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MONOPHONIC WIRELENGTH OF CIRCULANT NETWORKS INTO WHEELS AND FANS

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Abstract: This paper presents the monophonic wirelength of circulant graph into wheels and fans. Also we present a monophonic algorithm to find the monophonic wirelength of family of circulant graphs into wheels and fans. Our monophonic algorithm produces the monophonic wirelength and cover a wide range of interconnection networks.

AMS Subject Classification: 05C, 05C12

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1. Introduction

By a graph $\Gamma = (V, E)$, we mean a connected, finite, undirected graph with neither loops nor multiple edges. For notations and terminology, refer [4]. The distance $d(x, y)$ between two vertices x and y in a graph G is the length of the shortest path from x to y in G.An edge $x_i x_j$ is a chord of a path $x_0, x_1, x_2, \cdots, x_n$ if $j \geq i + 2$. A monophonic path is a path if it contains no chord. The length

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of the longest $x-y$ monophonic path of a graph G is called the monophonic distance $d_m(x,y)$ for every x,y vertices in G. A monophonic path from x to y with length $d_m(x,y)$ is called an $x-y$ monophonic.For this refer [14].Consider a graph H,since other graphs or networks are embedded into it, as host graph and graphs or networks which are embedded in H are called guest graph. The embedding f of G to H is a bijective mapping from the vertex set of G to the vertex set of H and every edge $(x,y)\in E(G)$ is mapped to a path between $f(x)$ and $f(y)$ in H For this refer [8, 11]. If we find an embedding of G into H which produces the minimum wirelength $WL(G,H)$, such problem is called the wirelength problem. The *wirelength* of an embedding f of G into H is given as

$$WL_f(G,H)=\sum_{(x,y)\in E(G)} d_H(f(x),f(y))=\sum_{e\in E(G)} EC_f(G,H(e)),$$

where $EC_f(G,H(e))$ is the maximum number of edges of G that are embedded on e known as the **edge congestion** of f of G into H.

2. Preliminaries

We use definitions, lemmas and theorems from [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] for this work.

Definition 1. [1] A graph denoted by W_n of order n is called a wheel graph if it has an outer cycle of n-1 vertices and these n-1 vertices are connected to a single vertex known as hub. Wheel graphs have an unique role in interconnection network designs and circuit layout.

Definition 2. [1] A graph denoted by F_n of order n is called a fan graph if it has an outer path of n-1 vertices and these n-1 vertices are connected to a single vertex known as the core.

Lemma 3. *(see [11], Congestion Lemma) Let G be an k-regular graph and let $f:G\rightarrow H$ be an embedding.Let the graph $H-E$ has the components H_i, i=1,2 and $G_i=f^{-1}(H_i)$ then the edge cut E of H has the following properties:*

1. *The path $P_f(f(x),f(y))$ has no edges in E for every edge $(x,y)\in G_i, i=$ 1, 2.*

2. *The path $P_f(f(x),f(y))$ has exactly one edge in E for every edge (x,y) in G with $x\in G_1$ and $y\in G_2$.*

3. G_1 is a maximum subgraph of G.

Then $EC_f(E)$ is minimum and $EC_f(E) = k|V(G_1)| - 2|E(G_1)|$.

Lemma 4. *(see [11], Partition Lemma) Let $f : G \to H$ be an embedding. Let $\{E_1, E_2, \cdots, E_p\}$ be a partition of $E(H)$ such that each E_i is an edge cut of H. Then $WL_f(G,H) = \sum_{i=1}^{p} EC_f(E_i)$.*

Lemma 5. *(see [1], k-partition Lemma) Let f be an embedding of G into H. Let $\{E_1, E_2, \cdots, E_p\}$ be a partition of $k[E(H)]$ such that each E_i is an edge cut of H. Then $WL_f(G,H) = \frac{1}{k}\sum_{i=1}^{p} EC_f(E_i)$.*

Maximum Subgraph Problem (see [7]) The problem of finding a subset of vertices of a given graph, such that the number of edges in the sub graph induced by this subset is maximal among all induced sub graphs with the same number of vertices. Mathematically, for a given m, if

$$I_G(m) = max_{A \subseteq V, |A|=m} |I_G(A)|,$$

where $I_G(A) = \{(u,v) \in E : u, v \in A\}$, then the problem is to find $A \in V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$.

