i) *If K has characteristic two, then $R/I_{r,k,d}$ does not have the WLP.*

(ii) *If the characteristic of K is not two, then $R/I_{r,k,d}$ has the WLP if and only if k is even.*

Proof. For simplicity, write $I = I_{r,k,d}$ and $A = R/I_{r,k,d}$.

Claim (a) follows easily from the observation that A has socle degree $k-1$ and that up to degree $k-1$ the ideal I is radical, so multiplication by a general linear form is injective in degree $\leq k-1$.

To show claim (b) we first describe bases of $(A)_{k-1}$ and $(A)_k$, respectively. We choose the residue classes of the elements in the following two sets.

$$\begin{aligned} B_{k-1} &= \{x_i^j x_m^{k-1-j} \mid 1 \leq i < m \leq r,\ 1 \leq j \leq k-2\} \cup \{x_i^{k-1} \mid 1 \leq i \leq r\} \\ B_k &= \{x_i^j x_m^{k-j} \mid 1 \leq i < m \leq r,\ 1 \leq j \leq k-1\}. \end{aligned}$$

Counting we get

$$h_A(k-1) = (k-2)\binom{r}{2} + r \leq (k-1)\binom{r}{2}, \tag{3.2}$$

where the inequality follows from $r \geq d = 3$.

Now we assume that k is odd. In this case we claim that A does not have the WLP. Because of Inequality (3.2), this follows once we have shown that, for each linear form $L \in R$, the multiplication map $\phi_k : (A)_{k-1} \xrightarrow{\times L} (A)_k$ is not injective.

To show the latter assertion we exhibit a non-trivial element in its kernel. Write $L = a_1x_1 + \ldots + a_rx_r$ for some $a_1, \ldots, a_r \in K$. We define the polynomial $f \in R$ as

$$f = \sum_{j_i + j_m = k-1} (-1)^{\max\{j_i, j_m\}} (a_ix_i)^{j_i} (a_mx_m)^{j_m}.$$

Note that f is not in I. We claim that $L \cdot f$ is in I. Indeed, since all monomials involving three distinct variables are in I, a typical monomial in $L \cdot f$ mod I is of the form

$$(a_ix_i)^{j_i}(a_mx_m)^{k-j_i}.$$

It arises in exactly two ways in Lf, namely as $(a_ix_i) \cdot (a_ix_i)^{j_i-1}(a_mx_m)^{k-j_i}$ and as $(a_mx_m) \cdot (a_ix_i)^{j_i}(a_mx_m)^{k-1-j_i}$. Using that $k-1$ is even, it is easy to see that these two monomials occur in f with different signs. It follows that the above multiplication map is not injective.

If k is even, but char $K = 2$, then the same analysis again shows that ϕ_k is not injective. Hence, for the remainder of the proof we may assume that the characteristic of K is not two.

Assume k is even. Then we claim that $L = x_1 + \cdots + x_r$ is a Lefschetz element. To this end we first show that the multiplication map $\phi_k : (A)_{k-1} \xrightarrow{\times L} (A)_k$ is injective.

Let f be any element in the vector space generated by B_{k-1}. Pick three of the variables $x_1, \ldots, x_r$ and call them x, y, z. Below we explicitly list all the terms in f that involve only the variables x, y, z:

$$\begin{aligned} f = \ & a_0x^{k-1} + a_1x^{k-2}y + \cdots + a_{k-2}xy^{k-2} + \\ & \qquad\qquad b_1x^{k-2}z + \cdots + b_{k-2}xz^{k-2} + b_{k-1}z^{k-1} + \\ & c_0y^{k-1} + c_1y^{k-2}z + \cdots + c_{k-2}yz^{k-2} \\ & + \cdots . \end{aligned}$$

As above, we see that each monomial in $L \cdot f$ arises from exactly two of the monomials in f. Hence the condition $L \cdot f \in I$ leads to the following three systems of equations.

Focussing only on the variables x, y we get:

$$\begin{aligned} a_0 + a_1 &= 0 \\ a_1 + a_2 &= 0 \\ &\vdots \\ a_{k-3} + a_{k-2} &= 0 \\ a_{k-2} + c_0 &= 0. \end{aligned}$$

It follows that $a_i = (-1)^i a_0$ and

$$c_0 = (-1)^{k-2} a_0 = a_0 \tag{3.3}$$

because k is even. Considering the variables x, z we obtain:

$$\begin{aligned} a_0 + b_1 &= 0 \\ b_1 + b_2 &= 0 \\ &\vdots \\ b_{k-2} + b_{k-1} &= 0, \end{aligned}$$

hence

$$b_i = (-1)^i a_0. \tag{3.4}$$

Finally, using the variables y, z we get:

$$\begin{aligned} c_0 + c_1 &= 0 \\ &\vdots \\ c_{k-1} + c_{k-2} &= 0 \\ c_{k-2} + b_{k-1} &= 0. \end{aligned}$$

Combining this, it follows that

$$-a_0 = b_{k-1} = -c_0 = a_0.$$

Since we assumed that the characteristic of K is not two, we conclude that the three linear systems above have only the trivial solution. Since the variables x, y, z were chosen arbitrarily, we see that the map ϕ_k is injective, as claimed.

According to Lemma 2.1 it remains to show that the multiplication map $\phi_{k+1} : (A)_k \xrightarrow{\times L} (A)_{k+1}$ is surjective. Note that the residue classes of the elements of the form $x_i^j x_m^{k+1-j}$ with $2 \leq j \leq k-1$, $1 \leq i < m \leq r$ form a basis of $(A)_{k+1}$. Setting for simplicity $x := x_i, y = x_m$ it is enough to show that, for each $j = 2, \ldots, k-1$, the residue class of $x^j y^{k+1-j}$ is in the image of ϕ_{k+1}.

We induct on $j \geq 2$. If $j = 2$, then we get modulo I that $L \cdot xy^{k-1} \equiv x^2 y^{k-1}$, thus $\overline{x^2 y^{k-1}} \in \operatorname{im} \phi_{k+1}$, as claimed. Let $3 \leq j \leq k-1$, then, modulo I, we get $L \cdot x^{j-1} y^{k-j} \equiv x^j y^{k-j} + x^{j-1} y^{k-j+1}$. Since by induction $\overline{x^{j-1} y^{k-j+1}} \in \operatorname{im} \phi_{k+1}$, we also obtain $\overline{x^j y^{k-j}} \in \operatorname{im} \phi_{k+1}$. This completes the proof. $\square$

The above result and our computer experiments suggest that the larger d becomes, the rarer it is that $R/I_{r,k,d}$ has the WLP. Based on computer experiments we expect the following to be true.

Conjecture 3.4. *Consider the algebra $R/I_{r,k,d}$. Then*

(a) *If $d = 4$, then it has the WLP if and only if $k \bmod 4$ is 2 or 3.*
(b) *If $d = 5$, then the WLP fails.*

(c) *If $d = 6$, then the WLP fails.*

We summarize our results in case $k = d = 3$.

Example 3.5. Consider the ideal

$$I_{r,3,3} = (x_1^3, x_2^3, \ldots, x_r^3, (\text{all squarefree monomials of degree 3})).$$

Then the corresponding inverse system is $(x_1^2x_2^2,\quad x_1^2x_3^2,\quad \ldots,\quad x_{r-1}^2x_r^2)$. Furthermore, the Hilbert function of $R/I_{r,3,3}$ is

$$1 \quad r \quad \binom{r+1}{2} \quad r(r-1) \quad \binom{r}{2} \quad 0$$

and $R/I_{r,3,3}$ fails to have the WLP because the map from degree 2 to degree 3 by a general linear form is not injective.

Remark 3.6. By truncating, we get a compressed level algebra with socle degree 3 that fails to have the WLP. We expect that there are compressed level algebras with larger socle degree that fail to have the WLP. However, we do not know such an example.

4. Monomial almost complete intersections in any codimension

In the paper [10] the first and second authors asked the following question (Question 4.2, page 95): For any integer $n \geq 3$, find the minimum number $A(n)$ (if it exists) such that *every* Artinian ideal $I \subset K[x_1, \ldots, x_n]$ with number of generators $\mu(I) \leq A(n)$ has the WLP. In Example 7.10 below, we show that $A(n)$ does not exist in positive characteristic. In any case, in [4] it was shown for $n = 3$ and characteristic zero that $A(3) = 3$ (also using a result of [6]), as noted in the introduction. A consequence of the main result of this section, below, is that in any number of variables and any characteristic there is an almost complete intersection that fails to have the WLP. Hence the main open question that remains is whether, in characteristic zero, all complete intersections have the WLP (as was shown for $n = 3$ in [6]), i.e. whether $A(n) = n$ in characteristic zero.

We begin by considering ideals of the form

$$I_{r,k} = (x_1^k, \ldots, x_r^k, x_1 \ldots x_r) \subset K[x_1, \ldots, x_r]. \tag{4.1}$$

Notice that this is a special case of the class of ideals described in Section 3. It is not too difficult to determine the graded Betti numbers.

Proposition 4.1. *The minimal free resolution of $I_{r,k}$ has the form*

$$0 \to R(-r+(r-1)(1-k))^{\binom{r}{r-1}} \to \begin{array}{c} R(-(r-1)k))^{\binom{r}{r-1}} \\ \oplus \\ R(-r+(r-2)(1-k))^{\binom{r}{r-2}} \end{array} \to \cdots \to \begin{array}{c} R(-3k)^{\binom{r}{3}} \\ \oplus \\ R(-r+2(1-k))^{\binom{r}{2}} \end{array} \to$$

$$\begin{array}{c} R(-2k)^{\binom{r}{2}} \\ \oplus \\ R(-r+(1-k))^{r} \end{array} \to \begin{array}{c} R(-k)^{r} \\ \oplus \\ R(-r) \end{array} \to I_{r,k} \to 0.$$

Proof. Since $I_{r,k}$ is an almost complete intersection, we can link it using the complete intersection $\mathfrak{a} = (x_1^k, \ldots, x_r^k)$ to an Artinian Gorenstein ideal, J. However, since both $I_{r,k}$ and $\mathfrak{a}$ are monomial, so is J. But it was first shown by Beintema [2] that any monomial Artinian Gorenstein ideal is a complete intersection. Hence we get by direct computation that $(x_1^k, \ldots, x_r^k) : x_1x_2 \cdots x_r = (x_1^{k-1}, \ldots, x_r^{k-1})$. Then use the mapping cone and observe that there is no splitting. $\square$

Before we come to the main result of this section, we prove a preliminary result about the Hilbert function of complete intersections that will allow us to apply Proposition 2.5.

