Zhang and Liang *Advances in Difference Equations* (2015) 2015:324
DOI 10.1186/s13662-015-0665-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Monotone iterative technique for impulsive fractional evolution equations with noncompact semigroup

Lingzhong Zhang[1*] and Yue Liang[2]

*Correspondence: zhanglingzhong1208@163.com
[1]School of Science and Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), Jiangnan University, Wuxi, 214122, People's Republic of China
Full list of author information is available at the end of the article

Abstract
This paper deals with the existence of mild solutions for a class of Caputo fractional impulsive evolution equation with nonlocal condition and noncompact semigroup. By using a monotone iterative technique in the presence of coupled lower and upper L-quasi-solutions and using Sadovskii's fixed point theorem, some existence theorems are obtained. The discussion is based on operator semigroup theory.

MSC: 34K45; 35F25

Keywords: impulsive fractional evolution equation; coupled mild l-quasi-solutions; mild solutions; mixed monotone iterative technique

1 Introduction

Fractional differential equation is a new and important branch of differential equation theory. It is a valuable tools in the modeling of many phenomena in various fields of engineering, physics, economics, *etc.* Actually, it has been an important area of investigation in recent years; see [1–5]. Particularly, the existence of solutions to fractional evolution equations has been studied by many authors, see [6–16]. In [6, 7], El-Borai introduced a concept of mild solutions to fractional evolution equations in terms of probability density functions. Recently, it was developed by Zhou and Wang *et al.* in [8–16]. They introduced two characteristic solution operators and gave a suitable concept on the mild solution by applying Laplace transform and probability density functions. But all these papers did not consider the impulse effects. To study fractional evolution equation with impulsive conditions, many authors made the preparatory works. Particularly, Wang *et al.* [17] presented the concept of the mild solution of impulsive fractional evolution equations in a Banach space X,

$$\begin{cases} D^q u(t) + Au(t) = f(t, u(t)), & t \in J := [0, a], t \neq t_k, \\ u(t_k^+) = u(t_k^-) + y_k, & k = 1, 2, \ldots, m, \\ u(0) = u_0 + g(u), \end{cases} \tag{1}$$

where D^q denotes the Caputo fractional derivative of order $q \in (0, 1)$, $-A : D(A) \subset X \to X$ generates a compact C_0-semigroup in X, f, g are given functions, y_k, u_0 are the elements of X, $a > 0$ is a fixed constant, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively. By using fixed point theorems of compact operator, they derived many existence and uniqueness results concerning the mild solutions for system (1).

On the other hand, as far as we know that there are few papers studied the fractional evolution equations with noncompact semigroup. Recently, Wang *et al.* [18] discussed the local existence of mild solutions for nonlocal problem of fractional evolution equations under the situation that $-A$ generates a noncompact analytic semigroup. Chen *et al.* [19] investigated the existence of saturated mild solutions for the initial value problem of fractional evolution equations under the situation that $-A$ generates an equicontinuous C_0-semigroup.

In this paper, in the case of a noncompact semigroup, we consider the following nonlocal problems of impulsive fractional evolution equations in an ordered Banach space E

$$\begin{cases} D^q u(t) + Au(t) = f(t, u(t), u(t)), & t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-) = I_k(u(t_k), u(t_k)), & k = 1, 2, \dots, m, \\ u(0) = u_0 + g(u, u), \end{cases} \tag{2}$$

where D^q denotes the Caputo fractional derivative of order $q \in (0, 1)$, $-A : D(A) \subset E \to E$ generates a C_0-semigroup $S(t)$ $(t \geq 0)$ in E, f and g are given functions will be specified later, $u_0 \in E$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a$, $I_k : E \times E \to E$ $(k = 1, 2, \dots, m)$ are the impulsive functions. Utilizing a monotone iterative technique in the presence of coupled lower and upper L-quasi-solutions and using Sadovskii's fixed point theorem, we obtain some existence results concerning the coupled mild L-quasi-solutions and mild solutions for system (2).

In present work, we only assume that $-A$ generates a positive C_0-semigroup in Theorem 1 and Theorem 3, which is noncompact and nonanalytic. In Theorem 2, $-A$ generates a positive and equicontinuous C_0-semigroup, but the other conditions on f, g, and I_k are much weaker than existing results.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given on the fractional calculus and the measure of noncompactness. The definition of coupled lower and upper L-quasi-solutions of the system (2) is also given in this section. In Section 3, we study the existence of coupled mild L-quasi-solutions and mild solutions for the system (2). Particularly, a uniqueness result is also obtained in this section. An example is given in Section 4 to illustrate the effectiveness of our results.

2 Preliminaries

Let X be a Banach space with norm $\|\cdot\|$, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a C_0-semigroup $S(t)$ $(t \geq 0)$ in X. It is well known that there exist $\overline{M} > 0$ and $\delta \in \mathbb{R}$ such that

$$\|S(t)\| \leq \overline{M} e^{\delta t}, \quad t \geq 0. \tag{3}$$

From (3), it is clear that there exists a constant $C > 0$ such that $\|S(t)\| \leq C$ for any $t \in [0, a]$.

Definition 1 A C_0-semigroup $S(t)$ $(t \geq 0)$ in X is said to be positive, if the inequality $S(t)x \geq 0$ holds for $x \geq 0$ and $t \geq 0$.

It is clear that for any $M \geq 0$, $-(A + MI)$ also generates a C_0-semigroup $S_1(t) = e^{-Mt}S(t)$ $(t \geq 0)$ in X. $S_1(t)$ $(t \geq 0)$ is a positive C_0-semigroup if $S(t)$ $(t \geq 0)$ is positive. For more details about the positive C_0-semigroup, please see [20].

Let us recall the following known definitions in fractional calculus. For more details, see [3, 5, 8–10] and the references therein.

Definition 2 The fractional integral of order $\sigma > 0$ with the lower limits zero for a function f is defined by

$$I^{\sigma}f(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}f(s)\,ds, \quad t > 0,$$

where Γ is the gamma function.

The Caputo fractional derivative of order $n-1 < \sigma < n$ with the lower limits zero for a function $f \in C^n[0,\infty)$ can be written as

$$D^{\sigma}f(t) = \frac{1}{\Gamma(n-\sigma)}\int_0^t (t-s)^{n-\sigma-1}f^{(n)}(s)\,ds, \quad t > 0, n \in \mathbb{N}.$$

Remark 1

(1) The Caputo derivative of a constant is equal to zero.
(2) If f is an abstract function with values in X, then the integrals which appear in Definition 2 are taken in Bochner's sense.