Definition 6. [7] A connected undirected graph represented by $G(m, \pm S)$ where $S \subseteq \{1, 2, 3, \cdots, [m/2]\}, m \geq 3$ is said to be a circulant graph if it consists of the vertex set $V = \{0, 1, 2, \cdots, m-1\}$ and the edge set $E = \{(x,y) : |x-y| \cong s(modm), s \in S\}$.

Theorem 7. *[7]The number of edges in a maximum sub graph on k vertices of $G(n, \pm S)$ where $S \subseteq \{1, 2, 3, \cdots, j\}, 1 \leq j \leq [n/2], n \geq 3$ is given by*

$$\zeta = \begin{cases} k(k-1)/2, & k \leq j+1, \\ kj - j(j+1), & j+1 < k \leq n-j, \\ \frac{1}{2}\{(n-k)^2 + (4j+1)k - (2j+1)n\}, & n-j < k < n. \end{cases}$$

Theorem 8. *[7] A set of k consecutive vertices of $G(n, \pm 1); 1 \leq k \leq n$ induces a maximum subgraph of $G(n, \pm S)$ where $S = \{1, 2, 3, \cdots, j\}, 1 \leq j \leq [n/2], n \geq 3$.*

Theorem 9. *[8] The maximum subgraph on the set of all k vertices of $G(n, \{1, 2, \cdots, j\})$ for $k < j$ is complete graph on k vertices.*

3. Monophonic Wirelength Problem

Definition 10. Let $G(V, E)$ and $H(V, E)$ be finite graphs with n vertices. An embedding $f_m : G \to H$ is called a monophonic embedding if f_m maps each vertex of G into a vertex of H and each edge (x, y) of G is mapped to a monophonic path between $f_m(x)$ and $f_m(y)$ in H.

Definition 11. Let $f_m : G \to H$ be a monophonic embedding. The monophonic edge congestion of f_m of G into H is the maximum number of edges of the graph G that are embedded on an edge $e \in H$ and is given by $MEC_{f_m}(G, H) = maxMEC_{f_m}(G, H(e))$.

The monophonic wirelength problem of a graph G into H is the problem of finding a monophonic embedding $f_m : G \to H$ that produces the monophonic wire length $MWL(G, H)$.

Definition 12. Let $f_m : G \to H$ be a monophonic embedding. The monophonic wirelength $MWL(G, H)$of f_m is given as

$$MWL_{f_m}(G, H) = \sum_{(x,y)\in E(G)} d_m(f_m(x), f_m(y)).$$

Lemma 13. *(Monophonic Congestion Lemma) Let G be a k-regular graph with n vertices. Let H be a finite graph with n vertices. Let $f_m : G \to H$ be a monophonic embedding of G into H. Let the graph $H - E_j, j = 1, 2, \cdots, p$ have the components H_i, i =1,2 and $G_i = f_m^{-1}(H_i)$,where $E_j's$ are the edge cuts of H, form a partition in H and have the following properties:*

1. *For $m \geq 0$, there are m edges $(x, y) \in G_i, i = 1, 2$,such that the monophonic path $P_{f_m}(f_m(x), f_m(y))$ has exactly two edges in E_j.*
2. *The monophonic path $P_{f_m}(f_m(x), f_m(y))$ has exactly one edge in E_j for every $(x, y) \in G$ with $x \in G_1 \& y \in G_2$.*

where G_1 is a maximum subgraph of G. Then $MECf_m(E_j)$ is monophonic and the monophonic wirelength of f_m of G into H is given by $MWL_{f_m}(G, H) = \sum_{i=1}^{p} MEC_{f_m}(E_j)$ where $MEC_{f_m}(E_j) = r|V(G_1)| - 2|E(G_1)| + 2m, m \geq 0$.

