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Monotone iterative technique for a nonlinear fractional q -difference equation of Caputo type

Guotao Wang¹, Weerawat Sudsutad², Lihong Zhang¹ and Jessada Tariboon^{2*} 

*Correspondence:

jessada.t@sci.kmutnb.ac.th

²Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand
Full list of author information is available at the end of the article

Abstract

By establishing a comparison theorem and applying the monotone iterative technique combined with the method of lower and upper solutions, we investigate the existence of extremal solutions of the initial value problem for fractional q -difference equation involving Caputo derivative. An example is presented to illustrate the main result.

Keywords: Caputo q -fractional derivative; comparison theorem; monotone iterative sequences; extremal solutions

1 Introduction

The quantum calculus (calculus without limits or q -calculus) is not of recent appearance. It appeared as a connection between mathematics and physics. The quantum difference operator has a lot of applications in different mathematical areas, such as number theory, combinatorics, special functions, basic hyper-geometric functions, the calculus of variations, control theory, mechanics, and the theory of relativity. For the basic concepts of q -calculus, we refer the reader to [1]. Recently, the topic of quantum calculus has attracted the attention of several researchers and a variety of new results can be found in [2–11] and references cited therein.

The monotone iterative technique, combined with the method of lower and upper solutions, is an interesting and effective technique for proving the existence of solutions for initial and boundary value problems of nonlinear differential equations. The basic idea of this method is that by using the upper and lower solutions as an initial iteration, one can construct the monotone sequences for a corresponding linear equation and that converge monotonically to the extremal solutions of the nonlinear equation. So many authors developed the upper and lower solutions methods to solve fractional differential equations; for examples, see [12–26].

Motivated by the above-mentioned work, we investigate the existence of extremal solutions for the following initial value problem of a nonlinear fractional quantum difference equation:

$$\begin{cases} {}^C D_q^\alpha u(t) = f(t, u(t)), & t \in J, \\ u(a) = u_0, \end{cases} \quad (1.1)$$

where ${}^C D_q^\alpha$ denotes the Caputo q -fractional derivative of order α , $0 < \alpha \leq 1$, $J = [a, b]$, the function $f \in C(J \times \mathbb{R}, \mathbb{R})$, and the constant $u_0 \in \mathbb{R}$.

The aim of this paper is to extend the method of upper and lower solutions coupled with the monotone iterative technique to fractional q -difference equations. In order to apply the method, we establish a comparison theorem involving the Caputo q -fractional derivative, which plays a crucial role in this paper. To the best of our knowledge, it is the first crack at applying the method to a fractional q -difference equations of Caputo type.

The rest of paper is organized as follows: In Section 2, we present some preliminary notations, definitions and lemmas that we need in the sequel. In Section 3, we discuss the main result, while an example is presented to illustrate the main result in Section 4.

2 Preliminaries

We give some notations, definitions, and preliminary facts which are used throughout the paper.

To begin with, we give some properties of a q -shifting operator ${}_a \Phi_q(m) = qm + (1 - q)a$ that can be found in [9].

Property 2.1 [9] For any $a, m, n \in \mathbb{R}$, and for all positive integer k the following properties hold:

- (i) ${}_a \Phi_q^k(m) = {}_a \Phi_q^{k-1}({}_a \Phi_q(m))$ and ${}_a \Phi_q^k(m) = {}_a \Phi_q^k(m)$ with ${}_a \Phi_q^0(m) = m$.
- (ii) ${}_a(n - m)_q^{(0)} = 1$, ${}_a(n - m)_q^{(k)} = \prod_{i=0}^{k-1} (n - {}_a \Phi_q^i(m))$, $k \in \mathbb{N} \cup \{\infty\}$.
- (iii) ${}_a(n - m)_q^{(\gamma)} = n^{(\gamma)} \prod_{i=0}^{\infty} \frac{1 - \frac{a}{n} \Phi_q^i(m/n)}{1 - \frac{a}{n} \Phi_q^{\gamma+i}(m/n)}$ for $\gamma \in \mathbb{R}$.

We recall some basic concepts of q -calculus [9].