Lemma 1 [9] *A measurable function $h : [0,a] \to X$ is Bochner integrable if $\|h\|$ is Lebesgue integrable.*

For $x \in X$, we define two families $\{U(t)\}_{t\geq 0}$ and $\{V(t)\}_{t\geq 0}$ of the operators by

$$U(t)x = \int_0^{\infty} \eta_q(\theta)S(t^q\theta)x\,d\theta,$$
$$V(t)x = q\int_0^{\infty} \theta\eta_q(\theta)S(t^q\theta)x\,d\theta, \quad 0 < q < 1,$$

where

$$\eta_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\rho_q(\theta^{-\frac{1}{q}}),$$
$$\rho_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty).$$

η_q is a probability density function defined on $(0,\infty)$, which has the properties $\eta_q(\theta) \geq 0$ for all $\theta \in (0,\infty)$ and $\int_0^{\infty}\eta_q(\theta)\,d\theta = 1$. Clearly, if the semigroup $S(t)$ $(t \geq 0)$ is positive, then the operators $U(t)$ and $V(t)$ are also positive for all $t \geq 0$.

Lemma 2 *The operators $U(t)$ and $V(t)$ have the following properties.*

(i) *For any fixed $t \geq 0$ and any $x \in X$, one has*

$$\|U(t)x\| \leq C\|x\|, \qquad \|V(t)x\| \leq \frac{qC}{\Gamma(q+1)}\|x\| = \frac{C}{\Gamma(q)}\|x\|.$$

(ii) *The operators $U(t)$ and $V(t)$ are strongly continuous for all $t \geq 0$.*

(iii) *If $S(t)$ $(t \geq 0)$ is an equicontinuous semigroup, $U(t)$ and $V(t)$ are equicontinuous in X for $t > 0$.*

Proof From [8, 9], it is easy to prove (i) and (ii). Hence, we only prove (iii). For any $0 \leq t_1 < t_2 \leq a$, we have

$$\|U(t_2) - U(t_1)\| = \int_0^\infty \eta_q(\theta)\|S(t_2^q\theta) - S(t_1^q\theta)\|\,d\theta$$

and

$$\|V(t_2) - V(t_1)\| = q\int_0^\infty \theta\eta_q(\theta)\|S(t_2^q\theta) - S(t_1^q\theta)\|\,d\theta.$$

According to the equicontinuity of $S(t)$ for $t > 0$, we see that $\|U(t_2) - U(t_1)\|$ and $\|V(t_2) - V(t_1)\|$ tend to zero as $t_2 - t_1 \to 0$, which means that the operators $U(t)$ and $V(t)$ are equicontinuous in X for $t > 0$. □

We denote by $C(J,X)$ the Banach space of all continuous X-value functions on interval J with the norm $\|u\|_C = \max_{t\in J}\|u(t)\|$. Let $\alpha_X(\cdot)$ and $\alpha_{C(J,X)}(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set in X and $C(J,X)$, respectively. Let $B \subset X$ be a bounded set. It is well known that $0 \leq \alpha_X(B) < \infty$. $\alpha(B) \equiv 0$ if and only if the set B is precompact. For more details of the definition and properties of measure of noncompactness, see [21]. For any $B \subset C(J,X)$ and $t \in J$, set $B(t) = \{u(t) : u \in B\} \subset X$. If B is bounded in $C(J,X)$, $B(t)$ is bounded in X, and $\alpha_X(B(t)) \leq \alpha_{C(J,X)}(B)$. A mapping $Q : B \to B$ is said to be condensing, if $\alpha_{C(J,X)}(Q(B)) < \alpha_{C(J,X)}(B)$. For the measure of noncompactness, the following lemmas will be used in this paper.

Lemma 3 [22] *Let $B \subset C(J,X)$ be bounded and equicontinuous. Then $\alpha_X(B(t))$ is continuous on J and*

$$\alpha_{C(J,X)}(B) = \max_{t\in J}\alpha_X(B(t)) = \alpha_X(B(J)),$$

where $B(J) = \{u(t) : u \in B, t \in J\}$.

Lemma 4 [23] *Let $B = \{u_n\} \subset C(J,X)$ be countable. If there exists $\psi \in L^1(J)$ such that $\|u_n(t)\| \leq \psi(t)$ a.e. $t \in J$, $n = 1, 2, \ldots$, then $\alpha_X(B(t))$ is Lebesgue integrable on J and*

$$\alpha_X\left(\left\{\int_J u_n(t)\,dt : n \in \mathbb{N}\right\}\right) \leq 2\int_J \alpha_X(B(t))\,dt.$$

Lemma 5 [24] *Let $B \subset C(J,X)$ be bounded. Then there exists a countable subset B_0 of B such that $\alpha_{C(J,X)}(B) \leq 2\alpha_{C(J,X)}(B_0)$.*

Lemma 6 [25] (Sadovskii's fixed point theorem) *Let X be a Banach space and Ω be a nonempty bounded convex closed set in X. If $Q : \Omega \to \Omega$ is a condensing mapping, then Q has a fixed point in Ω.*

Let E be an ordered Banach space with the norm $\|\cdot\|$ and the partial order $\leq$, whose positive cone $K = \{x \in E : x \geq 0\}$ is normal. Let $PC(J,E) = \{u : J \to E : u(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k$ and $u(t_k^+)$ exists, $k = 1,2,\ldots,m\}$. Evidently, $PC(J,E)$ is a Banach space with the norm $\|u\|_{PC} = \sup_{t\in J}\|u(t)\|$. $PC(J,E)$ is also an ordered Banach space with partial order $\leq$ reduced by the positive cone $K_{PC} = \{u \in PC(J,E) : u(t) \geq 0, t \in J\}$. We use E_1 to denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. Let $J' = J\backslash\{t_1,t_2,\ldots,t_m\}$. An abstract function $u \in PC(J,E) \cap C^1(J',E) \cap C(J',E_1)$ is called a solution of the system (2) if $u(t)$ satisfies all the equalities in (2).

