# MONOPOLIST PRICING WITH DYNAMIC DEMAND AND PRODUCTION COST*

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This paper deals with pricing of a new product over time by a monopolist who maximizes the discounted profit stream. The interdependency of cost and demand on cumulative production makes the problem inherently dynamic. Cost is assumed to be declining with cumulative production (learning curve effect), while demand is a function of price and cumulative sales, representing word-of-mouth and saturation effects.

The paper addresses this problem in a general framework that includes several previous results as special cases, and provides new insights in other situations. While the learning curve and word-of-mouth effect cause prices to be lower than the price that maximizes immediate revenues, the saturation factor has the opposite effect. The price path over time is affected by these factors and the interest rate. We characterize the price path under several different situations and interpret the results for policy guidelines.
(Pricing; Dynamic Pricing; Learning Curve; Diffusion of Innovations)

## 1. Introduction

Considerations of the effects of current price and sales on future sales and production cost did not receive much attention in the pricing literature until recently. In the last several years, there has been increasing interest in this subject. This paper addresses the monopolist pricing problem in a general framework that includes several previous results as special cases, and provides new insight in other situations.

While most pricing research focuses on steady-state situations in which the pricing rule is based on marginal cost equalling marginal revenues, it is the

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existence of factors that cause sales and costs to vary over time and makes the pricing rule over the product life cycle different. These factors include: (1) 'learning curve' cost decline on the production side, the process by which unit cost declines as more and more experience is gained. This phenomenon has been studied in the literature (Alchian (1959), Hirsch (1952), Arrow (1962), Rosen (1972), and Spence (1981)), although it gained popular recognition more recently. (See, for example, Abell and Hammond (1979), Henderson (1980), Porter (1980), and Conley (1981).) (2) 'Diffusion' effect on the demand side. By diffusion we refer to a host of phenomena that causes the likelihood of purchase to increase, as a result of a higher market penetration. These phenomena include increased awareness through word of mouth (Rogers and Shoemaker (1971)), and product self-advertising, reduction in uncertainty and improved reputation of the product (Jeuland (1981b), Kalish (1983), Schmalensee (1982)), and other factors (Bass (1969), (1980)). (3) Saturation effects for durable goods, i.e., the fact that for durable goods as more units are sold the remaining unfulfilled demand decreases. (4) Uncertainty about future technological developments and competitive reaction to follow such developments in the future (Teng and Thompson (1980), Eliashberg and Jeuland (1982), Clarke and Doland (1982), and Bass and Rao (1982)).

We restrict the following work to deal with the first three phenomena, i.e., the case of a monopolist where there is no direct competition (e.g. patent protection). We also assume that consumers do not change their behavior by developing price expectations. Thus, we look at the problem of maximizing the present value of the profit stream, given that marginal cost of production decreases with cumulative output, and that the demand for the product depends on current price, and cumulative sales, representing the diffusion and saturation effects.

Since the three factors mentioned earlier cause future profits to depend on past sales, the monopolist has to trade off between current profits and future profits. The learning curve cost decline alone creates an incentive to reduce price and produce more, so that future costs will be lower. Similarly, a positive diffusion effect creates an incentive to price lower than the immediate profit maximizing price, and accelerate the diffusion of the product that way. On the other hand, saturation in the case of durable goods creates the opposite effect. Here, the monopolist wants to price higher, and reduce price as more is sold, thereby charging a higher price for those who are willing to pay more for early adoption. The interactions of the above factors, combined with the level of discount rate—that weighs current and future profits—creates a host of possible pricing patterns.

There are two types of questions that we try to answer. First, how will price at each point in time be, relative to the price that will maximize the immediate profits at that point? Second, what would the price pattern over time look like? Since cost data are seldom available publicly, it is difficult for the outsider to compare actual prices to myopic prices. However, price patterns over time are observable, and we will compare our results with pricing patterns observed in industry.

The paper is structured as follows: In §2 we formulate the general model, characterize the solution, and discuss the two questions of interest in light of the different factors. In §3 we analyze subclasses of the general formulation. We first look at the effects of learning curve alone, where sales are a function of price only, in order to isolate this factor from demand complications. Then we analyze the subclass where sales can be written as a product of the effect of price and the diffusion/saturation factor. This subclass includes many of the previously published studies (Robinson and Lakhani (1975), Dolan and Jeuland (1981), Jeuland and Dolan (1982), Spremann (1981), Teng and Thompson (1980)). Finally we look at another subclass, that is more suitable for durable goods. This is followed with a summary of the results and conclusions in §4.

## 2. Model Formulation

We begin the analysis by specifying the model with only a few robust assumptions. This allows us to examine the most general and robust conclusions. Subsequently, we look at more specific cases and obtain stronger results.

Let $x(t)$ be the experience at time $t$, i.e., the total volume produced (sold) by that time. The sales incurred from $t$ to $t + 1$ is clearly $x(t + 1) - x(t)$. We will use a continuous approximation, by looking at very small time periods; the sales rate then is the time derivative of $x(t)$: $s(t) = dx(t)/dt \overset{\Delta}{=} \dot{x}(t)$. (We adopt the dot notation for the derivative with respect to time.)

Sales is a function of price, $p(t)$, as well as the cumulative sales $x(t)$. We write then: $s(t) = \dot{x}(t) = f(x(t), p(t))$. The unit cost, or marginal cost, is a function of experience $x(t)$. Write $c = c(x(t))$.[1]

We assume that the increasing price will result in reduced sales, i.e., $\partial f(x, p)/\partial p < 0$, and that there are learning economies, i.e. $dc(x)/dx \leqslant 0$.

The effect of the experience on sales, $\partial f(x, p)/\partial x$, is not uniform however. At introduction, for a 'good' product, it will have a positive effect due to word of mouth, improved reputation, etc. This will be contrasted, in the case of durable goods, with the negative effect on demand, since each additional unit sold removes the customer from the market, thus reducing demand. We will study these situations in what follows.

The monopolist's problem is to determine a pricing strategy, $p(t)$, that will maximize the discounted profits over the planning horizon. Let $r$ be the discount rate, and $T$ the end of the planning horizon; then the sum of the discounted profits is:

$$\pi = \int_0^T e^{-rt} \cdot \left( p(t) - c\big(x(t)\big)\right) \cdot s(t)\, dt$$

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[1] Modeling marginal cost as a function of experience alone is common. See e.g. Spence (1981), Dolan and Jeuland (1981), Jeuland and Dolan (1982), Bass (1980) and others. Haley (1981) and Clarke et al. (1982) model it as a function of experience and rate of production. For a discussion of these issues see Alchian (1959).

and the monopolist problem is:

$$\underset{p(t)}{\text{Max}} \quad \pi = \int_0^T e^{-rt}\big(p(t) - c(x(t))\big) \cdot \dot{x}(t)\, dt \tag{1}$$

$$\text{s.t.} \quad \dot{x}(t) = f(x, p); \quad x(0) = x_0.$$

This is a dynamic optimization problem, and we use the maximum principle to characterize the optimal policy.[2] We introduce the constraint into the objective function, by multiplying them with the shadow prices, $e^{-rt} \cdot \lambda(t)$, to form the Hamiltonian:

$$H(p, x, \lambda) = e^{-rt} \cdot \big[ p - c(x) + \lambda \big] \cdot f(x, p).$$

