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MONOMIAL IDEALS WHOSE POWERS HAVE A LINEAR RESOLUTION

JÜRGEN HERZOG, TAKAYUKI HIBI and XINXIAN ZHENG

Introduction

In this paper we consider graded ideals in a polynomial ring over a field and ask when such an ideal has the property that all of its powers have a linear resolution. Recall that a graded module M is said to have a linear resolution if all entries in the matrices representing the differentials in a graded minimal free resolution of M are linear forms. If an ideal I has a linear resolution, then necessarily all generators of I have the same degree, say t. In that case, one also says that I has a t-linear resolution.

It is known [7] that polymatroidal ideals have linear resolutions and that powers of polymatroidal ideals are again polymatroidal (see [2] and [8]). In particular they have again linear resolutions. In general however, powers of ideals with linear resolution need not to have linear resolutions. The first example of such an ideal was given by Terai. He showed that over a base field of characteristic $\neq 2$ the Stanley Reisner ideal $I = (abd, abf, ace, adc, aef, bde, bcf, bce, cdf, def)$ of the minimal triangulation of the projective plane has a linear resolution, while I^2 has no linear resolution. The example depends on the characteristic of the base field. If the base field has characteristic 2, then I itself has no linear resolution.

Another example, namely $I = (def, cef, cdf, cde, bef, bcd, acf, ade)$ is given by Sturmfels [13]. Again I has a linear resolution, while I^2 has no linear resolution. The example of Sturmfels is interesting because of two reasons: 1. it does not depend on the characteristic of the base field, and 2. it is a linear quotient ideal. Recall that an equigenerated ideal I is said to have *linear quotients* if there exists an order $f_1, \dots, f_m$ of the generators of I such that for all $i = 1, \dots, m$ the colon ideals $(f_1, \dots, f_{i-1}) : f_i$ are generated by linear forms. It is quite easy to see that such an ideal has a linear resolution (independent on the characteristic of the base field). However the example of Sturmfels also shows that powers of a linear quotient ideal need not to be again

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linear quotient ideals.

On the other hand it is known (see [3] and [9]) that the regularity of powers I^n of a graded ideal I is bounded by a linear function $an + b$, and is a linear function for large n. For ideals I whose generators are all of degree d one has the bound $\operatorname{reg}(I^n) \leq nd + \operatorname{reg}_x(R(I))$, as shown by Römer [12]. Here $R(I)$ is the Rees ring of I which is naturally bigraded, and $\operatorname{reg}_x(R(I))$ is the x-regularity of $R(I)$. The definition of x-regularity is given in Section 1. It follows from this formula that each power of I has a linear resolution if $\operatorname{reg}_x(R(I)) = 0$.

In this paper we will show (Theorem 3.2) that if $I \subset K[x_1, \ldots, x_n]$ is a monomial ideal with 2-linear resolution, then each power has a linear resolution. Our proof is based on the formula of Römer. In the first section we give a new and very short proof of his result, and remark that if there is a term order such that the initial ideal of the defining ideal P of the Rees ring $R(I)$ is generated by monomials which are linear in the variables $x_1, \ldots, x_n$, then $\operatorname{reg}_x(R(I)) = 0$.

In Section 2 we recall a result of Fröberg [6] where he gives a combinatorial characterization of squarefree monomial ideals 2-linear resolution. A squarefree monomial ideal I may be viewed as the edge ideal of a graph G. By Fröberg, I has a linear resolution if and only if the complementary graph $\overline{G}$ is chordal. There is an interesting characterization of chordal graphs due to G. A. Dirac [4]. He showed that a graph is chordal if and only if it is the 1-skeleton of a quasi-tree. This characterization is essential for us in order to define the right lexicographical term order for which the initial ideal of P is linear in the x variables. We show this in the last section and use a description of the Graver basis of the egde ring of a graph due to Oshugi and Hibi [10]. Based on the same ideas and using polarization we also can treat monomial ideals which are not necessarily squarefree.

