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Monotone Iterative Technique for Nonlinear Periodic Time Fractional Parabolic Problems

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Abstract

In this paper, existence and uniqueness of weak solutions for a linear parabolic problem with conformable derivative are proved, the existence of weak periodic solutions for conformable fractional parabolic nonlinear differential equation is proved by using a more generalized monotone iterative method combined with the method of upper and lower solutions. We prove the convergence of monotone sequence to weak periodic minimal and maximal solutions. Moreover, the conformable version of the Lions-Magness and Aubin–Lions lemmas are also proved.

Keywords: Nonlinear equation, parabolic equations, conformable fractional derivative, upper and lower solutions, monotone iterative method, conformable Aubin-Lions lemma .
2010 MSC: 35R11, 34K37, 47J35, 58J35, 35K55, 35K91.

1. Introduction

Let $\Omega \subset \mathrm{R}^N$ be a bounded domain with boundary $\partial\Omega, Q = (0,T)\times\Omega$ and $\Gamma = [0,T]\times\partial\Omega$

In this paper, we consider the following fractional parabolic periodic boundary valued problem (PBVP for short)

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$$\begin{cases} T_t^\alpha(u)(t,x) - \Delta u(t,x) = f(t,x,u) \text{ in } Q, \\ u(t,x) = 0, \quad \text{on } \Gamma, \\ u(0,x) = u(T,x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where T_t^α is the conformable fractional derivative of order $\alpha \in (0,1]$ with respect to t and Δ is Laplace operator. The nonlinear right-hand side $f : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is f is measurable in $(t,x) \in Q$ for each $u \in \mathbb{R}$ and continuous in u for a.e. $(t,x) \in Q$.

The derivative of non-integer order has been an interesting research topic for several centuries. The idea was motivated by the question, "What does it mean by $\frac{d^n f}{dx^n}$, if $n = \frac{1}{2}$?", asked by L'Hopital in 1695 in his letters to Leibniz. Since then, the mathematicians tried to answer this question for centuries in several points of view. Various types of fractional derivatives were introduced: Riemann-Liouville, Caputo, Hadamard, Grünwald-Letnikov, Marchaud, and the last definitions are Caputo-Fabrizio derivative, Atangana–Baleanu derivative and Conformable fractional derivative [22]. It is well known that fractional order differential equations provide an excellent setting for capturing, in a model framework, real-world problems in many disciplines, such as chemistry, physics, engineering, biology and ecology [29, 41, 35, 50, 30, 24, 27, 25] . In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Podlubny [41], Kilbas et al. [29], Zhou et al. [50], and the recent papers [49, 30, 4, 27, 28, 23, 26, 8, 21, 38] and the references therein.

Since the last century, periodic parabolic problems have been the subject of extensive study, see for example [19, 17, 6, 14, 7, 48, 18].

Recently, the authors in [36] and [37] have studied the positive mild solutions of (1.1) with Caputo fractional derivative by using the characteristics of positive operators, semigroups and the monotone iterative scheme. Binh et al. [12] have considered the initial inverse problem for a diffusion equation with a conformable derivative in a general bounded domain. Tuan et al. [44] have studied a backward problem for a nonlinear diffusion equation with a conformable derivative in the case of multidimensional and discrete data. Tuan et al. [45] have considered an inverse problem of recovering the initial value for a generalization of time-fractional diffusion equation, where the time derivative is replaced by a regularized hyper-Bessel operator. The authors in [46] have investigated an inverse problem to determine an unknown source term for fractional diffusion equation with Riemann–Liouville derivative. Au et al. [10] studied the ill-posed property in the sense of Hadamard for an inverse nonlinear diffusion equation with conformable time derivative.

The method of upper and lower solutions coupled with monotone iterative technique offers an effective and flexible mechanism for proving theoretical as well as existence and comparison results for a variety of nonlinear differential problems, see [20, 39, 31].

In this paper we develop the generalized monotone iterative method combined with the method of upper and lower solutions for the nonlinear fractional periodic parabolic differential problems. Here, we prove the existence and uniqueness of the weak solution for a linear parabolic equation with conformable derivative. We show that the monotone sequences, which are solutions of the linear fractional parabolic equation converge to the minimal and maximal periodic solutions of the nonlinear equation, these comparison results are used to establish the last result. In addition, the conformable version of the Lions-Magness and Aubin - Lions lemmas are also proved and used to establish our results.

