

## Monotone interaction of walk and graph: recurrence versus transience\*

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### Abstract

We consider recurrence versus transience for models of random walks on growing in time, connected subsets  $G_t$  of some fixed locally finite, connected graph, in which monotone interaction enforces such growth as a result of visits by the walk (or probes it sent), to the neighborhood of the boundary of  $G_t$ .

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## 1 Introduction

There has been much interest in studies of random walks in random environment (see [6]). Of particular challenge are problems in which the walker affects its environment, as in reinforced random walks. In this context even the most fundamental question of recurrence versus transience is often open. The recurrence of linearly edge reinforced random walks on graphs of bounded degree, under large enough reinforcement strength, and its transience on the integer lattice graphs  $\mathbb{Z}^d$ ,  $d \geq 3$  under small enough reinforcement, have been recently solved (see [2, 4, 11]). In contrast, for once edge-reinforced random walk, both M. Keane question about the recurrence on  $\mathbb{Z}^2$  (for any reinforcement strength), and the conjectured sensitivity of such recurrence to the strength of the reinforcement in  $\mathbb{Z}^d$ ,  $d \geq 3$  (see e.g. [7]), remain open. We consider here certain general models of such time-varying, highly non-reversible evolution (containing in particular the class of once edge-reinforced random walks). Specifically, similarly to [3] we study discrete time simple random walk (SRW)  $\{X_t\}$  on connected graphs  $G_t \uparrow G_\infty \subseteq \bar{G}$  (for some given, locally finite, possibly non-simple, connected graph  $\bar{G}$ , adopting the notation  $\mathbb{D}_t$  in case  $\bar{G} = \mathbb{Z}^d$ ). That is, starting at given  $G_0$  and initial site  $X_0 \in G_0$ , the sequence  $\{X_t\}$  is adapted to some filtration  $\{\mathcal{F}_t\}$  with  $\{G_t\}$  being  $\mathcal{F}$ -previsible (most often using  $\mathcal{F}_t = \sigma(X_s, s \leq t)$  the canonical filtration of the SRW), and having  $X_t = x$ , one passes to  $X_{t+1}$  along a uniformly chosen edge adjacent to  $x$  in  $G_t$ .

Our companion paper [3] deals with  $\{G_t\}$  growing independently of  $\{X_t\}$ , a situation in which universality is to be expected (c.f. [3, Conj. 1.2, 1.8 and 1.10]) and the analogous

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conjectures made in [1] for the corresponding strictly positive and finite conductances model). In contrast, rich and often counter intuitive behavior occurs when focusing on genuine monotone interaction between the path  $\{X_0, \dots, X_t\}$  of the walk and the growth  $\mathbb{G}_{t+1} \setminus \mathbb{G}_t$  of the graphs. In this context, our Lemma 1.4 provides an equivalent condition for transience/recurrence of the SRW on growing  $\{\mathbb{G}_t\}$ . We examine here its consequences for various monotone interactions. In particular, in *open by touch* type interactions a local or bounded number of edges is added to  $\mathbb{G}_t$  as result of each visit to its boundary sites (see Defn. 1.6). We then expect the SRW on  $\mathbb{G}_t$  to inherit the transience of  $\overline{\mathbb{G}}$  when starting at large enough  $\mathbb{G}_0$  (see Prop. 1.9 and Conjecture 1.12), while it should be recurrent when  $\mathbb{G}_0$  is small and  $\mathbb{G}_t$  almost regular (see Defn. 1.16 and Prop. 1.18, but beware the counter example of transience provided in Prop. 1.21). This recurrence should be related to such interaction requiring order of surface-area visits to the graph's boundary. In the same direction we find sharp transition between transience and recurrence at lower than surface-area growth of the number of boundary visits for *expanding glassy spheres* type interactions. These are of almost regular shape due to global growth upon completion of the required number of visits to the current graph's boundary, see Defn. 1.13 and Prop. 1.14. Finally, we consider *probing simple random walk* where a variable/fixed number of guided/unguided probes is sent from walker's current location, with each probe adding a site at the graph's boundary. In this setting guided probes may flip the walk between transience and recurrence, whereas for unguided probes the SRW supposedly inherits the transience/recurrence of the underlying graph  $\mathbb{Z}^d$  (see Prop. 1.22 and Conjecture 1.24).

Recall that for any time-homogeneous Markov chain  $\{Z_t\}$  on countable state space  $\overline{\mathbb{G}}$ , a zero-one law applies for the recurrence of state  $z \in \overline{\mathbb{G}}$ , namely for the event  $\{N_z = \infty\}$  and  $N_z := \sum_t \mathbb{I}_{\{z\}}(Z_t)$ . Further, such recurrence, i.e.  $\mathbb{P}_z(N_z = \infty) = 1$ , is equivalent to  $\mathbb{E}_z(N_z) = \infty$  and to  $\mathbb{P}_z(Z_t = z \text{ for some } t) = 1$ . In contrast, neither such equivalence, nor zero-one law apply in our more general setting of monotonically interacting SRW on growing graphs. For example, both zero-one law and equivalence break for suitable choices of  $\{1, \dots, \infty\}$ -valued random variable  $K$ , taking  $\mathbb{G}_t$ ,  $t \leq K$  a single edge adjacent to the origin and  $\mathbb{G}_t = \mathbb{Z}^3$  for  $t > K$ . This prompts our selection hereafter of the following definition of sample-path recurrence.

**Definition 1.1.** A site  $x \in \overline{\mathbb{G}}$  is recurrent for the sample path of SRW  $\{X_t\}$  on  $\{\mathbb{G}_t\}$  if  $\{X_t = x \text{ i.o.}\}$ . Otherwise we say that the site  $x \in \overline{\mathbb{G}}$  is transient for this sample path of the SRW on  $\{\mathbb{G}_t\}$ .

Let  $\deg_{\mathbb{G}}(z)$  denote the degree of vertex  $z$  in graph  $\mathbb{G}$ ,  $d^{\mathbb{G}}(x, y)$  the graph distance in  $\mathbb{G}$  between  $x, y \in \mathbb{G}$  and  $\mathbb{B}^{\mathbb{G}}(z, r)$  the corresponding (closed) ball of radius  $r$  and center  $z$  in  $(\mathbb{G}, d^{\mathbb{G}})$ , with  $\mathbb{B}_r$  denoting the projection on  $\mathbb{Z}^d$  of the (closed) Euclidean ball of radius  $r$  centered at the origin. Hereafter, we set  $X_0 = 0$ , assuming that  $\mathbb{B}^{\overline{\mathbb{G}}}(0, 1) \subset \mathbb{G}_0$ , and in view of the following lemma, focus without loss of generality on sample path recurrence of this distinguished vertex.

**Lemma 1.2.** For any SRW  $\{X_t\}$  on monotone increasing connected graphs  $\{\mathbb{G}_t\}$ , on the event  $\{X_t = 0, \text{ f.o.}\}$  of transience of 0 we have that a.s.  $d^{\mathbb{G}_t}(0, X_t) \rightarrow \infty$  when  $t \rightarrow \infty$ . Consequently, with probability one, 0 is transient for the sample path of the SRW if and only if every vertex of  $\overline{\mathbb{G}}$  is transient for this path.

