Progress of Theoretical Physics, Vol. 121, No. 6, June 2009

Monopole Operators in $\mathcal{N} = 4$ Chern-Simons Theories and Wrapped M2-Branes

Yosuke IMAMURA

Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan

(Received March 5, 2009)

Monopole operators in Abelian $\mathcal{N} = 4$ Chern-Simons theories described by circular quiver diagrams are investigated. The magnetic charges of non-diagonal $U(1)$ gauge symmetries form the $SU(p) \times SU(q)$ root lattice where p and q are numbers of untwisted and twisted hypermultiplets, respectively. For monopole operators corresponding to the root vectors, we propose a correspondence between the monopole operators and states of a wrapped M2-brane in the dual geometry.

Subject Index: 103, 121, 125

§1. Introduction

Since the proposal of Bagger-Lambert-Gusstavson (BLG) model,[1)–5)] three-dimensional supersymmetric Chern-Simons theories have attracted considerable interest as low-energy effective theories of multiple M2-branes in various backgrounds. BLG model is $SU(2) \times SU(2)$ Chern-Simons theory with bifundamental matter fields that possesses $\mathcal{N} = 8$ supersymmetry. This is the first example of interacting Chern-Simons theory with $\mathcal{N} \geq 4$ supersymmetry. Following the BLG model, various Chern-Simons theories with $\mathcal{N} \geq 4$ have been constructed,[6)–13)] and their properties have been studied extensively.

In this paper, we discuss a field-operator correspondence in AdS_4/CFT_3. In general, field-operator correspondence claims that there is one-to-one correspondence between gauge-invariant operators in CFT and fields in the AdS space, and is one of the most important claims of AdS/CFT correspondence. For the $\mathcal{N} = 6$ Chern-Simons theory, Aharony-Bergman-Jafferis-Maldacena (ABJM) model,[9)] we need to take account of monopole operators to reproduce the desired moduli spaces.[9)] Namely, some of Kaluza-Klein modes on the gravity side correspond to local operators carrying magnetic charges. (See also Refs. 14) and 15) for similar analyses for BLG model.) Monopole operators in ABJM model are further investigated in Refs. 16)–18).

This is also the case in theories with less supersymmetries. In the case of $\mathcal{N} = 2$ quiver gauge theories that describe M2-branes in toric Calabi-Yau four-folds, the relation between mesonic operators and holomorphic monomial functions, which are specified by the charges in toric $U(1)$ symmetries, was proposed in Ref. 19). In the reference, a simple prescription to establish concrete correspondence between Kaluza-Klein modes and mesonic operators is given by utilizing brane crystals.[19)–21)] When this method was proposed, it had not been realized that the quiver gauge theories are actually quiver Chern-Simons theories. After the importance of the existence of Chern-Simons terms was realized, this proposal was confirmed[22)–24)] for a special kind of brane crystal, which can be regarded as "M-theory lift" of brane tilings.[25)–27)] Monopole operators enter the correspondence again as well as the case of ABJM model. The results in Refs. 22)–24) indicate, however, that the set of primary operators corresponding to the supergravity Kaluza-Klein modes includes only a special kind of monopole operators, "diagonal" monopole operators, which carry only the diagonal $U(1)$ magnetic charge and are constructed by combining the dual photon fields and chiral matter fields. (We now consider Abelian quiver Chern-Simons theories and assume that the gauge group for each vertex is $U(1)$.)

The other monopole operators, which we call nondiagonal monopole operators, have no correspondents in the bulk Kaluza-Klein modes. In Ref. 28), it is suggested that such nondiagonal monopole operators correspond to M2-branes wrapped on two-cycles in the internal space. The purpose of this work is to study this correspondence in more detail for $\mathcal{N}=4$ Abelian quiver Chern-Simons theories described by circular quiver diagrams.[8),13)]

Because we consider Abelian Chern-Simons theory, whose gauge group is the product of $U(1)$, the dual geometry has large curvature. By this reason, we mainly focus only on the charges of global symmetries, which are quantized and are hopefully reproduced on the gravity side correctly. We do not attempt to reproduce the conformal dimension of monopole operators by using the gravity description.

This paper is organized as follows. In the next section, we briefly explain the relation between the dual photon field and monopole operators in quiver Chern-Simons theories. In §3, we review the radial quantization method used in Refs. 29) and 30) to compute the conformal dimension and global $U(1)$ charges of monopole operators. After explaining the $\mathcal{N}=4$ Chern-Simons theory in §4 and the structure of the dual geometry in §5, we discuss the duality between nondiagonal primary monopole operators and wrapped M2-branes in §6. The last section is devoted to conclusions and discussion.

§2. Monopole operators and the dual photon field

To briefly review the basic facts about monopole operators, let us consider a generic $\mathcal{N}=2$ quiver Chern-Simons theory described using a connected quiver diagram with n vertices. We label vertices and edges by a and I, respectively. We assume that the gauge group of every vertex is $U(1)$. We denote the gauge group for vertex a by $U(1)_a$, and its Chern-Simons level by k_a. We impose the constraint

$$\sum_{a=1}^{n} k_a = 0, \tag{2·1}$$

on the levels to obtain moduli space, which can be regarded as the background of an M2-brane. The action includes the following Chern-Simons terms.