Lemma 4.2. *Let $R = K[x_1, \dots, x_s]$ with $s \geq 2$, and let*

$$I_s = (x_1^s, \dots, x_s^s) \quad \text{and} \quad J_s = (x_1^{s+1}, x_2^{s+1}, x_3^s, \dots, x_s^s).$$

Note that the midpoint of the Hilbert function of R/I_s is $\binom{s}{2}$ and that of R/J_s is $\binom{s}{2} + 1$. Then

$$h_{R/I_s}\left(\tbinom{s}{2}\right) - h_{R/I_s}\left(\tbinom{s}{2} - 1\right) \leq h_{R/J_s}\left(\tbinom{s}{2} + 1\right) - h_{R/J_s}\left(\tbinom{s}{2} + 2\right).$$

Proof. The lemma is trivial to verify when $s = 2$ or $s = 3$, so we assume $s \geq 4$ for this proof. Observe that both quantities are positive, but one involves a difference to the left of the midpoint of the Hilbert function, while the other involves a difference to the right. We will use this formulation, although there exists others thanks to the symmetry of the Hilbert function of an Artinian complete intersection.

Our approach will be via basic double linkage. We will use the formula in Lemma 2.6 without comment. In fact, J_s is obtained from I_s by a sequence of two basic double links:

$$\begin{aligned} I_s \quad & \rightsquigarrow x_1 \cdot I_s + (x_2^s, \dots, x_s^s) := G = (x_1^{s+1}, x_2^s, \dots, x_s^s) \\ & \rightsquigarrow x_2 \cdot G + (x_1^{s+1}, x_3^s, \dots, x_s^s) = J_s. \end{aligned}$$

Note that G is a complete intersection of codimension s and that the ideals $C_1 := (x_2^s, \dots, x_s^s)$ and $C_2 := (x_1^{s+1}, x_3^s, \dots, x_s^s)$ are complete intersections of codimension $s - 1$. The midpoints of the *h-vectors* of R/C_1 and R/C_2 are $\frac{(s-1)^2}{2}$ and $\frac{(s-1)^2+1}{2}$ respectively. We now compute Hilbert functions (and notice the shift, and that the lines for R/C_1 and R/C_2 are the first difference of those Hilbert functions, i.e. the h-vectors):

$$\begin{array}{c|cccccccc} R/I_s & & 1 & s & \dots & h_{R/I_s}\left(\binom{s}{2} - 1\right) & h_{R/I_s}\left(\binom{s}{2}\right) & \dots \\ R/C_1 & 1 & s-1 & \dots & \dots & \Delta h_{C_1}\left(\binom{s}{2}\right) & \Delta h_{C_1}\left(\binom{s}{2} + 1\right) & \dots \\ \hline R/G & 1 & s & \dots & \dots & A & B & \dots \end{array}$$

where

$$(4.2) \qquad \begin{aligned} h_{R/G}\left(\tbinom{s}{2}\right) &= A = h_{R/I_s}\left(\tbinom{s}{2} - 1\right) + \Delta h_{R/C_1}\left(\tbinom{s}{2}\right), \\ h_{R/G}\left(\tbinom{s}{2} + 1\right) &= B = h_{R/I_s}\left(\tbinom{s}{2}\right) + \Delta h_{R/C_1}\left(\tbinom{s}{2} + 1\right), \end{aligned}$$

and

$$\begin{array}{c|cccccccc} R/G & & 1 & s & \dots & A & B & \dots \\ R/C_2 & 1 & s-1 & \dots & \dots & \Delta h_{C_2}\left(\binom{s}{2} + 1\right) & \Delta h_{C_2}\left(\binom{s}{2} + 2\right) & \dots \\ \hline R/J_s & 1 & s & \dots & \dots & C & D & \dots \end{array}$$

where

$$\begin{aligned} h_{R/J_s}\left(\tbinom{s}{2} + 1\right) &= C = h_{R/I_s}\left(\tbinom{s}{2} - 1\right) + \Delta h_{R/C_1}\left(\tbinom{s}{2}\right) + \Delta h_{R/C_2}\left(\tbinom{s}{2} + 1\right), \\ h_{R/J_s}\left(\tbinom{s}{2} + 2\right) &= D = h_{R/I_s}\left(\tbinom{s}{2}\right) + \Delta h_{R/C_1}\left(\tbinom{s}{2} + 1\right) + \Delta h_{R/C_2}\left(\tbinom{s}{2} + 2\right). \end{aligned}$$

Now observe that the complete intersection G has odd socle degree $s(s-1)+1$; hence $A = B$. Then it follows from (4.2) that

$$(4.3) \qquad h_{R/I_s}\left(\tbinom{s}{2}\right) - h_{R/I_s}\left(\tbinom{s}{2} - 1\right) = \Delta h_{R/C_1}\left(\tbinom{s}{2}\right) - \Delta h_{R/C_1}\left(\tbinom{s}{2} + 1\right).$$

Thus we obtain

$$
\begin{array}{rcl}
h_{R/J_s}(\binom{s}{2}+1) - h_{R/J_s}(\binom{s}{2}+2) & = & h_{R/I_s}(\binom{s}{2}-1) - h_{R/I_s}(\binom{s}{2}) \\
& & +\Delta h_{R/C_1}(\binom{s}{2}) - \Delta h_{R/C_1}(\binom{s}{2}+1) \\
& & +\Delta h_{R/C_2}(\binom{s}{2}+1) - \Delta h_{R/C_2}(\binom{s}{2}+2) \\
& = & \Delta h_{R/C_2}(\binom{s}{2}+1) - \Delta h_{R/C_2}(\binom{s}{2}+2).
\end{array}
\tag{4.4}
$$

Combining (4.3) and (4.4), we see that it remains to show that

$$
\Delta h_{R/C_1}\left(\binom{s}{2}\right) - \Delta h_{R/C_1}\left(\binom{s}{2}+1\right) \leq \Delta h_{R/C_2}(\binom{s}{2}+1) - \Delta h_{R/C_2}(\binom{s}{2}+2). \tag{4.5}
$$

By the symmetry of the h-vectors of R/C_1 and R/C_2 we see that this is equivalent to showing

$$
\Delta h_{R/C_1}\left(\binom{s-1}{2}\right) - \Delta h_{R/C_1}\left(\binom{s-1}{2}-1\right) \leq \Delta h_{R/C_2}(\binom{s-1}{2}) - \Delta h_{R/C_2}(\binom{s-1}{2}-1). \tag{4.6}
$$

Now, $\Delta h_{R/C_i}$ $(i = 1, 2)$ is the Hilbert function of an Artinian monomial complete intersection in R, namely R/C_i', where C_i' is obtained from C_i by adding the missing variable. Furthermore, if we replace C_2' by $D_2 = (x_1, x_2^{s+1}, x_3^s, \ldots, x_s^s)$, we have that R/C_2' and R/D_2 have the same Hilbert function, and $D_2 \subset C_1'$.

But such ideals have the Weak Lefschetz property ([12], [14]). In particular, if L is a general linear form, then the left-hand side of (4.6) is the Hilbert function of $R/(C_1'+(L))$ in degree $\binom{s-1}{2}$ and the right-hand side is the Hilbert function of $R/(D_2 + (L))$ in the same degree. Because of the inclusion of the ideals, (4.6) follows and so the proof is complete. □

We now come to the main result of this section. The case $r = 3$ was proven by Brenner and Kaid [4]. Note that when $r \leq 2$, all quotients of R have the WLP by a result of [6].

Theorem 4.3. *Let $R = K[x_1, \ldots, x_r]$, with $r \geq 3$, and consider*

$$I_{r,r} = (x_1^r, \ldots, x_r^r, x_1x_2\cdots x_r).$$

Then the level Artinian algebra $R/I_{r,r}$ fails to have the WLP.

Proof. Specifically, we will check that the multiplication on $R/I_{r,r}$ by a general linear form fails surjectivity from degree $\binom{r}{2} - 1$ to degree $\binom{r}{2}$, even though the value of the Hilbert function is non-increasing between these two degrees.

The proof is in two steps.

Step 1. We first prove this latter fact, namely that

$$h_{R/I_{r,r}}(d-1) \geq h_{R/I_{r,r}}(d) \quad \text{for } d = \tbinom{r}{2}.$$

To do this, we again use basic double linkage. We observe that

$$
\begin{aligned}
J_1 := (x_1^r, x_2^{r-1}, \dots, x_r^{r-1}, x_1) &\rightsquigarrow x_2 \cdot (x_1^r, x_2^{r-1}, \dots, x_r^{r-1}, x_1) + (x_1^r, x_3^{r-1}, \dots, x_r^{r-1}) \\
&= (x_1^r, x_2^r, x_3^{r-1}, \dots, x_r^{r-1}, x_1x_2) := J_2 \\
&\rightsquigarrow x_3 \cdot (x_1^r, x_2^r, x_3^{r-1}, \dots, x_r^{r-1}, x_1x_2) + (x_1^r, x_2^r, x_4^{r-1}, \dots, x_r^{r-1}) \\
&= (x_1^r, x_2^r, x_3^r, x_4^{r-1}, \dots, x_r^{r-1}, x_1x_2x_3) := J_3 \\
&\vdots \\
&\rightsquigarrow x_r \cdot (x_1^r, \dots, x_{r-1}^r, x_r^{r-1}, x_1 \cdots x_{r-1}) + (x_1^r, \dots, x_{r-1}^r) \\
&= (x_1^r, \dots, x_r^r, x_1 \cdots x_r) := J_r = I_{r,r}
\end{aligned}
$$

and we note that the first ideal, J_1, is just $(x_2^{r-1}, \dots, x_r^{r-1}, x_1)$. Furthermore, this ideal is a complete intersection with socle degree $(r-1)(r-2)$. The midpoint of the Hilbert function is in degree $\binom{r-1}{2}$.

Let $C_1 = (x_1^r, x_3^{r-1}, \dots, x_r^{r-1})$ and $C_2 = (x_1^r, x_2^r, x_4^{r-1}, \dots, x_r^{r-1})$. We note that the first difference of the Hilbert function of R/C_1 is symmetric with odd socle degree, so the values in degrees $\binom{r-1}{2}$ and $\binom{r-1}{2}+1$ are equal. The key point, though, is that by applying Lemma 4.2 with $s = r-1$, we obtain that

$$h_{R/J_1}(\tbinom{r-1}{2}) - h_{R/J_1}(\tbinom{r-1}{2} - 1) \leq h_{R/C_2}(\tbinom{r-1}{2} + 1) - h_{R/C_2}(\tbinom{r-1}{2} + 2). \tag{4.7}$$

We are interested in the values of the Hilbert function of $R/I_{r,r}$ in degrees $\binom{r}{2} - 1$ and $\binom{r}{2}$. Since $I_{r,r}$ is obtained from J_1 by a sequence of $r-1$ basic double links, and since each one involves a shift by 1 of the Hilbert function, this corresponds (first) to an examination of the Hilbert function of R/J_1 in degrees $\binom{r}{2} - 1 - (r-1) = \binom{r-1}{2} - 1$ and $\binom{r-1}{2}$. The former is smaller than the latter, but we do not need to know the precise values.