Definition 3 Let $L \geq 0$ be a constant. If the functions $v_0, w_0 \in PC(J,E) \cap C^1(J',E) \cap C(J',E_1)$ satisfy

$$\begin{cases} D^q v_0(t) + Av_0(t) \leq f(t, v_0(t), w_0(t)) + L(v_0(t) - w_0(t)), \quad t \in J', \\ \Delta v_0|_{t=t_k} \leq I_k(v_0(t_k), w_0(t_k)), \quad k = 1,2,\ldots,m, \\ v_0(0) \leq u_0 + g(v_0, w_0), \end{cases} \tag{4}$$

$$\begin{cases} D^q w_0(t) + Aw_0(t) \geq f(t, w_0(t), v_0(t)) + L(w_0(t) - v_0(t)), \quad t \in J', \\ \Delta w_0|_{t=t_k} \geq I_k(w_0(t_k), v_0(t_k)), \quad k = 1,2,\ldots,m, \\ w_0(0) \geq u_0 + g(w_0, v_0), \end{cases} \tag{5}$$

we call v_0, w_0 coupled lower and upper L-quasi-solutions of the system (2). If we only choose = in (4) and (5), we call v_0, w_0 coupled L-quasi-solutions of the system (2). Furthermore, if we choose = in (4) and (5) and let $\tilde{u} := v_0 = w_0$, then we call $\tilde{u}$ a solution of the system (2).

In this paper we adopt the following definition of mild solutions of the system (2), which comes from [17].

Definition 4 By a mild solution of the system (2), we mean that a function $u \in PC(J,E)$ satisfies the following integral equation

$$u(t) = \begin{cases} U(t)[u_0 + g(u,u)] + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s), u(s))\,ds, \quad t \in [0,t_1], \\ U(t)[u_0 + g(u,u)] + U(t-t_1)I_1(u(t_1), u(t_1)) \\ \quad + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s), u(s))\,ds, \quad t \in [t_1,t_2], \\ \ldots, \\ U(t)[u_0 + g(u,u)] + \sum_{i=1}^m U(t-t_i)I_i(u(t_i), u(t_i)) \\ \quad + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s), u(s))\,ds, \quad t \in [t_m,a]. \end{cases} \tag{6}$$

In the proof of the main results, we also need the following generalized Gronwall-Bellman inequality, which can be found in [26].

Lemma 7 *Suppose $b \geq 0$, $\beta > 0$, and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq \infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\beta-1} u(s)\,ds$$

on this interval, then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds, \quad 0 \leq t < T.$$

Remark 2 In Lemma 7, if $a(t) \equiv 0$ for all $0 \leq t < T$, we easily see that $u(t) = 0$.

3 Main results

In this section, we always assume that E is an ordered Banach space, whose positive cone K is normal with normal constant N, $A : D(A) \subset E \to E$ is a closed linear operator. Let us list the following hypotheses:

(H_1) $f \in C(J \times E \times E, E)$ and there exist $M > 0$ and $L \geq 0$ such that

$$f(t, x_2, y_2) - f(t, x_1, y_1) \geq -M(x_2 - x_1) + L(y_2 - y_1),$$

for any $t \in J$, $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$ and $v_0(t) \leq y_2 \leq y_1 \leq w_0(t)$;

(H_2) $I_k \in C(E \times E, E)$ satisfies

$$I_k(x_1, y_1) \leq I_k(x_2, y_2), \quad k = 1, 2, \ldots, m,$$

for any $t \in J$, $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$ and $v_0(t) \leq y_2 \leq y_1 \leq w_0(t)$;

(H_3) $g : [v_0, w_0] \times [v_0, w_0] \to E$ is continuous and satisfies

$$g(x_1, y_1) \leq g(x_2, y_2),$$

for any $v_0 \leq x_1 \leq x_2 \leq w_0$ and $v_0 \leq y_2 \leq y_1 \leq w_0$;

(H_4) there exists a constant $L_1 > 0$ such that

$$\alpha_E\big(\{f(t, x_n, y_n) + f(t, y_n, x_n)\}\big) \leq L_1 \alpha_E\big(\{x_n\} + \{y_n\}\big),$$

for $t \in J$, increasing monotonic sequence $\{x_n\} \subset [v_0(t), w_0(t)]$ and decreasing sequence $\{y_n\} \subset [v_0(t), w_0(t)]$;

(H_5) $\{g(x_n, y_n)\}$ is precompact for any monotone sequence $\{x_n\}, \{y_n\} \subset [v_0, w_0]$.

Theorem 1 *Let $-A$ generate a positive C_0-semigroup $S(t)$ ($t \geq 0$). Assume that the system* (2) *has coupled lower and upper L-quasi-solutions v_0 and w_0 with $v_0 \leq w_0$. If the assumptions* (H_1)-(H_5) *hold, the system* (2) *has minimal and maximal coupled mild L-quasi-solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0.*

Proof Let $u_0 \in E$ be fixed. Define an operator $Q : [v_0, w_0] \times [v_0, w_0] \to PC(J, E)$ by

$$Q(u,v)(t) = \begin{cases} U_1(t)[u_0 + g(u,v)] + \int_0^t (t-s)^{q-1} V_1(t-s)[f(s,u(s),v(s)) \\ \quad + (M+L)u(s) - Lv(s)]\,ds, \quad t \in [0, t_1], \\ \quad \times U_1(t)[u_0 + g(u,v)] + U_1(t-t_1) I_1(u(t_1), v(t_1)) + \int_0^t (t-s)^{q-1} \\ V_1(t-s)[f(s,u(s),v(s)) + (M+L)u(s) - Lv(s)]\,ds, \quad t \in [t_1, t_2], \\ \ldots, \\ U_1(t)[u_0 + g(u,v)] + \sum_{i=1}^m U_1(t-t_i) I_i(u(t_i), v(t_i)) + \int_0^t (t-s)^{q-1} \\ \quad \times V_1(t-s)[f(s,u(s),v(s)) + (M+L)u(s) - Lv(s)]\,ds, \quad t \in [t_m, a], \end{cases} \tag{7}$$

where

$$U_1(t) = \int_0^\infty \eta_q(\theta)S_1\bigl(t^q\theta\bigr)\,d\theta, \qquad V_1(t) = q\int_0^\infty \theta\eta_q(\theta)S_1\bigl(t^q\theta\bigr)\,d\theta,$$
$$S_1(t) = e^{-Mt}S(t) \quad (t \geq 0).$$

It is clear that $U_1(t)$ and $V_1(t)$ are positive operators if $S(t)$ $(t \geq 0)$ is a positive C_0-semigroup.

For any $t \in J$, $v_0(t) \leq x_1(t) \leq x_2(t) \leq w_0(t)$ and $v_0(t) \leq y_2(t) \leq y_1(t) \leq w_0(t)$, from the assumptions ($H_1$)-($H_3$) and the positive property of the operators $U_1(t)$ and $V_1(t)$ for $t \geq 0$, it follows that $Q(x_1, y_1)(t) \leq Q(x_2, y_2)(t)$ for all $t \in J$, which means that Q is a mixed monotone operator.