*Proof.* Fixing  $r$  finite, let  $A_t = \{X_{t+u} = 0, \text{ some } u \geq 0\}$  and  $\Gamma_{t,r} = \{d^{\mathbb{G}_t}(0, X_t) \leq r\}$ . Since  $X_0 = 0$ , and  $\mathbb{G}_t \subseteq \overline{\mathbb{G}}$  are non-decreasing (so in particular  $\mathbb{B}^{\mathbb{G}_t}(0, r) \subseteq \mathbb{B}^{\mathbb{G}}(0, r)$ , for any  $t \geq 0$ ), and  $\overline{\mathbb{G}}$  is locally finite, it follows that

$$\mathbb{E}_{X_t}(\mathbb{I}_{A_t} | \mathcal{F}_t) \geq M_r^{-r} \mathbb{I}_{\Gamma_{t,r}},$$

with  $M_r := \max_{z \in \mathbb{B}_{\bar{\mathbb{G}}}(0,r)} \{\deg_{\bar{\mathbb{G}}}(z)\}$  finite. When  $t \rightarrow \infty$  we have that  $\mathbb{I}_{A_t} \rightarrow \mathbb{I}_{\{X_t=0 \text{ i.o.}\}}$  and

$$\liminf_{t \rightarrow \infty} \mathbb{I}_{\Gamma_{t,r}} = \mathbb{I}_{\{X_t \in \mathbb{B}_{\mathbb{G}_t}(0,r) \text{ i.o.}\}},$$

hence by Lévy's upward theorem, w.p.1. if  $\{X_t \in \mathbb{B}_{\mathbb{G}_t}(0,r) \text{ i.o.}\}$  then  $\{X_t = 0 \text{ i.o.}\}$ . Taking  $r \rightarrow \infty$  we deduce that w.p.1. transience at 0 of the sample path implies finitely many visits of  $X_t$  to  $\mathbb{B}_{\mathbb{G}_t}(0,r)$  for each  $r$ , hence both transience of every  $x \in \bar{\mathbb{G}}$  for this sample path and that  $d^{\mathbb{G}_t}(0, X_t) \rightarrow \infty$  when  $t \rightarrow \infty$ .  $\square$

**Definition 1.3.** With  $\deg_{\bar{\mathbb{G}}}(z)$  the maximal possible degree of vertex  $z$ , let

$$\partial\mathbb{G}_t := \{z \in \mathbb{G}_t : \deg_{\mathbb{G}_t}(z) < \deg_{\bar{\mathbb{G}}}(z)\}.$$

The boundary  $\partial\mathbb{G}_t$  is thus relative to  $\bar{\mathbb{G}}$ , consisting of all vertices of  $\mathbb{G}_t$  whose degree may yet increase as  $\mathbb{G}_t \uparrow \mathbb{G}_\infty$ . It is in general neither the vertex, nor the edge boundary of  $\mathbb{G}_t$  (indeed,  $\partial\mathbb{G}_t$  may be non-empty even when all vertices of  $\bar{\mathbb{G}}$  are in  $\mathbb{G}_t$ ).

Equipped with the preceding definition, we characterize transience via summability of  $p_n := \mathbb{P}(A_n | \mathcal{F}_{\eta_n})$ , for events

$$A_n := \{\exists s \in [\eta_n, \sigma_n) : X_s = 0\},$$

and the following  $\mathcal{F}_t$ -stopping times  $\{\eta_n, \sigma_n\}$ , starting at  $\eta_0 = 0$ :

$$\sigma_n := \inf\{t \geq \eta_n : X_t \in \partial\mathbb{G}_{\eta_n}\}, \quad n \geq 0 \tag{1.1}$$

$$\eta_{n+1} := \inf\{t \geq \sigma_n : X_t \notin \partial\mathbb{G}_t\}. \tag{1.2}$$

**Lemma 1.4.** Let  $S := \sum_n p_n$ .

(a) The sample path of  $\{X_t\}$  is a.s. recurrent on  $S = \infty$ ;

(b) Conversely, if srw on the fixed graph  $\bar{\mathbb{G}}$  is transient, then the sample path of  $\{X_t\}$  is a.s. transient on  $S < \infty$ .

**Remark 1.5.** In [1, Sections 4,5] it is shown that a monotonically interacting strictly positive and finite conductance model on a tree  $\bar{\mathbb{G}}$  tends to inherit the recurrence (and transience) property of its starting and ending conductances (in particular, this applies for  $\bar{\mathbb{G}} = \mathbb{Z}$ ). However, this approach, based on using flows to construct suitable sub or super martingales, is limited in scope to trees (indeed [1, Section 6] provides a counter example to such conclusion in case  $\bar{\mathbb{G}} = \mathbb{Z}^2$ ). In contrast, while less explicit, Lemma 1.4 applies for any  $\bar{\mathbb{G}}$ . Further, the advantage of this lemma lies in  $p_n$  being the probability that a srw on fixed graph  $\bar{\mathbb{G}}$  starting at the random position  $X_{\eta_n}$  visits 0 before  $\partial\mathbb{G}_{\eta_n}$ , hence amenable to the use of classical hitting probability estimates for random walk on a fixed graph.

*Proof.* Recall Paul Lévy's extension of Borel-Cantelli lemma (see [5, Theorem 5.3.2]), that a.s.  $S = \infty$  if and only if  $\{A_n \text{ i.o.}\}$  which immediately yields part (a). Further,  $\deg_{\mathbb{G}_t}(0) = \deg_{\bar{\mathbb{G}}}(0)$  for all  $t$  (by our assumption that  $\mathbb{B}_{\bar{\mathbb{G}}}(0,1) \subset \mathbb{G}_0$ ), hence  $X_s \neq 0$  whenever  $s \in [\sigma_n, \eta_n)$  and the a.s. transience of 0 for  $\{X_t\}$  in case  $S < \infty$  follows, provided  $\sigma_n < \infty$  for all  $n$ . To rule out having with positive probability  $\{\sigma_n = \infty \text{ and } X_t = 0 \text{ for infinitely many } t \geq \eta_n\}$ , note that by our assumption of transience of the srw on  $\bar{\mathbb{G}}$ , the former can not occur if  $\partial\mathbb{G}_{\eta_n} = \emptyset$ . So, assuming hereafter that  $\partial\mathbb{G}_{\eta_n}$  is non-empty, conditional on  $\mathcal{F}_{\eta_n}$ , if the irreducible srw on the fixed connected graph  $\mathbb{G}_{\eta_n}$  visits 0 i.o., then it a.s. would also enter  $\partial\mathbb{G}_{\eta_n}$  in finite time, namely having  $\sigma_n < \infty$ .  $\square$

Of particular interest to us are the *open by touch* type interaction models, in which graph growth occurs only upon the walker's visits of the graph's boundary sites.

**Definition 1.6.** We say that  $Y_t \in \mathbb{G}_t \uparrow \mathbb{G}_\infty \subseteq \overline{\mathbb{G}}$  is an open by touch (oBT) interaction model, if  $\mathbb{G}_{t+1} = \mathbb{G}_t$  except when  $Y_t \in \partial\mathbb{G}_t$ , at which times all edges of  $\mathbb{B}^{\overline{\mathbb{G}}}(Y_t, 1)$  are added to  $\mathbb{G}_{t+1}$ . More generally, in a partially open by touch (poBT) interaction we add to  $\mathbb{G}_{t+1}$ , with uniformly bounded away from zero probability, one (or more) of the edges adjacent to  $Y_t \in \partial\mathbb{G}_t$ , in a first open by touch (foBT) interaction such addition of edges adjacent to  $x \in \partial\mathbb{G}_t$  occurs only at the first visit of  $x$  by  $Y_t$ , whereas in a remotely open by touch (roBT) we only require that the collection of edges  $\mathbb{A}_t$  added to  $\mathbb{G}_t$  when  $Y_t \in \partial\mathbb{G}_t$  be of uniformly bounded cardinality.