Our observation in the paragraph preceding (4.7) shows that when we add the first difference of the (shifted) Hilbert function of R/C_1 to get $h_{R/J_2}(\binom{r-1}{2})$ and $h_{R/J_2}(\binom{r-1}{2}+1)$, the difference between these two values is the same as the difference between the values of the Hilbert function of R/J_1 in degrees $\binom{r-1}{2} - 1$ and $\binom{r-1}{2}$, with the latter being larger. However, the point of (4.7) is that when we then add the (shifted) first difference of the Hilbert function of R/C_2, we overcome this difference and already have a Hilbert function with the value in degree $\binom{r-1}{2}+1$ larger than that in degree $\binom{r-1}{2}+2$. Since each subsequent Hilbert function has the (shifted) value in the smaller of the corresponding degrees larger than the value in the second, we finally obtain the same for the desired Hilbert function, namely that of $R/I_{r,r}$. This concludes step 1.

Step 2. To prove that $R/I_{r,r}$ fails surjectivity from degree $\binom{r}{2} - 1$ to degree $\binom{r}{2}$, we will use Proposition 2.5 (d). Note that

$$\bar{I}_{r,r} \cong (x_1^r, \dots, x_{r-1}^r, (x_1 + \cdots + x_{r-1})^r, x_1 \cdots x_{r-1} \cdot (x_1 + \cdots + x_{r-1})).$$

We now claim that it is enough to verify that there is a homogeneous form $F \in \bar{R} \cong K[x_1, \cdots, x_{r-1}]$ of degree $\binom{r-1}{2}$ such that

$$F \cdot x_1 x_2 \cdots x_{r-1}(x_1 + \cdots + x_{r-1}) \in (x_1^r, \dots, x_{r-1}^r) \tag{4.8}$$

and

$$F \cdot (x_1 + ... + x_{r-1})^r \in (x_1^r, ..., x_{r-1}^r). \tag{4.9}$$

Indeed, clearly $x_1^r, \dots, x_{r-1}^r$ is a regular sequence in $\bar{I}_{r,r}$ that extends to a minimal generating set for $I_{r,r}$. So Proposition 2.5 (d) shows that it is enough to find a form F of degree $r(r-1) - \left(\binom{r}{2} + r - 1\right) = \binom{r-1}{2}$, non-zero modulo $(x_1^r, \dots, x_{r-1}^r)$, such that (4.8) and (4.9) hold. Hence our claim holds.

The heart of the proof is to show that the specific polynomial

$$F = \sum_{\substack{i_1+\cdots+i_{r-1}=\binom{r-1}{2} \\ 0\le i_j \le r-2, \quad i_j \ne i_\ell}} (-1)^{sgn(i_1,\cdots,i_{r-1})} x_1^{i_1} \cdots x_{r-1}^{i_{r-1}}$$

satisfies (4.8) and (4.9). Note that F is the determinant of a Vandermonde matrix. F simply consists of a sum of terms, all with coefficient 1 or -1, obtained as follows. Each term consists of a product of different powers of the $r-1$ variables (remember that we are in the quotient ring). Namely, for *each* permutation, σ, of $(0, 1, 2, ..., r-2)$, we look at the term

$$(-1)^{sgn(\sigma)} \cdot A$$

where A is the monomial obtained by taking the i-th variable to the power given by the i-th entry in σ. For example, if $r = 5$ and $\sigma = (2, 0, 3, 1)$ then $sgn(\sigma) = -1$ so we have the term $-x_1^2 x_3^3 x_4$.

We first check (4.8). In order for the product to be contained in $(x_1^r, \dots, x_{r-1}^r)$, we need that every term in the product that does not contain at least one exponent $\ge r$ be canceled by another term in the product. That is, we have $F \cdot x_1 x_2 ... x_{r-1}(x_1 + ... + x_{r-1}) \in (x_1^r, ..., x_{r-1}^r)$ if and only if

$$\sum_{m=1}^{r-1} \left(\sum_{\substack{i_1+\cdots+i_{r-1}=\binom{r-1}{2} \\ 0\le i_j \le r-2, \quad i_j \ne i_\ell, \quad i_m \ne r-2}} (-1)^{sgn(i_1,\cdots,i_{r-1})} x_1^{i_1+1} \cdots, x_m^{i_m+2}, \cdots, i_{r-1}^{i_{r-1}+1} \right) = 0.$$

Notice that we have ruled out $i_m = r-2$ since otherwise $i_m + 2 = r$ and that term is automatically in the desired ideal. But then the hypotheses $i_1 + \cdots + i_{r-1} = \binom{r-1}{2}$, $0 \le i_j \le r-2$ and $i_j \ne i_\ell$ imply that there exists a unique integer n with $1 \le n \le r-1$ such that $i_m + 2 = i_n + 1$. Hence the summand

$$(-1)^{sgn(i_1,\cdots,m,\cdots,n,\cdots,i_{r-1})} x_1^{i_1+1} \cdots, x_m^{i_m+2}, \cdots, x_{r-1}^{i_{r-1}+1}$$

is cancelled against the summand

$$(-1)^{sgn(i_1,\cdots,m+1,\cdots,n-1,\cdots,i_{r-1})} x_1^{i_1+1} \cdots, x_m^{i_m+2}, \cdots, x_{r-1}^{i_{r-1}+1}.$$

(For notational convenience we have assumed $m < n$ but this is not at all important.)

We now prove (4.9). We have

$$F \cdot (x_1 + ... + x_{r-1})^r = F \cdot \left(\sum_{\substack{j_1+\cdots+j_{r-1}=r \\ j_i \ge 0}} \frac{r!}{j_1! \cdots j_{r-1}!} x_1^{j_1} \cdots x_{r-1}^{j_{r-1}} \right) =$$

$$\left(\sum_{\substack{j_1+\cdots+j_{r-1}=r \\ j_i \ge 0}} \frac{r!}{j_1! \cdots j_{r-1}!} x_1^{j_1} \cdots x_{r-1}^{j_{r-1}} \right) \cdot \left(\sum_{\substack{i_1+\cdots+i_{r-1}=\binom{r-1}{2} \\ 0\le i_j \le r-2, \quad i_j \ne i_\ell}} (-1)^{sgn(i_1,\cdots,i_{r-1})} x_1^{i_1} \cdots x_{r-1}^{i_{r-1}} \right) =$$

$$\sum_{\substack{j_1+\cdots+j_{r-1}=r\\ j_i\geq 0}}\left(\sum_{\substack{i_1+\cdots+i_{r-1}=\binom{r-1}{2}\\ 0\leq i_j\leq r-2, \quad i_j\neq i_\ell}}\frac{(-1)^{sgn(i_1,\cdots,i_{r-1})}r!}{j_1!\cdots j_{r-1}!}x_1^{i_1+j_1}\cdots x_{r-1}^{i_{r-1}+j_{r-1}}\right).$$

Therefore

$$F\cdot(x_1+...+x_{r-1})^r\in(x_1^r,...,x_{r-1}^r)$$

if and only if

$$G:=\sum_{\substack{j_1+\cdots+j_{r-1}=r\\ r-1\geq j_i\geq 0}}\left(\sum_{\substack{i_1+\cdots+i_{r-1}=\binom{r-1}{2}\\ 0\leq i_\ell\leq \min(r-2,r-1-j_\ell), \quad i_j\neq i_\ell}}\frac{(-1)^{sgn(i_1,\cdots,i_{r-1})}r!}{j_1!\cdots j_{r-1}!}x_1^{i_1+j_1}\cdots x_{r-1}^{i_{r-1}+j_{r-1}}\right)=0.$$

Given an $(r-1)$-uple of non-negative integers $\underline{j}:=(j_1,\cdots,j_{r-1})$ such that $j_1+\cdots+j_{r-1}=r$, we set

$$C_{\underline{j}}:=\frac{r!}{j_1!\cdots j_{r-1}!}.$$

Notice that two $(r-1)$-uples of non-negative integers $(j_1,\cdots,j_{r-1})$ and $(j'_1,\cdots,j'_{r-1})$ with $j_1+\cdots+j_{r-1}=r=j'_1+\cdots+j'_{r-1}$ verify

$$\frac{r!}{j_1!\cdots j_{r-1}!}=\frac{r!}{j'_1!\cdots j'_{r-1}!}\Leftrightarrow\{j_1,\cdots,j_{r-1}\}=\{j'_1,\cdots,j'_{r-1}\}. \tag{4.10}$$

Given an $(r-1)$-uple of non-negative integers $\underline{j}:=(j_1,\cdots,j_{r-1})$ such that $j_1+\cdots+j_{r-1}=r$ and an $(r-1)$-uple $\underline{\alpha}:=(\alpha_1,\cdots,\alpha_{r-1})$, we define (from now on $\#(B)$ means the cardinality of the set B):

$$N_{\underline{\alpha},\underline{j}}:=\#(A(\underline{\alpha})_{\underline{j}})$$

where $A(\underline{\alpha})_{\underline{j}}$ is the set of monomials $\pm C_{\underline{j}}x_1^{\alpha_1}\cdots x_{r-1}^{\alpha_{r-1}}$ in G of multidegree $\underline{\alpha}$ and coefficient $\pm C_{\underline{j}}$. To prove (4.9), it is enough to see that $N_{\underline{\alpha},\underline{j}}$ is even and half of the elements of $A(\underline{\alpha})_{\underline{j}}$ have coefficient $+C_{\underline{j}}$ and the other half have coefficient $-C_{\underline{j}}$. Let us prove it. Without loss of generality we can assume that $\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_{r-1}$ (we re-order the variables, if necessary). We will see that for any monomial in $A(\underline{\alpha})_{\underline{j}}$ there is a well determined way to associate another monomial in $A(\underline{\alpha})_{\underline{j}}$ with the opposite sign. Indeed, the monomials in $A(\underline{\alpha})_{\underline{j}}$ have degree $\binom{r-1}{2}+r=\binom{r}{2}+1$ and, moreover, $0\leq\alpha_\ell\leq r-1$ for all $1\leq\ell\leq r-1$. Therefore, there exist integers $1\leq p<q\leq r-1$ such that $\alpha_p=\alpha_q$.