Let $h(t) \triangleq D^q v_0(t) + Av_0(t) + Mv_0(t)$. Then $h \in PC(J,E)$ and $h(t) \leq f(t, v_0(t), w_0(t)) + (M + L)v_0(t) - Lw_0(t)$. Hence for any $t \in [0, t_1]$, from (4) and (7), we have

$$\begin{aligned} v_0(t) &= U_1(t)v_0(0) + \int_0^t (t-s)^{q-1}V_1(t-s)h(s)\,ds \\ &\leq U_1(t)\bigl[u_0 + g(v_0, w_0)\bigr] + \int_0^t (t-s)^{q-1}V_1(t-s)\bigl[f\bigl(s, v_0(s), w_0(s)\bigr) \\ &\quad + (M+L)v_0(s) - Lw_0(s)\bigr]\,ds \\ &= Q(v_0, w_0)(t). \end{aligned}$$

For any $t \in (t_1, t_2]$, from (4) and (7), we have

$$\begin{aligned} v_0(t) &= U_1(t)v_0(0) + U_1(t-t_1)\Delta v_0|_{t=t_1} + \int_0^t (t-s)^{q-1}V_1(t-s)h(s)\,ds \\ &\leq U_1(t)\bigl[u_0 + g(v_0, w_0)\bigr] + U_1(t-t_1)I_1\bigl(v_0(t_1), w_0(t_1)\bigr) \\ &\quad + \int_0^t (t-s)^{q-1}V_1(t-s)\bigl[f\bigl(s, v_0(s), w_0(s)\bigr) + (M+L)v_0(s) - Lw_0(s)\bigr]\,ds \\ &= Q(v_0, w_0)(t). \end{aligned}$$

Similarly, we can obtain $v_0(t) \leq Q(v_0, w_0)(t)$ for any $t \in (t_k, t_{k+1}]$, $k = 2, 3, \ldots, m$. That is, $v_0 \leq Q(v_0, w_0)$. A similar argument can prove that $Q(w_0, v_0) \leq w_0$. So, $Q : [v_0, w_0] \times [v_0, w_0] \to [v_0, w_0]$ is a mixed monotone operator. By the continuity of f, we easily prove that $Q : [v_0, w_0] \times [v_0, w_0] \to [v_0, w_0]$ is continuous.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = Q(v_{n-1}, w_{n-1}), \qquad w_n = Q(w_{n-1}, v_{n-1}), \quad n = 1, 2, \ldots. \tag{8}$$

Then from the mixed monotonicity of Q, it follows that

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0. \tag{9}$$

Next, we prove that the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly convergent on J.

For convenience, let $B = \{v_n : n \in \mathbb{N}\} + \{w_n : n \in \mathbb{N}\}$, $B_1 = \{v_n : n \in \mathbb{N}\}$, $B_2 = \{w_n : n \in \mathbb{N}\}$, $B_{10} = \{v_{n-1} : n \in \mathbb{N}\}$, and $B_{20} = \{w_{n-1} : n \in \mathbb{N}\}$. Then $B = B_1 + B_2$, $B_1 = Q(B_{10}, B_{20})$, and

$B_2 = Q(B_{20}, B_{10})$. From $B_{10} = B_1 \cup \{v_0\}$ and $B_{20} = B_2 \cup \{w_0\}$, it follows that $\alpha_E(B_{10}(t)) = \alpha_E(B_1(t))$ and $\alpha_E(B_{20}(t)) = \alpha_E(B_2(t))$ for $t \in J$. Let $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, and let $\varphi(t) := \alpha_E(B(t))$ for $t \in J$. Going from J_0 to J_m interval by interval, we show that $\varphi(t) \equiv 0$ for $t \in J$.

For $t \in J_0$, from assumptions (H_4), (H_5), and (7), we have

$$\begin{aligned}
\varphi(t) &= \alpha_E\bigl(B(t)\bigr) = \alpha_E\bigl(B_1(t) + B_2(t)\bigr) = \alpha_E\bigl(Q(B_{10}, B_{20})(t) + Q(B_{20}, B_{10})(t)\bigr) \\
&\leq \alpha_E\bigl(\bigl\{U_1(t)\bigl[2u_0 + g(v_{n-1}, w_{n-1}) + g(w_{n-1}, v_{n-1})\bigr]\bigr\}\bigr) \\
&\quad + \alpha_E\biggl(\biggl\{\int_0^t (t-s)^{q-1} V_1(t-s)\bigl[\bigl(f\bigl(s, v_{n-1}(s), w_{n-1}(s)\bigr) + f\bigl(s, w_{n-1}(s), v_{n-1}(s)\bigr)\bigr) \\
&\quad + M\bigl(v_{n-1}(s) + w_{n-1}(s)\bigr)\bigr]\,ds\biggr\}\biggr) \\
&\leq C\bigl(\alpha_E\bigl(\bigl\{g(v_{n-1}, w_{n-1})\bigr\}\bigr) + \alpha_E\bigl(\bigl\{g(w_{n-1}, v_{n-1})\bigr\}\bigr)\bigr) \\
&\quad + \frac{2C}{\Gamma(q)} \int_0^t (t-s)^{q-1}\bigl[\alpha_E\bigl(\bigl\{f\bigl(s, v_{n-1}(s), w_{n-1}(s)\bigr) + f\bigl(s, w_{n-1}(s), v_{n-1}(s)\bigr)\bigr\}\bigr) \\
&\quad + M\alpha_E\bigl(\bigl\{v_{n-1}(s) + w_{n-1}(s)\bigr\}\bigr)\bigr]\,ds \\
&\leq \frac{2C(M + L_1)}{\Gamma(q)} \int_0^t (t-s)^{q-1} \alpha_E\bigl(B_{10}(s) + B_{20}(s)\bigr)\,ds \\
&\leq \frac{2C(M + L_1)}{\Gamma(q)} \int_0^t (t-s)^{q-1} \varphi(s)\,ds.
\end{aligned}$$

Hence by Lemma 7, $\alpha_E(B(t)) = \varphi(t) \equiv 0$ for $t \in J_0$, which means that $\{v_n(t)\} + \{w_n(t)\}$ is precompact in E for $t \in J_0$. In particular, $\alpha_E(B_{10}(t_1)) = 0$ and $\alpha_E(B_{20}(t_1)) = 0$. That is, $B_{10}(t_1)$ and $B_{20}(t_1)$ are precompact in E. Thus, $I_1(B_{10}(t_1), B_{20}(t_1))$, and $I_1(B_{20}(t_1), B_{10}(t_1))$ are precompact in E, and $\alpha_E(I_1(B_{10}(t_1), B_{20}(t_1))) = 0$, $\alpha_E(I_1(B_{20}(t_1), B_{10}(t_1))) = 0$.