**Remark 1.7.** The oBT interaction is both a poBT and a foBT interaction, with any poBT or foBT being in turn also a roBT interaction. The poBT is quite a general model, encompassing any once edge-reinforced walk  $Y_t$  on simple graph  $\mathbb{G}$  of uniformly bounded degrees, and rational  $q/p > 1$  reinforcement strength. Indeed, such process corresponds to  $\overline{\mathbb{G}}$  and  $\mathbb{G}_0$  having  $q > p \geq 1$ , respectively, parallel copies of each edge of  $\mathbb{G}$ , so  $\mathbb{B}^{\overline{\mathbb{G}}}(x, 1) = \mathbb{B}^{\mathbb{G}_t}(x, 1)$  for all  $t \geq 0$  (and  $Y_t$  evolves in  $\mathbb{G}_t$  as srw). With  $\mathbb{G}$  having uniformly bounded degrees, whenever  $Y_t$  is at the collection  $\partial\mathbb{G}_t$  of vertices from which emanates at least one non-reinforced edge, the probability of next taking one of the  $p$  copies of such an edge (and thereafter adding to  $\mathbb{G}_{t+1}$  its remaining  $(q - p)$  copies), is uniformly bounded away from zero.

In any oBT model the walker opens with a one step delay all edges of  $\overline{\mathbb{G}}$  adjacent to its current position, in effect performing srw on  $\overline{\mathbb{G}}$ , except at her first visit to certain sites. Thus, one may expect that oBT interactions inherit the transience of srw on  $\overline{\mathbb{G}}$ , as long as they follow that srw update rule, except maybe when at  $\partial\mathbb{G}_t$ . Indeed, we utilize such notion of extended simple random walks when studying this question (in the sequel).

**Definition 1.8.** We say that a roBT interaction model on  $\mathbb{G}_t \uparrow \mathbb{G}_\infty \subseteq \overline{\mathbb{G}}$  forms an extended simple random walk if  $Y_t$  follows the steps of srw on  $\overline{\mathbb{G}}$  except for allowing whenever  $Y_t \in \partial\mathbb{G}_t$  to have any  $\mathcal{F}_t$ -measurable mechanism for choosing  $Y_{t+1} \in \mathbb{G}_t \cap \mathcal{C}(Y_t)$ , for some fixed  $\mathcal{C}(x) := \mathbb{B}^{\overline{\mathbb{G}}}(x, r(x))$ ,  $c < 1$  and  $1 \leq r(x) \leq cd^{\overline{\mathbb{G}}}(x, 0)$ .

Specifically, with Lemma 1.4 applicable for extended simple random walks, we next prove that 0 is a.s. transient for the sample path in any poBT extended simple random walk, starting with  $\mathbb{D}_0 \subseteq \mathbb{Z}^d$ ,  $d \geq 3$  of fast enough diminishing density of closed edges. In contrast, [8] shows that the sample path  $\{Y_t\}$  is a.s. recurrent at  $Y_0 = 0$  for any evolution with  $\|Y_{t+1} - Y_t\|_1 = 1$  and  $\mathbb{E}[Y_{t+1} - Y_t | \mathcal{F}_t] = -\delta Y_t / \|Y_t\|_1$  for some  $\delta > 0$  at the first visit to a site  $Y_t \in \mathbb{Z}^d$ ,  $d \geq 3$ , while performing srw steps in all subsequent visits to a site (a model which is similar to an oBT extended random walk with  $r(x) = 1$  and  $\mathbb{D}_0 = \mathbb{B}(0, 1)$ ).

**Proposition 1.9.** If srw on  $\overline{\mathbb{G}}$  is transient, all sites (but not all edges), of  $\overline{\mathbb{G}}$  are in  $\mathbb{G}_0$  and

$$S_* := \sum_{x \in \partial\mathbb{G}_0} \sup_{y \in \mathcal{C}(x)} \{\mathbb{P}_y(\text{srw on } \overline{\mathbb{G}} \text{ ever hits } 0)\} < \infty, \tag{1.3}$$

then almost every sample path in any oBT extended simple random walk  $Y_t$  on  $\mathbb{G}_t \uparrow \mathbb{G}_\infty$  is transient. The same applies for any poBT provided  $\overline{\mathbb{G}}$  is of uniformly bounded degrees. In particular, in case  $\overline{\mathbb{G}} = \mathbb{Z}^d$ ,  $d \geq 3$ , denote by  $N(k)$  the number of vertices in  $\partial\mathbb{D}_0$  that are on the boundary of the box of side length  $k$  centered at 0. Then w.p.1. the sample path of poBT extended simple random walk  $Y_t$  on  $\mathbb{D}_t \uparrow \mathbb{D}_\infty$  is transient, provided

$$\sum_{k=1}^{\infty} N(k)k^{2-d} < \infty. \tag{1.4}$$

**Remark 1.10.** There is no analog of Proposition 1.9 for recurrent  $\overline{\mathbb{G}}$ . For example, taking  $\overline{\mathbb{G}} = \mathbb{Z}^2$  with  $Y_0 = 0$  in connected  $\mathbb{G}_0$  whose closed edges consist of exactly one among those touching sites  $(\pm s_t, 0)$  and  $(0, \pm s_t)$ , for  $s_t = \lceil (1 + c)^{t-1} \rceil$ ,  $t \geq 1$  and  $c < 1$  (hence of

fast diminishing density), one can create  $\text{obT}$  extended simple random walk with only four sample path,  $Y_t = (\pm s_t, 0)$  or  $Y_t = (0, \pm s_t)$ , all of which are transient.

**Remark 1.11.** Proposition 1.9 applies regardless of the manner and probability in which edges are added to  $\partial G_t$  in the  $\text{poBT}$  interaction, but this may be a somewhat delicate matter when starting with smaller graph  $G_0$ . For example, walking on sub-domains  $D_t$  of the recurrent  $\bar{G} = \mathbb{Z}^2$ , starting at  $D_0 = \mathbb{B}(0, 1)$ , Proposition 1.21 proves a.s. transience in  $\text{foBT}$  interaction for which only the right/up/down edges out of each site are added to  $D_{t+1}$  (upon first visit to the site by the  $\text{srw}$  on  $D_t$ ).

While Proposition 1.9 requires diminishing density of closed edges in  $D_0 \subseteq \mathbb{Z}^d$ ,  $d \geq 3$ , we believe that  $\text{srw}$  with  $\text{obT}$  interaction has a.s. transient sample path as soon as the open edges of  $D_0$  percolate in  $\mathbb{Z}^d$ . Specifically, we make the following conjecture.

**Conjecture 1.12.** For the  $\text{obT}$  interaction in  $\mathbb{Z}^d$ ,  $d \geq 3$ , upon starting its  $\text{srw}$  at  $0 \in D_0$ , if  $D_0$  is the infinite cluster of bond/site super-critical percolation, then the corresponding sample-path is a.s. transient.

Having seen the effect of the initial graph on recurrence versus transience for certain interacting walks and graphs, we turn to the implications of asymptotic regularity of  $G_t$ . To this end, we first consider *expanding glassy spheres* interactions, in which growth requires certain number of visits by the walk to the graph’s boundary, at which point a global expansion of the graph occurs.