We define $p_0:=\min\{p\mid\alpha_p=\alpha_{p+1}\}$. Now, we take an arbitrary monomial

$$\pm C_{\underline{j}}x_1^{\alpha_1}\cdots x_{p_0}^{\alpha_{p_0}}x_{p_0+1}^{\alpha_{p_0+1}}\cdots x_{r-1}^{\alpha_{r-1}}\in A(\underline{\alpha})_{\underline{j}}$$

where $\alpha_1=j_1+i_1,\ \dots,\ \alpha_{p_0}=j_{p_0}+i_{p_0},\ \alpha_{p_0+1}=j_{p_0+1}+i_{p_0+1},\ \dots,\ \alpha_{r-1}=j_{r-1}+i_{r-1}$. It will be cancelled with

$$\mp C_{\underline{j}}x_1^{\alpha_1}\cdots x_{p_0}^{\alpha_{p_0}}x_{p_0+1}^{\alpha_{p_0+1}}\cdots x_{r-1}^{\alpha_{r-1}}\in A(\underline{\alpha})_{\underline{j}}$$

where $\alpha_1=j_1+i_1,\ \dots\ ,\alpha_{p_0}=j_{p_0+1}+i_{p_0+1},\ \alpha_{p_0+1}=j_{p_0}+i_{p_0},\ \dots\ ,\ \alpha_{r-1}=j_{r-1}+i_{r-1}$ and we are done. □

Example 4.4. We illustrate the construction in Step 1 of the proof of Theorem 4.3 for the case $r = 5$. In the following table of Hilbert functions and h-vectors, we have

$$\begin{array}{rcl} J_1 & = & (x_1, x_2^4, x_3^4, x_4^4, x_5^4) \\ J_2 & = & x_2 \cdot J_1 + (x_1^5, x_3^4, x_4^4, x_5^4) = (x_1^5, x_2^5, x_3^4, x_4^4, x_5^4, x_1x_2) \\ J_3 & = & x_3 \cdot J_2 + (x_1^5, x_2^5, x_4^4, x_5^4) = (x_1^5, x_2^5, x_3^5, x_4^4, x_5^4, x_1x_2x_3) \\ J_4 & = & x_4 \cdot J_3 + (x_1^5, x_2^5, x_3^5, x_5^4) = (x_1^5, x_2^5, x_3^5, x_4^5, x_5^4, x_1x_2x_3x_4) \\ J_5 & = & x_5 \cdot J_4 + (x_1^5, x_2^5, x_3^5, x_4^5) = (x_1^5, x_2^5, x_3^5, x_4^5, x_5^5, x_1x_2x_3x_4x_5) \end{array}$$

In the following calculation, we have put in boldface the critical range of degrees.

Ideal	Hilbert function/h-vector																
J_1					1	4	10	20	31	**40**	**44**	40	31	20	10	4	1
(5,4,4,4)				1	4	10	20	32	43	**50**	**50**	43	32	20	10	4	1
J_2				1	5	14	30	52	74	**90**	**94**	83	63	40	20	8	2
(5,5,4,4)			1	4	10	20	33	46	56	**60**	**56**	46	33	20	10	4	1
J_3			1	5	15	34	63	98	130	**150**	**150**	129	96	60	30	12	3
(5,5,5,4)		1	4	10	20	34	49	62	70	**70**	**62**	49	34	20	10	4	1
J_4		1	5	15	35	68	112	160	200	**220**	**212**	178	130	80	40	16	4
(5,5,5,5)	1	4	10	20	35	52	68	80	85	**80**	**68**	52	35	20	10	4	1
J_5	1	5	15	35	70	120	180	240	285	**300**	**280**	230	165	100	50	20	5

It is interesting to note that experimentally we have verified that R/J_1 and R/J_2 have the WLP, while R/J_3, R/J_4 and R/J_5 do not. The algebras that fail to have the WLP all fail surjectivity in the range indicated in boldface. Only R/J_5 fails to have the WLP in any other degree, namely it fails injectivity in the preceding degree.

As mentioned above, we now have a partial answer to Question 4.2 of [10]. Recall that $A(n)$ is defined to be the minimum number (if it exists) such that *every* Artinian ideal $I \subset K[x_1, \dots, x_n]$ with number of generators $\mu(I) \leq A(n)$ has the WLP.

Corollary 4.5. *If $A(n)$ exists then it equals n.*

5. An almost monomial almost complete intersection

In order to illustrate the subtlety of the Weak Lefschetz Property, we now describe a class of ideals that is very similar to the class of ideals discussed in Section 4. That is, we consider, for each codimension $r \geq 3$, the ideal

$$\mathfrak{J}_r = (x_1^r, \dots, x_r^r, x_1 \cdots x_{r-1}(x_1 + x_r)).$$

We will compare the properties of this ideal with those of the ideal

$$I_{r,r} = (x_1^r, \dots, x_r^r, x_1 \cdots x_r).$$

Included in this subtlety is the fact that the WLP behavior changes with the characteristic. Notice that our results in positive characteristic do not depend on whether the field is finite or not.

Our first result shows that we cannot distinguish the two ideals by solely looking at their Hilbert functions.

Lemma 5.1. *$R/\mathfrak{J}_r$ and $R/I_{r,r}$ have the same Hilbert function.*

Proof. We will show that $\mathfrak{J}_r$ arises via a sequence of basic double links which are numerically equivalent to the one that produced $I_{r,r}$ in the first part of Theorem 4.3. Notice first that the ideals

$$(x_1^r, x_2^{r-1}, \dots, x_r^{r-1}, x_1) = (x_2^{r-1}, \dots, x_r^{r-1}, x_1) \quad \text{and} \quad (x_1^{r-1}, \dots, x_{r-1}^{r-1}, x_1 + x_r)$$

have the same Hilbert function. Notice also that this latter ideal is equal to

$$(x_1^{r-1}, \dots, x_{r-1}^{r-1}, x_r^r, x_1 + x_r).$$

In Step 1 of Theorem 4.3 we saw a sequence of basic double links starting with the ideal $(x_1^r, x_2^{r-1}, \dots, x_r^{r-1}, x_1)$ and ending with $(x_1^r, \dots, x_r^r, x_1, \dots, x_r)$. We will now produce a parallel sequence of basic double links starting with $(x_1^{r-1}, \dots, x_{r-1}^{r-1}, x_r^r, x_1 + x_r)$ and ending with $(x_1^r, \dots, x_r^r, x_1 \cdots x_{r-1}(x_1 + x_r))$, such that at each step the two sequences are numerically the same, and hence the resulting ideals at each step have the same Hilbert function.

$$\begin{aligned}
&(x_1^{r-1}, \dots, x_{r-1}^{r-1}, x_r^r, x_1 + x_r) \\
&\rightsquigarrow \quad x_{r-1} \cdot (x_1^{r-1}, \dots, x_{r-1}^{r-1}, x_r^r, x_1 + x_r) + (x_1^{r-1}, \dots, x_{r-2}^{r-1}, x_r^r) \\
&\qquad = (x_1^{r-1}, x_2^{r-1}, \dots, x_{r-2}^{r-1}, x_{r-1}^r, x_r^r, (x_1 + x_r) \cdot x_{r-1}) \\
&\rightsquigarrow \quad x_{r-2} \cdot (x_1^{r-1}, x_2^{r-1}, \dots, x_{r-2}^{r-1}, x_{r-1}^r, x_r^r, (x_1 + x_r) \cdot x_{r-1}) + (x_1^{r-1}, \dots, x_{r-3}^{r-1}, x_{r-1}^r, x_r^r) \\
&\qquad = (x_1^{r-1}, x_2^{r-1}, \dots, x_{r-3}^{r-1}, x_{r-2}^r, x_{r-1}^r, x_r^r, (x_1 + x_r) \cdot x_{r-2}x_{r-1}) := J_3 \\
&\qquad \vdots \\
&\rightsquigarrow \quad x_1 \cdot (x_1^{r-1}, x_2^r, \dots, x_{r-1}^r, x_r^r, (x_1 + x_r) \cdot x_2 \cdots x_{r-1}) + (x_2^r, \dots, x_r^r) \\
&\qquad = (x_1^r, \dots, x_r^r, x_1 \cdots x_{r-1}(x_1 + x_r)) = \mathfrak{J}_r.
\end{aligned}$$

This completes the proof. □

We will now show that the two algebras behave differently with respect to the WLP. Recall that $R/I_{r,r}$ does not have the WLP if $r \geq 3$. Studying $R/\mathfrak{J}_r$ when $r = 3$ is not too difficult:

Proposition 5.2. *For every field K, the algebra*

$$R/\mathfrak{J}_3 = K[x, y, z]/(x^3, y^3, z^3, xy(x + z))$$

has the WLP if and only if the characteristic of K is not three.

Proof. If the characteristic of K is three then for every linear form $\ell \in R$, ℓ^3 is in (x^3, y^3, z^3). Thus the residue class of ℓ^2 is in the kernel of the multiplication map

$$\times \ell : (R/\mathfrak{J}_3)_2 \to (R/\mathfrak{J}_3)_3.$$

This shows that $R/\mathfrak{J}_3$ does not have the WLP if $\operatorname{char} K = 3$.

Now assume that $\operatorname{char} K \neq 3$. Consider the linear form $L = x + y + z$. Then one checks that

$$(\mathfrak{J}_3, L)/(L) \cong ((x, y)^3, L)/(L),$$

which implies that the multiplication map

$$\times L : (R/\mathfrak{J}_3)_2 \to (R/\mathfrak{J}_3)_3$$

is surjective. Hence $R/\mathfrak{J}_3$ has the WLP in this case. □

The case when $r = 4$ is considerably more complicated.

Proposition 5.3. *For every field K, the algebra*

$$R/\mathfrak{J}_4 = K[w,x,y,z]/(w^4, x^4, y^4, z^4, wxy(w+z))$$

has the WLP if and only if the characteristic of K is not two or five.

Proof. The Hilbert function of $R/\mathfrak{J}_4$ is $1, 4, 10, 20, 30, 36, 34, \dots$. Hence, by Proposition 2.1, $R/\mathfrak{J}_4$ has the WLP if and only if, for a general form L, the multiplication maps

$$\times L : (R/\mathfrak{J}_4)_4 \to (R/\mathfrak{J}_4)_5 \quad \text{and} \quad \times L : (R/\mathfrak{J}_4)_5 \to (R/\mathfrak{J}_4)_6$$

are injective and surjective, respectively.

We first show that the latter map is surjective if $L := 2w + x + y + z$, provided the characteristic of K is neither 2 nor 5. Notice that this map is surjective if and only if $(R/(\mathfrak{J}_4, L)])_6 = 0$. Since

$$R/(\mathfrak{J}_4, L) \cong K[w,x,y]/(w^4, x^4, y^4, (2w+x+y)^4, wxy(w+x+y))$$

this is equivalent to the fact that $\dim_K((w^4, x^4, y^4, (2w+x+y)^4, wxy(w+x+y))_6 = 28$. To compute the dimension of $((w^4, x^4, y^4, (2w+x+y)^4, wxy(w+x+y))_6$, we consider the coefficients of the 28 degree 6 monomials in $K[w,x,y]$ occurring in each of the 30 polynomials fq, where f is one of the forms $w^4, x^4, y^4, (2w+x+y)^4, wxy(w+x+y)$ and q is one of the quadrics $w^2, wx, x^2, wy, xy, y^2$. Compute these coefficients assuming, temporarily, that $\operatorname{char} K = 0$, and record them in a 30×28 matrix M whose entries are integers. Using CoCoA we verified that the greatest common divisor of all the maximal minors of M is $320 = 2^8 \cdot 5$. This shows that the matrix M has rank 28 if and only if $\operatorname{char} K \neq 2, 5$.