$$S_{\rm CS} = \sum_{a=1}^{n} \frac{k_a}{4\pi} A_a dA_a. \tag{2·2}$$

We define another basis for n $U(1)$ gauge fields. We recombine A_a into n gauge fields

$$A_D, A_B, A'_1, \ldots, A'_{n-2}. \tag{2·3}$$

A_D is the gauge field of $U(1)_D$, the diagonal $U(1)$ subgroup. When we represent A_a as linear combinations of gauge fields in (2·3), A_D enters all of them with coefficient 1:

$$A_a = A_D + \cdots , \tag{2·4}$$

where $\cdots$ represents linear combinations of A_B and A'_i. By substituting this into (2·2), we obtain

$$S_{\rm CS} = \sum_{a=1}^{n} \frac{1}{2\pi} A_D dA_B + \cdots , \tag{2·5}$$

where $\cdots$ does not include A_D. Thanks to (2·1) we do not have $A_D dA_D$ term. A_B in (2·3) is defined by this equation as the gauge field appearing in the linear term of A_D, and is given by

$$A_B = \sum_{a=1}^{n} k_a A_a. \tag{2·6}$$

The diagonal gauge field A_D does not couple to matter fields and appears only in the Chern-Simons term (2·5), and the equation of motion of A_D is

$$dA_B = 0. \tag{2·7}$$

Owing to the "pure gauge" constraint (2·7), we can define the dual photon field a by

$$A_B = da. \tag{2·8}$$

The dual photon field is a periodic field with the period 2π,[31] and it is convenient to define operators in the form

$$e^{ima}, \quad m \in \mathbb{Z}. \tag{2·9}$$

Because the $U(1)_D$ field strength F_D is the canonical conjugate of the operator a, the operator (2·9) shifts the $U(1)_D$ flux by m. In other words, this operator carries the magnetic charge m for every $U(1)_a$. We call such operators diagonal monopole operators. General diagonal monopole operators can be constructed by combining e^{ima} and other magnetically neutral operators.

The relations (2·6) and (2·8) indicate that the dual photon field a is transformed under a gauge transformation $\delta A_a = d\lambda_a$ by

$$\delta a = \sum_{a=1}^{n} k_a \lambda_a. \tag{2·10}$$

This means that the operator e^{ima} carries electric $U(1)_a$ charge mk_a.

There also exist monopole operators that carry nondiagonal magnetic charges. Let m_a be the $U(1)_a$ magnetic charge of an operator. The equation of motion of A_a is

$$k_a F_a + j_a = 0, \tag{2·11}$$

where j_a is the matter contribution to the electric $U(1)_a$ current. By integrating this equation over a sphere enclosing the operator, we obtain

$$k_a m_a + Q_a = 0, \tag{2·12}$$

where Q_a is the matter contribution to the $U(1)_a$ charge of the operator. This is the Gauss law constraint guaranteeing the gauge invariance of the operator. (Because we consider only rotationally invariant operators, this integrated form is sufficient to guarantee their gauge invariance.)

The magnetic charges are constrained by the equation

$$\sum_{a=1}^{n} k_a m_a = 0, \tag{2·13}$$

which is obtained by integrating (2·7) or summing up (2·12) over a. Because of this constraint, the number of independent nondiagonal monopole charges is $n-2$. In the case of $\mathcal{N}=4$ theory, this number is indeed the same as two-cycles in the internal space.[28)]

§3. Radial quantization method

We use the radial quantization method[29),30)] to compute the conformal dimension and global $U(1)$ charges of monopole operators. We want to look for operators saturating the BPS-bound

$$\Delta \geq R, \tag{3·1}$$

where R is the charge of $U(1)_R$ subgroup of the $\mathcal{N}=2$ superconformal group.

We map a Euclidean three-dimensional CFT in $\mathbb{R}^3$ to the theory in $\boldsymbol{S}^2 \times \mathbb{R}$ by a conformal transformation. A monopole operator with magnetic charges m_a corresponds to a state in the Hilbert space defined in $\boldsymbol{S}^2$ with flux m_a through it. The conformal dimension of the operator is computed as the energy of the corresponding state. We can also obtain $U(1)$ charges of monopole operators as the charges of the corresponding states.

The fields in vector multiplets are treated as classical background fields. We expand fields in the chiral multiplets into spherical harmonics, and define creation and annihilation operators, which are used to construct the Hilbert space. Mode expansion of scalar and spinor fields in BPS monopole backgrounds is given in Ref. 30). Let $\mu \in \mathbb{Z}$ be the number of flux coupling to a chiral multiplet $\Phi = (\phi, \psi)$. The scalar component ϕ and the fermion component ψ are expanded by

$$\phi = \sum_{l=\frac{|\mu|}{2}}^{\infty} \sum_{m=-l}^{l} \alpha_{l,m} e^{-(l+1/2)\tau} Y^0_{l,m} + \sum_{l=\frac{|\mu|}{2}}^{\infty} \sum_{m=-l}^{l} \beta^\dagger_{l,m} e^{(l+1/2)\tau} Y^0_{l,m}, \tag{3·2}$$

$$\psi = \sum_{l=\frac{|\mu|+1}{2}}^{\infty} \sum_{m=-l}^{l} a_{l,m} e^{-(l+1/2)\tau} Y^+_{l,m} + \sum_{l=\frac{|\mu|-1}{2}}^{\infty} \sum_{m=-l}^{l} b^\dagger_{l,m} e^{(l+1/2)\tau} Y^-_{l,m}, \tag{3·3}$$

where $Y^0_{l,m}$ and $Y^{\pm}_{l,m}$ are spherical harmonics of scalar and spinor on the $\boldsymbol{S}^2$ with flux. Refer to Ref. 30) for more detail. To obtain the expansion above, we used the free field equations. The radial quantization method with the expansions (3·2) and (3·3) gives the tree level conformal dimensions for ϕ and ψ, and this cannot be justified in general $\mathcal{N} = 2$ theories in which the conformal dimension receives large quantum corrections. In the $\mathcal{N} = 4$ theory, the conformal dimensions of primary operators are protected by the non-Abelian R-symmetry, and we assume the applicability of the free field approximation in the computation below.

All the oscillators $\alpha_{l,m}$, $\beta_{l,m}$, $a_{l,m}$, and $b_{l,m}$ have the same indices l (angular momentum) and m (magnetic quantum number) associated with the rotational symmetry of $\boldsymbol{S}^2$. l must be non-negative, and when $\mu = 0$, the term including $b^\dagger_{-1/2,m}$ should be omitted. The energy of a quantum for each oscillator is

$$E_l = l + \frac{1}{2}, \tag{3·4}$$

for any of the four kinds of oscillator.