1. The x-condition

Let K be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring, $I \subset S$ an equigenerated graded ideal, that is, a graded ideal whose generators $f_1, \ldots, f_m$ are all of same degree d. Then the Rees ring

$$R(I) = \bigoplus_{j \geq 0} I^j t^j = S[f_1 t, \ldots, f_m t] \subset S[t]$$

is naturally bigraded with $\deg(x_i) = (1, 0)$ for $i = 1, \ldots, n$ and $\deg(f_i t) = (0, 1)$ for $i = 1, \ldots, m$.

Let $T = S[y_1, \ldots, y_m]$ be the polynomial ring over S in the variables $y_1, \ldots, y_m$. We define a bigrading on T by setting $\deg(x_i) = (1, 0)$ for $i =$

$1, \dots, n$, and $\deg(y_j) = (0, 1)$ for $j = 1, \dots, m$. Then there is a natural surjective homomorphism of bigraded K-algebras $\varphi\colon T \to R(I)$ with $\varphi(x_i) = x_i$ for $i = 1, \dots, n$ and $\varphi(y_j) = f_j t$ for $j = 1, \dots, m$.

Let

$$F_\bullet : 0 \to F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R(I) \to 0$$

be the bigraded minimal free T-resolution of $R(I)$. Here $F_i = \bigoplus_j T(-a_{ij}, -b_{ij})$ for $i = 0, \dots, p$. The *x-regularity* of $R(I)$ is defined to be the number

$$\operatorname{reg}_x(R(I)) = \max_{i,j}\{a_{ij} - i\}.$$

With the notation introduced one has the following result [12, Theorem 5.3(i)] of Römer.

THEOREM 1.1. $\operatorname{reg}(I^n) \le nd + \operatorname{reg}_x(R(I))$. *In particular, if* $\operatorname{reg}_x(R(I)) = 0$, *then each power of I admits a linear resolution.*

For the reader's convenience we give a simple proof of this theorem: For all n, the exact sequence $F_\bullet$ gives the exact sequence of graded S-modules

$$(1)\quad 0 \to (F_p)_{(*,n)} \longrightarrow (F_{p-1})_{(*,n)} \longrightarrow \cdots \longrightarrow (F_1)_{(*,n)} \longrightarrow (F_0)_{(*,n)} \longrightarrow R(I)_{(*,n)} \to 0.$$

We note that $R(I)_{(*,n)} = I^n(-dn)$, and that $T(-a, -b)_{(*,n)}$ is isomorphic to the free S-module $\bigoplus_{|u|=n-b} S(-a)y^u$. It follows that (1) is a (possibly non-minimal) graded free S-resolution of $I^n(-dn)$. This yields at once that $\operatorname{reg}(I^n(-dn)) \le \operatorname{reg}_x(R(I))$, and thus $\operatorname{reg}(I^n) \le nd + \operatorname{reg}_x(R(I))$.

We say that I satisfies the x-condition if $\operatorname{reg}_x(R(I)) = 0$.

COROLLARY 1.2. *Let $I \subset S$ be an equigenerated graded ideal, and let $R(I) = T/P$. Then each power of I has a linear resolution if for some term order $<$ on T the defining ideal P has a Gröbner basis G whose elements are at most linear in the variables $x_1, \dots, x_n$, that is,* $\deg_x(f) \le 1$ *for all* $f \in G$.

PROOF. The hypothesis implies that $\operatorname{in}(P)$ is generated by monomials $u_1, \dots, u_m$ with $\deg_x(u_i) \le 1$. Let $C_\bullet$ be the Taylor resolution of $\operatorname{in}(P)$. The module C_i has the basis e_σ with $\sigma = \{j_1 < i_2 < \cdots < j_i\} \subset [m]$. Each basis element e_σ has the multidegree (a_σ, b_σ) where $x^{a_\sigma} y^{b_\sigma} = \operatorname{lcm}\{u_{j_1}, \dots, u_{j_m}\}$. It follows that $\deg_x(e_\sigma) \le i$ for all $e_\sigma \in C_i$. Since the shifts of $C_\bullet$ bound the shifts of a minimal multigraded resolution of $\operatorname{in}(P)$, we conclude that $\operatorname{reg}_x(T/\operatorname{in}(P)) = 0$. On the other hand, by semi-continuity one always has $\operatorname{reg}_x(T/P) \le \operatorname{reg}_x(T/\operatorname{in}(P))$.