We now discuss the map $\times L : (R/\mathfrak{J}_4)_4 \to (R/\mathfrak{J}_4)_5$, where L is a general linear form. This map is injective if and only if $\dim_K(R/(\mathfrak{J}_4, L))_5 = 6$, which is equivalent to $\dim_K((\mathfrak{J}_4, L)/(L))_5 = 15$.

Assume first that the field K is infinite. Then an argument similar to the one in the proof of Proposition 2.2 shows we may assume that

$$L := tw + x + y - z,$$

where $t \in K$. Then

$$R/(\mathfrak{J}_4, L) \cong K[w,x,y]/(w^4, x^4, y^4, (tw+x+y)^4, wxy((t+1) \cdot w + x + y)).$$

To compute the dimension of $((w^4, x^4, y^4, (tw+x+y)^4, wxy((t+1) \cdot w + x + y)))_5$, we consider the coefficients of the 21 degree 5 monomials in $K[w,x,y]$ occurring in each of the 15 polynomials $f\ell$, where f is one of the forms $w^4, x^4, y^4, (tw+x+y)^4, wxy((t+1) \cdot w + x + y)$ and ℓ is one of the variables w, x, y. Compute these coefficients assuming, temporarily, that $\operatorname{char} K = 0$, and record them in a 15×21 matrix N whose entries are polynomials in $\mathbb{Z}[t]$. A CoCoA computation provides that all maximal minors of N are divisible by 10 and that one of the minors is $80t^4(t+1)^2$. It follows that the rank of N is 15 if and only if $\operatorname{char} K \neq 2, 5$. Hence we have shown that over an infinite field, for a general linear form L, the map $\times L : (R/\mathfrak{J}_4)_4 \to (R/\mathfrak{J}_4)_5$ is injective if and only if the characteristic of K is neither 2 nor 5. This also implies that $R/\mathfrak{J}_4$ does not have the WLP if K is a finite field of characteristic 2 or 5. Furthermore, every field whose characteristic is not 2 or 5 contains an element t such that $80t^4(t+1)^2$ is not zero. Hence the above arguments show that in this case there is a linear form L such that $\times L : (R/\mathfrak{J}_4)_4 \to (R/\mathfrak{J}_4)_5$ is injective.

Combining this with the first part of the proof, our assertion follows. $\square$

Remark 5.4. (i) For $R/\mathfrak{J}_4$, Proposition 5.3 provides an affirmative answer to Problem 2.3.

(ii) We expect that in characteristic zero, for each integer $r \geq 2$, the algebra $R/\mathfrak{J}_r$ has the WLP and that $L = 2x_1 + x_2 + \cdots + x_{r-1} - x_r$ is a Lefschetz element.

6. Monomial almost complete intersections in three variables

Now we consider ideals of the form

$$I = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$$

in $R = K[x,y,z]$, where $0 \leq \alpha < a$, $0 \leq \beta < b$ and $0 \leq \gamma < c$. This class of ideals was first considered in [3], Corollary 7.3.

Proposition 6.1. *If I is as above and is not a complete intersection then*

(i) *The inverse system for I is given by $(x^{a-1}y^{b-1}z^{\gamma-1}, x^{a-1}y^{\beta-1}z^{c-1}, x^{\alpha-1}y^{b-1}z^{c-1})$, where we make the convention that if a term has an exponent of -1 (e.g. if $\gamma = 0$), that term is removed.*

(ii) *In particular, if $\alpha, \beta, \gamma > 0$ then R/I has Cohen-Macaulay type 3. Otherwise it has Cohen-Macaulay type 2.*

(iii) *If $\alpha, \beta, \gamma > 0$, the socle degrees of R/I are $b+c+\alpha-3, a+c+\beta-3, a+b+\gamma-3$. In particular, R/I is level if and only if $a-\alpha = b-\beta = c-\gamma$.*

(iv) *Suppose that one of $\alpha, \beta, \gamma = 0$; without loss of generality say $\gamma = 0$. Then the corresponding socle degree in (iii), namely $a+b+\gamma-3$, does not occur. Now R/I is level if and only if $a-\alpha = b-\beta$, and c is arbitrary.*

(v) *Let $J = I : x^\alpha y^\beta z^\gamma$ be the ideal residual to I in the complete intersection (x^a, y^b, z^c). Then $J = (x^{a-\alpha}, y^{b-\beta}, z^{c-\gamma})$.*

(vi) *A free resolution of R/I is*

$$0 \to \begin{array}{c} R(-\alpha-b-c) \\ \oplus \\ R(-\beta-a-c) \\ \oplus \\ R(-\gamma-a-b) \end{array} \to \begin{array}{c} R(-\alpha-\beta-c) \\ \oplus \\ R(-\alpha-\gamma-b) \\ \oplus \\ R(-\beta-\gamma-a) \\ \oplus \\ R(-b-c) \\ \oplus \\ R(-a-c) \\ \oplus \\ R(-a-b) \end{array} \to \begin{array}{c} R(-\alpha-\beta-\gamma) \\ \oplus \\ R(-a) \\ \oplus \\ R(-b) \\ \oplus \\ R(-c) \end{array} \to R \to R/I \to 0. \tag{6.1}$$

This is minimal if and only if α, β, γ are all positive.

Proof. Part (i) follows by inspection. Then (ii), (iii) and (iv) follow immediately from (i). As before, (v) is a simple computation of the colon ideal, based on the fact [2] that J is a complete intersection, so it only remains to check the degrees. Having (v), it is a straightforward computation using the mapping cone to obtain (vi). $\square$

Theorem 6.2. *Assume that $K = \overline{K}$ is an algebraically closed field of characteristic zero. For $I = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$, if the WLP fails then $a+b+c+\alpha+\beta+\gamma \equiv 0 \pmod 3$.*

Proof. Let $\mathcal{E}$ be the syzygy bundle of I and let $L \cong \mathbb{P}^1$ be a general line. By [4] Theorem 3.3, if the WLP fails then $\mathcal{E}$ is semistable. Furthermore, the splitting type of $\mathcal{E}_{\text{norm}}$ must be $(1, 0, -1)$ (apply [4], Theorem 2.2 and the Grauert-Mülich theorem). Hence the twists of $\mathcal{E}|_L$ are three consecutive integers. Since the restriction of $\mathcal{E}$ to L is the (free) syzygy module corresponding to the restriction of the generators of I to $\mathbb{P}^1$, we see that the sum of the generators must be divisible by 3. $\square$

Corollary 6.3. *Assume that $K = \overline{K}$ is an algebraically closed field of characteristic zero. If R/I is level and the WLP fails then $a+b+c \equiv 0$ (mod 3) and $\alpha+\beta+\gamma \equiv 0$ (mod 3).*

Proof. Since R/I is level, by Proposition 6.1 we can write $a = \alpha + t$, $b = \beta + t$, $c = \gamma + t$ for some $t \geq 1$. By Theorem 6.2, we have

$$2(\alpha+\beta+\gamma) + 3t \equiv 0 \pmod 3.$$

It follows that $\alpha+\beta+\gamma \equiv 0 \pmod 3$, so again by Theorem 6.2 we also get $a+b+c \equiv 0 \pmod 3$. $\square$

Remark 6.4. The proof of Theorem 6.2 applies not only to monomial ideals. Indeed, for any almost complete intersection in $R = K[x,y,z]$, if the WLP fails then $\sum_{i=1}^4 d_i \equiv 0 \pmod 3$, where d_1, d_2, d_3, d_4 are the degrees of the minimal generators.

A very interesting class of ideals is the following, recalling the notation introduced in (3.1).

Corollary 6.5. *The algebra $A = R/I_{3,k,3}$ has the following properties.*

(a) *the socle degree is $e = 2k-2$.*
(b) *the peak of the Hilbert function occurs in degrees $k-1$ and k, and has value $3k-3$.*
(c) *The corresponding inverse system is $(x^{k-1}y^{k-1}, x^{k-1}z^{k-1}, y^{k-1}z^{k-1})$.*
(d) *Assuming $\operatorname{char} K \neq 2$, the WLP fails if and only if k is odd. Note that in this case $e \equiv 0$ (mod 4).*

Proof. Parts (a), (b) and (c) are immediate from Proposition 6.1. Part (d) is a special case of Theorem 3.3. $\square$

For the remainder of this section we focus on the WLP. If the non-pure monomial involves only two of the variables, then the ideal always has the WLP.

Lemma 6.6. *Adopt the above notation. If $\alpha = 0$ and K has characteristic zero, then R/I has the WLP.*

Proof. The assumption $\alpha = 0$ provides that R/I is isomorphic to $B \otimes C$, where $B = K[y,z]/(y^b, y^\beta z^\gamma, z^c)$ and $C = K[x]/(x^a)$. By Proposition 4.4 in [6], B and C have the Strong Lefschetz Property (SLP – see Example 8.1), hence A has the WLP by Proposition 2.2 of [7]. $\square$

If the non-pure monomial involves all three variables then, due to Theorem 6.1(iii), the ideal is level if and only if I is of the form

$$I_{\alpha,\beta,\gamma,t} = (x^{\alpha+t}, y^{\beta+t}, z^{\gamma+t}, x^\alpha y^\beta z^\gamma), \tag{6.2}$$

where $t \geq 1$ and, without loss of generality, $1 \leq \alpha \leq \beta \leq \gamma$.

Next, we analyze when the syzygy bundle of I is semistable. (The relevance of semistability to the WLP was introduced in [6] and generalized in [4].)

Lemma 6.7. *Assume that K is algebraically closed of characteristic zero. Then the syzygy bundle of I is semistable if and only if $\gamma \leq 2(\alpha+\beta)$ and $\frac{1}{3}(\alpha+\beta+\gamma) \leq t$.*

Proof. Brenner ([3], Corollary 7.3) shows that in general for an ideal $I = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$, the syzygy bundle is semistable if and only if

(i) $3\max\{a,b,c,\alpha+\beta+\gamma\} \leq a+b+c+\alpha+\beta+\gamma$, and

(ii) $\min\{\alpha+\beta+c, \alpha+b+\gamma, a+\beta+\gamma, a+b, a+c, b+a\} \geq \dfrac{a+b+c+\alpha+\beta+\gamma}{3}$.