**Definition 1.13.** Fix  $c \geq 1$ ,  $N(k) \geq 1$  and infinite (connected, locally finite), graph  $\bar{G}$ , setting  $\bar{B}_k := \mathbb{B}^{\bar{G}}(0, ck)$ . The *expanding glassy spheres (EGS) interaction* consists of  $\text{srw}$   $Z_t$  on  $G_t = \bar{B}_k$  for  $t \in [\tau_k, \tau_{k+1})$ , starting at  $Z_0 = 0$  and with  $\mathcal{F}^Z$ -stopping times  $\tau_1 := 0$ ,

$$\tau_{k+1} := \inf\{s > \tau_k : \sum_{t=\tau_k}^{s-1} \mathbb{1}_{\partial \bar{B}_k}(Z_t) = N(k)\}, \quad k \geq 1. \tag{1.5}$$

In case  $\bar{G} = \mathbb{Z}^d$ ,  $d \geq 1$ , we define such  $\text{EGS} \{Z_t\}$  as being confined to the projected Euclidean ball  $\mathbb{B}_{ck}$  until making the prescribed number of visits  $N(k)$  to its boundary, at which time this projected ball expands to  $\mathbb{B}_{c(k+1)}$ , and so on (instead of using the graph distance on  $\mathbb{Z}^d$  for defining such balls).

For a typical  $\bar{G}$  the size of  $\bar{B}_{k+1} \setminus \bar{B}_k$  grows unboundedly with  $k$ , in which case even for  $N(k) \equiv 1$ , the  $\text{EGS}$  interaction is not of the  $\text{roBT}$  type.

Employing Lemma 1.4 we determine the transition between recurrence and transience for  $\text{EGS}$  on  $\mathbb{Z}^d$ ,  $d \geq 2$  in terms of asymptotic growth of the prescribed hit counts  $\{N(k)\}$  (showing in particular that for  $d = 2$ , such  $\text{EGS}$  is always recurrent).

**Proposition 1.14.** For  $\bar{G}$  of bounded degrees and  $C_k := \{x \in \bar{B}_k : d^{\bar{G}}(x, \partial \bar{B}_k) = 1\}$ ,

$$\sum_{k=1}^{\infty} N(k) \inf_{x \in C_k} \mathbb{P}_x(\text{srw on } \bar{G} \text{ hits } 0 \text{ before } \partial \bar{B}_k) = \infty, \tag{1.6}$$

yields a.s. sample path recurrence for the corresponding  $\text{EGS}$ , whereas if

$$\sum_{k=1}^{\infty} N(k) \sup_{x \in C_k} \mathbb{P}_x(\text{srw on } \bar{G} \text{ hits } 0 \text{ before } \partial \bar{B}_{k+1}) < \infty, \tag{1.7}$$

then a.s. the sample path of the corresponding  $\text{EGS}$  is transient. In particular, for  $\bar{G} = \mathbb{Z}^d$ ,  $d \geq 2$ , we have zero-one law for transience/recurrence of the  $\text{EGS}$  interaction sample path which is a.s. transient if and only if

$$\sum_{k=1}^{\infty} N(k) k^{1-d} < \infty. \tag{1.8}$$

**Remark 1.15.** The assumed uniform bound on  $\deg_{\overline{\mathbb{G}}}(z)$  is only required for getting the factor  $N(k)$  within the sum on LHS of (1.6). To this end it suffices to have a uniform (in  $k$  and  $z$ ), upper bound on the expected hitting time of  $\mathcal{C}_k$  by the SRW on  $\mathbb{B}_k$  starting at  $z \in \partial\mathbb{B}_k$ . Some such condition is relevant for recurrence/transience of EGS. Indeed, consider  $c = 1$  and  $\overline{\mathbb{G}}$  arranged in layers, with each site in  $k$ -th layer (i.e. of distance  $k$  from 0), having  $\ell_k^\pm = \ell_k q_k p_k^\pm \geq 1$  edges to  $(k \pm 1)$ -th layer (with  $\ell_0^- = 0$ ), and  $\ell_k^0 = \ell_k(1 - q_k)p_k^-$  edges to its own layer. For EGS on such graph,  $t \mapsto d_{\overline{\mathbb{G}}}(0, Z_t)$  evolves up to holding times within layers, as a modified birth-death chain  $W_s$  on  $\mathbb{Z}_+$ , starting at  $W_0 = 0$ , moving with probability  $p_k^\pm$  from  $k$  to  $(k \pm 1)$ , but opening edge  $(k, k + 1)$  only after an independent Binomial( $N(k), q_k$ ) steps from  $k$  to  $(k - 1)$  are made by  $\{W_s\}$ . Conditions (1.6) and (1.7) amount to divergence and convergence, respectively, of  $\sum_k N(k) \mathbb{P}_{k-1}(W_s \text{ hits } 0 \text{ before } k)$ , which for uniformly bounded  $\{q_k\}$  is indeed a sharp criterion for recurrence/transience of  $\{W_s\}$  (and thereby of the EGS). However, the latter series is missing the factor  $q_k$ , so for unbounded  $\{q_k\}$  it is often wrong for determining transience versus recurrence of  $\{W_s\}$ .

Building on the insight provided by Proposition 1.14, we show that regularity, as defined below, of the random graphs  $\mathbb{D}_t$  produced by an ROBT interaction model on  $\mathbb{Z}^d$ ,  $d \geq 2$ , implies the a.s. sample path recurrence for the corresponding SRW.

**Definition 1.16.** We say that growing domains  $\{\mathbb{D}_t\}$  in  $\mathbb{Z}^d$ ,  $d \geq 2$ , admit a (non-random)  $\gamma$ -almost regular shape  $\mathbb{K}$ , if  $\mathbb{D}_t \supseteq f(t)\mathbb{K} \cap \mathbb{Z}^d$  for all  $t$  large and some non-decreasing  $f(\cdot) \geq 1$ , such that  $d^{\mathbb{D}_t}(z, f(t)\mathbb{K}) \leq \gamma \log f(t)$  for all  $z \in \mathbb{D}_t$ .

**Remark 1.17.** Some regularity of  $\mathbb{D}_t$  has been observed in simulations of specific ROBT interaction models. Unfortunately, there is no general proof of such, nor is it known for which interactions such regularity is strong enough to fit Definition 1.16.

**Proposition 1.18.** There exist  $c_d > 0$ , such that if  $0 \in \mathbb{D}_0$  a finite connected domain in  $\mathbb{Z}^d$ ,  $d \geq 2$  and the ball  $\mathbb{B}$  is  $c_d$ -almost regular shape for the growing domains  $\mathbb{D}_t$ , then 0 is a.s. recurrent for the sample path of any SRW  $\{R_t\}$  of ROBT interaction with  $\mathbb{D}_t$ .

**Remark 1.19.** If the range of an ROBT interaction model  $\{R_s, s \leq t\}$  contains the whole  $f(t)\mathbb{B}$  ball within  $\mathbb{D}_t$ , one may be tempted to conclude that recurrence of  $\{R_t\}$  then trivially follows since every site of  $\mathbb{Z}^d$  is for sure being visited at least once. However, as mentioned before, unlike SRW on fixed graph, in case of growing domains the walk may nevertheless w.p.1 return to 0 only finitely many times. So, Proposition 1.18 provides a non-trivial conclusion, even in this setting.