For $t \in J_1$, from above argument, we have

$$\begin{aligned}
\varphi(t) &= \alpha_E\bigl(B(t)\bigr) = \alpha_E\bigl(B_1(t) + B_2(t)\bigr) = \alpha_E\bigl(Q(B_{10}, B_{20})(t) + Q(B_{20}, B_{10})(t)\bigr) \\
&\leq \alpha_E\bigl(\bigl\{U_1(t)\bigl[2u_0 + g(v_{n-1}, w_{n-1}) + g(w_{n-1}, v_{n-1})\bigr]\bigr\}\bigr) \\
&\quad + \alpha_E\biggl(\biggl\{\int_0^t (t-s)^{q-1} V_1(t-s)\bigl[\bigl(f\bigl(s, v_{n-1}(s), w_{n-1}(s)\bigr) + f\bigl(s, w_{n-1}(s), v_{n-1}(s)\bigr)\bigr) \\
&\quad + M\bigl(v_{n-1}(s) + w_{n-1}(s)\bigr)\bigr]\,ds\biggr\}\biggr) \\
&\quad + C\alpha_E\bigl(\bigl\{I_1\bigl(v_{n-1}(t_1), w_{n-1}(t_1)\bigr) + I_1\bigl(w_{n-1}(t_1), v_{n-1}(t_1)\bigr)\bigr\}\bigr) \\
&\leq \frac{2C(M + L_1)}{\Gamma(q)} \int_0^t (t-s)^{q-1} \varphi(s)\,ds.
\end{aligned}$$

Again by Lemma 7, $\alpha_E(B(t)) = \varphi(t) \equiv 0$ for $t \in J_1$, which means that $\{v_n(t)\} + \{w_n(t)\}$ is precompact in E for $t \in J_1$. Particularly, we obtain that $\alpha_E(B_{10}(t_2)) = 0$, $\alpha_E(B_{20}(t_2)) = 0$ and $\alpha_E(I_2(B_{10}(t_2), B_{20}(t_2))) = 0$, $\alpha_E(I_2(B_{20}(t_2), B_{10}(t_2))) = 0$.

Continuing such a process interval by interval up to J_m, we can prove that $\alpha_E(B(t)) = \varphi(t) \equiv 0$ on every J_k, $k = 0, 1, 2, \ldots, m$. Hence for any $t \in J$, $\{v_n(t)\} + \{w_n(t)\}$ is precompact in E. So, $\{v_n(t)\}$ and $\{w_n(t)\}$ are convergent, respectively. Let

$$\underline{u}(t) = \lim_{n\to\infty} v_n(t), \qquad \overline{u}(t) = \lim_{n\to\infty} w_n(t), \quad t \in J. \tag{10}$$

Letting $n \to \infty$ in (8), by the continuity of the operator Q, we obtain

$$\underline{u}(t) = Q(\underline{u}, \overline{u})(t), \qquad \overline{u}(t) = Q(\overline{u}, \underline{u})(t), \quad t \in J. \tag{11}$$

Evidently, $\{v_n(t)\}, \{w_n(t)\} \subset PC(J,E)$, so $\underline{u}(t)$, $\overline{u}(t)$ are bounded integrable on J and $\underline{u}, \overline{u} \in PC(J,E)$. Combining this with the monotonicity (9), we obtain $v_0(t) \le \underline{u}(t) \le \overline{u}(t) \le w_0(t)$ for $t \in J$. By the mixed monotonicity of Q, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal coupled fixed points of Q on $[v_0, w_0]$. and therefore, they are the minimal and maximal coupled mild L-quasi-solutions of the system (2) between v_0 and w_0. □

Let the following condition be satisfied:

$(\mathrm{H}_4)'$ There exists a constant $\overline{L}_1 > 0$ such that

$$\alpha_E\bigl(\{f(t,x_n,y_n)\}\bigr) \le \overline{L}_1\bigl(\alpha_E(\{x_n\}) + \alpha_E(\{y_n\})\bigr),$$

for any $t \in J$ and monotone sequences $\{x_n\}, \{y_n\} \subset [v_0(t), w_0(t)]$.

For any $t \in J$, increasing sequence $\{x_n\} \subset [v_0(t), w_0(t)]$ and decreasing sequence $\{y_n\} \subset [v_0(t), w_0(t)]$, by $(\mathrm{H}_4)'$, we see that

$$\begin{aligned}
\alpha_E\bigl(\{f(t,x_n,y_n) + f(t,y_n,x_n)\}\bigr) &\le \alpha_E\bigl(\{f(t,x_n,y_n)\}\bigr) + \alpha_E\bigl(\{f(t,y_n,x_n)\}\bigr) \\
&\le 2\overline{L}_1\bigl(\alpha_E(\{x_n\}) + \alpha_E(\{y_n\})\bigr) \\
&\le 2\overline{L}_1\bigl(\alpha_E(\{x_n\} + \{y_n\}) + \alpha_E(\{y_n\} + \{x_n\})\bigr) \\
&= 4\overline{L}_1\alpha_E\bigl(\{x_n\} + \{y_n\}\bigr).
\end{aligned}$$

Let $L_1 = 4\overline{L}_1$. Then the assumption $(\mathrm{H}_4)'$ implies (H_4). Therefore, by Theorem 1, we obtain the following existence result.