Applying this to our ideal I condition (i) reads as

$$3\max\{\gamma+t, \alpha+\beta+\gamma\} \leq 2(\alpha+\beta+\gamma)+3t.$$

Hence, it is equivalent to

$$\gamma \leq 2(\alpha+\beta) \tag{6.3}$$

and

$$\frac{1}{3}(\alpha+\beta+\gamma) \leq t. \tag{6.4}$$

Condition (ii) reads in our case as

$$\min\{\alpha+\beta+\gamma+t, \alpha+\beta+t\} \geq t+\frac{2}{3}(\alpha+\beta+\gamma),$$

which is equivalent to

$$3t \geq 2\gamma-\alpha-\beta.$$

Using Inequality (6.3) one checks that the last condition is implied by Inequality (6.4). This completes the argument. □

Suppose we are given $1 \leq \alpha \leq \beta$ and want to choose γ, t such that the syzygy bundle of I is semistable. Then there is only a finite number of choices for γ since we must have $\beta \leq \gamma \leq 2(\alpha+\beta)$, whereas we have infinitely many choices for t as the only condition is $t \geq \frac{1}{3}(\alpha+\beta+\gamma)$.

If the syzygy bundle of I is not semistable, then R/I must have the WLP ([4], Theorem 3.3). Combining this with Corollary 6.3 and Lemma 6.7, our computer experiments suggest the following characterization of the presence of the WLP in characteristic zero.

Conjecture 6.8. *Let $I \subset R = K[x,y,z]$ be a level Artinian almost complete intersection, i.e., I is of the form*

$$(x^{\alpha+t}, y^{\beta+t}, z^{\gamma+t}, x^\alpha y^\beta z^\gamma),$$

where $t > 0$ and, without loss of generality, $0 \leq \alpha \leq \beta \leq \gamma$. Assume that K is an algebraically closed field of characteristic zero. Then:

(a) *R/I has the WLP if any of the following conditions is satisfied:*
  (i) *$\alpha = 0$,*
  (ii) *$\alpha+\beta+\gamma$ is not divisible by 3,*
  (iii) *$\gamma > 2(\alpha+\beta)$,*
  (iv) *$t < \frac{1}{3}(\alpha+\beta+\gamma)$.*

(b) *Assume that $1 \leq \alpha \leq \beta \leq \gamma \leq 2(\alpha+\beta)$, $\alpha+\beta+\gamma \equiv 0 \pmod 3$, and $t \geq \frac{1}{3}(\alpha+\beta+\gamma)$. Then R/I fails to have the WLP if and only if t is even and either of the following two conditions is satisfied:*
  (i) *α is even, $\alpha = \beta$ and $\gamma-\alpha \equiv 3 \pmod 6$;*
  (ii) *α is odd and*
  $$\alpha = \beta \text{ and } \gamma-\alpha \equiv 0 \pmod 6,$$
  or
  $$\beta = \gamma \text{ and } \gamma-\alpha \equiv 0 \pmod 3.$$

Furthermore, in all of the above cases, the Hilbert function has "twin peaks."

Note that part (a) is true by Lemma 6.6, Corollary 6.3, and Lemma 6.7. Part (b) will be discussed in the following section, including the fact that there are exactly two known counterexamples, which were pointed out to us by David Cook II after reading the first version of this paper.

7. A proof of half of Conjecture 6.8

We are going to establish sufficiency of the numerical conditions given in Conjecture 6.8(b) for failure of the WLP. First we establish the claim about the twin peaks of the Hilbert function.

Lemma 7.1. *Consider the ideal $I_{\alpha,\beta,\gamma,t} = (x^{\alpha+t}, y^{\beta+t}, z^{\gamma+t}, x^\alpha y^\beta z^\gamma)$ which (by Proposition 6.1) defines a level algebra. Assume that $t \geq \max\left\{\frac{2\gamma-\alpha-\beta}{3}, \frac{\alpha+\beta+\gamma}{3}\right\}$ and that $0 < \alpha \leq \beta \leq \gamma$. Assume furthermore that $\alpha+\beta+\gamma \equiv 0 \pmod 3$. Then the values of the Hilbert function of $R/I_{\alpha,\beta,\gamma,t}$ in degrees $\frac{2(\alpha+\beta+\gamma)}{3}+t-2$ and $\frac{2(\alpha+\beta+\gamma)}{3}+t-1$ are the same.*

Proof. By Proposition 6.1, we know the minimal free resolution of $R/I_{\alpha,\beta,\gamma,t}$, which we can use to compute the Hilbert function in any degree. We first claim that in the specified degrees, this computation has no contribution from the last and the penultimate free modules in the resolution. To do this, it is enough to check that the degree $\frac{2(\alpha+\beta+\gamma)}{3}+t-1$ component of any summand in the penultimate free module is zero. The first three summands correspond to the observation that

$$-\frac{\alpha}{3}-\frac{\beta}{3}-\frac{\gamma}{3}-1<0.$$

Since $\alpha \leq \beta \leq \gamma$ and $a = \alpha+t$, $b = \beta+t$, and $c = \gamma+t$, we have only to check that

$$\frac{2(\alpha+\beta+\gamma)}{3}+t-1-\alpha-t-\beta-t<0.$$

This is equivalent to the inequality on t in the hypotheses.

Now rather than explicitly computing the Hilbert functions in the two degrees, it is enough to express them as linear combinations of binomial coefficients and show that the difference is zero, using the formula $\binom{p}{2}-\binom{p-1}{2} = p-1$. This is a routine computation. ☐

Theorem 7.2. *Consider the level algebra $R/I_{\alpha,\beta,\gamma,t} = R/(x^{\alpha+t}, y^{\beta+t}, z^{\gamma+t}, x^\alpha y^\beta z^\gamma)$. We make the following assumptions:*

- $0 < \alpha \leq \beta \leq \gamma \leq 2(\alpha+\beta)$;
- $t \geq \dfrac{\alpha+\beta+\gamma}{3}$;
- $\alpha+\beta+\gamma \equiv 0 \pmod 3$.

Then there is a square matrix, M, with integer entries, having the following properties.

(a) *M is a $\left(t+\dfrac{\alpha+\beta-2\gamma}{3}\right) \times \left(t+\dfrac{\alpha+\beta-2\gamma}{3}\right)$ matrix.*

(b) *If $\det M \equiv 0 \pmod p$, where p is the characteristic of K, then $R/I_{\alpha,\beta,\gamma,t}$ fails to have the WLP. This includes the possibility that $\det M = 0$ as an integer.*

(c) *If $\det M \not\equiv 0 \pmod p$ then $R/I_{\alpha,\beta,\gamma,t}$ satisfies the WLP.*

Proof. We note first that the second bullet in the hypotheses implies (using the first bullet) that the following inequality also holds:

$$t > \frac{2\gamma-\alpha-\beta-3}{3}$$

We will use this fact without comment in this proof.

Thanks to Lemma 7.1, the values of the Hilbert function of $R/I_{\alpha,\beta,\gamma,t}$ in degrees $\frac{2(\alpha+\beta+\gamma)}{3}+t-2$ and $\frac{2(\alpha+\beta+\gamma)}{3}+t-1$ are the same. Hence thanks to Proposition 2.1, checking whether or not the WLP holds is equivalent to checking whether multiplication by a general linear form between these degrees is an isomorphism or not.

We will use Proposition 2.5. Let L be a general linear form, let $\bar{R} = R/(L) \cong K[x,y]$ and let $\bar{I}$ be the image of $I_{\alpha,\beta,\gamma,t}$ in $\bar{R}$. Note that $\bar{I} \cong (x^{\alpha+t}, y^{\beta+t}, \ell^{\gamma+t}, x^\alpha y^\beta \ell^\gamma)$, where ℓ is the restriction to $\bar{R}$ of z, and thanks to Lemma 2.2 we will take $\ell = x+y$.

Of course $x^{\alpha+t}, y^{\beta+t}$ is a regular sequence. Hence it suffices to check whether or not there is an element $F \in \bar{R}$ of degree

$$f := \alpha+t+\beta+t-\left[\frac{2(\alpha+\beta+\gamma)}{3}+t-1+2\right] = \frac{\alpha+\beta-2\gamma}{3}+t-1 = \frac{\alpha+\beta+\gamma}{3}-\gamma+t-1,$$

non-zero modulo $(x^{\alpha+t}, y^{\beta+t})$, such that $F\cdot\bar{I} \subset (x^{\alpha+t}, y^{\beta+t})$. The latter condition is equivalent to

$$F\cdot(x+y)^{\gamma+t} \in (x^{\alpha+t}, y^{\beta+t}) \text{ and } F\cdot x^\alpha y^\beta (x+y)^\gamma \in (x^{\alpha+t}, y^{\beta+t}). \tag{7.1}$$

Claim:

(1) $\gamma \geq 2(\alpha+\beta)$ *if and only if* $F\cdot(x+y)^{\gamma+t}$ *is automatically in* $(x^{\alpha+t}, y^{\beta+t})$.

(2) $t \leq \dfrac{\alpha+\beta+\gamma}{3}$ *if and only if* $F\cdot x^\alpha y^\beta(x+y)^\gamma$ *is automatically in* $(x^{\alpha+t}, y^{\beta+t})$.

Indeed, the first inequality is equivalent to $\deg(F\cdot(x+y)^{\gamma+t}) \geq (\alpha+t)+(\beta+t)-1$, so every term of $F\cdot(x+y)^{\gamma+t}$ is divisible by either $x^{\alpha+t}$ or $y^{\beta+t}$. The second inequality is equivalent to $\deg(F\cdot x^\alpha y^\beta(x+y)^\gamma) \geq (\alpha+t)+(\beta+t)-1$. This establishes the claim. Thanks to our hypotheses, then, the conditions in (7.1) add constraints on the possibilities for F. We want to count these constraints.

Let $F = \lambda_0 x^f + \lambda_1 x^{f-1}y + \lambda_2 x^{f-2}y^2 + \cdots + \lambda_{f-2}x^2y^{f-2} + \lambda_{f-1}xy^{f-1} + \lambda_f y^f$. We now consider how many conditions (7.1) imposes on the λ_i. Consider the first product, which has degree $\frac{\alpha+\beta+\gamma}{3}+2t-1$. A typical term in $F\cdot(x+y)^{\gamma+t}$ is some scalar times x^iy^j, where $i+j = \frac{\alpha+\beta+\gamma}{3}+2t-1$. The set of all pairs (i,j) for which x^iy^j is not in $(x^{\alpha+t}, y^{\beta+t})$ is

$$\{(i,j)\} = \left\{\left(\alpha+t-1, \frac{\beta+\gamma-2\alpha}{3}+t\right), \dots, \left(\frac{\alpha+\gamma-2\beta}{3}+t, \beta+t-1\right)\right\} \tag{7.2}$$

Since each such term has to vanish, this imposes a total of $\frac{2\alpha+2\beta-\gamma}{3}$ conditions on the λ_i. Similarly, consider the second product, which has degree $\frac{4\alpha+4\beta+\gamma}{3}+t-1$. A typical term in $F\cdot x^\alpha y^\beta(x+y)^\gamma$ is some scalar times x^iy^j, where $i+j = \frac{4\alpha+4\beta+\gamma}{3}+t-1$. The set of all pairs (i,j) for which x^iy^j is not in the ideal $(x^{\alpha+t}, y^{\beta+t})$ is

$$\{(i,j)\} = \left\{\left(\alpha+t-1, \frac{\alpha+4\beta+\gamma}{3}\right), \dots, \left(\frac{4\alpha+\beta+\gamma}{3}, \beta+t-1\right)\right\} \tag{7.3}$$

This imposes a total of $t - \frac{\alpha+\beta+\gamma}{3}$ conditions, since we need all of these terms to vanish. Combining, we have a total of $t+\frac{\alpha+\beta-2\gamma}{3} = f+1$ conditions. Since there are $f+1$ variables λ_i, the coefficient matrix is the desired square matrix. Now it is clear that $\det M = 0$ (regardless of the characteristic) if and only of the corresponding homogeneous system has a non-trivial solution, i.e. there is a polynomial F as desired, if and only if $R/I_{\alpha,\beta,\gamma,t}$ fails to have the WLP. $\square$

We can specifically give the matrix described in the last result.