**Remark 1.20.** For any  $\mathbb{K} \subset \mathbb{R}^d$  and  $f > 0$ , let  $(f\mathbb{K})_d = f\mathbb{K} \cap \mathbb{Z}^d$  denote the corresponding lattice projection. Proposition 1.18 then holds for any  $\mathbb{K}$  such that for some  $c$  finite,

$$\liminf_{f \rightarrow \infty} \inf_{\{z \in (f\mathbb{K})_d : d(z, \partial(f\mathbb{K})_d) \geq c \log f\}} \frac{1}{\log f} \log \mathbb{P}_z(\text{SRW hits } 0 \text{ before } \partial(f\mathbb{K})_d) > -d. \quad (1.9)$$

Transience w.p.1. is proved in [1, Sect. 6] for the following monotone increasing conductance model on edges of  $\mathbb{Z}^2$ : starting at  $t = 0$  with walker at the origin and conductance 1 at each edge, upon walker's first visit of each vertex, the conductances of its adjacent edges to the right/up/down are increased to 2. Adapting the arguments of [1, Sect. 6], we next provide examples of a.s. transient FOBT interaction between SRW  $\{E_t\}$  and the corresponding growing domains  $\{\mathbb{D}_t\}$  in  $\mathbb{Z}^2$ , emphasizing the role of the initial graph  $\mathbb{D}_0$ .

**Proposition 1.21.** Consider the SRW  $\{E_t\}$  on  $\mathbb{D}_t \subseteq \mathbb{Z}^2$ , that starts from  $E_0 = 0 \in \mathbb{D}_0$  and opens only the three right/up/down edges adjacent to each site that it first visits (where after each such opening the walk stays put for one step before choosing its next position, now on  $\mathbb{D}_{t+1}$ ).

(a) If  $\mathbb{D}_0$  consists of the vertices of  $\mathbb{Z}^2$  with each edge of  $\mathbb{Z}^2$  independently chosen to be

in  $\mathbb{D}_0$  with same probability  $p \in [0, 1)$ , then the sample path of  $E_t$  is  $\mathbb{P}_p$ -a.s. transient.  
 (b) Alternatively, the sample path of  $\{E_t\}$  is a.s. transient whenever  $k^{-r}|\mathbb{D}_0 \cap [-k, k]^2| \rightarrow 0$  as  $k \rightarrow \infty$ , for some constant  $r < 3/4$ .

Our final result deals with a.s. transience for the *probing simple random walk* (PSRW),  $\{K_t\}$  on growing domains  $\{\mathbb{D}_t\}$  in  $\mathbb{Z}^d$ ,  $d \geq 2$ . Starting at  $K_0 = 0$  and  $\mathbb{D}_0 = \{0\}$ , such PSRW is allowed to send at time  $t$  some  $\mathcal{F}_t$ -adapted number of probes  $m(t)$ , with each probe adding precisely one site to  $\mathbb{D}_t$  (and opening all relevant edges connecting those sites with the existing graph), prior to the walk's move from  $K_t$  to  $K_{t+1} \in \mathbb{D}_{t+1}$ . The aim of the PSRW is to guarantee a.s. transience of its sample path with minimal asymptotic running average number of probes  $\bar{m}_t := t^{-1} \sum_{s=1}^t m(s)$ . Conversely, the PSRW may aim at a.s. recurrence of its sample path with a maximal asymptotic running average number of probes. In different versions of this problem the PSRW may or may not have control on the probes locations and the number of probes being used in each step.

**Proposition 1.22.**

- (a) For  $\mathbb{Z}^d$ ,  $d \geq 2$  and any  $\epsilon > 0$ , there exist  $\mathcal{F}_t$ -adapted  $\{m(t)\}$  and choices of the  $m(t)$  probe positions at  $\mathbb{Z}^d$ -distance-one from  $\mathbb{D}_t$ , such that eventually  $\bar{m}_t < \epsilon$  and the sample path of  $K_t$  is a.s. transient. There also exist (some other) such probe numbers and locations for which eventually  $\bar{m}_t > \epsilon^{-1}$  and the sample path of  $K_t$  is a.s. recurrent.
- (b) Suppose each probed site is chosen according to the hitting measure of  $\mathbb{D}_t^c$  by a SRW on  $\mathbb{Z}^d$  which starts at the current position  $K_t$  of the PSRW. Then there exist finite constants  $c_d$  and  $\mathcal{F}_t$ -adapted process  $\{m(t)\}$  such that a.s.  $\limsup_t \bar{m}_t < c_d$ , the PSRW sample path is transient in case  $d \geq 3$ , and recurrent with  $\liminf_t \bar{m}_t$  arbitrarily large, in case  $d = 2$ .

**Remark 1.23.** Two obvious open problems are whether part (b) of Proposition 1.22 holds for any  $c_d > 0$ ,  $d \geq 3$ , and whether in this context one can also select a process  $m(t)$  yielding a.s. sample path recurrence when  $d \geq 3$  and transience when  $d = 2$ .

We end with the following conjecture and related open problems.

**Conjecture 1.24.** In the setting of part (b) of Proposition 1.22 there exist  $\mathcal{F}_t$ -adapted  $m(t)$  which is uniformly bounded above by non-random integer  $\lambda_d$ , and a PSRW having a.s. transient sample path when  $d \geq 3$ , and a.s. recurrent sample path with  $m(t) \geq 1$ , when  $d = 2$ .

If this conjecture is valid, does it apply for  $\lambda_d = 1$  and does the same apply even for constant  $m(t) = \lambda_d$  (i.e. removing all control from the PSRW)?

**2 Proof of Propositions 1.9, 1.14 and 1.18**

*Proof of Proposition 1.9.* First consider an oBT extended simple random walk. Recall Remark 1.5 that  $p_n$  is the probability that SRW on  $\bar{\mathbb{G}}$  starting at  $Y_{\eta_n}$  visits 0 before  $\partial\mathbb{G}_{\eta_n}$ , and Definition 1.8 that  $Y_{\eta_n} \in \mathcal{C}(Y_{\eta_{n-1}})$ . Hence, setting

$$g(x) := \sup_{y \in \mathcal{C}(x)} \{\mathbb{P}_y(\text{SRW on } \bar{\mathbb{G}} \text{ ever hits } 0)\},$$

we have that for any  $n \geq 1$ ,

$$p_n \leq \mathbb{P}_{Y_{\eta_n}}(\text{SRW on } \bar{\mathbb{G}} \text{ ever hits } 0) \leq g(Y_{\eta_{n-1}}).$$

By assumption, all sites of  $\bar{\mathbb{G}}$  are already in  $\mathbb{G}_0$ , and with the oBT interaction enforcing that  $Y_t \notin \partial\mathbb{G}_{t+1}$ , the distinct sites  $\{Y_{\eta_{n-1}}, n \geq 1\}$  are all in  $\partial\mathbb{G}_0$ . Hence,  $S = \sum_{n \geq 1} p_n$  is bounded above by the assumed finite term  $S_*$  of (1.3) and the a.s. sample path transience of  $\{Y_t\}$  follows by part (b) of Lemma 1.4. In case of an poBT interaction, the same derivation yields the bound

$$S \leq \sum_{x \in \partial\mathbb{G}_0} L_x g(x),$$

where  $L_x$  denotes the number of visits by  $\{Y_t\}$  to  $x \in \partial G_0$ , up to the possibly infinite stopping time  $\theta_x := \inf\{t \geq 0 : \mathbb{B}^{\overline{G}}(x, 1) \subseteq G_t\}$ . The POBT interaction adds at least one edge to  $G_{t+1}$  upon each visit to  $Y_t \in \partial G_t$  with probability at least  $\epsilon > 0$ . Hence,  $\mathbb{E}[L_x | \mathcal{F}_0] \leq \epsilon^{-1} \text{deg}_{\overline{G}}(x)$ . In particular, almost surely,  $S$  is finite if

$$\mathbb{E}[S | \mathcal{F}_0] \leq \epsilon^{-1} \sum_{x \in \partial G_0} \text{deg}_{\overline{G}}(x) g(x) < \infty,$$

which, for  $\overline{G}$  of uniformly bounded degrees, follows from finiteness of  $S_*$ .