2. Monomial ideals with 2-linear resolution

Let K be a field and $I \subset S = K[x_1, \dots, x_n]$ be a squarefree monomial ideal generated in degree 2. We may attach to I a graph G whose vertices are the elements of $[n] = \{1, \dots, n\}$, and $\{i, j\}$ is an edge of G if and only if $x_i x_j \in I$. The ideal I is called the *edge ideal* of G and denoted $I(G)$. Thus the assignment $G \mapsto I(G)$ establishes a bijection between graphs and squarefree monomial ideals generated in degree 2.

The *complementary graph* $\overline{G}$ of G is the graph whose vertex set is again $[n]$ and whose edges are the non-edges of G. A graph G is called *chordal* if each cycle of length > 3 has a chord.

We recall the following result of Fröberg [6, Theorem 1] (see also [14])

THEOREM 2.1 (Fröberg). *Let G be graph. Then $I(G)$ has a linear resolution if and only if $\overline{G}$ is chordal.*

For our further considerations it is important to have a characterization of chordal graphs which is due to Dirac [4]: let Δ be simplicial complex, and denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. A facet $F \in \mathcal{F}(\Delta)$ is called a *leaf* if either F is the only facet of Δ, or there exists $G \in \mathcal{F}(\Delta)$, $G \neq F$ such that $H \cap F \subset G \cap F$ for each $H \in \mathcal{F}(\Delta)$ with $H \neq F$. A vertex i of Δ is called a *free vertex* if i belongs to precisely one facet.

Faridi [5] calls Δ a tree if each simplicial complex generated by a subset of the facets of Δ has leaf, and Zheng [15] calls Δ a *quasi-tree* if there exists a labeling $F_1, \dots, F_m$ of the facets such that for all i the facet F_i is a leaf of the subcomplex $\langle F_1, \dots, F_i \rangle$. We call such a labeling a *leaf order*. It is obvious that any tree is a quasi-tree, but the converse is not true. For us however the quasi-trees are important, because of

THEOREM 2.2 (Dirac, [4] Theorem 1 and 2). *A graph G is chordal if and only if G is the 1-skeleton of a quasi-tree.*

As a consequence of Theorem 2.1 and 2.2 we obtain

PROPOSITION 2.3. *Let $I \subset S = K[x_1, \dots, x_n]$ be a squarefree monomial ideal with 2-linear resolution. Then after suitable renumbering of the variables we have: if $x_i x_j \in I$ with $i \neq j$, $k > i$ and $k > j$, then either $x_i x_k$ or $x_j x_k$ belongs to I.*

PROOF. We consider I as the egde ideal of the graph G. Then by Theorem 2.1 and Theorem 2.2 the complementary graph $\overline{G}$ is the 1-skeleton of a quasi-tree Δ. Let $F_1, \dots, F_m$ be a leaf order of Δ. Let i_1 be the number of free vertices of the leaf F_m. We label the free vertices of F_m by $n, n-1, \dots, n-i_1+1$, in any order. Next F_{m-1} is a leaf of $\langle F_1, \dots, F_{m-1} \rangle$. Say, F_{m-1} has i_2 free vertices. Then we label the free vertices of F_{m-1} by $n-i_1, \ldots, n-(i_1+i_2)+1$, in any order. Proceeding in this way we label all the vertices of Δ, that is, those of G, and then choose the numbering of the variables of S according to this labeling.