Corollary 1 *Let $-A$ generate a positive C_0-semigroup $S(t)$ $(t \ge 0)$. Assume that the system* (2) *has coupled lower and upper L-quasi-solutions v_0 and w_0 with $v_0 \le w_0$. If the conditions* (H_1)-(H_3), $(\mathrm{H}_4)'$, *and* (H_5) *hold, the system* (2) *has minimal and maximal coupled mild L-quasi-solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0.*

Now, we discuss the existence of mild solution for the system (2) between the minimal and maximal coupled mild L-quasi-solutions $\underline{u}$ and $\overline{u}$. We assume that:

(H_6) there exists a constant $L_2 > 0$ such that

$$\alpha_E\bigl(\{f(t,x_n,y_n)\}\bigr) \le L_2\bigl(\alpha_E(\{x_n\}) + \alpha_E(\{y_n\})\bigr),$$

for any $t \in J$ and countable subsets $\{x_n\}, \{y_n\} \subset [v_0(t), w_0(t)]$;

(H_7) there exist $M_k > 0$, $k = 1, 2, \dots, m$ with $\sum_{k=1}^{m} M_k < \frac{1}{4C}$ such that

$$\alpha_E\big(\{I_k\big(x_n(t_k), y_n(t_k)\big)\}\big) \le M_k\big[\alpha_E\big(\{x_n(t_k)\}\big) + \alpha_E\big(\{y_n(t_k)\}\big)\big], \quad k = 1, 2, \dots, m,$$

for any countable subsets $\{x_n\}, \{y_n\} \subset [v_0, w_0]$;

(H_8) $\{g(x_n, y_n)\}$ is precompact for any countable subsets $\{x_n\}, \{y_n\} \subset [v_0, w_0]$.

Then we obtain the following existence result.

Theorem 2 *Let $-A$ generate a positive and equicontinuous C_0-semigroup $S(t)$ $(t \ge 0)$ in E. Assume that the system* (2) *has coupled lower and upper L-quasi-solutions v_0 and w_0 with $v_0 \le w_0$. If the conditions* (H_1)-(H_3) *and* (H_6)-(H_8) *hold, the system* (2) *has minimal and maximal coupled mild L-quasi-solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, and has at least one mild solution between $\underline{u}$ and $\overline{u}$.*

Proof It is clear that (H_6) implies (H_4)$'$ and (H_8) implies (H_5). Hence, by Corollary 1, the system (2) has minimal and maximal coupled mild L-quasi-solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0. Next, we prove the existence of mild solutions of the system (2) between $\underline{u}$ and $\overline{u}$. Let $Tu = Q(u, u)$. Then $T : [v_0, w_0] \to [v_0, w_0]$ is continuous. If $u \in PC(J, E)$ is a fixed point of the operator T, then $u = Tu = Q(u, u)$. By Definitions 3 and 4 and the definition of the operator Q, u is the mild solution of the system (2). For any $D \subset [v_0, w_0]$, we see that $T(D) \subset [v_0, w_0]$ is bounded and equicontinuous. So, by Lemma 5, there exists a countable set $D_0 = \{x_n\} \subset D$ such that $\alpha_{C(J,E)}(T(D)) \le 2\alpha_{C(J,E)}(T(D_0))$. For $t \in J_0 = [0, t_1]$, by the definition of T, we have

$$\begin{aligned}
\alpha_E\big(T(D_0)(t)\big) &= \alpha_E\big(Q(D_0, D_0)(t)\big) \\
&\le \alpha_E\big(\{U_1(t)\big[u_0 + g(x_n, x_n)\big]\}\big) \\
&\quad + \alpha_E\left(\left\{\int_0^t (t-s)^{q-1} V_1(t-s)\big[f\big(s, x_n(s), x_n(s)\big) + Mx_n(s)\big]\,ds\right\}\right) \\
&\le C\alpha_E\big(\{g(x_n, x_n)\}\big) \\
&\quad + \frac{2qC}{\Gamma(q+1)} \int_0^t (t-s)^{q-1} \alpha_E\big(\{f\big(s, x_n(s), x_n(s)\big) + Mx_n(s)\}\big)\,ds \\
&\le \frac{2qC(M + 2L_2)}{\Gamma(q+1)} \int_0^t (t-s)^{q-1} \alpha_E\big(D_0(s)\big)\,ds \\
&\le \frac{2C(M + 2L_2)a^q}{\Gamma(q+1)} \alpha_{C(J,E)}(D).
\end{aligned}$$

For $t \in J_k = (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, by the definition of T, we have

$$\begin{aligned}
\alpha_E\big(T(D_0)(t)\big) &= \alpha_E\big(Q(D_0, D_0)(t)\big) \\
&\le \alpha_E\big(\{U_1(t)\big[u_0 + g(x_n, x_n)\big]\}\big) + \alpha_E\left(\left\{\sum_{i=1}^{k} U_1(t-t_i) I_i\big(x_n(t_i), x_n(t_i)\big)\right\}\right) \\
&\quad + \alpha_E\left(\left\{\int_0^t (t-s)^{q-1} V_1(t-s)\big[f\big(s, x_n(s), x_n(s)\big) + Mx_n(s)\big]\,ds\right\}\right)
\end{aligned}$$

$$\le 2C\sum_{i=1}^{k} M_i\alpha_E\big(D_0(t_i)\big) + \frac{2qC(M+2L_2)}{\Gamma(q+1)}\int_0^t (t-s)^{q-1}\alpha_E\big(D_0(s)\big)\,ds$$
$$\le \left(2C\sum_{k=1}^{m} M_k + \frac{2C(M+2L_2)a^q}{\Gamma(q+1)}\right)\alpha_{C(J,E)}(D).$$

Hence for any $t \in J$, we have

$$\alpha_E\big(T(D_0)(t)\big) \le \left(2C\sum_{k=1}^{m} M_k + \frac{2C(M+2L_2)a^q}{\Gamma(q+1)}\right)\alpha_{C(J,E)}(D).$$

Since $T(D_0)$ is bounded and equicontinuous, by Lemma 3, we have

$$\alpha_{C(J,E)}\big(T(D)\big) \le 2\alpha_{C(J,E)}\big(T(D_0)\big) = 2\max_{t\in J}\alpha_E\big(T(D_0)(t)\big)$$
$$\le \left(4C\sum_{k=1}^{m} M_k + \frac{4C(M+2L_2)}{\Gamma(q+1)}a^q\right)\alpha_{C(J,E)}(D).$$

(i) If $4C\sum_{k=1}^m M_k + \frac{4C(M+2L_2)}{\Gamma(q+1)}a^q < 1$, then the $T : [v_0, w_0] \to [v_0, w_0]$ is a condensing mapping. By Lemma 6, T has at least one fixed point u in $[v_0, w_0]$.