The q -derivative of a function f on the interval $[a, b]$ is defined by

$$({}_a D_q f)(t) = \frac{f(t) - f({}_a \Phi_q(t))}{(1 - q)(t - a)}, \quad t \neq a, \quad ({}_a D_q f)(a) = \lim_{t \rightarrow a} ({}_a D_q f)(t),$$

and the q -derivative of higher order is given by

$$({}_a D_q^0 f)(t) = f(t), \quad ({}_a D_q^k f)(t) = {}_a D_q^{k-1} ({}_a D_q f)(t), \quad k \in \mathbb{N}.$$

The q -derivatives of a product and ratio of functions f and g on $[a, b]$ are

$$\begin{aligned} {}_a D_q (fg)(t) &= f(t) {}_a D_q g(t) + g({}_a \Phi_q(t)) {}_a D_q f(t) \\ &= g(t) {}_a D_q f(t) + f({}_a \Phi_q(t)) {}_a D_q g(t) \end{aligned}$$

and

$${}_a D_q \left(\frac{f}{g} \right) (t) = \frac{g(t) {}_a D_q f(t) - f(t) {}_a D_q g(t)}{g(t)g({}_a \Phi_q(t))}, \quad g(t)g({}_a \Phi_q(t)) \neq 0.$$

The q -integral of a function f defined on the interval $[a, b]$ is given by

$$({}_a I_q f)(t) = \int_a^t f(s) {}_a d_q s = (1 - q)(t - a) \sum_{i=0}^{\infty} q^i f({}_a \Phi_q^i(t)), \quad t \in [a, b],$$

with

$$({}_aI_q^0 f)(t) = f(t), \quad ({}_aI_q^k f)(t) = {}_aI_q^{k-1}({}_aI_q f)(t), \quad k \in \mathbb{N}.$$

The fundamental theorem of calculus applies to the operator ${}_aD_q$ and ${}_aI_q$, that is,

$$({}_aD_q {}_aI_q f)(t) = f(t),$$

and if f is continuous at $t = a$, then

$$({}_aI_q {}_aD_q f)(t) = f(t) - f(a).$$

The formula for q -integration by parts on the interval $[a, b]$ is

$$\int_a^b f(s) ({}_aD_q g(s))_a d_q s = (fg)(t)|_a^b - \int_a^b g({}_a\Phi_q(s)) ({}_aD_q f(s))_a d_q s.$$

Let us give the definitions of the Riemann-Liouville fractional q -derivative and the q -integral on the interval $[a, b]$ and their properties [9].

Definition 2.2 [9] The fractional q -derivative of Riemann-Liouville type of order $\alpha \geq 0$ on the interval $[a, b]$ is defined by $({}_aD_q^\alpha f)(t) = f(t)$ and

$$({}_aD_q^\alpha f)(t) = ({}_aD_q^l {}_aI_q^{l-\alpha} f)(t), \quad \alpha > 0, \tag{2.1}$$

where l is the smallest integer greater than or equal to α .

Definition 2.3 [9] Let $\alpha \geq 0$ and f be a function defined on $[a, b]$. The fractional q -integral of Riemann-Liouville type is given by $({}_aI_q^\alpha f)(t) = f(t)$ and

$$({}_aI_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - {}_a\Phi_q(s))_q^{(\alpha-1)} f(s)_a d_q s, \quad \alpha > 0, t \in [a, b], \tag{2.2}$$

where the q -gamma function is defined by

$$\Gamma_q(\alpha) = \frac{{}_0(1-q)_q^{(\alpha-1)}}{{}_0(1-q)_q^{\alpha-1}}. \tag{2.3}$$

Obviously, $\Gamma_q(\alpha + 1) = [\alpha]_q \Gamma_q(\alpha)$.

From [9], we have the following formulas:

$${}_aD_q^\alpha (s - a)^\beta (t) = \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\beta - \alpha + 1)} (t - a)^{\beta - \alpha}, \tag{2.4}$$

$${}_aI_q^\alpha (s - a)^\beta (t) = \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\beta + \alpha + 1)} (t - a)^{\beta + \alpha}. \tag{2.5}$$

Lemma 2.4 [9] *Let $\alpha, \beta \in \mathbb{R}^+$ and f be a continuous function on $[a, b]$. The Riemann-Liouville fractional q -integral has the following semi-group property:*

$${}_a I_q^\beta {}_a I_q^\alpha f(t) = {}_a I_q^\alpha {}_a I_q^\beta f(t) = {}_a I_q^{\alpha+\beta} f(t). \tag{2.6}$$

Lemma 2.5 [9] *Let f be a q -integrable function on $[a, b]$. Then the following equality holds:*

$${}_a D_q^\alpha {}_a I_q^\alpha f(t) = f(t), \quad \alpha > 0, t \in [a, b]. \tag{2.7}$$

Lemma 2.6 [9] *Let $\alpha > 0$ and p be a positive integer. Then for $t \in [a, b]$ the following equality holds:*

$${}_a I_q^\alpha {}_a D_q^p f(t) = {}_a D_q^p {}_a I_q^\alpha f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} {}_a D_q^k f(a). \tag{2.8}$$

Next, the definition of Caputo fractional q -derivative is as follows.