Corollary 7.3. *The matrix in Theorem 7.2 has the form*

$$M = \begin{bmatrix} \binom{\gamma}{\frac{\alpha+\beta+\gamma}{3}} & \binom{\gamma}{\frac{\alpha+\beta+\gamma}{3}-1} & \cdots & \binom{\gamma}{\gamma-t+2} & \binom{\gamma}{\gamma-t+1} \\ & & \vdots & & \\ \binom{\gamma}{t-1} & \binom{\gamma}{t-2} & \cdots & \binom{\gamma}{\frac{2\gamma-\alpha-\beta}{3}+1} & \binom{\gamma}{\frac{2\gamma-\alpha-\beta}{3}} \\ \binom{\gamma+t}{t+\beta-1} & \binom{\gamma+t}{t+\beta-2} & \cdots & \binom{\gamma+t}{\frac{2(\beta+\gamma)-\alpha}{3}+1} & \binom{\gamma+t}{\frac{2(\beta+\gamma)-\alpha}{3}} \\ \binom{\gamma+t}{t+\beta-2} & \binom{\gamma+t}{t+\beta-3} & \cdots & \binom{\gamma+t}{\frac{2(\beta+\gamma)-\alpha}{3}} & \binom{\gamma+t}{\frac{2(\beta+\gamma)-\alpha}{3}-1} \\ & & \vdots & & \\ \binom{\gamma+t}{t+\frac{\beta+\gamma-2\alpha}{3}} & \binom{\gamma+t}{t-1+\frac{\beta+\gamma-2\alpha}{3}} & \cdots & \binom{\gamma+t}{\gamma-\alpha+2} & \binom{\gamma+t}{\gamma-\alpha+1} \end{bmatrix}$$

Proof. This is a tedious computation, but is based entirely on the proof of Theorem 7.2. The top "half" of the matrix corresponds to the second product in (7.1), and the bottom "half" of the matrix corresponds to the first product. Each row in the top "half" corresponds to one ordered pair in (7.3), and each row in the bottom "half" corresponds to one ordered pair in (7.2). $\square$

The following corollary establishes the sufficiency of the numerical conditions given in Conjecture 6.8.

Corollary 7.4. *Let K be an arbitrary field and $R = K[x, y, z]$. Consider the ideal $I_{\alpha,\beta,\gamma,t} = (x^{\alpha+t}, y^{\beta+t}, z^{\gamma+t}, x^\alpha y^\beta z^\gamma)$, where $1 \le \alpha \le \beta \le \gamma$. Assume that one of the following three cases holds:*

(1) $(\alpha, \beta, \gamma, t) = (\alpha, \alpha, \alpha + 3\lambda, t)$ *with α even, λ odd, $t \ge \alpha + \lambda$ even and $1 \le \lambda \le \alpha$;*
(2) $(\alpha, \beta, \gamma, t) = (\alpha, \alpha, \alpha + 6\mu, t)$ *with α odd, $t \ge \alpha + 2\mu$ even, and $0 \le \mu \le \frac{\alpha-1}{2}$; or*
(3) $(\alpha, \beta, \gamma, t) = (\alpha, \alpha + 3\rho, \alpha + 3\rho, t)$ *with α odd, $t \ge \alpha + 2\rho$ even, and $\rho \ge 0$.*

Then $R/I_{\alpha,\beta,\gamma,t}$ fails to have the WLP.

Proof. Possibly after an extension of the base field, we may assume that K is an infinite field. One can verify quickly (using the constraints on the invariants given in the theorem) that the hypotheses of Theorem 7.2 hold here in all three cases (the parity is important in some instances). Hence it is only a matter of identifying M, via Corollary 7.3, and checking that in all the cases mentioned, $\det M = 0$. We first consider Case (1).

By applying Corollary 7.3, we obtain the $(t-2\lambda)\times(t-2\lambda)$ matrix M below, corresponding to $t-2\lambda$ homogeneous equations in $t-2\lambda$ unknowns.

$$M = \begin{bmatrix} \binom{\alpha+3\lambda}{\alpha+\lambda} & \binom{\alpha+3\lambda}{\alpha+\lambda-1} & \cdots & \binom{\alpha+3\lambda}{\alpha-t+\lambda+2} & \binom{\alpha+3\lambda}{\alpha-t+\lambda+1} \\ \binom{\alpha+3\lambda}{\alpha+\lambda+1} & \binom{\alpha+3\lambda}{\alpha+\lambda} & \cdots & \binom{\alpha+3\lambda}{\alpha-t+\lambda+3} & \binom{\alpha+3\lambda}{\alpha-t+\lambda+2} \\ & & \vdots & & \\ \binom{\alpha+3\lambda}{t-1} & \binom{\alpha+3\lambda}{t-2} & \cdots & \binom{\alpha+3\lambda}{2\lambda+1} & \binom{\alpha+3\lambda}{2\lambda} \\ \binom{\alpha+t+3\lambda}{t+\lambda} & \binom{\alpha+t+3\lambda}{\lambda-1} & \cdots & \binom{\alpha+t+3\lambda}{3\lambda+2} & \binom{\alpha+t+3\lambda}{3\lambda+1} \\ \binom{\alpha+t+3\lambda}{t+\lambda+1} & \binom{\alpha+t+3\lambda}{t+\lambda} & \cdots & \binom{\alpha+t+3\lambda}{3\lambda+3} & \binom{\alpha+t+3\lambda}{3\lambda+2} \\ & & \vdots & & \\ \binom{\alpha+t+3\lambda}{\alpha+t-1} & \binom{\alpha+t+3\lambda}{\alpha+t-2} & \cdots & \binom{\alpha+t+3\lambda}{\alpha+2\lambda+1} & \binom{\alpha+t+3\lambda}{\alpha+2\lambda} \end{bmatrix}$$

This system has a non-trivial solution (giving the existence of the desired form F) if and only if M has determinant zero.

We will show that under our assumptions, this determinant is indeed zero. Observe that if M is flipped about the central vertical axis, and then the top portion and the bottom are (separately) flipped about their respective central horizontal axes, then we restore the matrix M. Since $t-2\lambda$ is even, the first step can be accomplished with $\frac{t-2\lambda}{2}$ interchanges of columns. The top portion contains $t-\lambda-\alpha$ rows and the bottom portion contains $\alpha-\lambda$ rows. Both are odd, so the second step can be done with $\frac{t-\lambda-1-\alpha}{2}$ interchanges and the last one with $\frac{\alpha-\lambda-1}{2}$ interchanges. All together we have $t-2\lambda-1$ interchanges of rows/columns, which is an odd number. Therefore $\det M = -\det M$, and so $\det M = 0$.

Case (2) is similar and is left to the reader. Case (3), however, is somewhat different. Now we will restrict to $K[y,z]$ rather than $K[x,y]$. We obtain

$$\bar{I} = ((y+z)^{\alpha+t}, y^{\alpha+3\rho+t}, z^{\alpha+3\rho+t}, (y+z)^{\alpha}\cdot y^{\alpha+3\rho}z^{\alpha+3\rho})$$

and we have to check whether there is a form $F \in K[y,z]$ of degree $t+2\rho-1$ (obtained after a short calculation) such that

$$F\cdot(y+z)^{\alpha+t} \in (y^{\alpha+3\rho+t}, z^{\alpha+3\rho+t}) \text{ and } F\cdot(y+z)^{\alpha}\cdot y^{\alpha+3\rho}z^{\alpha+3\rho} \in (y^{\alpha+3\rho+t}, z^{\alpha+3\rho+t}).$$

The calculations again follow the ideas of Theorem 7.2, and we obtain the following $(t+2\rho)\times(t+2\rho)$ matrix of integers:

$$
\begin{bmatrix}
0 & 0 & \binom{\alpha}{0} & \binom{\alpha}{1} & \dots & \dots & \binom{\alpha}{\alpha} & 0 & 0 & \dots & 0 & \dots & 0 \\
 & & \vdots & & \vdots & \vdots & & & & \vdots & & \vdots & \\
0 & 0 & 0 & 0 & \binom{\alpha}{0} & \binom{\alpha}{1} & \dots & \dots & \dots & \binom{\alpha}{\alpha} & 0 & \dots & 0 \\
\binom{\alpha+t}{\alpha+\rho} & \binom{\alpha+t}{\alpha+\rho+1} & \dots & \dots & \binom{\alpha+t}{\alpha+t} & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
\binom{\alpha+t}{\alpha+\rho-1} & \binom{\alpha+t}{\alpha+\rho} & \dots & \dots & \binom{\alpha+t}{\alpha+t-1} & \binom{\alpha+t}{\alpha+t} & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\
 & & \vdots & & & \vdots & & & \vdots & & \vdots & & \\
\binom{\alpha+t}{0} & \binom{\alpha+t}{1} & \dots & \dots & \dots & \dots & \dots & \binom{\alpha+t}{\alpha+t} & 0 & \dots & \dots & \dots & 0 \\
0 & \binom{\alpha+t}{0} & \dots & \dots & \dots & \dots & \dots & \binom{\alpha+t}{\alpha+t-1} & \binom{\alpha+t}{\alpha+t-1} & \dots & \dots & \dots & 0 \\
 & & \vdots & & & \vdots & & & \vdots & & & \vdots & \\
0 & 0 & 0 & \binom{\alpha+t}{0} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \binom{\alpha+t}{t-\rho}
\end{bmatrix}
$$

We remark that in the top portion (i.e. the submatrix where the binomial coefficients have α as the top entry), the first row has a sequence of 2ρ zeroes before the $\binom{\alpha}{0}$, and the last row (of the top portion) has a sequence of 2ρ zeroes after the $\binom{\alpha}{\alpha}$. The top portion has $t-2\rho-\alpha$ rows, while the bottom portion (with binomial coefficients having $\alpha+t$ as top entry) has $\alpha+4\rho$ rows. In the same way as before (using the fact that t is even), it is easy to see that this matrix can be restored to itself with an odd number of row and column interchanges, and hence the determinant is zero. $\square$

Remark 7.5. Notice that to check the surjectivity of the multiplication by a linear form from degree $d-1$ to degree d, we have to check whether or not $(R/(I,L))_d$ is zero. Because of this, it is possible to obtain the result of Theorem 7.4 (including exactly the same matrix M) with a more direct computation, rather than using the liaison approach of Proposition 2.5. However, the computations seemed slightly more intricate, and we also felt that the existence of the form F might have other interesting applications.