Result 14. If there are partitions $\{E_1, E_2, \cdots, E_p\}$ of $kE(H)$, then by Lemma 5 $MEC_{f_m}(E_j) = \frac{1}{k}[r|V(G_1)| - 2|E(G_1)| + 2m], m \geq 0, j = 1, 2, \cdots, p.$

4. Monophonic Wirelength into Wheels and Fans

Theorem 15. *Let $f_m : G \to H$ be a monophonic embedding where G is an r-regular circulant graph $G(n, \pm S), S \subseteq \{1, 2, 3, \cdots, [n/2]\}$ and H is the wheel graph W_n. Then the wirelength of G into H induced by f_m is monophonic.*

Proof. Excluding the hub vertex of W_n, let $P_p = \{(p-1, p), (p+1, p+2), (n-1, p), (n-1, p+1)\}, 1 \leq p \leq n-1$,where the vertices are taken $mod(n-1)$. Consider the edge set $\{(p-1, p), (n-1, p)/1 \leq p \leq n-1\}$ which represents the edges of W_n exactly once. Therefore $\{P_1, P_2, \cdots, P_{n-1}\}$ form a partition of the edge set of W_n twice.

Let A_{p_1} and A_{p_2} be the components of $W_n - P_p$ for every p. Let us take $A_{p_1} = \{p, p+1\}$. Under the monophonic embedding f_m, let $G_{p_1} = f_m^{-1}(A_{p_1})$ and $G_{p_2} = f_m^{-1}(A_{p_2})$. Then an edge of G is induced by G_{p_1}. Hence each P_p satisfies the properties stated in Lemma 14. Therefore $MECf_m(P_p)$ is monophonic and hence by Lemma 5 the wirelength of f_m from G to W_n is monophonic. □

Theorem 16. *The monophonic wirelength of an r-regular graph G with n vertices, into the wheel graph W_n is given by $MWL[G, W_n] = WL[G, W_n] + m, m \geq 0$.*

Proof. As explained in Theorem 15, the cut edge $\{P_1, P_2, \cdots, P_{n-1}\}$ is a partition of edge set of W_n twice.Hence by Lemma 5,

$$\begin{aligned} MEC_{f_m}(E_j) &= \frac{1}{2}[r|V(G_1)| - 2|E(G_1)| + 2m] \\ &= \frac{1}{2}[r|V(G_1)| - 2|E(G_1)|] + \frac{1}{2}2m \end{aligned}$$

Hence $MWL[G, W_n] = WL[G, W_n] + m, m \geq 0, j = 1, 2, \cdots, p$ □

Monophonic Embedding Algorithm I

Aim: To find a monophonic embedding $f_m : G \to H$ that produces the monophonic wirelength $MWLf_m(G, H)$ where G is the family of circulant graph with $2n$ vertices of r-regular and H is the wheel graph W_{2n}.

Monophonic Algorithm: Case (i) Name the vertices of $G[2n, \pm S], S \subseteq \{1, 2, 3, \cdots, n\}$ as a cycle from $0, 1, 2, \cdots, 2n-1$.

Case (ii) Name the vertices of W_{2n} as an outer cycle from $0, 1, 2, \cdots, 2n-2$ and the hub vertex $2n-1$.

Case(i).

Input: *Preimage- The family of circulant graphs

$$G[2n, \{1, 2, 3, \cdots, n-1\}], \quad n \geq 3.$$

Image- The family of wheel graphs W_{2n}.