The maximum principle states that the optimal solution, $p^*(t)$, to system 1, has to maximize the Hamiltonian, $H$, at each instant $t$, with the sales $\dot{x}^*$ and the shadow $\lambda^*$ follows the differential equations:

$$\dot{x}^* = f(x^*, p^*); \quad x^*(0) = x_0, \tag{2a}$$

$$\dot{\lambda}^* = r\lambda^* - \frac{\partial H^*}{\partial x}$$

$$= r\lambda^* - \frac{\partial f(x^*, p^*)}{\partial x} \cdot (p^* - c(x^*) + \lambda^*) + c'(x^*) \cdot f(x^*, p^*);$$

$$\lambda^*(T) = 0. \tag{2b}$$

(In what follows we will eliminate the function arguments where there is no confusion, in order to improve clarity.) We assume that the optimal solution exists at every time $t$,[3] and therefore the derivative of the Hamiltonian with respect to $p$ must vanish on the optimal path $p^*$:

$$\frac{\partial H}{\partial p} = 0 \Rightarrow p^* = c^* - \lambda^* - \frac{f^*}{\partial f^*/\partial p}. \tag{2c}$$

(2c) can be rewritten as:

$$p^* = \frac{\eta}{\eta - 1}(c^* - \lambda^*) \tag{3}$$

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[2] See Arrow and Kurz (1970) for a review of dynamic optimization techniques with special focus on economic problems.

[3] See Appendix for a discussion of this assumption.

where $\eta$ is the elasticity of demand ($\eta = -(d\dot{x}/dp) \cdot (p/\dot{x})$). Note that equation (3) resembles the classical static monopolist's pricing rule, that price markup over marginal cost depends on demand elasticities as above, except that here we modify marginal costs by subtracting $\lambda^*$, the shadow price. Recall that the shadow price, $\lambda(t)$, is the net benefit of having the constraint relaxed by one unit. In our context, it is the dollar value (at time $t$) of having one more unit produced. It is exactly because of the experience curve and the influence of experience on demand that producing one more unit now will affect future benefits. Thus, for example, if there is only learning in production, producing one more unit now reduces future costs, so $\lambda(t)$ is positive and marginal cost has to be modified to account for that, causing price to be lowered. (Alternatively, equation (2c) above says that current marginal revenues plus future marginal benefit must equal marginal cost.[4]) So we see that the sign of the shadow price, $\lambda(t)$, will determine whether the price will be lower or higher than a myopic monopolist facing the same conditions. If $\lambda(t) > 0$, i.e. there are future benefits for overproducing, then price will be lower and vice versa. To see how the shadow price evolves over time, we substitute (2c) into (2b), and rearrange terms to get:

$$\dot{\lambda} = r\lambda + \dot{c} - \frac{f_x p}{\eta} \; ; \qquad \lambda(T) = 0$$

which has the solution:

$$\lambda(t) = \int_t^T \left( \frac{f_x p}{\eta} - \dot{c} \right) e^{-r(\tau - t)} \, d\tau.$$

By examining the equation for $\lambda(t)$, we see that if the 'diffusion' effect on the demand side is uniformly positive (i.e. $f_x \geqslant 0$), then the integrand is uniformly positive, and so is $\lambda(t)$. Moreover, for the case $r = 0$, $\lambda(t)$ is monotonically decreasing. This is consistent with intuition, since in this case there is an incentive to sacrifice profits now in order to benefit later. On the other extreme, if $\dot{c}(x) = 0$, i.e. no cost decline, and negative diffusion effect on demand (i.e. $f_x < 0$), then $\lambda(t) < 0$, and the price will be higher than the myopic monopolist's price.

The second-order condition for a maximum which we use in what follows is:

$$\left. \frac{\partial^2 H}{\partial p^2} \right|_{p^*} \leqslant 0 \Leftrightarrow \frac{\partial f^*}{\partial p} (p^* - c^* - \lambda^*) + 2 \frac{\partial f^*}{\partial p} \leqslant 0. \tag{4}$$

[4] See also Simon (1982) for an interpretation of shadow prices in the context of strategic pricing.

Substituting $p^*$ from (2c) we get:

$$\frac{\partial^2 H}{\partial p^2}\bigg|_{p^*} = f_p^*\left[2 - \frac{f_{pp}^* \cdot f^*}{f_p^{*2}}\right] \leqslant 0 \Leftrightarrow \left[2 - \frac{f_{pp}^* \cdot f^*}{f_p^{*2}}\right] \geqslant 0 \tag{5}$$

(where $f_p = (\partial f(x, p)/\partial p)$, $f_{pp} = \partial^2 f(x, p)/\partial p^2$, etc. In what follows we will eliminate the $*$ notation, and all equations refer to the optimal solution, unless otherwise stated.)

A second question of interest is whether price increases or decreases over time. In order to analyze that, we take the time derivative of the optimal price given by (2c):[5]

$$\dot{p} = c'(x) \cdot \dot{x} - \dot{\lambda} - \frac{d}{dt}\left[\frac{f}{f_p}\right].$$

Substituting (2b) for $\dot{\lambda}$, and rearranging terms, we get:

$$\dot{p}\left[2 - \frac{f \cdot f_{pp}}{f_p^2}\right] = -r\lambda - 2\frac{f_x \cdot f}{f_p} + f^2\frac{f_{px}}{f_p^2}. \tag{6}$$

Recall from (5) that on the optimal path the term in brackets on the left-hand side of the equation is positive. Therefore, the sign of the time derivative of price is the same sign as the right-hand side. It is evident that for the case of no discount (i.e. $r = 0$), the first term vanishes, and the dependence of price derivative on the cost is only implicit through the terminal condition $p(T)$ $= (\eta/(\eta - 1)) \cdot c(x(T))$ (from (3)), since in this case (2a) and (6) characterize the optimal solution completely. Thus, in the zero interest case, the price derivative depends on demand specification $f$. Hence, for demand functions where the sign of $f_x$ is the sign of $f_{xp}$ uniformly, price will be monotonic regardless of the cost function. In other words, if demand increases with penetration ($f_x > 0$), and this increase is stronger for higher price ($f_{xp} > 0$), then price increases, and vice versa. We will see several examples later on.

### 3.  Subclasses of the General Formulation

While the general formulation discussed in the last section is useful to gain insight into the factors affecting the optimal price, we can obtain stronger results for specific cases. First, we consider the case where demand exhibits no 'learning' or saturation, i.e. sales depend on price alone. This case pinpoints the effects of the learning curve alone. Next, we consider the case where sales depend on experience in a specific form, i.e. multiplicative form. Finally, we consider several models that are not included in the above characterizations.

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[5]See Appendix for a discussion of conditions where this can be 'safely' done.

### 3.1.  Static Demand with Cost Decline

In this section we investigate the effects of learning curve cost decline alone, where sales are a function of current price alone. This situation characterizes nondurable products, where word of mouth is not important, e.g. established products.

In this case (2a)–(2c) reduces to:

$$\text{(a) } \dot{x} = f(p); \qquad x(0) = x_0, \qquad \text{(b) } \dot{\lambda} = r\lambda + c'(x) \cdot f,$$

$$\text{(c) } p = \frac{\eta}{\eta - 1} \left( c(x) - \lambda \right),$$

(7)

and the time derivative of the optimal price:

$$\dot{p}\left(2 - \frac{ff''}{f'^2}\right) = -r\lambda.$$

(8)

Clearly, for $r = 0$, the problem reduces to a static problem, where the whole planning period is looked at as one period, and price is set for the whole period, so that marginal revenues equal marginal cost at the end of the period. Note that $\lambda(t)$ in this case is: $\lambda(t) = c(x(t)) - c(x(T))$. This means that if one or more units are sold at time $t$, then the benefit is that all subsequent units will be less expensive. The difference amounts to selling one more unit at the end, and one less at time $t$, giving rise to the above benefit.