Definition 2.7 [10, 11] *The fractional q -derivative of Caputo type of order $\alpha \geq 0$ on the interval $[a, b]$ is defined by $({}^C D_q^\alpha f)(t) = f(t)$ and*

$$({}^C D_q^\alpha f)(t) = ({}_a I_q^{n-\alpha} {}_a D_q^n f)(t), \quad \alpha > 0, \tag{2.9}$$

where n is the smallest integer greater than or equal to α .

Lemma 2.8 [10, 11] *Let $\alpha > 0$ and n be the smallest integer greater than or equal to α . Then for $t \in [a, b]$ the following equality holds:*

$${}_a I_q^\alpha {}^C D_q^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma_q(k+1)} {}_a D_q^k f(a). \tag{2.10}$$

The relation between the Caputo fractional q -derivative and the Riemann-Liouville fractional q -derivative is given by

$${}_a D_q^\alpha f(t) = {}_a D_q^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{{}_a D_q^k f(a)}{\Gamma_q(k+1)} (t-a)^k \right]. \tag{2.11}$$

Let $C(J, \mathbb{R}) = \{u : J \rightarrow \mathbb{R} : u(t) \text{ is continuous}\}$ with the norm $\|u\|_C = \sup_{t \in J} |u(t)|$.

The following result will play a very important role in this paper.

Lemma 2.9 *If $p \in C(J, \mathbb{R})$ and satisfies the following relations:*

$$\begin{cases} {}^C D_q^\alpha p(t) \geq -\lambda p(t), & t \in [a, b], \\ p(a) \geq 0, \end{cases} \tag{2.12}$$

where $\lambda > -\Gamma_q(\alpha + 1)/(b - a)^q$ is a constant, then $p(t) \geq 0$ for all $t \in J$.

Proof The proof is similar to the argument of Lemma 2.1 in [15]. Assume that $p(t) \geq 0, \forall t \in (a, b]$ is not true. Then, by $p(a) \geq 0$, there exist points $t_0, t_1 \in (a, b]$ such that $p(t_0) = 0, p(t_1) < 0, p(t) \geq 0$ for $t \in (a, t_0)$ and $p(t) < 0$ for $t \in (t_0, t_1]$. Let t_2 be the first minimal point of $p(t)$ on $[t_0, t_2]$. We shall show that $\lambda > 0$ and $\lambda \leq 0$.

First of all, let $\lambda > 0$. From (2.12), it follows that, for $t_0 < t \leq t_1$,

$${}^C D_q^\alpha p(t) \geq 0.$$

Using (2.11), Definition 2.2, and equation (2.4), we can compute

$$\begin{aligned} {}^C D_q^\alpha p(t) &= {}_a D_q^\alpha [p(t) - p(a)] \\ &= {}_a D_{qa} I_q^{1-\alpha} [p(t) - p(a)] \\ &= {}_a D_q [{}_a I_q^{1-\alpha} p(t) - p(a) {}_a I_q^{1-\alpha} (1)(t)] \\ &= {}_a D_q \left[{}_a I_q^{1-\alpha} p(t) - \frac{p(a)(t-a)^{1-\alpha}}{\Gamma_q(2-\alpha)} \right] \\ &= {}_a D_q H(t), \end{aligned}$$

where

$$H(t) = {}_a I_q^{1-\alpha} p(t) - \frac{p(a)(t-a)^{1-\alpha}}{\Gamma_q(2-\alpha)}.$$

The fundamental theorem of q -calculus applies to the operator ${}_{t_0} I_q$, and ${}_{t_0} D_q$, we have

$${}_{t_0} I_q {}^C D_q^\alpha p(t) = {}_{t_0} I_q {}_{t_0} D_q H(t) = H(t) - H(t_0) \geq 0. \tag{2.13}$$