The conformal dimension of the monopole operator corresponding to the Fock vacuum is computed as the zero-point energy. By using an appropriate regularization, we obtain the contribution of the oscillators of ϕ and ψ as

$$\Delta_0 = \frac{|\mu|}{4}. \tag{3·5}$$

We can also compute $U(1)$ charges of the monopole operator. If a $U(1)$ charge of the fermion ψ in a chiral multiplet is q, the contribution of the chiral multiplet to the zero point charge is

$$Q_0 = -\frac{|\mu|q}{2}. \tag{3·6}$$

Excited states are constructed by acting creation operators on the Fock vacuum. If we assume that the R-charge of chiral multiplets is not corrected from the classical value, only creation operator saturating the BPS-bound (3·1) is $\beta^\dagger_{0,0}$, and it exists only when $\mu = 0$. We can use only this operator to construct excited BPS states.

Δ_0 and Q_0 in a quiver gauge theory are obtained by summing up the contribution of all chiral multiplets. Let Q_{aI} be the $U(1)_a$ charge of chiral multiplet Φ_I. We consider a monopole operator with magnetic $U(1)_a$ charge m_a. The flux coupling to Φ_I is given by

$$\mu_I = \sum_{a=1}^{n} m_a Q_{aI}. \tag{3·7}$$

The energy of the Fock vacuum is

$$\Delta_0 = \frac{1}{4}\sum_I |\mu_I|. \tag{3·8}$$

The summation is taken over all the bifundamental chiral multiplets. For a $U(1)$ symmetry, if the charge of chiral multiplet Φ_I is q_I, the zero-point charge of the $U(1)$

symmetry is

$$Q_0 = -\frac{1}{2}\sum_I |\mu_I| q_I. \tag{3·9}$$

For the R-symmetry, $q_I = -1/2$, and Q_0 coincides with Δ_0. Namely, the BPS-bound (3·1) is saturated by the vacuum state. General BPS states are constructed by exciting the Fock vacuum with the creation operators $\beta^\dagger_{I,0,0}$, which exist only for chiral fields with $\mu_I = 0$.

§4. $\mathcal{N} = 4$ Chern-Simons theory

Let us consider an Abelian $\mathcal{N} = 4$ Chern-Simons theory described using a circular quiver diagram[8),13)] with period n shown in Fig. 1.

We label hypermultiplets by the integer I in order in the quiver diagram. I is defined only modulo n and $I = 1$ and $I = n + 1$ are identified. In terms of the language of $\mathcal{N} = 2$ supersymmetry, a hypermultiplet I consists of two chiral multiplets, h_I and $\widetilde{h}_I$. We use half odd integers to label vertices, and denote $U(1)$ gauge symmetry coupling to h_I and h_{I+1} by $U(1)_{I+\frac{1}{2}}$. $U(1)_{I+\frac{1}{2}} \times U(1)_{I-\frac{1}{2}}$ charges of h_I and $\widetilde{h}_I$ are $(+1,-1)$ and $(-1,+1)$, respectively.

There are two kinds of hypermultiplet, which are called untwisted and twisted hypermultiplets.[8)] Let us define numbers s_I associated with hypermultiplets, which are 0 for untwisted hypermultiplets and 1 for twisted hypermultiplets.

$$s_I = 0\text{: untwisted hypermultiplet,} \quad s_I = 1\text{: twisted hypermultiplet.} \tag{4·1}$$

We use indices $a, b, \ldots$ to label untwisted hypermultiplets and $\dot{a}, \dot{b}, \ldots$ for twisted hypermultiplets. Namely, a ($\dot{a}$) runs over the integers I such that $s_I = 0$ ($s_I = 1$).

This theory possesses the R-symmetry

$$\mathrm{Spin}(4)_R = SU(2)_A \times SU(2)_B, \tag{4·2}$$

and flavor symmetry

$$U(1)_A \times U(1)_B. \tag{4·3}$$

We denote the generators of $SU(2)_A$, $SU(2)_B$, $U(1)_A$, and $U(1)_N$ by T_i, $\widetilde{T}_i$ ($i = 1, 2, 3$), P, and $\widetilde{P}$, respectively.

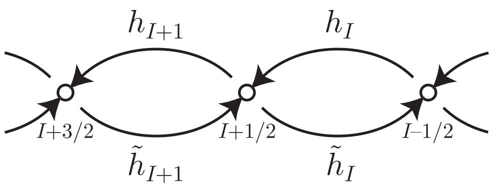

Fig. 1. A part of a circular quiver diagram of an $\mathcal{N} = 4$ supersymmetric Chern-Simons theory is shown. Arrows represent chiral multiplets.

Table I. The conformal dimension and charges of scalar components of multiplets are shown.

	Δ	T_3	P	$\widetilde{T}_3$	$\widetilde{P}$
h_a	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0
$\widetilde{h}_a$	$\frac{1}{2}$	$\frac{1}{2}$	-1	0	0
$h_{\dot{a}}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1
$\widetilde{h}_{\dot{a}}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	-1

Scalar fields in untwisted and twisted hypermultiplets are transformed by $SU(2)_A$ and $SU(2)_B$, respectively. We can form the doublets as

$$h_a^\alpha = \begin{pmatrix} h_a \\ \widetilde{h}_a^* \end{pmatrix}, \quad h_{\dot{a}}^{\dot{\alpha}} = \begin{pmatrix} h_{\dot{a}} \\ \widetilde{h}_{\dot{a}}^* \end{pmatrix}, \tag{4·4}$$

where α and $\dot{\alpha}$ are $SU(2)_A$ and $SU(2)_B$ spinor indices, respectively. The conformal dimension Δ and the charges T_3, P, $\widetilde{T}_3$, and $\widetilde{P}$ of scalar fields are shown in Table I. The R-charge of the $\mathcal{N}=2$ superconformal subgroup is

$$R = T_3 + \widetilde{T}_3, \tag{4·5}$$

and all the scalar components of the chiral multiplets saturate the BPS-bound

$$\Delta \geq T_3 + \widetilde{T}_3. \tag{4·6}$$

In order for the theory to possess $\mathcal{N}=4$ supersymmetry, the levels should be given by

$$k_{I+\frac{1}{2}} = k(s_{I+1} - s_I), \quad k \in \mathbb{Z}. \tag{4·7}$$

We refer to the integer k simply as the "level" of the theory.