For $r > 0$, however, the results change. It is characterized as follows.

THEOREM 1.  *If demand is a function of price alone, and costs decrease with experience, then*:

1. *Optimal price decreases monotonically over time for $r > 0$.*

2. *Total quantity sold and average price are greater and lower respectively than for a myopic monopolist.*

3. *The present value of the shadow price, $e^{-rt} \cdot \lambda(t)$, decreases monotonically over time.*

PROOF.  See Appendix.[6]

This result states that the effect of learning economies is to reduce price and produce more, in order to benefit by lowering future production cost. Although early production is discounted relatively to the price charged by a

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[6] A similar result was obtained independently by Clarke et al. (1982) for a somewhat more general cost function. They have also investigated a situation where the fixed cost declines with experience, but not the marginal unit cost. Interestingly enough, the results reverse in this case, and price monotonically increases over time. This, however, is a less realistic situation in my opinion. Previously, special cases of Theorem 1 were obtained by Jeuland and Dolan (1982), for a particular cost decline, and Spence (1981) for the no interest case. It is easy to verify that the theorem actually applies to a wider class of problems, where there is an exogenous shift in demand, i.e. $\dot{x} = k(t) \cdot f(p)$. This generalizes the work of Bass and Bultez (1982). See Appendix.
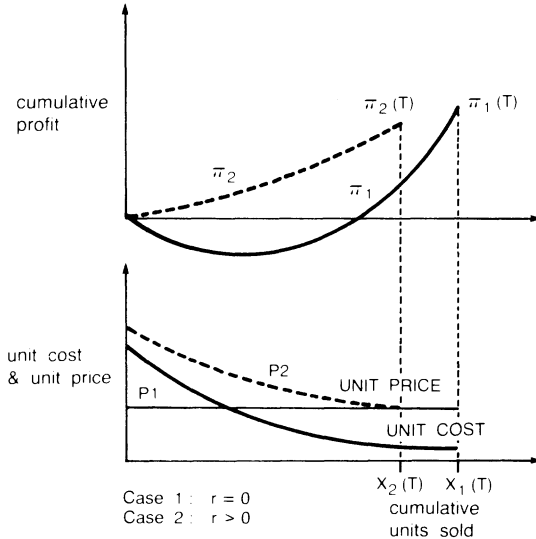
FIGURE 1.   Optimal Price and Profits for $r = 0$ and $r > 0$.

myopic monopolist, this will always result in a monotonically decreasing price over time (for $r > 0$). Moreover, for the case $r = 0$, we say that the shadow price, $\lambda(t)$, is monotonically decreasing, resulting in a monotonically decreasing price discount relative to a myopic monopolist facing the same conditions, and a monotonically increasing profit margin. While this will not be generally true for the positive interest rate, it may hold for low discount rates.

EXAMPLE 1.   Consider a demand function $\dot{x} = k \cdot e^{-dp}$. Then we get:

$$p^* = 1/d + c - \lambda, \qquad \dot{p}^* = -r\lambda < 0.$$

For $r = 0$, $p^* = 1/d + c(x(T))$, where

$$x(T) = x_0 + \int_0^T \dot{x}\, dt = x_0 + \int_0^T k e^{-d \cdot (1/d + c(x(T)))}\, dt = x_0 + T \cdot k \cdot e^{-1 + dc(x(T))}$$

which is an algebraic equation for $x(T)$. Therefore, the firm may actually lose money at the beginning, depending on whether $c(x_0) - c(x(T)) > 1/d$ or not. In the $r > 0$ case, price starts higher, and decreases monotonically (see Figure 1).

## 3.2.   *Multiplicative Separable Functions*

We turn next to investigate the case where the interaction between price and cumulative sales takes a specific form, i.e. sales, $s(x, p) = f(x) \cdot g(p)$. This subclass of models is interesting for two reasons: (1) It is a naturally simple way to model the interaction between price and experience. (2) Several

previous studies that examined pricing over the life cycle have used such functional forms. For example, Dolan and Jeuland (1981) have used $\dot{x} = [N - x][a + bx] \cdot e^{-dp}$. A similar model was used by Robinson and Lakhani (1975).

The main implication of this class of formulation is that demand elasticity, i.e. the percentage response in sales to a percentage change in price, is independent of experience, $x(t)$. This assumption is particularly bothersome for durable goods, since in this case the potential population actually changes as a result of past sales.[7] However, for repeat purchase goods this class of models might be a useful approximation where the experience, $x$, may stand for a measure of reputation, for example.

In this case, the system of equation that characterizes the optimal solution, (2), reduces the following:

$$\text{(a) } \dot{x} = f(x) \cdot g(p); \qquad x(0) = x_0,$$

$$\text{(b) } \dot{\lambda} = r\lambda + c'(x) \cdot f \cdot g + \frac{f' \cdot g^2}{g'} ; \qquad \lambda(T) = 0, \qquad (9)$$

$$\text{(c) } p = c - \lambda - \frac{g}{g'} ,$$

and the derivative of price:

$$\dot{p}\left[2 - \frac{gg''}{g'^2}\right] = -r\lambda - \frac{f'g^2}{g'} . \qquad (10)$$

3.2.1. *Separable Demand, No Discounting.* Consider first the case of zero discount rate. Although this case may sound unrealistic, it is useful to look at it, since the results are interesting and it serves as an approximation to a low discount rate case. The following result holds for this case:

THEOREM 2. *For the separable demand and zero discount rate, optimal monopolist price is increasing when $f'(x) > 0$, and decreasing when $f'(x) < 0$.*

PROOF.  Obvious from (10) above.[8]

This result implies that optimal price increases as long as additional sales increase future demand, and vice versa. This is independent of the cost function (as a result of zero discount rate assumption). In other words, for a

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[7]See Jeuland (1979), Kalish (1980) and Jeuland (1981b) for more detailed discussions of the limitation of these types of models for durable goods.

[8]The theorem is still valid for the more general demand, where there are exogenous shifts: $\dot{x} = f(x) \cdot g(p) \cdot h(t)$ (see Appendix).

period of positive effects of sales on demand, price is initially low to stimulate early adopters, which in turn will stimulate demand. Price will monotonically increase to the point where this 'word-of-mouth' effect diminishes. On the other hand, if there is a negative effect of sales now on subsequent demand, price is initially relatively high, skimming more profits from those who are willing to pay for early adoption, decreasing monotonically over time.

This result is a generalization of Dolan and Jeuland (1981) and Jeuland and Dolan (1982). They have investigated a situation with particular functional form for $f(x)$ and $g(p)$. Here we show that these results apply to *any* model regardless of the particular specification, as long as it is separable, i.e. it is not a peculiarity of the functional form they have chosen.

A typical situation previously examined is the case of a durable good where at introduction positive word of mouth stimulates demand, while later in the life cycle, saturation takes over and demand increases. In this case optimal price under the above conditions will start relatively low, increase as long as the word-of-mouth effect overcomes the saturation effect, and decrease thereafter. The case where word of mouth is not strong enough to overcome the saturation effect, or when it has an adverse effect (bad product), price will monotonically decline. Finally, for a nondurable product, where there is no saturation effect, price will start relatively low, and monotonically increase as the product gains reputation.