Suppose there exist $x_i x_j \in I$ and $k > i, j$ such that $x_i x_k \notin I$ and $x_j x_k \notin I$. Let r be the smallest number such that $\Gamma = \langle F_1, \ldots, F_r \rangle$ contains the vertices $1, \ldots, k$. Then k is a free vertex of F_r in Γ. Since $x_i x_k \notin I$ and $x_j x_k \notin I$, we have that $\{i, k\}$ and $\{j, k\}$ are edges of Γ, and since k is a free vertex of F_r in Γ it follows that i and j are vertices of F_r. Therefore $\{i, j\}$ is an edge of F_r and hence of Γ. However, this contradicts the assumption that $x_i x_j \in I$.

We now consider a monomial ideal I generated in degree 2 which is not necessarily squarefree. Let $J \subset I$ be the ideal generated by all squarefree monomials in I. Then $I = (x_{i_1}^2, \ldots, x_{i_k}^2, J)$.

LEMMA 2.4. *Suppose I has a linear resolution. Then J has a linear resolution.*

PROOF. Polarizing (see [1, Lemma 4.2.16]) the ideal $I = (x_{i_1}^2, \ldots, x_{i_k}^2, J)$ yields the ideal $I^* = (x_{i_1} y_1, \ldots, x_{i_k} y_k, J)$ in $K[x_1, \ldots, x_n, y_1, \ldots, y_k]$. We consider I^* as the edge ideal of the graph G^* with the vertices $-k, \ldots, -1, 1, \ldots, n$, where the vertices $-i$ correspond to the variables y_i and the vertices i to the variables x_i. Let G be the restriction of G^* to the vertices $1, \ldots, n$. In other words, $\{i, j\}$ with $1 \leq i < j$ is an edge of G if and only it is an edge of G^*. Then it is clear that J is the edge ideal of G.

Assuming that I has a linear resolution implies that I^* has a linear resolution since I^* is an unobstructed deformation of I. It follows that $\overline{G^*}$ is chordal, by Theorem 2.1. Obviously the restriction of a chordal graph to a subset of the vertices is again chordal. Hence $\overline{G}$ is chordal, and so again by Theorem 2.1 we get that J has a linear resolution.

In the situation of Lemma 2.4 let $J = I(G)$, and let Δ be the quasi-tree whose 1-skeleton is $\overline{G}$, see Theorem 2.1 and Theorem 2.2.

PROPOSITION 2.5. *If $I = (x_{i_1}^2, \ldots, x_{i_k}^2, J)$ has a linear resolution, then i_j is a free vertex of Δ for $j = 1, \ldots, k$, and no two of these vertices belong to the same facet.*

PROOF. We refer to the notation in the proof of Lemma 2.4. Our assumption implies that $\overline{G^*}$ is chordal. Let Δ^* the quasi-tree whose 1-skeleton is $\overline{G^*}$.

Suppose that i_j is not a free vertex of Δ. Then there exist edges $\{i_j, r\}$ and $\{i_j, s\}$ in $\overline{G}$ such that $\{r, s\}$ is not an edge in $\overline{G}$. Then $\{i_j, r\}$ and $\{i_j, s\}$ are also edges in $\overline{G^*}$, and $\{r, s\}$ is not an edge in $\overline{G^*}$. Since $x_{i_j} y_j \in I^*$, it follows that $\{i_j, -j\}$ is not an edge in G^*, and since $x_r y_j$ and $x_s y_j$ do not belong to I^* it follows that $\{-j, r\}$ and $\{-j, s\}$ are edges of $\overline{G^*}$. Thus $\{i_j, r\}, \{r, -j\}, \{-j, s\}, \{s, i_j\}$ is circuit of length 4 with no chords, a contradiction.