Consequently, for $t \in (t_0, t_1]$, it follows that

$$\begin{aligned} H(t) - H(t_0) &= \left[{}_a I_q^{1-\alpha} p(t) - \frac{p(a)(t-a)^{1-\alpha}}{\Gamma_q(2-\alpha)} \right] - \left[{}_a I_q^{1-\alpha} p(t_0) - \frac{p(a)(t_0-a)^{1-\alpha}}{\Gamma_q(2-\alpha)} \right] \\ &= \left[\frac{1}{\Gamma_q(1-\alpha)} \int_a^t {}_a (t - {}_a \Phi_q(s))_q^{(-\alpha)} p(s)_a d_q s - \frac{p(a)(t-a)^{1-\alpha}}{\Gamma_q(2-\alpha)} \right] \\ &\quad - \left[\frac{1}{\Gamma_q(1-\alpha)} \int_a^{t_0} {}_a (t_0 - {}_a \Phi_q(s))_q^{(-\alpha)} p(s)_a d_q s - \frac{p(a)(t_0-a)^{1-\alpha}}{\Gamma_q(2-\alpha)} \right] \\ &= \frac{1}{\Gamma_q(1-\alpha)} \int_a^t {}_a (t - {}_a \Phi_q(s))_q^{(-\alpha)} p(s)_a d_q s \\ &\quad - \frac{1}{\Gamma_q(1-\alpha)} \int_a^{t_0} {}_a (t_0 - {}_a \Phi_q(s))_q^{(-\alpha)} p(s)_a d_q s \\ &\quad - \frac{p(a)}{\Gamma_q(2-\alpha)} [(t-a)^{1-\alpha} - (t_0-a)^{1-\alpha}] \\ &\leq \frac{1}{\Gamma_q(1-\alpha)} \int_a^{t_0} {}_a (t - {}_a \Phi_q(s))_q^{(-\alpha)} p(s)_a d_q s \\ &\quad + \frac{1}{\Gamma_q(1-\alpha)} \int_{t_0}^t {}_a (t - {}_a \Phi_q(s))_q^{(-\alpha)} p(s)_a d_q s \end{aligned}$$

$$\begin{aligned} & -\frac{1}{\Gamma_q(1-\alpha)} \int_a^{t_0} {}_a(t_0 - {}_a\Phi_q(s))_q^{(-\alpha)} p(s)_a d_qs \\ &= \frac{1}{\Gamma_q(1-\alpha)} \int_a^{t_0} [{}_a(t - {}_a\Phi_q(s))_q^{(-\alpha)} - {}_a(t_0 - {}_a\Phi_q(s))_q^{(-\alpha)}] p(s)_a d_qs \\ & \quad + \frac{1}{\Gamma_q(1-\alpha)} \int_{t_0}^t {}_a(t - {}_a\Phi_q(s))_q^{(-\alpha)} p(s)_a d_qs. \end{aligned}$$

Since $[{}_a(t - {}_a\Phi_q(s))_q^{(-\alpha)} - {}_a(t_0 - {}_a\Phi_q(s))_q^{(-\alpha)}] < 0$ for $0 \leq s \leq t_0$ and $p(t) < 0$, for all $t \in (t_0, t_1]$, we have

$$\int_{t_0}^t {}^C D_{t_0}^\alpha p(s) d_qs < 0,$$

which contradicts (2.13). Hence, we obtain the result $\lambda > 0$.

Finally, we assume $0 \geq \lambda > -\Gamma_q(1 + \alpha)/(b - a)^\alpha$. Since the Riemann-Liouville fractional q -integral ${}_a I_q^\alpha$ is a monotone operator, thus, applying the fractional q -integral order α on both sides of problem (2.12), by using Lemmas 2.8 and 2.4, we have