The Higgs branch moduli space of this theory is analyzed in Ref. 32). See also Refs. 33) and 34). When $k=1$, it is the product of two orbifolds

$$\mathcal{M}_{p,q} = \mathbb{C}^2/\mathbb{Z}_p \times \mathbb{C}^2/\mathbb{Z}_q, \tag{4·8}$$

where p and q are the numbers of untwisted and twisted hypermultiplets, respectively. The complex coordinates of the $\mathbb{C}^2/\mathbb{Z}_p$ factor can be spanned by

$$M = h_a \widetilde{h}_a, \quad X = e^{-ia} \prod_a h_a, \quad \widetilde{X} = e^{ia} \prod_a \widetilde{h}_a. \tag{4·9}$$

The operator M is independent of the index a due to the F-term conditions. By definition, these three operators satisfy $M^p = X\widetilde{X}$, and this is nothing but the defining equation of the orbifold $\mathbb{C}^2/\mathbb{Z}_p$. The generator of the orbifold group $\mathbb{Z}_p$, which keeps the operators in (4·9) invariant, is

$$e^{2\pi i P/p} \in U(1)_A. \tag{4·10}$$

The other factor $\mathbb{C}^2/\mathbb{Z}_q$ in (4·8) is parameterized by

$$N = h_a \widetilde{h}_a, \quad Y = e^{ia} \prod_{\dot{a}} h_{\dot{a}}, \quad \widetilde{Y} = e^{-ia} \prod_{\dot{a}} \widetilde{h}_{\dot{a}}, \tag{4·11}$$

and these satisfy $N^q = Y\widetilde{Y}$, the defining equation of $\mathbb{C}^2/\mathbb{Z}_q$. The generator of $\mathbb{Z}_q$ is

$$e^{-2\pi i\widetilde{P}/q} \in U(1)_B. \tag{4·12}$$

When $k \geq 2$, the electric charges of the operator e^{ia} become k times those for $k = 1$. In this case, we formally define (M, X, Y) and $(\widetilde{M}, \widetilde{X}, \widetilde{Y})$ by (4·9) and (4·11) with $e^{\pm ia}$ replaced by $e^{\pm ia/k}$. Because $e^{\pm ia/k}$ is ill-defined owing to the fractional coefficient in the exponent, we need to combine these formal operators so that the coefficient in the exponent becomes integral. This is equivalent to imposing the invariance under

$$(X, Y, \widetilde{X}, \widetilde{Y}) \to (\omega_k X, \omega_k^{-1} Y, \omega_k^{-1}\widetilde{X}, \omega_k \widetilde{Y}). \tag{4·13}$$

This transformation is realized by

$$e^{2\pi i(P/kp - \widetilde{P}/kq)} \in U(1)_A \times U(1)_B. \tag{4·14}$$

This means that the moduli space is orbifold of (4·8) divided by $\mathbb{Z}_k$ generated by (4·14). As the result, we obtain the orbifold

$$\mathcal{M}_{p,q,k} = ((\mathbb{C}^2/\mathbb{Z}_p) \times (\mathbb{C}^2/\mathbb{Z}_q))/\mathbb{Z}_k. \tag{4·15}$$

§5. Internal space

The gravity dual of the $\mathcal{N} = 4$ Chern-Simons theory is $AdS_4 \times X_7$ with

$$X_7 = \mathcal{M}_{k,p,q}|_{r=1} = (\boldsymbol{S}^7/(\mathbb{Z}_p \times \mathbb{Z}_q))/\mathbb{Z}_k. \tag{5·1}$$

The homologies $H_i(X_7, \mathbb{Z})$ of this manifold are[28)]

$$\begin{aligned} &H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z}_k, \quad H_2 = \mathbb{Z}^{p+q-2}, \quad H_3 = (\mathbb{Z}_{kp}^{q-1} \times \mathbb{Z}_{kq}^{p-1} \times \mathbb{Z}_{kpq})/(\mathbb{Z}_p \times \mathbb{Z}_q), \\ &H_4 = 0, \quad H_5 = \mathbb{Z}^{p+q-2} \times \mathbb{Z}_k, \quad H_6 = 0, \quad H_7 = \mathbb{Z}. \end{aligned} \tag{5·2}$$

To discuss the relation between monopole operators and wrapped M2-branes in X_7, we need to know where the two- and three-cycles are in the manifold X_7. For this purpose, it is convenient to represent X_7 as a $\boldsymbol{T}^2$ fibration over $B = \boldsymbol{S}^5$ by using the global symmetry $U(1)_A \times U(1)_B$ to define $\boldsymbol{T}^2$ fibers as orbits.

Let us first describe the covering space $\widetilde{X}_7 = \boldsymbol{S}^7$ as $\boldsymbol{T}^2$ fibration over B. Each of $U(1)_A$ and $U(1)_B$ has fixed submanifold $\boldsymbol{S}^3 \subset \boldsymbol{S}^7$. We denote those for $U(1)_A$ and $U(1)_B$ by $\boldsymbol{S}^3_A$ and $\boldsymbol{S}^3_B$, respectively. $\boldsymbol{S}^3_A$ and $\boldsymbol{S}^3_B$ are projected into two $\boldsymbol{S}^2 \subset B$, $\boldsymbol{S}^2_A$ and $\boldsymbol{S}^2_B$, linking with each other. By the $\mathbb{Z}_p \subset U(1)_A$ orbifolding and blowing up the resultant orbifold singularity, $\boldsymbol{S}^2_A$ is split into p loci in B, which we refer to as x_a $(a = 1, \ldots, p)$. Similarly, the $\mathbb{Z}_q \subset U(1)_B$ orbifolding and the blow-up generate q loci, $y_{\dot{a}}$ $(\dot{a} = \dot{1}, \ldots, \dot{q})$. (Fig. 2) (Although we blew up the singularities to define the loci x_a and $y_{\dot{a}}$, we only consider the singular limit.) We use indices a and $\dot{a}$ for the loci just like the two types of hypermultiplets. As mentioned in Ref. 28), by a certain duality between M2-branes in the orbifold and a D3-fivebrane system in type IIB string theory, the loci are mapped to fivebranes, and each hypermultiplet arises