Specializing to  $\overline{G} = \mathbb{Z}^d$ ,  $d \geq 3$ , of uniformly bounded degree, recall Definition 1.8 that here  $\|y\|_1 \geq \|x\|_1 - \|x - y\|_1 \geq (1 - c)\|x\|_1$  for any  $y \in \mathcal{C}(x)$ . Thus, by the elementary potential theory formula

$$\mathbb{P}_y(\text{SRW on } \mathbb{Z}^d \text{ ever hits } 0) \leq c_d \|y\|_1^{2-d},$$

for some finite  $c_d$  and all  $y$  (see [9, Proposition 1.5.9]), we get that  $g(x) \leq \kappa_d \|x\|_1^{2-d}$ , for some  $\kappa_d$  finite, with condition (1.4) implying that  $S_*$  is finite.  $\square$

*Proof of Proposition 1.14.* Recall that for the EGS interaction, the underlying graph  $G_t$  is sequentially changed from  $\overline{B}_k$  to  $\overline{B}_{k+1}$  at the increasing stopping times  $\tau_{k+1}$ ,  $k \geq 1$ , of (1.5). In particular, the stopping times  $\{\eta_n\}$  of (1.2) in which the walk returns from  $\partial G_t$ , are such that  $G_{\eta_n} = \overline{B}_k$  whenever  $\eta_n \in [\tau_k, \tau_{k+1})$ . To each  $k \geq 1$  correspond  $L_k \in [1, N(k)]$  such stopping times. Further, when on  $\partial \overline{B}_k$  for some  $t \in [\tau_k, \tau_{k+1})$ , either the walk stays on  $\partial \overline{B}_k$  at time  $t + 1$  or else, it is then at  $C_k$ . Hence,  $Z_{\eta_n} \in C_k$  for all but the smallest of these stopping times (namely, where  $\eta_n = \tau_k$  for the given  $k \geq 2$ ), in which case  $Z_{\eta_n} \in \overline{B}_{k-1}$  is within distance one of  $\partial \overline{B}_{k-1}$ . Consequently,  $S$  of Lemma 1.4 is bounded above by the lhs of (1.7). Further, if  $\sup_z \text{deg}_{\overline{G}}(z) \leq \overline{\text{deg}}$  finite, then conditional on  $\mathcal{F}_t^Z$ , upon each visit of  $\partial \overline{B}_k$  by  $Z_t$  (i.e. time  $\sigma_n$ ), we have that  $Z_{t+1}$  is not in  $\partial \overline{B}_k$  with probability at least  $\epsilon := 1/\overline{\text{deg}}$ . It follows that the collection  $\{L_k\}$  stochastically dominates the independent Binomial( $N(k), \epsilon$ ) variables  $\{L'_k\}$ , hence  $S$  stochastically dominates the lhs of (1.6) with  $N(k)$  replaced there by  $L'_k$ . The latter is the monotone upward limit  $T_\infty$  of a series  $T_n = \sum_{k=1}^n q_k L'_k$ , with  $q_k \in [0, 1]$  non-random and condition (1.6) amounting to  $\mathbb{E}T_n \uparrow \infty$ . With  $\text{var}(T_n) \leq \mathbb{E}T_n$ , we have that  $T_n/\mathbb{E}T_n \rightarrow 1$  in probability, hence (1.6) yields that a.s.  $S \geq T_\infty = \infty$ . Our thesis about the a.s. transience and recurrence of the corresponding EGS thus follows from Lemma 1.4 (we note in passing that the assumption of  $\overline{G}$  transient is only used in part (b) of Lemma 1.4 for dealing with  $\partial G_{\eta_n} = \emptyset$ , which can not occur for EGS).

In case of  $\overline{G} = \mathbb{Z}^d$ ,  $d \geq 3$ ,  $c \geq 1$ , upon replacing  $\overline{B}_k$  by  $B_{ck}$  it remains only to verify that our conditions (1.6) and (1.7) are equivalent to the divergence, respectively convergence, of  $\sum_k N(k)k^{1-d}$ . This follows by potential theory, since

$$k^{d-1} \mathbb{P}_x(\text{SRW on } \mathbb{Z}^d \text{ hits } 0 \text{ before } \partial B_{ck}), \tag{2.1}$$

is bounded above and below away from zero, uniformly over  $k \geq 1$  and  $x \in B_{ck}$  whose graph distance from  $\partial B_{ck}$  is between 1 and  $2dc$ . We note in passing that having here  $x$  within constant distance of  $\partial B_{ck}$ , the standard error term turns out to be  $O(1)$  (see formula of [9, Proposition 1.5.10]), so for the stated uniform lower bound it must be refined by using asymptotics of Green's function (cf. [10, Page 96]). In case  $d = 2$ , the probabilities appearing in (2.1) are similarly bounded below by  $C/(k \log k)$  for some  $C > 0$ , all  $k$  and relevant  $x$  (see [9, Propostion 1.6.7]; here the error term is refined using the asymptotics of potential kernel, cf. [10, Page 104]). With  $N(k) \geq 1$ , it follows that in this case (1.6) holds, yielding the a.s. recurrence of the EGS, in agreement with the



divergence of  $\sum_k N(k)k^{1-d}$  for  $d = 2$ . □

*Proof of Proposition 1.18.* With  $\xi_r := \inf\{t \geq 0 : \mathbb{D}_t \cap \mathbb{B}_r^c \neq \emptyset\}$ , denoting the first time the tip of  $\mathbb{D}_t$  reaches the sphere of radius  $r$  around 0, we construct  $L = O(m^d/\log m)$  stopping times  $\sigma_1 < \sigma_2 < \dots < \sigma_L$  within the time interval  $[\xi_m, \xi_{2m-1}]$ , such that for some constant  $\delta < 1$ ,

$$m^{d-1+\delta} \mathbb{P}(R_s = 0 \text{ for some } s \in [\sigma_\ell, \sigma_{\ell+1}) \mid \mathcal{F}_{\sigma_\ell}) \tag{2.2}$$

is bounded away from zero, uniformly in  $\ell$  and  $m$ . Similarly to the proof of Lemma 1.4, upon considering the union of these events over all dyadic  $m = 2^k$ , the a.s. sample path recurrence of the ROBT interacting SRW  $\{R_t\}$  then follows by Paul Lévy's extension of Borel-Cantelli. To this end, since the Euclidean ball  $\mathbb{B}$  is  $\gamma$ -almost regular for the growing domains  $\mathbb{D}_t$ , it follows that for all  $t$  large and some non-increasing  $f(\cdot) \geq 1$ ,

$$\mathbb{B}_{f(t)} \subseteq \mathbb{D}_t \subseteq \mathbb{B}_{f(t)+\gamma \log f(t)}. \tag{2.3}$$

We set  $w := \gamma \log(2m)$ , the maximal fluctuation  $\gamma \log f(t)$  in shape of  $\mathbb{D}_t$  when  $t \leq \xi_{2m-1}$  (hence  $f(t) \leq 2m$ ). From (2.3) one has that  $\mathbb{D}_{\xi_{2m-1}} \supseteq \mathbb{B}_{2m-w}$  (as otherwise  $2m-w > f \geq 2m - \gamma \log f$  for some  $f = f(\xi_{2m-1}) \leq 2m$ , contradictory to our choice of  $w$ ). There are thus at least  $C'm^d$  edges in  $\mathbb{D}_{\xi_{2m-1}} \setminus \mathbb{D}_{\xi_m}$ , for some universal constant  $C' > 0$  and all  $m$ . Further, domain growth occurs in ROBT only when  $R_t \in \partial\mathbb{D}_t$ , and each such boundary visit entails adding at most  $C'/C$  edges to  $\mathbb{D}_{t+1}$  for some universal constant  $C > 0$  (see Defn. 1.6). So for each  $m$  there are at least  $Cm^d$  such boundary visits within  $[\xi_m, \xi_{2m-1}]$ .