Suppose i_j and i_l are free vertices belonging to the same facet of Δ. Then $\{i_j, i_l\}$ is an edge in $\overline{G^*}$, and we also have that $\{i_j, -l\}$, $\{i_l, -j\}$ and $\{-j, -l\}$ are egdes of $\overline{G^*}$ since $x_{i_j} y_l$, $x_{i_l} y_j$ and $y_j y_l$ do not belong to I^*. On the other hand, $\{i_j, -j\}$ and $\{i_l, -l\}$ are not edges of $\overline{G^*}$ since $x_{i_j} y_j$ and $x_{i_l} y_l$ belong to I^*. Therefore $\{i_j, i_l\}, \{i_l, -j\}, \{-j, -l\}, \{-l, i_j\}$ is the circuit of length 4 with no chords, a contradiction.

COROLLARY 2.6. *Suppose I has a linear resolution and $x_i^2 \in I$. Then with the numbering of the variables as given in Proposition* 2.3 *the following holds: for all $j > i$ for which there exists k such that $x_k x_j \in I$, one has $x_i x_j \in I$ or $x_i x_k \in I$.*

PROOF. Suppose $x_i^2 \in I$ and there exists a $j > i$ for which there exists k such that $x_k x_j \in I$, but $x_i x_j$ and $x_i x_k$ both do not belong to I. Then $k \neq i$, because $x_i^2 \in I$.

If $k \neq j$, then $\{k, j\}$ is not an edge of Δ, and $\{i, j\}$, $\{i, k\}$ both are edges of Δ. This implies that i is not a free vertex of Δ, contradicting Proposition 2.5.

If $k = j$, then $x_j^2 \in I$ and j is a free vertex of Δ, by Proposition 2.5. But since $x_i x_j \notin I$ we have that $\{i, j\}$ is an edge of Δ. This implies that i and j belong to the same facet, again a contradiction to Proposition 2.5.

3. Monomial ideals satisfying the x-condition

In the previous section we have seen that if I is a monomial ideal generated in degree 2 which has a linear resolution then it satisfies the conditions $(*)$ and $(**)$ listed in the next theorem.

THEOREM 3.1. *Let $I \subset S = K[x_1, \ldots, x_n]$ be an ideal which is generated by quadratic monomials and suppose that I possesses the following properties* $(*)$ *and* $(**)$*:*

$(*)$ *if $x_i x_j \in I$ with $i \neq j$, $k > i$ and $k > j$, then either $x_i x_k$ or $x_j x_k$ belongs to I;*

$(**)$ *if $x_i^2 \in I$ and $j > i$ for which there is k such that $x_k x_j \in I$, then either $x_i x_j \in I$ or $x_i x_k \in I$.*

Let $R(I) = T/P$ be the Rees ring of I. Then there exists a lexicographic order $<_{\text{lex}}$ on T such that the reduced Gröbner basis G of the defining ideal P with respect to $<_{\text{lex}}$ consists of binomials $f \in T$ with $\deg_x(f) \leq 1$.

PROOF. Let Ω denote the finite graph with the vertices $1, \ldots, n, n+1$ whose edge set $E(\Omega)$ consists of those edges and loops $\{i, j\}$, $1 \leq i \leq j \leq n$, with $x_i x_j \in I$ together with the edges $\{1, n+1\}, \{2, n+1\}, \ldots, \{n, n+1\}$. Let $K[\Omega] \subset S[x_{n+1}]$ denote the *edge ring* of Ω studied in, e.g., [10] and [11]. Thus $K[\Omega]$ is the affine semigroup ring generated by those quadratic monomials $x_i x_j$, $1 \leq i \leq j \leq n+1$, with $\{i, j\} \in E(\Omega)$. Let $T = K[x_1, \ldots, x_n, \{y_{\{i,j\}}\}_{\substack{1 \leq i \leq n,\, 1 \leq j \leq n \\ \{i,j\} \in E(\Omega)}}]$ be the polynomial ring and define the surjective homomorphism $\pi : T \to K[\Omega]$ by setting $\pi(x_i) = x_i x_{n+1}$ and $\pi(y_{\{i,j\}}) = x_i x_j$. The *toric ideal* of $K[\Omega]$ is the kernel of π. Since the Rees ring $R(I)$ is isomorphic to the edge ring $K[\Omega]$ in the obvious way, we will identify the defining ideal P of the Rees ring with the toric ideal of $K[\Omega]$.