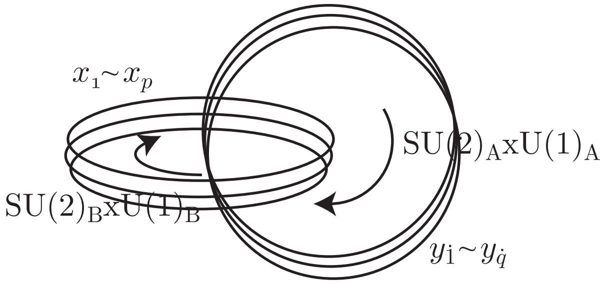

Fig. 2. The loci x_a and $y_{\dot a}$ in the base manifold $\boldsymbol{S}^5$ are shown. On the loci x_a, $SU(2)_A \times U(1)_A$ acts as isometry, while $SU(2)_B \times U(1)_B$ does as transverse rotations. For $y_{\dot a}$, the roles of these symmetries are exchanged.

at the intersection of each fivebrane and D3-branes. Through this duality, we have a natural one-to-one correspondence between the loci and the hypermultiplets.

We define α-, β-, and γ-cycles in the $\boldsymbol{T}^2$ fiber, as cycles corresponding to the generators $e^{2\pi i P/p}$ in (4·10), $e^{-2\pi i \widetilde{P}/q}$ in (4·12), and $e^{2\pi i (P/kp - \widetilde{P}/kq)}$ in (4·14), respectively. The α- (β-)cycle shrinks on the loci x_a ($y_{\dot a}$). The two-cycle homology $H_2(X_7, \mathbb{Z})$ is generated by

$$[x_a, x_b]^\alpha, \quad [y_{\dot a}, y_{\dot b}]^\beta, \tag{5·3}$$

where $[x_a, x_b]$ represents a segment in B connecting two loci, x_a and x_b, and the superscript α means the lift of the segment to the two-cycle in X_7 by combining the α-cycle. $[y_{\dot a}, y_{\dot b}]^\beta$ is defined similarly. It is convenient to define the formal bases $\boldsymbol{x}_a$ and $\boldsymbol{y}_{\dot a}$ by $[x_a, x_b] = \boldsymbol{x}_a - \boldsymbol{x}_b$ and so on. The general two-cycles are in the form

$$\Sigma_2 = \sum_a c_a \boldsymbol{x}_a^\alpha + \sum_{\dot a} c_{\dot a} \boldsymbol{y}_{\dot a}^\beta, \quad c_a, c_{\dot a} \in \mathbb{Z}, \tag{5·4}$$

with the coefficients satisfying

$$\sum_a c_a = \sum_{\dot a} c_{\dot a} = 0. \tag{5·5}$$

A set of generating three-cycles of $H_3(X_7, \mathbb{Z})$ is

$$[x_a, x_b]^{\alpha\gamma} = \boldsymbol{x}_a^{\alpha\gamma} - \boldsymbol{x}_b^{\alpha\gamma}, \quad [y_{\dot a}, y_{\dot b}]^{\alpha\gamma} = \boldsymbol{y}_{\dot a}^{\alpha\gamma} - \boldsymbol{y}_{\dot b}^{\alpha\gamma}, \quad [x_a, y_{\dot b}]^{\alpha\gamma} = \boldsymbol{x}_a^{\alpha\gamma} - \boldsymbol{y}_{\dot b}^{\alpha\gamma}. \tag{5·6}$$

The superscripts "$\alpha\gamma$" denote the lift of segments to three-cycles by combining α- and γ-cycles with the segments. The three-cycle homology group is defined as the set of elements in the form

$$\Sigma_3 = \sum n_a \boldsymbol{x}_a^{\alpha\gamma} + \sum n_{\dot a} \boldsymbol{y}_{\dot a}^{\alpha\gamma}, \quad n_a, n_{\dot a} \in \mathbb{Z}, \tag{5·7}$$

with the coefficients constrained by

$$\sum_a n_a + \sum_{\dot a} n_{\dot a} = 0, \tag{5·8}$$

and the identification relations

$$k\boldsymbol{v}_a^{\alpha\gamma} = k\boldsymbol{w}_{\dot{a}}^{\alpha\gamma} = 0, \tag{5·9}$$

where $\boldsymbol{v}_a$ and $\boldsymbol{w}_{\dot{a}}$ are defined by

$$\boldsymbol{v}_a = -q\boldsymbol{x}_a + \sum_{\dot{b}=\dot{1}}^{\dot{q}} \boldsymbol{y}_{\dot{b}}, \quad \boldsymbol{w}_{\dot{a}} = \sum_{b=1}^{p} \boldsymbol{x}_b - p\boldsymbol{y}_{\dot{a}}. \tag{5·10}$$

§6. Monopole operators and M2-branes

Monopole operators are labeled by n magnetic charges $m_{I+\frac{1}{2}} \in \mathbb{Z}$. We define the group of nondiagonal magnetic charges as the set of charges $m_{I+\frac{1}{2}}$ with identification

$$(m_{\frac{1}{2}}, \cdots, m_{n-\frac{1}{2}}) \sim (m_{\frac{1}{2}} + 1, \cdots, m_{n-\frac{1}{2}} + 1) \tag{6·1}$$

removing the diagonal $U(1)$ charge. To realize this identification automatically, we use the relative charges μ_I defined by

$$\mu_I = m_{I+\frac{1}{2}} - m_{I-\frac{1}{2}}. \tag{6·2}$$

This can be regarded as the effective flux for hypermultiplet I. By definition, μ_I is constrained by

$$\sum_I \mu_I = 0. \tag{6·3}$$

Equation (2·13) imposes further constraint

$$0 = \sum_I k_{I+\frac{1}{2}} m_{I+\frac{1}{2}} = -k \sum_I s_I \mu_I. \tag{6·4}$$

Equations (6·3) and (6·4) can be rewritten as

$$\sum_a \mu_a = \sum_{\dot{a}} \mu_{\dot{a}} = 0. \tag{6·5}$$

The integer μ_I satisfying (6·5) forms the $SU(p) \times SU(q)$ root lattice.