$$p(t) - p(a) + \lambda {}_a I_q^\alpha p(t) \geq 0,$$

for $t \in (a, b]$, from $p(a) \geq 0$, it follows that

$$p(t_2) + \lambda {}_a I_q^\alpha p(t_2) \geq 0. \tag{2.14}$$

For $t = t_2$, we can calculate

$$\begin{aligned} {}_a I_q^\alpha p(t_2) &= \frac{1}{\Gamma_q(\alpha)} \int_a^{t_2} {}_a(t_2 - {}_a\Phi_q(s))_q^{(\alpha-1)} p(s)_a d_qs \\ &= \frac{1}{\Gamma_q(\alpha)} \int_a^{t_0} {}_a(t_2 - {}_a\Phi_q(s))_q^{(\alpha-1)} p(s)_a d_qs \\ & \quad + \frac{1}{\Gamma_q(\alpha)} \int_{t_0}^{t_2} {}_a(t_2 - {}_a\Phi_q(s))_q^{(\alpha-1)} p(s)_a d_qs \\ &\geq \frac{1}{\Gamma_q(\alpha)} \int_{t_0}^{t_2} {}_a(t_2 - {}_a\Phi_q(s))_q^{(\alpha-1)} p(s)_a d_qs \\ &\geq \frac{1}{\Gamma_q(\alpha)} \int_{t_0}^{t_2} {}_a(t_2 - {}_a\Phi_q(s))_q^{(\alpha-1)} p(t_2)_a d_qs \\ &= \frac{(t_2 - t_0)^\alpha}{\Gamma_q(\alpha + 1)} p(t_2) \\ &\geq \frac{(b - a)^\alpha}{\Gamma_q(\alpha + 1)} p(t_2), \end{aligned}$$

which implies that

$$\lambda {}_a I_q^\alpha p(t_2) \leq \frac{\lambda(b - a)^\alpha}{\Gamma_q(\alpha + 1)} p(t_2). \tag{2.15}$$

Hence, using (2.14) and (2.15), we obtain

$$\left(1 + \frac{\lambda(b-a)^\alpha}{\Gamma_q(\alpha+1)}\right)p(t_2) \geq 0,$$

since $1 + \lambda(b-a)^\alpha/\Gamma_q(\alpha+1) > 0$, thus, this contradicts the negative property of $p(t_2)$. The proof is completed. □

3 Main results

Consider the linear initial value problem

$$\begin{cases} {}^C D_q^\alpha u(t) + \lambda u(t) = h(t), & 0 < \alpha < 1, \\ u(a) = u_0, & u_0 \in \mathbb{R}, \quad t \in J, \end{cases} \tag{3.1}$$

where λ is a given constant and $h \in C(J, \mathbb{R})$.

Lemma 3.1 *Let $v_0, w_0 : J \rightarrow \mathbb{R}$ be continuous functions. Assume that v_0, w_0 are lower and upper solutions of (3.1), respectively, and $v_0 \leq w_0$, for all $t \in J$. If*

$$\frac{\lambda(b-a)^\alpha}{\Gamma_q(\alpha+1)} < 1, \tag{3.2}$$

then the linear initial value problem (3.1) has a unique solution $u \in [v_0, w_0]$ on J .

Proof The proof consist of two steps.

Step I. We shall show that if u is a solution of (3.1), then $v_0 \leq u \leq w_0$.

Let $p = u - v_0$, we get

$$\begin{cases} {}^C D_q^\alpha p(t) \geq -\lambda p(t), \\ p(a) \geq 0. \end{cases}$$

By Lemma 2.9, $p(t) \geq 0$, for $t \in J$, that is, $u \geq v_0$. In the same way, if we set $r = w_0 - u$, then we can show that $u \leq w_0$. Thus, $v_0 \leq u \leq w_0$.

Step II. To prove that problem (3.1) has a unique solution.

Problem (3.1) is equivalent to the following integral equation:

$$u(t) = u_0 - \lambda {}_a I_q^\alpha u(t) + {}_a I_q^\alpha h(t), \quad \forall t \in J.$$

Let the operator

$$Au(t) = u_0 - \lambda {}_a I_q^\alpha u(t) + {}_a I_q^\alpha h(t), \quad \forall t \in J.$$

For any $u, v \in C(J, \mathbb{R})$, using (2.5), we obtain

$$\|Au(t) - Av(t)\| \leq \lambda {}_a I_q^\alpha (1)(b) \|u - v\|_C = \frac{\lambda(b-a)^\alpha}{\Gamma_q(\alpha+1)} \|u - v\|_C.$$

By using (3.2),

$$\|Au - Av\|_C < \|u - v\|_C.$$

Hence the operator A is a contraction in view of the condition (3.2). Consequently, by Banach's fixed point theorem, the operator A has a unique fixed point. That is, problem (3.1) has a unique solution. This completes the proof. \square

Next, we give the definitions of lower and upper solutions of problem (1.1).