We introduce the lexicographic order $<_{\text{lex}}$ on T induced by the ordering of the variables as follows: (i) $y_{\{i,j\}} > y_{\{p,q\}}$ if either $\min\{i, j\} < \min\{p, q\}$ or ($\min\{i, j\} = \min\{p, q\}$ and $\max\{i, j\} < \max\{p, q\}$) and (ii) $y_{\{i,j\}} > x_1 > x_2 > \cdots > x_n$ for all $y_{\{i,j\}}$. Let G denote the reduced Gröbner basis of P with respect to $<_{\text{lex}}$.

It follows (e.g., [11, p. 516]) that the Graver basis of P coincides with the set of all binomials f_Γ, where Γ is a primitive even closed walk in Ω. (In [11] a finite graph with no loop is mainly discussed. However, all results obtained there are valid for a finite graph allowing loops with the obvious modification.)

Now, let f be a binomial belonging to G and

$$\Gamma = (\{w_1, w_2\}, \{w_2, w_3\}, \ldots, \{w_{2m}, w_1\})$$

the primitive even closed walk in Ω associated with f. In other words, with setting $y_{\{i,n+1\}} = x_i$ and $w_{2m+1} = w_1$, one has

$$f = f_\Gamma = \prod_{k=1}^{m} y_{\{w_{2k-1}, w_{2k}\}} - \prod_{k=1}^{m} y_{\{w_{2k}, w_{2k+1}\}}.$$

What we must prove is that, among the vertices $w_1, w_2, \ldots, w_{2m}$, the vertex $n+1$ appears at most one time. Let $y_{\{w_1,w_2\}}$ be the biggest variable appearing in f with respect to $<_{\text{lex}}$ with $w_1 \leq w_2$. Let $k_1, k_2, \ldots$ with $k_1 < k_2 < \cdots$ denote the integers $3 \leq k < 2m$ for which $w_k = n+1$.

Case I: Let k_1 be even. Since $\{n+1, w_1\} \in E(\Omega)$, the closed walk

$$\Gamma' = (\{w_1, w_2\}, \{w_2, w_3\}, \ldots, \{w_{k_1-1}, w_{k_1}\}, \{w_{k_1}, w_1\})$$

is an even closed walk in Ω with $\deg_x(f_{\Gamma'}) = 1$. Since the initial monomial $\text{in}_{<_{\text{lex}}}(f_{\Gamma'}) = y_{\{w_1,w_2\}} y_{\{w_3,w_4\}} \cdots y_{\{w_{k_1-1},w_{k_1}\}}$ of $f_{\Gamma'}$ divides $\text{in}_{<_{\text{lex}}}(f_\Gamma) = \prod_{k=1}^{m} y_{\{w_{2k-1},w_{2k}\}}$, it follows that $f_\Gamma \notin G$ unless $\Gamma' = \Gamma$.

Case II: Let both k_1 and k_2 be odd. This is impossible since Γ is primitive and since the subwalk

$$\Gamma'' = (\{w_1, w_2\}, \ldots, \{w_{k_1-1}, w_{k_1}\}, \{w_{k_2}, w_{k_2+1}\}, \ldots, \{w_{2m}, w_1\})$$

of Γ is an even closed walk in Ω.