As for the sign of the shadow price, $\lambda$, note that in this case we can integrate (9b) to get:

$$\lambda(t) = c(t) - c(T) - \int_t^T \frac{f' g^2}{g'} \, d\tau$$

which simply states that marginal benefit for producing more units now is the cost savings plus the benefits due to effects on demand. Thus in the third case above, where $f'(x) > 0$ uniformly, clearly $\lambda(t)$ is positive and decreasing, so the difference between marginal cost and marginal revenues is big initially, and monotonically decreases to 0. The opposite case where $f'(x) < 0$ uniformly, there is a tradeoff between future cost savings and adverse effects on demand. However if there is no cost decline, the shadow price will be negative and increasing to 0, which now implies pricing higher than the marginal revenues equal marginal cost rule.

EXAMPLE 2.   Consider again $g(p) = e^{-dp}$. (9b) turns out to be:

$$\dot{\lambda} = \dot{c} + \frac{f'}{f} \cdot \frac{\dot{x}}{(-d)} \Rightarrow \lambda(t) = \int_t^T -\dot{c} + \frac{\dot{\ln}(f)}{d} \, dt$$

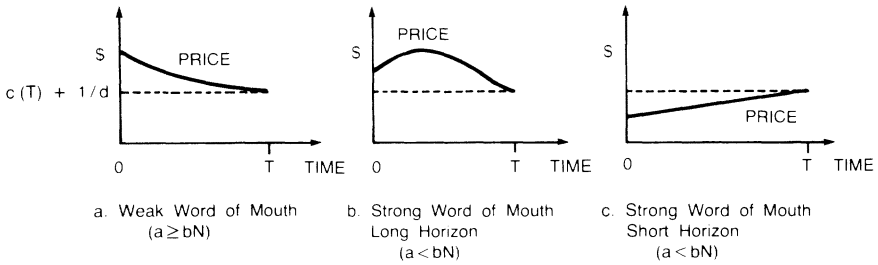$$= -c(x(T)) + c(x(t)) + \frac{1}{d} \frac{\ln(f(x(T)))}{\ln(f(x(t)))} \, .$$

FIGURE 2. Optimal Price for $\dot{x} = [N - x][a + bx] \cdot e^{-dp}$. Note that $c(T)$ will be generally different across these situations. Therefore, these figures should not be compared quantitatively.

So

$$p^* = c - \lambda + \frac{1}{d} = c(x(T)) + \frac{1}{d} \cdot \frac{\ln(f(x(t)))}{\ln(f(x(T)))} + \frac{1}{d}.$$

In the case $f(x) = (N - x)(a + bx)$, which Dolan and Jeuland (1981) have investigated, this result reduces to their result. It is easy to verify that in this case $f'(x) > 0$ as long as $x/N < \frac{1}{2}(1 - a/bN)$. Thus, if $a < bN$,[9] increasing price is optimal until the above point, and then price should decrease (see Figures 2(b) and (c)). If, however, $a \geqslant bN$, then $f'(x) < 0 \ \forall t$, which implies price decreases monotonically (see Figure 2(a)).

3.2.2. *Separable Demand, Positive Discount.* The more realistic case of $r > 0$ complicates the analysis. Clearly, the higher the interest rate, the more valuable early profits are in comparison with future profits. Therefore price will be shifted from the nondiscount price towards the myopic monopolist's price, trading more profits now for less profits later. At the extreme, if $r$ is very large, price will be close to that of a myopic monopolist. As we have shown earlier in the absence of demand-experience interaction, this causes prices to follow cost, i.e. decrease over time. However, when experience interacts with demand, the resultant price depends on the tradeoff between the two factors. For cases where the two factors reinforce each other, e.g. negative word of mouth, the resultant price will indeed decrease over time as we will show. For the more interesting case of positive word of mouth, however, the resultant price path depends on the tradeoff between the two factors.

To unbundle these interactions, we shall examine first the situation where there are no learning economies, and then combine the two. (In order to improve clarity, only several 'typical' situations are presented here, while the proofs and comments on other cases are deferred to the Appendix.)

THEOREM 3. *For the separable demand, with constant cost over time, if $f''(x) \leqslant 0 \ \forall t \in [0, T]$, then optimal price is characterized as follows:*
  (a) *If $f'(x(T)) > 0 \Rightarrow \dot{p} > 0 \ \forall t \in [0, T]$.*

[9] In empirical studies this condition usually holds. See Bass (1969), (1980).

(b) *If* $f'(x_0) > 0$, $f'(x(T)) < 0 \Rightarrow p$ *is increasing and then decreasing.*

(c) *If* $f'(x_0) < 0 \Rightarrow p$ *is monotonically decreasing, or increasing and then decreasing.*

PROOF.   See Appendix.

Note that the inclusion of interest rates still keeps the qualitative results similar to the no interest case. Consider again the 'Bass'-type diffusion effect, $f(x) = [N - x][a + bx]$; $f''(x) = -2b < 0$ here. So if the planning horizon is such that $f'(x) > 0$ for the whole period, price will monotonically increase. If the planning horizon is longer, price will increase and then decrease. However, if the word of mouth is weak to start with, i.e. $bN \leqslant a$, price will either monotonically decrease, or increase and then decrease.

Finally, for the case where we add experience, we can characterize the following situation:

THEOREM 4.   *For the separable demand with learning cost decline and positive discount, if* $f''(x) \geqslant 0$, *the following characterize optimal price:*

(a) *If* $f'(x(T)) < 0 \Rightarrow \dot{p} < 0 \; \forall t \in [0, T]$.

(b) *If* $f'(x(0)) < 0$ *and* $f'(x(T)) > 0 \Rightarrow p^*$ *is decreasing and then increasing.*

(c) *If* $f'(x(0)) > 0 \Rightarrow$ *price is either monotonically increasing or decreasing and then increasing.*

PROOF.   See Appendix.

Note that the results given in Theorem 4 above are for the less interesting case, since in many situations we expect $f''(x) < 0$ (e.g. the Bass model). In what follows we will discuss the other case and summarize the separable demand function.

3.2.3.   *Summary of Separable Demand Pricing.*   By examining several special cases we have been able to isolate how each of the factors—cost decline, demand diffusion-saturation, and discount rate—affect price over time. We have shown that if discount rate is negligible, then price increases as long as it is beneficial to 'subsidize' adopters, i.e. demand increases due to more units sold and vice versa. This policy makes intuitive sense, since it will also tend to smooth production: increase price in anticipation of increasing demand, thus mitigating the sales increase and vice versa.