There are $n-2$ independent charges and we would like to identify these with the wrapping charges of M2-branes. Indeed, the two-cycle Betti number of the internal space X_7 is $b_2 = n - 2$ and coincides with the number of nondiagonal magnetic charges. We want to establish not only the coincidence of the numbers of charges but also the one-to-one correspondence between the magnetic charge μ_I and two-cycles in (5·4). A natural guess consistent with (5·5) is

$$\Sigma_2[\mu_I] = \sum_a \mu_a \boldsymbol{x}_a^{\alpha} + \sum_{\dot{a}} \mu_{\dot{a}} \boldsymbol{y}_{\dot{a}}^{\beta}. \tag{6·6}$$

Let us consider magnetic operators that are primary in the sense of $\mathcal{N} = 2$ superconformal symmetry. This means that we look for operators saturating (4·6).

Table II. The conformal dimension and global $U(1)$ charges of monopole operators $\mathfrak{m}_{ab}$ and $\mathfrak{m}_{\dot{a}\dot{b}}$ are shown.

	Δ	T_3	P	$\widetilde{T}_3$	$\widetilde{P}$
$\mathfrak{m}_{ab}$	1	1	0	0	0
$\mathfrak{m}_{\dot{a}\dot{b}}$	1	0	0	1	0

The zero-point contribution to the conformal dimension and the R-charge are

$$\Delta_0 = R_0 = \frac{1}{2}\sum_I |\mu_I|. \tag{6·7}$$

For simplicity, we consider operators with minimum values of R_0. Because μ_I is constrained by (6·5), the minimum R_0 is 1 for monopoles with one relative charge $+1$ and one relative charge -1. The indices of the two nonvanishing relative charges should be both undotted or both dotted. Namely, the following are the two sets of monopole operators

$$\mathfrak{m}_{ab}: \quad \mu_c = -\delta_{ca} + \delta_{cb}, \quad \mu_{\dot{c}} = 0, \tag{6·8}$$

$$\mathfrak{m}_{\dot{a}\dot{b}}: \quad \mu_c = 0, \quad \mu_{\dot{c}} = -\delta_{\dot{c}\dot{a}} + \delta_{\dot{c}\dot{b}}. \tag{6·9}$$

The conformal dimensions and global $U(1)$ charges of these operators are given in Table II.

Because two sets are discussed in a parallel manner, we focus only on the operators $\mathfrak{m}_{ab}$ in the following.

The magnetic charges of monopole operators $\mathfrak{m}_{ab}$ form $SU(p)$ root system. Indeed, the intersection among the cycles (6·6) for $\mathfrak{m}_{ab}$ forms the $SU(p)$ Cartan matrix. In the dual geometry, this $SU(p)$ can be identified with the gauge symmetry on the coincident p D6-branes, which arises from the $\mathbb{C}^2/\mathbb{Z}_p$ singularity through the $U(1)_A$ orbit compactification to type IIA string theory. If we identify the wrapped M2-branes with the nondiagonal components of the $SU(p)$ vector multiplets on the D6-branes, we can interpret the charge $T_3[\mathfrak{m}_{ab}] = 1$ as the R-charge of a scalar field in the vector multiplet. Because $SU(2)_A$ from the type IIA perspective is the transverse rotation around the D6-branes, the scalar components of the vector multiplet belong to the $SU(2)_A$ triplet. There is one component with $T_3 = 1$, and is identified with the operator $\mathfrak{m}_{ab}$.

In general, the vacuum state does not give the gauge-invariant operators. For the operator to be gauge-invariant, the Gauss law constraint (2·12) must be satisfied. The (absolute) magnetic charge $m_{I+\frac{1}{2}}$ of the monopole operator $\mathfrak{m}_{ab}$ is given by

$$m_{I+\frac{1}{2}}[\mathfrak{m}_{ab}] = d + \left[a > I + \frac{1}{2} > b\right], \tag{6·10}$$

where d is an arbitrary integer representing the diagonal magnetic charge, and the inequality in the bracket stands for 1 (0) if it is true (false). Because the quiver diagram is circular, we cannot say which of the given two indices, say a and b, is greater or smaller. However, we can say if three indices are in the descending order or not. In this sense, the bracket in (6·10) is well defined.

To satisfy (2·12), we need to add an appropriate set of chiral multiplets. Gauge invariant monopole operators are given by

$$\mathfrak{M}_{ab} = \begin{cases} \mathfrak{m}_{ab}\mathcal{O}_{\rm neutral} \prod_{a>\dot{c}>b} h_{\dot{c}}^{k(d+1)} \prod_{b>\dot{c}>a} h_{\dot{c}}^{kd}, & (d \geq 0) \\ \mathfrak{m}_{ab}\mathcal{O}_{\rm neutral} \prod_{a>\dot{c}>b} \widetilde{h}_{\dot{c}}^{-k(d+1)} \prod_{b>\dot{c}>a} \widetilde{h}_{\dot{c}}^{-kd}, & (d \leq -1) \end{cases} \tag{6·11}$$

where $\mathcal{O}_{\rm neutral}$ is an electrically and magnetically neutral operator. The products are taken with respect to $\dot{c}$ satisfying the inequalities in the sense that we mentioned above. Note that we cannot use h_a and h_b because when $\mu_I \neq 0$ the corresponding chiral multiplet does not include oscillators saturating the BPS-bound. Owing to the F-term conditions, $\mathcal{O}_{\rm neutral}$ can be written in terms of M in (4·9) and N in (4·11) as

$$\mathcal{O}_{\rm neutral} = M^m N^n, \quad m, n = 0, 1, 2, \ldots. \tag{6·12}$$