Hereafter we fix  $\epsilon > 0$  small, set  $w_\epsilon = (1 + 2\epsilon)w$ ,  $L = Cm^d/w_\epsilon$  and consider the stopping times  $\{\sigma_\ell, 1 \leq \ell \leq L\}$ , with  $\sigma_\ell$  denoting the  $w_\epsilon \ell$ -th smallest  $t \geq \xi_m$  such that  $R_t \in \partial\mathbb{D}_t$ . Turning to prove the stated uniform probability lower bound for the corresponding events per (2.2), fix  $\sigma = \sigma_\ell$  and  $f = f(\sigma) \in \mathcal{F}_\sigma$ . Recall Defn. 1.6 that  $d^{\mathbb{D}_\sigma}(R_\sigma, \mathbb{B}_f) \leq w$ , hence there exists a path in  $\mathbb{D}_\sigma$  of length at most  $w_\epsilon - 1$  leading from  $R_\sigma$  to some specific  $x \in \mathbb{B}_f$  such that  $d^{\mathbb{B}_f}(x, \partial\mathbb{B}_f) \geq \epsilon w$ . Setting  $\delta := (1 + 2\epsilon)\gamma \log(2d)$ , the event  $\mathcal{A}_\ell$  that the SRW  $\{R_{\sigma+s}, s \geq 0\}$  on  $\mathbb{D}_{\sigma+s}$  takes this specific path has probability at least  $(2d)^{-w_\epsilon} = (2m)^{-\delta}$ . Since  $\sigma_{\ell+1} \geq \sigma_\ell + w_\epsilon$  and  $\mathbb{B}_f \subseteq \mathbb{D}_{\sigma_\ell}$ , the event considered in (2.2) contains the intersection of  $\mathcal{A}_\ell$  and the event that starting at position  $x$  the SRW on  $\mathbb{B}_f$  visits 0 before reaching  $\partial\mathbb{B}_f$ . Clearly,  $f \geq m - \gamma \log f \geq m - w \geq m/2$  (by (2.3)). Here  $x \in \mathbb{B}_f$  is of Euclidean distance at least  $C_0 \log m$  from  $\partial\mathbb{B}_f$  for some constant  $C_0 = C_0(\epsilon, \gamma, d) > 0$ , all  $m$  and  $\ell$ . Hence, by potential theory, the probability that SRW starting at  $x$  visits 0 before  $\partial\mathbb{B}_f$ , is bounded below by  $\kappa_d m^{1-d}$  for some  $\kappa_d > 0$ , all  $d \geq 2$ ,  $m$  and any such  $x$  (see (2.1) in case  $d \geq 3$ , and text following it for how to handle  $d = 2$ ). In conclusion, as claimed, uniformly in  $\ell$  and  $m$ ,

$$\mathbb{P}(R_s = 0 \text{ for some } s \in [\sigma_\ell, \sigma_{\ell+1}) \mid \mathcal{F}_{\sigma_\ell}) \geq \kappa_d m^{1-d} \mathbb{P}(\mathcal{A}_\ell \mid \mathcal{F}_{\sigma_\ell}) \geq \kappa_d m^{1-d} (2m)^{-\delta}. \tag{2.4}$$

□

### 3 Proof of Propositions 1.21 and 1.22

*Proof of Proposition 1.21.* We say that the time  $m \geq 0$  of first visit to some  $E_m$  is *left-neighbor-non-visited* (LNNV), if  $\{(-1, 0) + E_m, E_m\}$  is unvisited by  $\{E_t, t < m\}$ . We further couple our FOBT walk  $\{E_t\}$  to the SRW  $\{R(t)\}$  on  $\mathbb{Z}^2$ , both starting at  $(0, 0)$ , so that  $E_{t+1} - E_t = R(t+1) - R(t)$  except if the edge to the left of  $E_t$  is not in  $\mathbb{D}_t$ , in which case with probability 1/4 both walks have the same right/up/down increment, while with probability 1/12 each,  $E_{t+1} - E_t$  is the right/up/down increment, while  $R(t+1) - R(t) = (-1, 0)$ . Clearly,  $t \mapsto (E_t - R(t))_1$  is then non-decreasing. Moreover, independently of

$\{E_t, R(t), t \leq m\}$ , with probability  $1/4$  the value of  $(E_t - R(t))_1$  increases by at least one at each LNNV-time  $m$  for which the edge to the left of  $E_m$  is not in  $\mathbb{D}_0$ . Fixing  $\epsilon > 0$ , let  $\mathcal{A}_n$  denote the event that there exist  $n^{3/4-2\epsilon}$  LNNV times  $m \in [0, n]$  with the edge to the left of  $E_m$  not being in  $\mathbb{D}_0$ . If  $\mathbb{P}(\mathcal{A}_n) \geq 1 - Cn^{-1}$  for some  $C$  finite, then a.s.  $(E_n - R(n))_1 \geq 0.1n^{3/4-2\epsilon}$  for all  $n$  large (by Borel-Cantelli lemma it holds along dyadic  $n_k = 2^k$ , which by monotonicity of  $(E_n - R(n))_1$  extends to all  $n$  large). Since  $n^{-1/2-\epsilon}|R(n)| \rightarrow 0$ , taking  $\epsilon < 1/12$  yields the stated a.s. sample path transience of  $\{E_t\}$ .

Adapting [1, Sect. 6], we proceed to show that indeed  $\mathbb{P}(\mathcal{A}_n) \geq 1 - Cn^{-1}$  for some  $C$  finite and all  $n$ . To this end, first analogously to [1, Lemma 6.1], we know that  $\inf_n (\log n) \mathbb{P}_{(0,0)}(\mathcal{I}_n) \geq C$ , for some  $C$  positive and events

$$\mathcal{I}_n := \bigcap_{t \leq n} \left\{ R(t) \notin \{(-1, 0), (0, 0)\} \right\}.$$

Next, fixing  $n$ , similarly to [1, Lemma 6.4] we call  $m \in [n^{2\epsilon}, n]$  a *tan time* if  $R[m - \lfloor n^\epsilon \rfloor, m]$  avoids  $\{(-1, 0), (0, 0)\} + R(m)$ , and  $R[0, m - \lfloor n^\epsilon \rfloor]$  avoids  $F + R(m)$ , for the funnel

$$F := \{(x, y) : x \geq -1, |y| \leq \log^3(n\sqrt{x+2})\}.$$

Equipped with this modification of tan time, it is easy to adapt the proof of [1, Lemma 6.6], yielding that the number of  $n^\epsilon$ -separated tan times within  $[0, n]$ , exceeds  $n^{3/4-2\epsilon}$  with probability at least  $1 - Cn^{-2}$ . Then, following the proof of [1, Lemma 6.9], we deduce that under our coupling, for some  $C$  finite, with probability at least  $1 - Cn^{-1}$ , whenever  $m < n$  is a tan time for  $\{R(t)\}$ , at least one of  $[m - n^\epsilon, m]$  must be a LNNV time for  $\{E_t\}$ . Consequently, with such probability there are at least  $n^{3/4-2\epsilon}$  LNNV times  $m \in [0, n]$ . It thus suffices to show that a uniformly bounded away fraction of these times has edge left of  $E_m$  that is not in  $\mathbb{D}_0$ .