Since discount rate means that money now is preferred to future income, this policy is shifted slightly towards that of a myopic monopolist. At the extreme, if $r$ is very large, price would be practically identical to that of the myopic monopolist, because mostly immediate profits would count. Such a myopic policy follows the well-known marginal revenues equal marginal cost rule. Since cost declines with production, price would fall along with cost. Thus, it is the interaction of these two factors that will determine price in the realistic setting. If the two factors reinforce each other, when demand is decreasing with more sales for example (which is the case of durable goods later in the life cycle), then price decline is called for by the two factors: cost and demand. If, on the other hand, word of mouth has a strong effect in the introduction of such a durable, then low penetration price is possible, then increasing, followed by the decreasing price later on. Finally, if the demand is
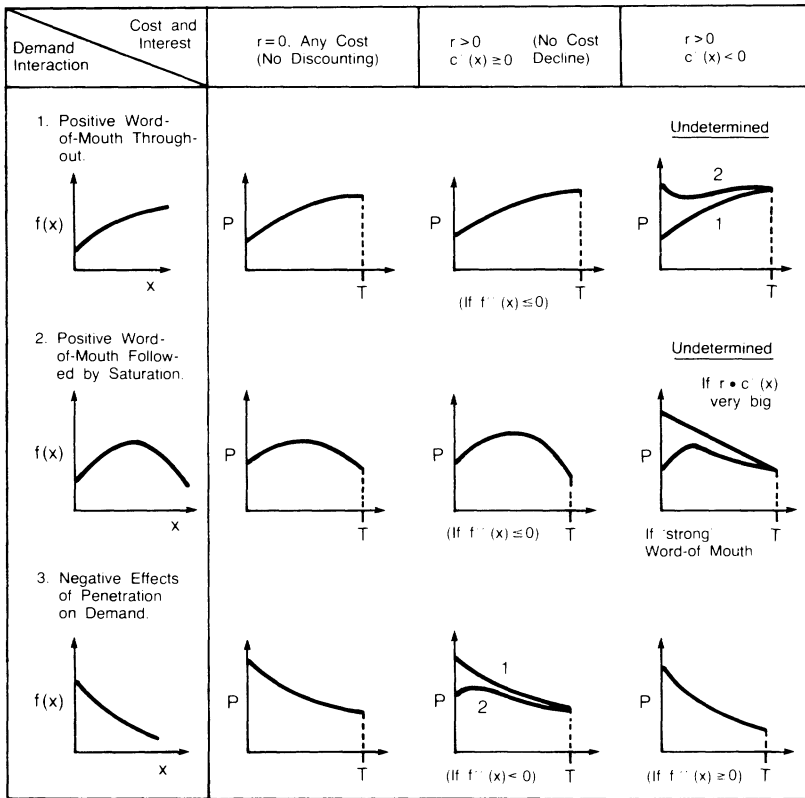
FIGURE 3. Summary of Pricing over Time: Separable Demand Function, under Different Cost/Demand Experience Interactions.

increasing with more sales, price change over time depends on the particular case: how high the interest rate is, the rate of the cost decline, and the intensity of the 'word-of-mouth' factor. In case there is no cost decline, e.g., price indeed will monotonically increase (see Figure 3).

Finally, consider the shadow prices. If the 'word-of-mouth' effect is positive throughout, then both cost decline and word of mouth creates an incentive to 'invest' in over production. Indeed, the shadow price is positive, which means that price is lower than the myopic monopolist's price facing the same conditions, or in other words, price is below the price that will maximize immediate revenues at each point in time. Moreover, the discounted value of the shadow price is monotonically decreasing to 0 at the end of the planning horizon.

On the other extreme, if each additional unit sold has an adverse effect on demand, durable goods e.g., then there is a conflict between the cost benefit on the one hand, and the demand decline on the other, so one cannot tell in general whether price will be above or below the profit maximizing price. However, if there is not cost decline, price will always be above the immediate profit maximizing price.

### 3.3.  *Other Functional Forms*

As pointed out earlier, the separable demand function has several limita-tions. In particular, it is limited for modeling durable goods. In this section we will analyze two functional forms that seem more appropriate in the context of durable goods.

For a *perfectly durable* good (i.e., goods that last forever), the market size is limited by the number of individuals who are willing to purchase the durable at a particular price. Therefore, it is logical to characterize $N(p)$ as the number of units sold over the life cycle of the product, i.e. the total number of individuals who are willing to buy at a certain price. Sales rates or the timing of adoption, however, is not instantaneous. In general, the sales rate is a function of the remaining potential at a given price, multiplied by the likelihood of purchase. In the case where the product is new, the likelihood of purchase increases as a function of cumulative sales since information about the product is spread by word of mouth and the product *self-advertising*, thereby reducing the risk associated with buying the product.[10]

Mathematically we have $\dot{x} = [N(p) - x] \cdot h(x)$, where $[N(p) - x]$ is the remaining potential, while $h(x)$ is the conditional likelihood of purchase.[11] For a good product we expect $h'(x) \geq 0$, i.e. the conditional likelihood of purchase increases with more products sold.

These models have an appealing interpretation as follows. Each individual in the population is characterized by a reservation price for the product, which is the highest price he would be willing to pay for the product. Given an actual price $p$, $N(p)$ represents the total number of users whose reservation price is above $p$. As price decreases over time, more and more people find price acceptable. However if price increases, some who have already bought may find out that the selling price $p$ is above their valuation of the product. In this case, they may find it beneficial to sell the durable on the secondary market.[12] Therefore the relevant market potential for the firm is still $[N(p) - x]$. At the extreme case, if price is increased so that $N(p(t)) < x(t)$, then sales halt (unless they are willing to buy back at this price, e.g., DeBeers has recently bought back diamonds in order to keep prices high (*Fortune* (1982))). The model can be modified as follows:

$$\dot{x} = \begin{cases} [N(p) - x] \cdot h(x) & \text{for} \quad N(p) \geq x, \\ 0 & \text{otherwise.} \end{cases}$$

However for simplicity, we will proceed by assuming that the optimal price

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[10] Product self-advertising means that the product itself by being out there (e.g. a new car) generates awareness. See Kalish (1980) for an analysis of such a model, where word-of-mouth effect is modeled as $[a + bx]$.

[11] Note that this form cannot be written as $f(x) \cdot g(p)$, therefore it is not multiplicatively separable. In fact, the elasticity of demand here crucially depends on the level of penetration, $x$. See Kalish (1980) for more details.

[12] As an example, consider the reselling of silver and gold by individuals when gold and silver prices skyrocketed in 1979.

will never be increased to the point where sales actually halt, in which case we can proceed without this modification.[13]

The case where $r = 0$ represents perfect price discrimination. In other words, the firm will price initially at the highest reservation price. Once these are sold, it reduces the price to the second-highest reservation price, and so on until the point where price equals marginal cost. Thus, regardless of $h(x)$, if there is no time preference for money, the price is monotonically decreasing with perfect price discrimination, i.e., there is no consumer surplus.

The situation is different for $r > 0$ when the firm prefers more cash flow early on. Therefore, it has the incentive to reduce price at the beginning. In what follows, we will analyze this situation and find out whether there are conditions under which penetration pricing is still optimal. We start by analyzing the simpler case where $h(x) = a$, and then we look at the more complicated case.

3.3.1.   *Simple Price-Timing Model for Durables.*   Here we assume that the sales rate is proportional to the remaining potential population, i.e. $\dot{x} = [N(p) - x]a$. In this case the system of necessary conditions is:

$$\text{(a) } \dot{x} = \left[ N(p) - x \right]a; \qquad x(0) = x_0,$$

$$\text{(b) } \dot{\lambda} = r\lambda + \dot{x}\left( c'(x) - \frac{1}{N'(p)} \right); \qquad \lambda(T) = 0, \tag{11}$$

$$\text{(c) } p^* = \left( c - \lambda - \frac{N - x}{N'} \right)$$

and the price derivative is:

$$\dot{p}\left[ 2 - \frac{(N - x)N''}{N'^2} \right] = -r\lambda + 2\frac{(N - x)a}{N'} \tag{12}$$

from which it can be verified that $\dot{p} < 0$ for $r = 0$. For $r > 0$ we can characterize the optimal solution as follows:

THEOREM 5.   *For* $\dot{x} = [N(p) - x]a$, *if* $c'(x) \leqslant 0$, *then the optimal pricing policy is monotonically decreasing over time.*

PROOF.   See Appendix.

So as expected, the introduction of positive interest rates does not change the monotonicity of price decline.