Output: A monophonic embedding f_m of $G[2n, \{1, 2, 3, \cdots, n-1\}]$ into W_{2n} is given by $f_m(x) = x$ with monophonic wirelength.

$$MWL[G[2n, \{1, 2, 3, \cdots, n-1\}], W_{2n}] = WL[G[2n, \{1, 2, 3, \cdots, n-1\}], W_{2n}] + \frac{1}{2}[(3|S| - 2)(|S|(2n - 5) - 1) + 2].$$

Proof. By Theorem 15, f_m is monophonic and hence the proof follows from Theorem 16, as here k =2 and

$$m = \frac{1}{2}[(3|S| - 2)(|S|(2n - 5) - 1) + 2].$$

Case(ii). Input: Preimage. The family of circulant graphs

$$G[2n, \{1, 2, 3, \cdots, n-2\}], \quad n \geq 4.$$

Image. The family of wheel graphs W_{2n}.

Output: A monophonic embedding f_m of $G[2n, \{1, 2, 3, \cdots, n-2\}]$ into W_{2n} is given by $f_m(x) = x$ with monophonic wirelength

$$MWL[G[2n, \{1, 2, 3, \cdots, n-2\}], W_{2n}] = WL[G(2n, \{1, 2, 3, \cdots, n-2\}), W_{2n}] + \frac{1}{2}[(3|S| - 2)(2|S|^2 - 1) + (4n - 1)|S| - 2)].$$

Proof. By Theorem 15, f_m is monophonic and hence the proof follows from Theorem 16 as $k = 2$ and $m = \frac{1}{2}[(3|S| - 2)(2|S|^2 - 1) + (4n - 1)|S| - 2)]$.

Case(iii). Input: Preimage. The family of circulant graphs

$$G[2n, \{1, 2, 3, \cdots, n\}], \quad n \geq 3.$$

Image. The family of wheel graphs W_{2n}.

Output: A monophonic embedding f_m of $G[2n, \{1, 2, 3, \cdots, n\}]$ into W_{2n} is given by $f_m(x) = x$ with monophonic wirelength

$$MWL[G[2n, \{1, 2, 3, \cdots, n\}], W_{2n}] = WL[G(2n, \{1, 2, 3, \cdots, n\}), W_{2n}] + (2n - 1)[(3|S|^2 - 17|S| + 25)].$$

Proof. By Theorem 15, f_m is monophonic and hence the proof follows from Theorem 16, as here $k = 2$ and $m = (2n - 1)[(3|S|^2 - 17|S| + 25)]$.

Theorem 17. *Let $f_m : G \to H$ be a monophonic embedding where G is an r-regular circulant graph $G(n, \pm S), S \subseteq \{1, 2, 3, \cdots, [n/2]\}$ and H is the fan graph F_n. Then the wirelength of G into H induced by f_m is monophonic.*

Proof. Excluding the core vertex of F_n, let

$$\begin{aligned} R_q =& \{(q-1, q), (q+1, q+2), (n-1, q), (n-1, q+1), 1 \leq q \leq n-4\}, \\ R_{n-3} =& \{(n-4, n-3), (n-1, n-3), (n-1, n-2)\}, \\ R_{n-2} =& \{(1, 2), (n-1, 0), (n-1, 1)\}, \\ R_{n-1} =& \{(n-3, n-2), (n-1, n-2), (n-1, 0)\} \end{aligned}$$

represents the edges of F_n exactly once. Also,the edge set

$$\{(q+1, q+2), (n-1, q+1)/1 \leq q \leq n-4\} \cup R_{n-2} \cup \{(n-1, n-2), (0, 1)\}$$

represents all edges of F_n exactly once.