By using charges given in Tables I and II, we obtain the following charges of $\mathfrak{M}_{ab}$:

$$T_3[\mathfrak{M}_{ab}] = 1 + m, \qquad m = 0, 1, 2, \ldots, \tag{6·13}$$

$$P[\mathfrak{M}_{ab}] = 0, \tag{6·14}$$

$$\widetilde{T}_3[\mathfrak{M}_{ab}] = \frac{1}{2}|\widetilde{P}[\mathfrak{M}_{ab}]| + n, \qquad n = 0, 1, 2, \ldots, \tag{6·15}$$

$$\widetilde{P}[\mathfrak{M}_{ab}] = \left(d + \frac{q_{[a,b]}}{q}\right) kq, \qquad d = 0, \pm 1, \pm 2, \ldots, \tag{6·16}$$

where $q_{[a,b]}$ is the number of twisted hypermultiplets between untwisted hypermultiplets a and b in the quiver diagram. Namely, by using the bracket used in (6·10), $q_{[a,b]}$ is given by

$$q_{[a,b]} = \sum_{\dot{c}} [a > \dot{c} > b]. \tag{6·17}$$

Let us interpret these charges in terms of wrapped M2-branes in the dual geometry. Wrapped M2-branes are localized on the $U(1)_A$ fixed submanifold. It is the Lens space $L_{kq} = \boldsymbol{S}_A^3/\mathbb{Z}_{kq}$, the γ-cycle fibration over $\boldsymbol{S}_A^2$. The symmetry group $SU(2)_B \times U(1)_B$ acts on L_{kq} as isometry. The interval kq of $\widetilde{P}$ eigenvalues in (6·16) is explained from the $\mathbb{Z}_{kq}$ orbifolding by the operator (4·14). The fractional shift $q_{[a,b]}/q$ in (6·16) is interpreted as the contribution of the Wilson line

$$\frac{q_{[a,b]}}{q} = \frac{1}{2\pi} \oint_{\boldsymbol{x}_a^{\alpha\gamma} - \boldsymbol{x}_b^{\alpha\gamma}} C_3 \qquad \text{mod } 1, \tag{6·18}$$

where C_3 is the three-form field in the 11-dimensional supergravity. For this relation to be acceptable, the torsion must be quantized by

$$\frac{1}{2\pi} \oint_{\boldsymbol{x}_a^{\alpha\gamma} - \boldsymbol{x}_b^{\alpha\gamma}} C_3 \in \frac{1}{q}\mathbb{Z}. \tag{6·19}$$

The geometry of the internal space, however, does not guarantee (6·19). Because the three-cycle $\boldsymbol{x}_a^{\alpha\gamma} - \boldsymbol{x}_b^{\alpha\gamma}$ generates $\mathbb{Z}_{kq}$ subgroup of the homology $H_3(X_7, \mathbb{Z})$, the right-hand side of (6·18) is quantized by

$$\frac{1}{2\pi}\oint_{\boldsymbol{x}_a^{\alpha\gamma}-\boldsymbol{x}_b^{\alpha\gamma}} C_3 \in \frac{1}{kq}\mathbb{Z}, \tag{6·20}$$

but this is not sufficient to guarantee (6·19).

The quantization (6·19) is explained in the following manner. The discrete torsion of C_3 represents the fractional M2-branes.[28),35)] Because we consider the case in which all the gauge groups are $U(1)$ and there are no fractional M2-branes, we should restrict the torsion to the ones corresponding to such situations. In Ref. 28), the relation between the torsion and the numbers of fractional M2-branes in the case of $\mathcal{N}=4$ Chern-Simons theories is studied, and the result shows that the absence of the fractional M2-branes requires

$$\frac{1}{2\pi}\int_{\boldsymbol{v}_a^{\alpha\gamma}} C_3 \in \mathbb{Z}, \quad \frac{1}{2\pi}\int_{\boldsymbol{w}_{\dot{a}}^{\alpha\gamma}} C_3 \in \mathbb{Z}. \tag{6·21}$$

Because $\boldsymbol{v}_a - \boldsymbol{v}_b = -q(\boldsymbol{x}_a - \boldsymbol{x}_b)$ follows from (5·10), the first quantization condition in (6·21) guarantees (6·19).

We can easily see that the spectrum of $\widetilde{T}_3$ in (6·15) is reproduced using a scalar wave function of the M2-brane collective motion in the Lens space L_{kq}. The spherical harmonics in L_{kq} is obtained from $\boldsymbol{S}^3$ spherical harmonics $Y_{l,m,m'}$ by restricting $\widetilde{P}$ eigenvalues by (6·16). $Y_{l,m,m'}$ has three indices, one angular momentum l and two magnetic quantum numbers m and m', which satisfy

$$-l \leq m, m' \leq l. \tag{6·22}$$

$Y_{l,m,m'}$ belongs to the spin (l,l) representation of the $\boldsymbol{S}^3$ rotational group $SO(4) \sim SU(2)^2$, and m and m' are acted by two $SU(2)$ factors separately. Let us choose $SU(2)_B \times U(1)_B \subset SO(4)$ so that $SU(2)_B$ and $U(1)_B$ act on m and m', respectively. Then m' is identified with the half of the $U(1)_B$ charge $\widetilde{P}$, and (l,m) with the $SU(2)_B$ quantum numbers. The inequality (6·22) means that for a given $\widetilde{P}$, allowed $SU(2)_B$ angular momenta are

$$l = \frac{1}{2}|\widetilde{P}|, \quad \frac{1}{2}|\widetilde{P}|+1, \quad \frac{1}{2}|\widetilde{P}|+2, \quad \ldots, \tag{6·23}$$

and this correctly reproduces (6·15).