Definition 3.2 A function $v_0 \in C([a, b], \mathbb{R})$ is called a lower solution of problem (1.1), if it satisfies

$$\begin{cases} {}^C D_q^\alpha v_0(t) \leq f(t, v_0), & t \in (a, b], \\ v_0(a) \leq u_0. \end{cases} \tag{3.3}$$

Definition 3.3 A function $w_0 \in C([a, b], \mathbb{R})$ is called an upper solution of problem (1.1), if it satisfies

$$\begin{cases} {}^C D_q^\alpha w_0(t) \geq f(t, w_0), & t \in (a, b], \\ w_0(a) \geq u_0. \end{cases} \tag{3.4}$$

In this paper, we will apply the monotone iterative method to present a result on the existence and uniqueness of the solution of problem (1.1).

Theorem 3.4 *Let the function $v_0, w_0 \in C(J, \mathbb{R})$. In addition assume that:*

- (H₁) v_0 and w_0 are lower and upper solutions of problem (1.1), respectively.
- (H₂) The function $f \in C(J, \mathbb{R})$ satisfies

$$f(t, v) - f(t, u) \geq -\lambda(v - u) \quad \text{for } v_0 \leq u \leq v \leq w_0,$$

where $\lambda > \Gamma_q(\alpha + 1)/(b - a)^\alpha$ is a constant.

Then there exist monotone iterative sequences $\{v_n\}$ and $\{w_n\}$, which converge uniformly on the interval J to the extremal solutions of (1.1) in $[v_0, w_0]$.

Proof For any $z \in [v_0, w_0]$, we consider the following linear IVP problem:

$$\begin{cases} {}^C D_q^\alpha u(t) = h_z(t) - \lambda u(t), & t \in (a, b], \\ u(a) = u_0, \end{cases} \tag{3.5}$$

where $h_z(t) = f(t, z(t)) + \lambda z(t)$ and λ is a given constant. Since v_0 and w_0 are lower and upper solutions of problem (1.1), by (H₂), we can get

$$\begin{aligned} {}^C D_q^\alpha v_0(t) &\leq f(t, v_0(t)) \leq f(t, z(t)) + \lambda z(t) - \lambda v_0(t) = h_z(t) - \lambda v_0(t), \\ v_0(a) &\leq u_0, \end{aligned}$$

and

$$\begin{aligned} {}^C D_q^\alpha w_0(t) &\geq f(t, w_0(t)) \geq f(t, z(t)) + \lambda z(t) - \lambda w_0(t) = h_z(t) - \lambda w_0(t), \\ w_0(a) &\geq u_0. \end{aligned}$$

Hence v_0 and w_0 are lower and upper solutions of problem (3.1). By Lemma 3.1, we know that problem (3.1) has a unique solution $u \in [v_0, w_0]$. Define an operator $A : [v_0, w_0] \rightarrow [v_0, w_0]$ by $u = Az$.

Next, we shall show that the operator A is nondecreasing. Let $z_1, z_2 \in [v_0, w_0]$, such that $z_1 \leq z_2$. Set $\eta = v_2 - v_1$, $v_1 = Az_1$, and $v_2 = Az_2$. By (H_2) , we obtain

$$\begin{aligned} {}^C D_q^\alpha \eta(t) &= {}^C D_q^\alpha v_2(t) - {}^C D_q^\alpha v_1(t) \\ &= f(t, z_2(t)) + \lambda z_2(t) - f(t, z_1(t)) - \lambda z_1(t) - \lambda(v_2(t) - v_1(t)) \\ &\geq -\lambda(z_2(t) - z_1(t)) + \lambda(z_2(t) - z_1(t)) - \lambda(v_2(t) - v_1(t)) \\ &= -\lambda\eta(t), \\ \eta(a) &\geq 0. \end{aligned}$$

By Lemma 2.9, $\eta(t) \geq 0$, for $t \in J$. That is, the operator A is nondecreasing.