(a). We reveal whether each of the i.i.d. Bernoulli( $p$ ) edges is in  $\mathbb{D}_0$  or not, only when our FORT walk  $\{E_t\}$  first visits one of the two ends of that edge. Hence, ordering the LNNV times  $m_1 < m_2 < \dots$ , since each LNNV time avoided both the FORT walk current position and the lattice site immediately to its left, we have not revealed up to time  $m_k$  whether the edge left to  $E_{m_k}$  is in  $\mathbb{D}_0$  or not. Thus, the joint law of events  $\{\text{edge left to } E_{m_k} \text{ is not in } \mathbb{D}_0\}$  stochastically dominates the corresponding i.i.d. Bernoulli( $1 - p$ ) variables. It then follows that with  $\mathbb{P}_p$ -probability at least  $1 - Cn^{-1}$ , for more than  $(1 - p)/2$  of the first  $n^{3/4-2\epsilon}$  LNNV times  $m_k$ , the edge left of  $E_{m_k}$  is not in  $\mathbb{D}_0$ , as claimed.

(b). With  $\{E_t, t \leq n\} \subset [-n, n]^2$ , taking  $\epsilon > 0$  small enough, our assumption that  $k^{-r}|\mathbb{D}_0 \cap [-k, k]^2| \rightarrow 0$  for  $r < 3/4 - 2\epsilon$  implies that at least half of the edges left to locations of the FORT walk at the first  $n^{3/4-2\epsilon}$  LNNV times, are not in  $\mathbb{D}_0$ , as claimed.  $\square$

*Proof of Proposition 1.22.*

(a). For any integers  $d, L \geq 2$ , consider the stretched lattice  $\mathbb{L}$ , consisting of vertices

$$\{(y_1, \dots, y_d) \in \mathbb{Z}^d : \text{at most one } y_i \text{ is not an integer multiple of } L\},$$

and the edges of  $\mathbb{Z}^d$  between them (i.e. connecting pairs of vertices from  $\mathbb{L}$  whose  $\mathbb{Z}^d$ -distance is one). Denoting by  $\mathbb{J} := (L\mathbb{Z})^d$  the subset of junction sites in  $\mathbb{L}$  and fixing  $d \geq 3$ , whenever  $K_t$  visits a site  $z$  of  $\mathbb{L}$  for the first time, it dispenses one probe per adjacent closed edge of  $\mathbb{L}$  (thereby using at most  $(2d - 1)$  probes if  $z \in \mathbb{J}$  and at most one probe otherwise). The resulting PSRW  $\{K_t\}$  has the same law as the SRW  $\{B_t\}$  on the fixed graph  $\mathbb{L}$  which is transient (having finite effective resistance between  $0$  and  $\infty$ ). The number of steps  $\tau_i$  it takes  $\{B_t\}$  to travel from any junction site in  $\mathbb{J}$  to one of its neighboring junction sites, are i.i.d. random variables whose mean being precisely the number of steps it takes the SRW on  $\mathbb{Z}$  to reach from  $0$  to  $\pm L$ . Thus,  $\mathbb{E}\tau_1$  is of order  $L^2$  (by

diffusivity of the  $\text{SRW}$  on  $\mathbb{Z}$ ). Further, during such  $\tau_i$  steps at most  $2dL$  vertices of  $\mathbb{L}$  are visited by our  $\text{PSRW}$ , hence at most  $(2d)^2L$  new probes are being used. Denoting by  $n(t)$  the number of visits made by  $\{K_s, s \leq t\}$  to the subset  $\mathbb{J}$ , recall that a.s.  $t^{-1}n(t) \rightarrow 1/\mathbb{E}\tau_1$  hence

$$\bar{m}_t \leq t^{-1}(2d)^2L(n(t) + 1),$$

is eventually bounded above by  $c_d/L$  for some non-random  $c_d$  finite and all  $L$  (which for  $L \rightarrow \infty$  is made as small as one wishes). In case  $d = 2$ , we modify the preceding construction by using our probes upon first visit of the  $\text{PSRW}$  to sites  $z \in \mathbb{J}$  *only* for opening its right/up/down adjacent edges in  $\mathbb{L}$ . The projection of the resulting  $\text{PSRW}$  to the subset  $\mathbb{J}$  of the stretched lattice  $\mathbb{L}$  is then a lazy version of the  $\text{FOBT}$  interaction model considered in Proposition 1.21 (for  $\mathbb{D}_0 = \{0\}$  and having here probability  $1 - 1/L$  of returning to the current position before reaching an adjacent junction site). Since by Proposition 1.21 the sample path of this  $\text{FOBT}$  walk on  $\mathbb{J}$  is a.s. transient, the same applies for our  $\text{PSRW}$ , while by the preceding reasoning  $\bar{m}_t$  is made as small as one wishes upon choosing  $L \rightarrow \infty$ .

As for the stated sample path recurrence, fix  $d, M \geq 1$  and the one-dimensional subspace  $\mathbb{O} := \mathbb{Z} \times \{0\} \times \dots \times \{0\}$  of  $\mathbb{Z}^d$ . Here we use  $2M$  probes each time step, placing these on the edges within  $\mathbb{O}$  that are adjacent to the currently symmetric open interval  $\mathbb{D}_t \subset \mathbb{O}$ . The resulting  $\text{PSRW}$  has  $\bar{m}_t = 2M$  as large as we wish and merely follows the path of the recurrent one-dimensional  $\text{SRW}$  on  $\mathbb{O}$ .

(b). Here a probe emitted at time  $t$  follows a  $\text{SRW}$  on  $\mathbb{D}_t$ , starting from the current position  $K_t$  of our  $\text{PSRW}$ . Fixing  $d \geq 2$ , we now opt to release at the first visit of our  $\text{PSRW}$  to each site  $z \in \mathbb{Z}^d$ , the  $\mathcal{F}_t$ -adapted minimal number of probes  $m(t)$  required for opening all  $2d$  edges of  $\mathbb{Z}^d$  adjacent to  $z$  (i.e.  $m(t) = \inf\{s \geq 1 : \mathbb{B}^{\mathbb{Z}^d}(z, 1) \subseteq \mathbb{D}_{t+s}\}$ ). It results with the sample path of  $\{K_t\}$  matching that of a  $\text{SRW}$  on  $\mathbb{Z}^d$ , which is thereby a.s. transient when  $d \geq 3$  and a.s. recurrent when  $d = 2$ . Further, the sequence  $\{m(t)\}$  of probe counts is stochastically dominated by the i.i.d. variables  $\xi_t - 1$ , with  $\xi_1$  following the  $2d$  coupon collector distribution (i.e. the number of independent, uniform samples from among  $2d$  distinct coupons one needs for possessing a complete set). Thus, by the  $\text{SLLN}$ , almost surely,

$$\limsup_{t \rightarrow \infty} \bar{m}_t \leq \mathbb{E}\xi_1 - 1 = c_d$$

(for  $c_d := 2d \sum_{\ell=1}^{2d-1} \ell^{-1}$ ). The same transience/recurrence holds even when extra  $M$  probes are emitted at *each* step of the  $\text{PSRW}$  (yielding  $\bar{m}_t \geq M$  arbitrarily large).  $\square$

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