Case III: Let k_1 be odd and let k_2 be even. Let C be the odd closed walk

$$C = (\{w_{k_1}, w_{k_1+1}\}, \{w_{k_1+1}, w_{k_1+2}\}, \ldots, \{w_{k_2-1}, w_{k_2}\})$$

in Ω. Since both $\{w_2, w_{k_1}\}$ and $\{w_{k_2}, w_1\}$ are edges of Ω, the closed walk

$$\Gamma''' = (\{w_1, w_2\}, \{w_2, w_{k_1}\}, C, \{w_{k_2}, w_1\})$$

is an even closed walk in Ω and the initial monomial $\mathrm{in}_{<_{\mathrm{lex}}}(f_{\Gamma'''})$ of $f_{\Gamma'''}$ divides $\mathrm{in}_{<_{\mathrm{lex}}}(f_\Gamma)$. Thus we discuss Γ''' instead of Γ.

Since Γ''' is primitive and since C is of odd length, it follows that none of the vertices of C coincides with w_1 and that none of the vertices of C coincides with w_2.

(III-a) First, we study the case when there is $p \geq 0$ with $k_1 + p + 2 < k_2$ such that $w_{k_1+p+1} \neq w_{k_1+p+2}$. Let W and W' be the walks

$$W = (\{w_{k_1}, w_{k_1+1}\}, \{w_{k_1+1}, w_{k_1+2}\}, \ldots, \{w_{k_1+p+1}, w_{k_1+p+2}\}),$$
$$W' = (\{w_{k_2}, w_{k_2-1}\}, \{w_{k_2-1}, w_{k_2-2}\}, \ldots, \{w_{k_1+p+3}, w_{k_1+p+2}\})$$

in Ω.

(III-a-1) Let $w_1 \neq w_2$. If either $\{w_2, w_{k_1+p+1}\}$ or $\{w_2, w_{k_1+p+2}\}$ is an edge of Ω, then it is possible to construct an even closed walk $\Gamma^\sharp$ in Ω such that $\mathrm{in}_{<_{\mathrm{lex}}}(f_{\Gamma^\sharp})$ divides $\mathrm{in}_{<_{\mathrm{lex}}}(f_{\Gamma'''})$ and $\deg_x(f_{\Gamma^\sharp}) = 1$. For example, if, say, $\{w_2, w_{k_1+p+2}\} \in E(\Omega)$ and if p is even, then

$$\Gamma^\sharp = (\{w_2, w_1\}, \{w_1, w_{k_2}\}, W', \{w_{k_1+p+2}, w_2\})$$

is a desired even closed walk.

(III-a-2) Let $w_1 \neq w_2$. Let $\{w_2, w_{k_1+p+1}\} \notin E(\Omega)$ and $\{w_2, w_{k_1+p+2}\} \notin E(\Omega)$. Since $\{w_{k_1+p+1}, w_{k_1+p+2}\}$ is an edge of Ω, by $(*)$ either $w_2 < w_{k_1+p+1}$ or $w_2 < w_{k_1+p+2}$. Let $w_2 < w_{k_1+p+2}$. Since $w_1 < w_2$ and $\{w_1, w_2\} \in E(\Omega)$, again by $(*)$ one has $\{w_1, w_{k_1+p+2}\} \in E(\Omega)$. If p is even, then consider the even closed walk

$$\Gamma^\flat = (\{w_1, w_2\}, \{w_2, w_{k_2}\}, W', \{w_{k_1+p+2}, w_1\})$$

in Ω. If p is odd, then consider the even closed walk

$$\Gamma^\flat = (\{w_1, w_2\}, \{w_2, w_{k_1}\}, W, \{w_{k_1+p+2}, w_1\})$$

in Ω. In each case, one has $\deg_x(f_{\Gamma^\flat}) = 1$. Since $y_{\{w_1,w_2\}} > y_{\{w_1,w_{k_1+p+2}\}}$, it follows that $\mathrm{in}_{<_{\mathrm{lex}}}(f_{\Gamma^\flat})$ divides $\mathrm{in}_{<_{\mathrm{lex}}}(f_{\Gamma'''})$.

(III-a-3) Let $w_1 = w_2$. Since $w_1 < w_{k_1+p+1}$, by (∗∗) either $\{w_1, w_{k_1+p+1}\} \in E(\Omega)$ or $\{w_1, w_{k_1+p+2}\} \in E(\Omega)$. Thus the same technique as in (III-a-2) can be applied.