Now let $v_n = Av_{n-1}$ and $w_n = Aw_{n-1}$ for $n = 1, 2, \dots$, then we have

$$v_0 \leq v_1 \leq \dots \leq v_n \leq \dots \leq w_n \leq \dots \leq w_1 \leq w_0. \tag{3.6}$$

It is easy to show that the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly bounded and equicontinuous on J . Hence, by Arzela-Ascoli's theorem, we have

$$\lim_{n \rightarrow \infty} v_n(t) = v^*(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} w_n(t) = w^*(t),$$

uniformly on $t \in J$ and the limit functions v^*, w^* satisfy problem (1.1). Furthermore, v^* and w^* satisfy the relation

$$v_0 \leq v_1 \leq \dots \leq v_n \leq v^* \leq w^* \leq \dots \leq w_n \leq \dots \leq w_1 \leq w_0.$$

Finally, we prove that v^* and w^* are extremal solutions of problem (1.1) in $[v_0, w_0]$.

Let $u \in [v_0, w_0]$ be any solution of (1.1). Then $Au = u$. Since $v_0 \leq u \leq w_0$ and considering the properties of A , i.e., A is nondecreasing, we obtain

$$v_n \leq u \leq w_n, \quad n = 1, 2, \dots \tag{3.7}$$

Taking the limit in (3.7) as $n \rightarrow \infty$, we have $v^* \leq u \leq w^*$. Therefore v^*, w^* are the extremal solutions of (1.1) in $[v_0, w_0]$. This completes the proof. \square

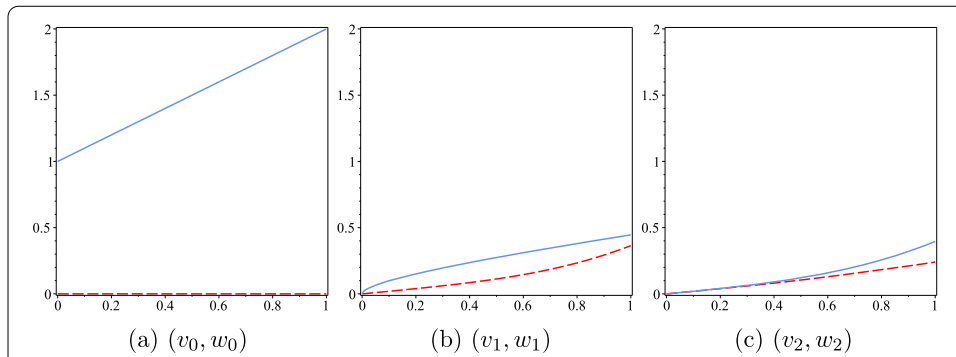


Figure 1 A plot of the lower and upper solutions of Problem (4.1).

4 Examples

Example 4.1 Consider the following initial value problem:

$$\begin{cases} {}^C_0D_q^\alpha u(t) = \frac{t^{1-\alpha}}{3\Gamma_q(1-\alpha)} + \frac{\Gamma_q(\alpha+1)}{4}(t-u(t))^3, & t \in (0, 1], \\ u(0) = 0, \end{cases} \tag{4.1}$$

where $\alpha = 1/2$, $q = 1/2$, $a = 0$, and $b = 1$. Taking $v_0(t) = 0$ and $w_0(t) = 1 + t$, it is easy to verify that v_0, w_0 are lower and upper solutions of (4.1), respectively, and $v_0 \leq w_0$. Then the assumption (H_1) of Theorem 3.4 holds.

The function f is given by

$$f(t, u) = \frac{t^{1-\alpha}}{3\Gamma_q(1-\alpha)} + \frac{\Gamma_q(\alpha+1)}{4}(t-u)^3,$$

which satisfies

$$f(t, v) - f(t, u) \geq -\frac{3\Gamma_q(\alpha+1)}{4}(v-u),$$

where $v_0 \leq u \leq v \leq w_0$. Then we get $\lambda = 3\Gamma_q(\alpha+1)/4$, which implies

$$\frac{\lambda(b-a)^\alpha}{\Gamma_q(\alpha+1)} = \frac{3}{4} < 1,$$

thus, the assumption (H_2) of Theorem 3.4 holds. Therefore, problem (4.1) satisfies all assumptions of Theorem 3.4. By Theorem 3.4, there exist monotone iterative sequences $\{v_n\}$ and $\{w_n\}$, which converge uniformly on interval $[0, 1]$ to the extremal solutions of (4.1) in $[v_0, w_0]$. The graphs of $\{v_n\}$ and $\{w_n\}$, for some values of n , are shown in Figure 1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

Author details

¹Department of Mathematics, Shanxi Normal University, Linfen, Shanxi 041004, P.R. China. ²Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand.

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