(ii) If $4C\sum_{k=1}^m M_k + \frac{4C(M+2L_2)}{\Gamma(q+1)}a^q \ge 1$, then divided $J = [0, a]$ into n equal parts. Let $\triangle_n : 0 = t'_0 < t'_1 < \cdots < t'_n = a$ and t'_i $(i = 1, 2, \ldots, n-1)$ be not the impulsive points such that

$$4C\sum_{k=1}^{m} M_k + \frac{4C(M+2L_2)}{\Gamma(q+1)}\|\triangle_n\|^q < 1. \tag{12}$$

By (i) and (12), the system (2) has a mild solution u_1 in $[0, t'_1]$. Again by (i) and (12), if (2) with $u(t'_1) = u_1(t'_1)$ as initial value, then it has a mild solution $u_2(t)$ in $[t'_1, t'_2]$, which satisfies $u_2(t'_1) = u_1(t'_1)$. Thus, the mild solution of (2) continuously extends from $[0, t'_1]$ to $[t'_1, t'_2]$. Continuing such a process interval by interval from $[0, t'_1]$ to $[t'_{n-1}, a]$, we obtain the mild solution $u \in PC(J, E)$ of the system (2), which satisfies $u(t) = u_i(t)$ for $t \in [t'_{i-1}, t'_i]$, $i = 1, 2, \ldots, n$.

Finally, since $u = Tu = Q(u, u)$, $v_0 \le u \le w_0$, by the mixed monotonicity of Q, we have

$$v_1 = Q(v_0, w_0) \le Q(u, u) \le Q(w_0, v_0) = w_1.$$

Similarly, $v_2 \le u \le w_2$. In general, $v_n \le u \le w_n$. Letting $n \to \infty$, we get $\underline{u} \le u \le \overline{u}$. Therefore, the system (2) has at least one mild solution between $\underline{u}$ and $\overline{u}$. □

Remark 3 Analytic semigroup and differentiable semigroup are equicontinuous semigroup [27]. In applications of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroup is an analytic semigroup. So Theorem 2 has extensive applicability.

Now, we discuss the uniqueness of mild solution for the system (2) in $[v_0, w_0]$. If we further assume that the following conditions hold:

(H_9) there exist $M_3 > 0$ and $L_3 > 0$ such that

$$f(t,x_2,y_2) - f(t,x_1,y_1) \leq M_3(x_2 - x_1) - L_3(y_2 - y_1),$$

for any $t \in J$, $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$ and $v_0(t) \leq y_2 \leq y_1 \leq w_0(t)$;

(H_{10}) $g : [v_0, w_0] \times [v_0, w_0] \to E$ is continuous and satisfies

$$g(x_1,y_1) \leq g(x_2,y_2),$$

for any $v_0 \leq x_1 \leq x_2 \leq w_0$ and $v_0 \leq y_2 \leq y_1 \leq w_0$; particularly, for any $x_1, x_2 \in [v_0, w_0]$ with $x_1 \leq x_2$, one has

$$g(x_2,x_1) - g(x_1,x_2) = 0,$$

then we obtain the following existence and uniqueness theorem.

Theorem 3 *Let $-A$ generate a positive C_0-semigroup $S(t)$ $(t \geq 0)$ in E. Assume that the system* (2) *has coupled lower and upper L-quasi-solutions v_0 and w_0 with $v_0 \leq w_0$. If the conditions* (H_1), (H_2), (H_5), (H_9), *and* (H_{10}) *hold, the system* (2) *has a unique mild solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 or w_0.*

Proof It is clear that (H_{10}) implies (H_3). we first prove that (H_1) and (H_9) imply (H_4). For any $t \in J$, let $\{x_n\} \subset [v_0(t), w_0(t)]$ be an increasing sequence and $\{y_n\} \subset [v_0(t), w_0(t)]$ be a decreasing sequence. For $m, n \in \mathbb{N}$ with $m > n$ and $t \in J$, by (H_1) and (H_9), we have

$$\begin{aligned} 0 &\leq f(t,x_m,y_m) - f(t,x_n,y_n) + M(x_m - x_n) - L(y_m - y_n) \\ &\leq (M + M_3)(x_m - x_n) - (L + L_3)(y_m - y_n). \end{aligned}$$

Combining this with the normality of cone K, we have

$$\begin{aligned} &\bigl\| f(t,x_m,y_m) - f(t,x_n,y_n) \bigr\| \\ &\quad\leq N \bigl\| (M + M_3)(x_m - x_n) - (L + L_3)(y_m - y_n) \bigr\| + M\|x_m - x_n\| + L\|y_m - y_n\| \\ &\quad\leq \bigl[N(M + M_3) + M\bigr]\|x_m - x_n\| + \bigl[N(L + L_3) + L\bigr]\|y_m - y_n\|. \end{aligned}$$

From this inequality and the definition of the measure of noncompactness, it follows that

$$\begin{aligned} \alpha\bigl(\bigl\{f(t,x_n,y_n)\bigr\}\bigr) &\leq \bigl[N(M + M_3) + M\bigr]\alpha\bigl(\{x_n\}\bigr) + \bigl[N(L + L_3) + L\bigr]\alpha\bigl(\{y_n\}\bigr) \\ &\leq \frac{L_4}{2}\bigl(\alpha\bigl(\{x_n\}\bigr) + \alpha\bigl(\{y_n\}\bigr)\bigr) \leq L_4\alpha\bigl(\{x_n\} + \{y_n\}\bigr), \end{aligned}$$

where $\frac{L_4}{2} = N(M + M_3) + M + N(L + L_3) + L$. Similarly, we can prove that there exists a constant $L_5 > 0$ such that

$$\alpha_E\bigl(\bigl\{f(t,y_n,x_n)\bigr\}\bigr) \leq L_5\alpha_E\bigl(\{x_n\} + \{y_n\}\bigr).$$

Thus, for any $t \in J$, increasing monotone sequence $\{x_n\} \subset [v_0(t), w_0(t)]$ and decreasing monotone sequence $\{y_n\} \subset [v_0(t), w_0(t)]$, we have

$$\begin{aligned}\alpha_E\big(\{f(t,x_n,y_n)+f(t,y_n,x_n)\}\big) &\le \alpha_E\big(\{f(t,x_n,y_n)\}\big)+\alpha_E\big(\{f(t,y_n,x_n)\}\big)\\&\le (L_4+L_5)\alpha_E\big(\{x_n\}+\{y_n\}\big).\end{aligned}$$

It implies that the condition (H_4) holds with $L_1 = L_4 + L_5$. Therefore, by Theorem 1, the system (2) has minimal and maximal coupled mild L-quasi-solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0. By the proof of Theorem 1, we know that (10) and (11) are valid. Going from J_0 to J_m interval by interval, we show that $\underline{u}(t) = \overline{u}(t)$ on every J_k.