Because $SU(2)_A \times U(1)_A$ acts on the L_{kq} as transverse rotations, the corresponding charges T_3 and P should be interpreted as spins of M2-branes. For $m=0$, we interpreted this above as the R-charge of a scalar field on the D6-branes. Thus, it seems natural to expect that the spectrum with $m \geq 1$ is also reproduced as the spin of the M2-brane in excited states.

Because the charge P, which is the D-particle charge from the type IIA perspective, vanishes, it may be possible to regard the excited M2-brane as an excited open string on the D6-branes. Indeed, if we approximate the D6-branes by the flat ones, there is the unique lowest energy state for each $T_3 \geq 1$, and this seems consistent with (6·13). This is of course a very rough argument because the D6-branes and the background geometry have large curvature.

§7. Conclusions and discussion

In this paper, we computed the conformal dimensions and the global $U(1)$ charges of primary monopole operators $\mathfrak{M}_{ab}$, which carry nondiagonal magnetic charges corresponding to roots of the $SU(p)$ algebra. In addition to the nondiagonal monopole charges, the operators are labeled by three integers d, $m \geq 0$, and $n \geq 0$. We identified these operators with M2-branes wrapped on two-cycles in the internal space, and we interpreted d and n with the quantum numbers associated with the orbital motions of wrapped M2-branes. We also proposed that the quantum number m may represent the spin of excited M2-branes.

In this paper, we considered Abelian Chern-Simons theories only. It is important to generalize the analysis to non-Abelian case. Then, we can take the large N limit, and more reliable analysis on the gravity side becomes possible. Furthermore, such a generalization enables us to study the relation between general discrete torsion and spectrum of monopole operators. If we take a general discrete torsion quantized by (6·20), the quantization of the momentum $\widetilde{P}$ is changed. This should be realized as the monopole spectrum.

A more challenging issue is the generalization to theories with less supersymmetries. In the case of $\mathcal{N} \leq 2$, the large quantum corrections are expected and the R-charges may be largely corrected. On the gravity side, two-cycles have in general a nonvanishing area, and in such a case, the computation on the gravity side predicts the conformal dimension of order $N^{1/2}$. It would be very interesting if we could explain this behavior as a result of dynamics in Chern-Simons theories.

Acknowledgements

I would like to thank S. Yokoyama for valuable discussions. This work was supported in part by a Grant-in-Aid for Young Scientists (B) (#19740122) from the Japan Ministry of Education, Culture, Sports, Science and Technology.

References

1) J. Bagger and N. Lambert, Phys. Rev. D **75** (2007), 045020.
2) J. Bagger and N. Lambert, Phys. Rev. D **77** (2008), 065008.
3) J. Bagger and N. Lambert, J. High Energy Phys. **02** (2008), 105.
4) A. Gustavsson, Nucl. Phys. B **811** (2009), 66.
5) A. Gustavsson, J. High Energy Phys. **04** (2008), 083.
6) D. Gaiotto and E. Witten, arXiv:0804.2907.
7) H. Fuji, S. Terashima and M. Yamazaki, Nucl. Phys. B **810** (2009), 354.
8) K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, J. High Energy Phys. **07** (2008), 091.
9) O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, J. High Energy Phys. **10** (2008), 091.

10) K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, J. High Energy Phys. **09** (2008), 002.
11) J. Bagger and N. Lambert, Phys. Rev. D **79** (2009), 025002.
12) M. Schnabl and Y. Tachikawa, arXiv:0807.1102.
13) Y. Imamura and K. Kimura, J. High Energy Phys. **10** (2008), 040.
14) N. Lambert and D. Tong, Phys. Rev. Lett. **101** (2008), 041602.
15) J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, J. High Energy Phys. **05** (2008), 038.
16) D. Berenstein and D. Trancanelli, Phys. Rev. D **78** (2008), 106009.
17) K. Hosomichi, K. M. Lee, S. Lee, S. Lee, J. Park and P. Yi, J. High Energy Phys. **11** (2008), 058.
18) I. Klebanov, T. Klose and A. Murugan, J. High Energy Phys. **03** (2009), 140.
19) S. Lee, S. Lee and J. Park, J. High Energy Phys. **05** (2007), 004.
20) S. Lee, Phys. Rev. D **75** (2007), 101901.
21) S. Kim, S. Lee, S. Lee and J. Park, Nucl. Phys. B **797** (2008), 340.
22) K. Ueda and M. Yamazaki, J. High Energy Phys. **12** (2008), 045.
23) Y. Imamura and K. Kimura, J. High Energy Phys. **10** (2008), 114.
24) A. Hanany and A. Zaffaroni, J. High Energy Phys. **10** (2008), 111.
25) A. Hanany and K. D. Kennaway, hep-th/0503149.
26) S. Franco, A. Hanany, K. D. Kennaway, D. Vegh and B. Wecht, J. High Energy Phys. **01** (2006), 096.
27) S. Franco, A. Hanany, D. Martelli, J. Sparks, D. Vegh and B. Wecht, J. High Energy Phys. **01** (2006), 128.
28) Y. Imamura and S. Yokoyama, Prog. Theor. Phys. **121** (2009), 915.
29) V. Borokhov, A. Kapustin and X. k. Wu, J. High Energy Phys. **11** (2002), 049.
30) V. Borokhov, A. Kapustin and X. k. Wu, J. High Energy Phys. **12** (2002), 044.
31) D. Martelli and J. Sparks, Phys. Rev. D **78** (2008), 126005.
32) Y. Imamura and K. Kimura, Prog. Theor. Phys. **120** (2008), 509.
33) M. Benna, I. Klebanov, T. Klose and M. Smedback, J. High Energy Phys. **09** (2008), 072.
34) S. Terashima and F. Yagi, J. High Energy Phys. **12** (2008), 041.
35) O. Aharony, O. Bergman and D. L. Jafferis, J. High Energy Phys. **11** (2008), 043.