Therefore $\{R_1, R_2, \cdots, R_n\}$ is a partition of the edge set of F_n twice. Let B_{q_1} and B_{q_2} be the two components of $F_n - R_q$ for every $1 \leq q \leq n-3$. Assume $B_{q_1} = \{q, q+1\}$. Under the monophonic embedding f_m,let $B_{q_1} = f_m^{-1}(G_{q_1})$ and $B_{q_2} = f_m^{-1}(G_{q_2})$. Then an edge of G is induced by G_{q_1}. Thus each R_q satisfies the properties given in Lemma 13 and by Lemma 5,$MECf_m(R_q)$ is monophonic and hence the wirelength of f_m from G to F_n is monophonic. □

Monophonic Embedding Algorithm II Aim: To find a monophonic embedding $f_m : G \to H$ that produces the monophonic wirelength $MWLf_m(G, H)$, where G is the family of circulant graph with $2n$ vertices of r-regular and H is the fan F_{2n}.

Monophonic Algorithm: Case (i) Name the vertices of $G[2n, \pm S], S \subseteq \{1, 2, 3, \cdots, n\}$ as a cycle from $0, 1, 2, \cdots, 2n-1$.

Case (ii) Name the vertices of F_{2n} as an outer cycle from $0, 1, 2, \cdots, 2n-2$ and the core vertex $2n-1$.

Case(i). Input: *Preimage- The family of circulant graphs

$$G[2n, \{1, 2, 3, \cdots, n-1\}], n \geq 3.$$

Image. The family of fan graphs F_{2n}.

Output: A monophonic embedding f_m of $G[2n, \{1, 2, 3, \cdots, n-1\}]$ into F_{2n} is given by $f_m(x) = x$ with monophonic wirelength.

$$MWL[G[2n, \{1, 2, 3, \cdots, n-1\}], F_{2n}] = WL[G(2n, \{1, 2, 3, \cdots, n-1\}), F_{2n}]$$

$$+\frac{1}{3}(n-1)[(4|S|^2-9|S|+5)].$$

Proof. The proof is obvious from Theorems 16 & 17 as $k=2$ and $m=\frac{1}{3}[(4|S|^2-9|S|+5)]$.

Case (i) Name the vertices of $G[2n,\pm S], S\subseteq\{1,2,3,\cdots,n\}$ as a cycle from $0,1,2,\cdots,2n-1$.

Case (ii) Name the vertices of F_{2n} as an outer cycle from $0,1,2,\cdots,2n-2$ and the core vertex $2n-1$.

Case(ii). Input: *Preimage- The family of circulant graphs

$$G[2n,\{1,2,3,\cdots,n-1\}], n\geq 3.$$

Image. The family of fan graphs F_{2n}.

Output:A monophonic embedding f_m of $G[2n,\{1,2,3,\cdots,n-2\}]$ into F_{2n} is given by $f_m(x)=x$ with monophonic wirelength.

$$MWL[G[2n,\{1,2,3,\cdots,n-2\}],F_{2n}]=WL[G(2n,\{1,2,3,\cdots,n-2\}),F_{2n}]$$
$$+\frac{1}{3}(n-3)[(4|S|^2+|S|-6)].$$

Proof. The proof is obvious from Theorems 16 & 17 as $k=2$ and $m=\frac{1}{3}(n-3)[(4|S|^2+|S|-6)]$.

Case(iii). Input: *Preimage- The family of circulant graphs $G[2n,\{1,2,3,\cdots,n\}], n\geq 2$.

Image. The family of fan graphs F_{2n}.

Output: A monophonic embedding f_m of $G[2n,\{1,2,3,\cdots,n\}]$ into F_{2n} is given by $f_m(x)=x$ with monophonic wirelength.

$$MWL[G[2n,\{1,2,3,\cdots,n\}],F_{2n}]=WL[G(2n,\{1,2,3,\cdots,n\}),F_{2n}]$$
$$+\frac{1}{6}(n-1)[(2|S|^2-7|S|+6)].$$

Proof. The proof is obvious from Theorems 16 & 17 as $k=2$ and $m=\frac{1}{6}(n-1)[(2|S|^2-7|S|+6)]$.

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