3.3.2.   *Durable Goods with Word of Mouth.*   As we described earlier, in this case we assume that prior cumulative sales influences the rate of adoption for

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[13] See Jeuland (1981b) for a detailed discussion of related issues, and one way of modeling it. See also Kalish (1983) for a somewhat different approach.

those who are potential buyers. The formulation is:

$$(a)\ \dot{x} = (N(p) - x) \cdot h(x); \qquad x(0) = x_0,$$

$$(b)\ \dot{\lambda} = r\lambda + \dot{x}\left(c'(x) - \frac{1}{N'(p)} + \frac{(N - x) \cdot h'}{N'h}\right), \qquad (13)$$

$$(c)\ p = c - \lambda - \frac{(N - x)}{N'} \quad \text{and}$$

$$\dot{p}\left[2 - \frac{(N - x) \cdot N''}{N'^2}\right] = -r\lambda + \frac{2\dot{x}}{N'}\left[1 - \frac{(N - x) \cdot h'}{2h}\right]. \qquad (14)$$

The case of negative word of mouth is the easier one to characterize as follows:

THEOREM 6. *For demand* $\dot{x} = [N(p) - x] \cdot h(x)$, *if* $h'(x) \le 0$, $h''(x) \ge 0$, *and* $c'(x) \le 0\ \forall x$, *then the optimal price is monotonically decreasing.*

PROOF. See Appendix.

Note that the above sufficient conditions are not binding, i.e., price will monotonically decrease even under weaker conditions; however these are not easily interpretable.

The more interesting case, where $h'(x) > 0$, which is more likely for good products, offers no uniform answers. In this case, price path would depend on the relative intensity of the 'word-of-mouth' effect vs. the 'price discrimination' effect, cost decline, and interest rate. If the 'word-of-mouth' effect is strong at introduction, however, then increasing introductory price is still optimal. Since it is difficult to characterize the general conditions for which this will be true, we will illustrate it by an example.

EXAMPLE 3. Consider a conditional probability of purchase, given by $h(x) = a + b \cdot x^\alpha$. (Such conditional likelihood of purchase was used by Easingwood et al. (1982).) For simplicity, consider the no interest case ($r = 0$), with $x(0) = 0$. Equation (14) becomes:

$$\dot{p} \cdot \left[2 - \frac{(N - x)N''}{N'^2}\right] = \frac{2\dot{x}}{N'}\left[1 - \frac{(N - x)b\alpha x^{\alpha - 1}}{2(a + bx^\alpha)}\right]. \qquad (15)$$

For $\alpha < 1$, $\lim_{x \to 0} x^{\alpha - 1} = \infty$. Since $\lim_{x \to 0}(N - x) > 0$ (otherwise there would be no sales), it follows (assuming continuity of price) that $\dot{p}(0) > 0$ for some interval $[0, t]$.

Finally, we will show that obtaining an increasing price under this formulation is likely as compared with the separable demand case. Rewrite (14) as

follows: (for $r = 0$)

$$\dot{p}\left[2 - \frac{(N - x)N''}{N'^2}\right] = \frac{(N - x)}{-N'} \cdot \left[\frac{\partial f}{\partial x} - h\right].\qquad(16)$$

Recall that in the separable demand situation under zero interest price was increasing iff $\partial f/\partial x > 0$. Here we see that the condition for price increase is stronger, i.e. $\partial f/\partial x > h$. This underlines the importance of using the appropriate functional forms in the different cases, as the policy implications could be substantially different.

With positive interest rates, price shifts towards the short-term profit maximizing policy. This policy is characterized by:

$$p = c - \frac{N - x}{N'} \Rightarrow \dot{p}\left[2 - \frac{(N - x)N''}{N'^2}\right] = \dot{x}\left(c'(x) + \frac{1}{N'}\right).$$

Note that the price derivative here is monotonically decreasing. Moreover, even if there is no cost decline, or even cost increase with production, price still decreases over time, as long as $c'(x) < -1/N'$ $\forall t \in [0, T]$. Contrast this again with the myopic price for the separable demand, where $\dot{p}[2 - gg''/g'^2]$ $= c'(x) \cdot \dot{x}$. Here we see that price increases or decreases with costs. So at the other extreme, where interest rates are very large, we see that with the durable model there is a stronger tendency for a decreasing price over time.

### 4. Conclusions

In their review of pricing models, Monroe and Della Bitta (1978) question the usefulness of classical pricing models for managerial applications. They state that there are no models for new product pricing decisions or mature products.

This work is an attempt to look at the intertemporal pricing implications due to what we consider to be the most important dynamic factors, i.e., word of mouth for new products, production cost decline with experience, and saturation (for durable goods in particular). We now state the assumptions and summarize the results and then discuss the limitations of these assumptions and directions for future research.

### 4.1. *Summary of Assumptions and Results*

There are three major assumptions made in our analysis. First, we consider the case of a monopoly that does not concern itself with future entry in its pricing policy. This assumption could be realistic in situations where there is patent protection or other high barriers to entry.

The second assumption is that consumers do not anticipate future prices or have expectations about them, thus altering their behavior. Rather, we assume that they only react to current price.

The third assumption is that production cost can be accurately represented as a function of experience or cumulative production.[14]

These assumptions allow us to set the problem as follows: the monopolist wants to determine the unit price over the planning horizon in order to maximize discounted profits, given that unit cost declines with cumulative production (learning curve), and that sales rate depends not only on price, but also on cumulative sales (diffusion and saturation effects).

There are two questions we have focused on: How would this optimal price compare with the price that will maximize immediate profits at the particular time period? Secondly, how will the price trajectory over time look?

The price that will be charged by a monopolist that maximizes immediate profits is the price that equates marginal revenues with marginal cost. We have shown in the characterization of the general solution that this price is somewhat altered in our case. The rule here is that immediate marginal revenues plus the net future marginal benefit should be equal to marginal cost. The future marginal benefit of an additional unit now is its effect on reducing future costs, as well as its effect on demand. Thus, in the case where there are only production cost declines, but static demand, future benefits are always positive, and therefore price will be lower than the price that will maximize immediate revenues, except for the last time period.

Similarly, on the sales side, if additional units sold never have adverse effect on future demand, which is the case for 'good' repeat-purchase goods, then this factor alone would cause net future benefits to be positive, and price to be below the profit maximizing price. Clearly, combining cost learning and 'positive' diffusion enhances this effect.

If, on the other hand, there are adverse effects of current sales on future sales, a durable good for example, then the future marginal benefit of selling one more unit is negative. Therefore, price will always be above the immediate profit maximizing price, except for the last time period. However, combining that with learning curve cost decline and/or with early positive word of mouth, then the interaction between these factors will determine the net future marginal benefit. The results cannot be generalized beyond this.

The second question, of price trajectory over time, is difficult to analyze for the general case. In order to understand the effects of the different factors on the price trajectory, we analyzed them one at a time. First, for the simple case where there is only learning on the production side, then for positive discount rate, price will monotonically decrease over time. Although price is below the profit maximizing price for each period, price still decreases over time.