For $t \in J_0$, we have

$$\begin{aligned}0 &\le \overline{u}(t)-\underline{u}(t) = Q(\overline{u},\underline{u})(t)-Q(\underline{u},\overline{u})(t)\\&= U_1(t)\big[u_0+g(\overline{u},\underline{u})\big]\\&\quad+\int_0^t (t-s)^{q-1}V_1(t-s)\big[f\big(s,\overline{u}(s),\underline{u}(s)\big)+(M+L)\overline{u}(s)-L\underline{u}(s)\big]\,ds\\&\quad-U_1(t)\big[u_0+g(\underline{u},\overline{u})\big]\\&\quad-\int_0^t (t-s)^{q-1}V_1(t-s)\big[f\big(s,\underline{u}(s),\overline{u}(s)\big)+(M+L)\underline{u}(s)-L\overline{u}(s)\big]\,ds\\&= U_1(t)\big[g(\overline{u},\underline{u})-g(\underline{u},\overline{u})\big]\\&\quad+\int_0^t (t-s)^{q-1}V_1(t-s)\big[f\big(s,\overline{u}(s),\underline{u}(s)\big)-f\big(s,\underline{u}(s),\overline{u}(s)\big)\\&\quad+(M+2L)\big(\overline{u}(s)-\underline{u}(s)\big)\big]\,ds\\&\le (M_3+L_3+M+2L)\int_0^t (t-s)^{q-1}V_1(t-s)\big(\overline{u}(s)-\underline{u}(s)\big)\,ds.\end{aligned}$$

From this and the normality of cone K, it follows that

$$\big\|\overline{u}(t)-\underline{u}(t)\big\| \le \frac{NC(M_3+L_3+M+2L)}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big\|\overline{u}(s)-\underline{u}(s)\big\|\,ds.$$

By Lemma 7, we obtain $\overline{u}(t) \equiv \underline{u}(t)$ on J_0. Particularly, $\overline{u}(t_1) = \underline{u}(t_1)$, so, $I_1(\overline{u}(t_1),\underline{u}(t_1)) = I_1(\underline{u}(t_1),\overline{u}(t_1))$.

For $t \in J_1$, we can prove that

$$\begin{aligned}\big\|\overline{u}(t)-\underline{u}(t)\big\| &\le \frac{NC(M_3+L_3+M+2L)}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big\|\overline{u}(s)-\underline{u}(s)\big\|\,ds\\&\le \frac{NC(M_3+L_3+M+2L)}{\Gamma(q)}\int_{t_1}^t (t-s)^{q-1}\big\|\overline{u}(s)-\underline{u}(s)\big\|\,ds.\end{aligned}$$

Again by Lemma 7, we obtain that $\overline{u}(t) \equiv \underline{u}(t)$ on J_1, and $I_2(\overline{u}(t_2),\underline{u}(t_2)) = I_2(\underline{u}(t_2),\overline{u}(t_2))$.

Continuing such a process interval by interval up to J_m, we see that $\overline{u}(t) \equiv \underline{u}(t)$ over the whole J. Hence, $\tilde{u} := \overline{u} = \underline{u}$ is the unique mild solution of the system (2) on $[v_0, w_0]$, which can be obtained by the monotone iterative procedure starting from v_0 or w_0. □

4 An example

Consider the impulsive fractional differential equation with nonlocal conditions of the form

$$\begin{cases} D_t^q u(t,y) + \frac{\partial^2}{\partial y^2} u(t,y) \\ \quad = f(t,y,u(t,y),u(t,y)), \quad q \in (0,1), (t,y) \in [0,1] \times [0,\pi], t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k,y),u(t_k,y)), \quad y \in [0,\pi], k = 1,2,\ldots,m, \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,1], \\ u(0,y) = u_0(y) + g(u(t,y),u(t,y)), \quad (t,y) \in [0,1] \times (0,\pi). \end{cases} \tag{13}$$

Let $E = L^2([0,\pi])$. Define $Au = \frac{\partial^2}{\partial y^2} u$ for $u \in D(A)$, where

$$D(A) = \left\{ x \in E : \frac{\partial x}{\partial y}, \frac{\partial^2 x}{\partial y^2} \in E, x(0) = x(\pi) = 0 \right\}.$$

Then $-A$ generates a positive C_0-semigroup $S(t)$ $(t \geq 0)$ in E, which is equicontinuous and $C = 1$.

Let $0 \leq w \in PC(J,E)$ satisfy the following conditions:

(P_1) $0 \leq I_k(0,w(t_k,y))$ and $I_k(w(t_k,y),0) \leq \Delta w|_{t=t_k}$, $k = 1,2,\ldots,m$, $y \in [0,\pi]$;

(P_2) $0 \leq u_0(y) + g(0,w(t,y))$ and $u_0(y) + g(w(t,y),0) \leq w(0,y)$, $(t,y) \in [0,1] \times (0,\pi)$;

(P_3) $Lw(t,y) \leq f(t,y,0,w(t,y))$ and $f(t,y,w(t,y),0) \leq D_t^q w(t,y) + (A - LI)w(t,y)$, $(t,y) \in [0,1] \times [0,\pi]$, $t \neq t_k$.

Then 0 and w are coupled lower and upper L-quasi-solutions of the systems (13).

Therefore, if the functions f, g, and I_k $(k = 1,2,\ldots,m)$ satisfy the conditions (H_1)-(H_5) on the interval $[0,w]$, the system (13) has minimal and maximal coupled mild L-quasi-solutions between 0 and w.

If the functions f, g, and I_k $(k = 1,2,\ldots,m)$ satisfy the conditions (H_1)-(H_3) and (H_6)-(H_8) on the interval $[0,w]$, the system (13) has at least mild L-quasi-solutions on $[0,w]$.

If the functions f, g, and I_k $(k = 1,2,\ldots,m)$ satisfy the conditions (H_1), (H_2), (H_5), (H_9), and (H_{10}) on the interval $[0,w]$, the system (13) has a unique mild solution on $[0,w]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

[1]School of Science and Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), Jiangnan University, Wuxi, 214122, People's Republic of China. [2]Science College, Gansu Agricultural University, Lanzhou, 730070, People's Republic of China.

Acknowledgements

Thanks to the reviewers for their helpful comments and suggestions. The second author is supported by the Sheng Tong-sheng Technological Innovation Fund of Gansu Agricultural University (Grant No. GSAU-STS-1423).

Received: 27 March 2015 Accepted: 9 October 2015 Published online: 19 October 2015

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