(III-b) Second, if $C = (\{n+1, j\}, \{j, j\}, \{j, n+1\})$, then in each of the cases $w_1 < w_2 < j$, $w_1 < j < w_2$ and $w_1 = w_2 < j$, by either (∗) or (∗∗), one has either has $\{w_1, j\} \in E(\Omega)$ or $\{w_2, j\} \in E(\Omega)$.

As the final conclusion of our considerations we obtain

THEOREM 3.2. *Let I be a monomial ideal generated in degree 2. The following conditions are equivalent:*

(a) *I has a linear resolution;*

(b) *I has linear quotients;*

(c) *Each power of I has a linear resolution.*

PROOF. The implication (c) ⇒ (a) is trivial, while (b) ⇒ (a) is a general fact. It follows from Proposition 2.3 and Corollary 2.6 that if I has a linear resolution, then the conditions (∗) and (∗∗) of Theorem 3.1 are satisfied, after a suitable renumbering of the variables. Hence by Corollary 1.2 each power of I has a linear resolution.

It remains to prove (a) ⇒ (b): Again we may assume that the conditions (∗) and (∗∗) hold. Let $G(I)$ be the unique minimal set of monomial generators of I. We denote by $[u, v]$ the greatest common divisor of u and v.

We show that the following condition (q) is satisfied: the elements of $G(I)$ can be ordered such that if $u, v \in G(I)$ with $u > v$, then there exists $w > v$ such that $w/[w, v]$ is of degree 1 and $w/[w, v]$ divides $u/[u, v]$. This condition (q) then implies that I has linear quotients.

The squarefree monomials in $G(I)$ will be ordered by the lexicographical order induced by $x_n > x_{n-1} > \cdots > x_1$, and if $x_i^2 \in G(I)$ then we let $u > x_i^2 > v$, where u is the smallest squarefree monomial of the form $x_k x_i$ with $k < i$, and where v is the largest squarefree monomial less than u.

Now, for any two monomials $u, v \in G(I)$ with $u > v$ corresponding to our order, we need to show that property (q) holds. There are three cases:

Case 1: $u = x_s x_t$ and $v = x_i x_j$ both are squarefree monomials with $s < t$ and $i < j$. Since $u > v$, we have $t \geq j$. If $t = j$, take $w = u$. If $t > j$, then by (∗), either $x_i x_t \in G(I)$ or $x_j x_t \in G(I)$. Accordingly, let $w = x_i x_t$ or $w = x_j x_t$.

Case 2: $u = x_t^2$ and $v = x_i x_j$ with $i < j$. Since $u > v$, we have $t > j$. Hence by (∗), either $x_i x_t \in G(I)$ or $x_j x_t \in G(I)$. Accordingly, let $w = x_i x_t$ or $w = x_j x_t$.

Case 3: $u = x_s x_t$ with $s \leq t$ and $v = x_i^2$. If $t = i$, then $s \neq t$ and take $w = u$. If $t > i$, then by (∗∗), we have either $x_i x_t \in G(I)$ or $x_i x_s \in G(I)$.

Both elements are greater than v in our order. Accordingly, let $w = x_i x_t$ or $w = x_i x_s$. Then again (q) holds.

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FACHBEREICH MATHEMATIK UND INFORMATIK
UNIVERSITÄT-GHS ESSEN
45117 ESSEN
GERMANY
E-mail: juergen.herzog@uni-essen.de

DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
OSAKA UNIVERSITY
TOYONAKA, OSAKA 560-0043
JAPAN
E-mail: hibi@math.sci.osaka-u.ac.jp

FACHBEREICH MATHEMATIK UND INFORMATIK
UNIVERSITÄT-GHS ESSEN
45117 ESSEN
GERMANY
E-mail: xinxian.zheng@uni-essen.de