Corollary 7.6. *Consider the level algebras of the form R/I with $I=(x^k,y^k,z^k,x^\alpha y^\alpha z^\alpha)$, α odd and $k\geq 2\alpha+1$ odd. Then R/I is level and fails to have the WLP.*

Example 7.7. Consider an ideal of the form $I=(x^{10},y^{10},z^{10},x^3y^3z^3)$ (i.e. we relax the condition in Corollary 7.6 that k be odd). Then $\det M = 78{,}408 = 2^3\cdot 3^4\cdot 11^2$. One can check on a computer program (e.g. CoCoA [5]) that in characteristic 2, 3 and 11, R/I_{10} does not have the WLP, while in characteristic 5, 7, 13, 17, $\dots$, R/I does have the WLP (as predicted by Theorem 7.2).

Corollary 7.8. *For any even socle degree there is a level monomial almost complete intersection which fails to have the WLP.*

Proof. In Corollary 7.6, simply consider the special cases $\alpha=1$ and $\alpha=3$. ¿From Proposition 6.1 we note that in the first case the socle degree is $2k-2$ and in the second case the socle degree is $2k$. $\square$

The following is a natural question to ask at this point:

Question 7.9. Is there is a monomial level almost complete intersection (or indeed any almost complete intersection) in three variables with odd socle degree and failing to have the WLP?

We now address this question in characteristic p. We begin with a simple example:

Example 7.10. Fix any prime p and consider the complete intersection $\mathfrak{a} = (x^p, y^p, z^p)$ in $R = K[x, y, z]$, where K has characteristic p. Note that $R/\mathfrak{a}$ has socle degree $3p-3$ and fails to have the WLP, since for a general linear form L, L^{p-1} is in the kernel of $(\times L)$. (This was observed for $p = 2$ in [6], Remark 2.9, and in [11], Remark 2.6, in arbitrary characteristic.) Now consider the ideal $I = (\mathfrak{a}, x^{p-1}y^{p-1}z^{p-1})$. This clearly is an almost complete intersection with socle degree $3p-4$ (an odd number), is level, and still fails to have the WLP.

Notice that similar examples exist whenever the number of variables is at least three.

This leads to the following refinement of our question:

Question 7.11. For fixed characteristic p, what odd socle degrees can occur for almost complete intersections without the WLP?

Obviously quotients of the ideal $\mathfrak{a}$ cannot give us any examples for socle degree $> 3p-3$, so we have to look to different powers of the variables. In [6] Remark 2.9 it was observed that in characteristic 2 the ideal (x^4, y^4, z^4) also fails to have the WLP. We are led to consider other powers, and we ask the following natural question.

Question 7.12. Given a prime p, consider ideals (x^k, y^k, z^k) in characteristic p. For which values of k does $R/(x^k, y^k, z^k)$ fail to have the WLP?

In characteristic zero, on the other hand, the situation seems to be different. David Cook II (University of Kentucky) has pointed out to us that when $(\alpha, \beta, \gamma, t)$ take the values $(2, 9, 13, 9)$ or $(3, 7, 14, 9)$, the corresponding algebras fail to possess WLP, even though these are not predicted by Conjecture 6.8. We subsequently conducted a computer search using CoCoA, and have found that these are the *only* counterexamples to the conjecture when the invariants α, β, γ and t are all ≤ 60.

We thus conjecture that the converse of Corollary 7.4 holds for all but these two exceptions, which is the only missing piece in characterizing the level monomial ideals in $K[x, y, z]$ that fail to have the WLP and establishing Conjecture 6.8.

Conjecture 7.13. *Assume that K has characteristic zero and that $(\alpha, \beta, \gamma, t) \neq (2, 9, 13, 9)$ or $(3, 7, 14, 9)$. Using the notation of Corollary 7.4, if $R/I_{\alpha,\beta,\gamma,t}$ fails to have the WLP then one of cases (1), (2) or (3) holds.*

Remark 7.14. According to Lemma 7.1, the Hilbert function of $R/I_{\alpha,\beta,\gamma,t}$ agrees in degrees $s = \frac{2(\alpha+\beta+\gamma)}{3} + t - 2$ and $s + 1$. Hence, by Proposition 2.1, Conjecture 7.13 and thus also Conjecture 6.8 is proven, if one shows that the multiplication map $(R/I_{\alpha,\beta,\gamma,t})_s \to (R/I_{\alpha,\beta,\gamma,t})_{s+1}$ by $x + y + z$ is injective or surjective, provided the conditions in (1), (2) or (3) of Corollary 7.4 all fail to hold.

Remark 7.15. We have been focusing on monomial complete intersections. A natural question is whether a "general" complete intersection of height three in characteristic p necessarily has the WLP even when the monomial one does not. If the field K is at least infinite, this is answered in the affirmative by the main result of [1].

8. Final Comments

In Section 4 we saw that for the ideal $I = (x_1^r, \dots, x_r^r, x_1 \cdots x_r) \subset R = K[x_1, \dots, x_r]$, the corresponding algebra R/I fails to have the WLP. On the other hand, we saw in Section 5 that making a very slight change to even one of the generators gave an algebra with the same Hilbert function, but possessing the WLP. In this section we analyze related phenomena and pose related questions.

Example 8.1. Stanley [12] and Watanabe [14] showed that a monomial complete intersection satisfies the *Strong Lefschetz Property (SLP)*. This property is a generalization of the WLP, and says that if L is a general linear form then for any i and any d, the multiplication $\times L^d : (R/I)_i \to (R/I)_{i+d}$ has maximal rank. This means that the algebra $R/(I, L^d)$ has the "expected" Hilbert function. By semicontinuity, the same is true when L^d is replaced by a general form of degree d.

It is of interest to find behavior of R/I that distinguishes multiplication by L^d from that by a general form of degree d. We have found such a phenomenon experimentally on CoCoA [5], although we have not given a theoretical justification. Let $R = K[x_1, x_2, x_3, x_4]$ and let $L \in R$ be a general linear form. Consider the ideals

$$\begin{aligned} I_N &= (x_1^N, x_2^N, x_3^N, x_4^N, L^N); \\ J_N &= (x_1^N, x_2^N, x_3^N, x_4^N, G) \end{aligned}$$

where G is a general form of degree N. By the above-cited result, R/I_N and R/J_N have the same Hilbert function, and it can be checked that in fact they have the same minimal graded Betti numbers. However, these algebras often have different behavior with respect to the WLP and with respect to minimal free resolutions! More precisely, we have the following experimental data, which we computed in CoCoA over the rational numbers.

N	I_N has the WLP?	J_N has the WLP?	Same resolution?
2	yes	yes	yes
3	no	yes	no
4	no	yes	yes
5	no	yes	yes
6	no	yes	yes
7	no	yes	no
8	no	yes	yes
9	no	yes	yes
10	no	yes	yes
11	no	yes	no
12	no	yes	?

Question 8.2. We end by posing some natural questions that remain to be addressed (in addition to the conjectures posed earlier).

(1) Do all the classes of algebras studied in this paper have unimodal Hilbert functions?

(2) For $r = 3$ we have the following questions. (Here "ACI" means "almost complete intersection.")
   (a) Are monomial ideals the only ACI's that fail to have the WLP?
   (b) Are there level ACI's without the WLP and having odd socle degree?
   (c) Do there exist ACI's that fail to have the WLP and are not level?

(d) Does every monomial ideal (not necessarily ACI) that is level of type two have the WLP?

(3) Do all ACI's have a unimodal Hilbert function?

References

[1] D. Anick, *Thin algebras of embedding dimension three*, J. Algebra **100** (1986), 235–259.
[2] M. Beintema, *A note on Artinian Gorenstein algebras defined by monomials*, Rocky Mountain J. Math. **23** (1993), 1–3.
[3] H. Brenner, *Looking out for stable syzygy bundles*, Adv. Math. **219** (2008), 401–427.
[4] H. Brenner and A. Kaid, *Syzygy bundles on* $\mathbb{P}^2$ *and the Weak Lefschetz Property*, Illinois J. Math. **51** (2007), 1299–1308.
[5] CoCoA: a system for doing Computations in Commutative Algebra, Available at `http://cocoa.dima.unige.it`.
[6] T. Harima, J. Migliore, U. Nagel and J. Watanabe, *The Weak and Strong Lefschetz Properties for Artinian* K*-Algebras*, J. Algebra **262** (2003), 99-126.
[7] J. Herzog and D. Popescu, *The strong Lefschetz property and simple extensions*, arXiv:math.AC/0506537.
[8] J. Kleppe, R. Miró-Roig, J. Migliore, and C. Peterson *Gorenstein liaison, complete intersection liaison invariants and unobstructedness*, Mem. Amer. Math. Soc. **154** (2001), no. 732.
[9] J. Migliore, "Introduction to Liaison Theory and Deficiency Modules," Progress in Mathematics **165**, Birkhäuser, 1998.
[10] J. Migliore and R. Miró-Roig, *Ideals of general forms and the ubiquity of the Weak Lefschetz Property*, J. Pure Appl. Algebra **182** (2003), 79–107.
[11] J. Migliore, U. Nagel and F. Zanello, *A characterization of Gorenstein Hilbert functions in codimension four with small initial degree*, Math. Res. Lett. **15** (2008), 331-349.
[12] R. Stanley, *Weyl groups, the hard Lefschetz theorem, and the Sperner property*, SIAM J. Algebraic Discrete Methods **1** (1980), 168–184.
[13] R. Stanley, *The number of faces of a simplicial convex polytope*, Adv. Math. **35** (1980), 236–238.
[14] J. Watanabe, *The Dilworth number of Artinian rings and finite posets with rank function*, Commutative Algebra and Combinatorics, Advanced Studies in Pure Math. Vol. 11, Kinokuniya Co. North Holland, Amsterdam (1987), 303–312.

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA
E-mail address: `Juan.C.Migliore.1@nd.edu`

Facultat de Matemàtiques, Department d'Algebra i Geometria, Gran Via des les Corts Catalanes 585, 08007 Barcelona, SPAIN
E-mail address: `miro@ub.edu`

Department of Mathematics, University of Kentucky, 715 Patterson Office Tower, Lexington, KY 40506-0027, USA
E-mail address: `uwenagel@ms.uky.edu`