A second special case is where interest rates equal zero. In this case, the whole planning horizon is like one period, since there is no preference for early profits. We have shown that under this situation, price will be increasing or decreasing depending on the demand function alone. For a wide variety of cases, if there is an adverse reaction of current sales on future sales, then price will monotonically decrease over time. On the other hand, if there is a positive

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[14]While the above assumptions are typical in the literature, the applicability of the results under different realistic situations will be discussed later.

diffusion effect for the whole period, then for the subclass of multiplicatively separable demand functions, price is increasing. Moreover, in such situations, price is increasing whenever demand is positively affected by additional units sold and vice versa. Thus, for the typical durable case, where a positive word-of-mouth effect early in the life cycle is replaced by an adverse saturation effect later on, price will increase initially, representing penetration pricing followed by decreasing price later on. We have also shown that such penetration pricing is possible under a different formulation, if initial word-of-mouth effect is strong enough.

Combining the several factors to represent a more realistic situation naturally does not provide clear-cut answers. The higher the interest rate, the closer price is to myopic profit maximizing price. This, combined with declining costs, calls for decreasing price over time. If the demand is adversely affected by more units sold, then these two tendencies reinforce each other, and in most cases studied here, price is monotonically decreasing over time. If, on the other hand, there is positive diffusion effect, then the two factors counteract. However, we have shown that for the constant unit cost situation, and separable demand functions, the qualitative results remain the same,[15] i.e., if there are positive diffusion effects over the whole period, then price increases over time. If initial positive effects are replaced by negative effects later on, then price increases and then decreases. For the general case, not much can be said.

### 4.2. *Applicability and Validity*

Any model is a simplified representation of the real world. A successful model, however, must include the most important and relevant factors. Clearly, there is no situation where our simplifying assumptions hold absolutely. However, in cases where this is a good approximation, the model's implication should be applicable. (This, of course, can be tested with actual data, as will be outlined below.)

First, consider the no competition assumption. We have assumed here that demand depends on the monopolist price alone. Clearly, in the case of competition, demand will depend on competitors' price as well. In some situations, patent protection as an example, the monopoly assumption may serve as a good approximation. If there are no such barriers to entry, then we have to consider the effects of these recommendations on the firm's competitive position. For example, we saw that learning curve cost decline causes the firm to reduce price, thus investing in higher return in the future, as cost will be lower. Similar results are obtained on the demand side where positive word-of-mouth effect can be regarded as reputation. So, in these cases, the monopoly model recommendation improves the firm's competitive position relative to a myopic pricing policy by creating better market reputation and a cost advantage. (This is conditional on the appropriability of these investments, i.e., to what extent there are spill-over effects in cost production and product reputation.)

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[15] See Theorem 3 for exact details and additional restrictions.

Consider, next, the consumer expectation problem. To what extent are people going to change their behavior by anticipating future prices? This could be a central issue in cases where prices change rapidly. If price is decreasing over time, many people delay purchases (e.g. calculators, micro-computers), while if price is increasing, many individuals make early pur-chases (e.g. it is a common phenomenon in countries with a high rate of inflation, where prices are increasing frequently; usually there are long lines of buyers prior to such increases). However if the resulting optimal price does not change dramatically over time, this assumption can still be a good approxima-tion for the real world.

The third assumption, that unit cost depends on cumulative output alone, is less restrictive in my judgment, although this has to be verified for each particular case.

Do the results of this study provide insight to planners for pricing products over time? Suppose the three assumptions above hold, then there are several policy implications that stand out:

1. Learning cost and positive diffusion effect cause prices to be lower than the immediate profit maximizing price.

2. Learning cost alone, with no sales dynamics, causes price decline over time.

3. Positive diffusion effects for the whole period, with constant cost, cause low introductory price which increases later on (see Theorem 3 for exact details).

4. Durable goods price tends to fall over time, thus price discriminating over time. If initial word-of-mouth effect is strong enough, a lower introductory price, increasing and then decreasing, is possible.

5. If the separable demand function is a good approximation, and interest rate is low, then the policy guideline is as follows: Price is increasing as long as there are positive effects of sales on future demand, and vice versa.

This policy guideline is also appealing since it stabilizes production cycles, i.e., price is low if demand is lower and increasing, price increases as long as demand is increasing, and vice versa. Thus, price increases in anticipation of high demand, reducing production, and vice versa.

These conclusions are consistent with and unify the current state of knowl-edge. In particular, consider the two different types of models that were recently analyzed by Bass and Bultez (1982), and Dolan and Jeuland (1982) respectively. Bass and Bultez postulate that the diffusion effect is a function of time since introduction, while in the other formulation it is a function of the number of adoptors. Here we show that the results for particular parameter-ization reported in these two papers are actually true for general classes of problems; i.e., if changes in demand over time are a function of time alone then price would decrease monotonically given production cost decline. If, however, the expansion in demand is a result of the number of adopters, then it may actually be optimal to price low and increasing at introduction. We have also shown that for durable goods increasing price at introduction is not as likely as might be perceived by using the separable model, although it is a possibility if the "diffusion" effect is strong enough.

As for the bottom line, does it make a big difference in term of the present value of profits? Robinson and Lakhani (1975) report substantial differences between a myopic policy and the optimal policy, while Bass and Bultez (1982) report small differences. We can provide some insight into understanding under what conditions there would be big differences. Recall that $\lambda(t)$, the shadow price, measures the marginal effect of the number of adopters at time $t$ on the objective function, the discounted profits. While a full analysis of the parameters that affect the sensitivity of deviation from optimal policy on profits is not presented here, the following are some rules of thumb: (1) The higher the discount rate, the smaller the difference between myopic policies and optimal policies. This is intuitive, since if future profits are heavily discounted, then most that counts are early immediate profits. (2) The higher the rate of cost decline, particularly at introduction, the larger the difference in the bottom line. Again, this is intuitive, since if cost decline is large, then every unit produced affects future profits by lowering future costs. (3) The larger the positive diffusion effect is, the larger the difference between the two policies. Again, this is logical similarly to the above. Therefore, it is not all that surprising that Bass and Bultez report small differences, since they have used $r = 0.30$, which is a large discount rate (note that the relevant interest rate here is the real interest rate, after subtracting the inflation rate). Also in their formulation there are no diffusion effects, since the shifts in demand are a function of time, and not of the number of adopters. While the discount rate in Robinson and Lakhani's example was high as well (0.40), and the cost decline was similar, their formulation has the diffusion effect which made the difference.

In any real situation there are other marketing variables at the firm's disposal. Advertising can substitute for word of mouth, warranties can substitute for reducing uncertainties, etc. Therefore stimulating early adopters need not be done by price alone. However, since these factors exhibit decreasing return to scale, it is the author's conjecture that even in their presence the qualitative results presented above will still hold. That is, even if advertising is set at the optimal level, it is still more efficient to somewhat subsidize early adopters if their effect on increasing future demand is positive. This conjecture has support in Kalish (1983). It is also the author's conjecture that violation of the competition assumption, and the consumer expectations assumption will not change the qualitative nature of the results, but this is left for future work.

Finally, do we observe these pricing patterns in practice? Although a full-scale empirical analysis remains to be done, casual observations are consistent with the above implications:

· We observe price decline in situations where cost decline is a major factor (calculators, semiconductors).

· We observe low introductory prices for repeat purchase goods (coupons, free samples, specials).

· In durable goods, where 'penetration' pricing is less likely, we observe mainly price decline over time (televisions, Rubik's cube, cameras).

· In cases where initial word-of-mouth is crucial, we observe low introductory price (medical instrumentation, e.g., cat scanners).

### 4.3. Future Research

The main direction for future research is the introduction of competition. This is a very difficult task, due to the complex interaction between competitors over time, and uncertainty about future entry. However, several attempts have already been made (see Teng and Thompson (1980), Eliashberg and Jeuland (1982), Clarke and Dolan (1982), Spence (1981), and Bass and Rao (1982)).

A second direction is relaxing the assumption that cost declines with cumulative production alone. A recent attempt in this direction has been published recently (Clarke et al. (1982)). The 'nonanticipating consumer' assumption can be relaxed as well. There are several ways to introduce consumers' expectations (see, e.g., Bulow (1981), Schmalensee (1979)). Finally, an important direction for future research is the inclusion of other marketing mix variables, e.g. advertising. (Some recent attempts are Horsky and Simon (1982), Kalish (1983), Spremann (1981), and Teng and Thompson (1980).) In this context, heterogeneous population with respect to advertising and social-communication can be explored (see Jeuland (1981a)).

### Appendix.   Optimal Solution Characterization and Proof of Theorems

We assume $f(x, p)$ and $c(x)$ to be twice continuously differentiable, $f_p < 0$, $f \geq 0$ $\forall p \in (-\infty, \infty)$. We assume that a solution exists. Since the problem is unconstrained, the necessary condition at the optimal price is $H_p = 0$. (A general existence proof is difficult for this unconstrained problem. However, it is natural that any well-posed realistic problem has a solution indeed.) Moreover, at the optimal path, a necessary condition for a maximum is $H_{pp} \leq 0$, therefore it follows that (5) is true. We will use (5) frequently in the proofs.

The solution is not necessarily continuous, let alone differentiable. However, we assume the solution is twice differentiable in some of the proofs. For cases where the Hamiltonian is concave, however, then the necessary conditions above uniquely define the optimal solution, which ensures continuity and differentiability of the optimal solution. This condition can be checked for particular cases. It is conjectured that for most well-posed realistic problems the solution will be continuous and 'differentiable'.

Consider the more general case where we allow for exogenous shifts in demand over time, $\dot{x} = h(t) \cdot f(x, p)$. Characterization of optimal solution and price derivative for this system is similar to (2):

$$\text{(a) } \dot{x} = h(t) \cdot f(x, p); \qquad x(0) = x_0,$$

$$\text{(b) } \dot{\lambda} = r\lambda + \left( \frac{f_x}{f_p} + c' \right) \cdot \dot{x}; \qquad \lambda(T) = 0, \tag{A1}$$

$$\text{(c) } p = c - \lambda - \frac{f}{f_p}, \qquad \text{(d) } \dot{p}\left[ 2 - \frac{f_{pp}f}{f_p^2} \right] = -r\lambda - \left[ \frac{2f_x}{f_p} - \frac{f \cdot f_{px}}{f_p^2} \right] \cdot \dot{x}.$$

In the proofs that follow, we shall see that Theorems 1 and 2 hold for the more general system above.

PROOF OF THEOREM 1*.   Set all $f_x \equiv 0$ in (A1) above.

I. Since $c'(x) < 0$, $\Rightarrow \lambda(t) > 0 \ \forall t \in [0, t)$. Therefore, by (A1)(d) $\dot{p} < 0$.

II. Since $\lambda(t) > 0 \ \forall t \in (0, T)$, for a given $x$, the net present value maximizer (NPV) will price lower than the myopic monopolist (MYP). Therefore, at $t = 0$, $\dot{x}_{\mathrm{NPV}} > \dot{x}_{\mathrm{MYP}}$. Suppose by contradiction, $\exists \tilde{t}$ s.t. $x_{\mathrm{NPV}}(\tilde{t}) = x_{\mathrm{MYP}}(\tilde{t})$, then $\exists x_0 < \hat{x} < \tilde{x}(t)$ s.t. $\dot{x}_{\mathrm{NPV}}\big|_{x = \hat{x}} < \dot{x}_{\mathrm{MYP}}\big|_{x = \hat{x}}$, a contradiction. Since for any given $x$, $p_{\mathrm{NPV}}(x) < p_{\mathrm{MYP}}(x)$, it is clear that average NPV price is lower than average MYP price.

III. $e^{-rt} \cdot \lambda(t) = \int_t^T c'(x) \cdot \dot{x} \, d\tau$, which is clearly decreasing.   □

PROOF OF THEOREM 2*.   (A1)(d) becomes:

$$ \dot{p}\left[ 2 - \frac{gg''}{g'^2} \right] = - \frac{f'g^2}{g'} \cdot h(t). \quad \square $$

PROOF OF THEOREM 3.   From (10) it is obvious that $\mathrm{sign}(\dot{p}(T)) = \mathrm{sign}(f'(x(T)))$. Taking the time derivative of (10), and substituting $\dot{p} = 0$, $\dot{c} = 0$, we get that the second derivative of the optimal path at such point is:

$$ \ddot{p}\left[ 2 - \frac{g'' \cdot g}{g'^2} \right] = - \frac{f'' \cdot g^2}{g'} \cdot \dot{x}. $$

If $f''(x) \leqslant 0$, then:

(a) $f'(x(T)) > 0 \Rightarrow \dot{p}(T) > 0$. Assuming twice differentiability of $p(t)$, if $\dot{p}(t)$ changes signs, $\ddot{p} \leqslant 0$ at that point, $\Rightarrow \dot{p}(T) \leqslant 0$, contradiction. $\Rightarrow \dot{p}(t) > 0$ $\forall t \in [0, T]$.

(b) $\dot{p}(T) < 0$. Let $\tilde{t}$ be the point that $f'(x(\tilde{t})) = 0$.

$$ \lambda(\tilde{t}) = \int_{\tilde{t}}^T e^{-r(\tau - \tilde{t})} \left( - \frac{f'g^2}{g'} \right) d\tau < 0 \Rightarrow \dot{p}(\tilde{t}) > 0 $$

by (10). From (a), $\dot{p}(t) > 0 \ \forall t \in [0, t]$. Assuming regularity, $p(t)$ cannot change signs more than once after that, which completes the proof.

(c) Here $\dot{p}(T) < 0$, but one change of sign cannot be ruled out.

Note also that if $f''(x) > 0 \ \forall x$, then by a similar argument:

(a) If $f'(x(T)) < 0 \Rightarrow \dot{p}(t) < 0 \ \forall t \in [0, T]$.

(b) If $f'(x(0)) < 0$ and $f'(x(T)) > 0$, then $p(t)$ is decreasing and then increasing.

(c) If $f'(x(0)) > 0$, then $p(t)$ is either monotonic increasing, or decreasing and then increasing.   □

PROOF OF THEOREM 4.    In case we get that the second derivative of price at $\dot{p} = 0$ is:

$$\ddot{p}\left[2 - \frac{g'' \cdot g}{g'^2}\right] = -\frac{f'' \cdot g^2}{g'} \cdot \dot{x} - r \cdot c'(x) \cdot \dot{x}.$$

The theorem follows now as 3 above.    □

PROOF OF THEOREM 5.    This is a special case of Theorem 6.    □

PROOF OF THEOREM 6.    Substitute $\lambda$ from (13c) into (14), taking time derivative, and substituting $\dot{p} = 0$, we get:

$$\ddot{p} \cdot \left[2 - \frac{(N - x)N''}{N'^2}\right]$$

$$= -\dot{x}\left[r \cdot \left(c'(x) + \frac{1}{N'}\right) + \frac{1}{N'}\left(2h - 4h'(N - x) + (N - x)^2 \cdot h''\right)\right].$$

Since all terms in brackets are negative, $\ddot{p}_{|\dot{p}=0} > 0$. From (14), $\dot{p}(T) < 0$, so $\dot{p}(t)$ cannot change signs under the regularity conditions.    □

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