The paper is organized as follows. Section 2 provides the definitions and preliminary results to be used in the article. In section 3, we prove some lemmas needed in the proof of our main theorems. In Section 4 main results are stated and proved: comparison results are obtained and existence of weak solution to linear equation and the extremal periodic solutions for the nonlinear equations (1.1) are proved.

2. Definitions and preliminaries

In the literature, there are many definitions of fractional derivative. To mention some:

1. Riemann - Liouville definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$ {}_{a}^{RL}D_t^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx. $$

2. Caputo definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$ {}_{a}^{C}D_t^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx. $$

3. Caputo-Fabrizio definition [16]:

$$ {}_{a}^{CF}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(x) e^{-\frac{\alpha(t-x)}{1-\alpha}} dx, \quad 0 < \alpha < 1, t > a. $$

4. Atangana-Baleanu-Caputo definition [9]:

$$ {}_{b}^{ABC}D_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha} \int_b^t f'(x) E_{\alpha} \left(-\alpha \frac{(t-x)^{\alpha}}{1-\alpha} \right) dx, \quad 0 < \alpha < 1, t > b. $$

However, the following are the setbacks of one definition or the other:

1. The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0$, if α is not a natural number.
2. All fractional derivatives do not satisfy the known formula of the derivative of the product

$$ D_a^{\alpha}(fg) = f D_a^{\alpha}(g) + g D_a^{\alpha}(f). $$

3. All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions:

$$ D_a^{\alpha}(f/g) = \frac{g D_a^{\alpha}(f) - f D_a^{\alpha}(g)}{g^2}. $$

4. All fractional derivatives do not satisfy the chain rule:

$$ D_a^{\alpha}(f \circ g)(t) = f^{(\alpha)}(g(t)) g^{(\alpha)}(t). $$

5. All fractional derivatives do not satisfy: $D^{\alpha} D^{\beta} f = D^{\alpha+\beta} f$, in general.

In [22], the authors gave a new definition of fractional derivative which is a natural extension to the usual first derivative as follows:

Definition 2.1 ([22])**.** Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$. Then for all $t > 0, \quad \alpha \in (0, 1]$, let

$$ T^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, $$

T^{α} is called the **conformable fractional derivative** of f of order α. If f is α-differentiable in some $(0, b), b > 0$, and $\lim_{t \to 0^+} T^{\alpha}(f)(t)$ exists, then let

$$ T^{\alpha}(f)(0) = \lim_{t \to 0^+} T^{\alpha}(f)(t). $$

Now we shall present some properties of this new derivative, more properties of this type of derivation are given in [22, 1, 47, 11, 2, 3, 13].

Proposition 2.1 ([22]).

1. $T^{\alpha}(af + bg) = aT^{\alpha}(f) + bT^{\alpha}(g)$ for all real constant a, b.
2. $T^{\alpha}(fg) = fT^{\alpha}(g) + gT^{\alpha}(f)$.
3. $T^{\alpha}\left(\frac{f}{g}\right) = \frac{gT^{\alpha}(f) - fT^{\alpha}(g)}{g^2}$.

4. $T^\alpha(c) = 0$ with c is a constant.
5. $T^\alpha(f \circ g)(t) = f'(g(t))T^\alpha g(t)$, for f differentiable at $g(t)$.
6. If, in addition, f is differentiable, then $T^\alpha(f) = t^{1-\alpha} f'(t)$.

Definition 2.2 (Fractional Integral[22]). Let $a \geq 0$ and $t \geq a$. Also, let f be a function defined on $(a, t]$ and $\alpha \in \mathbb{R}$. Then, the $\alpha-$fractional integral of f is defined by,

$$\mathcal{I}_a^\alpha(f)(t) := \int_a^t f(x) d_\alpha x = \int_a^t x^{\alpha-1} f(x) dx,$$

if the Riemann improper integral exists. When $a = 0$ we write $\mathcal{I}_0^\alpha(f)(t) = \mathcal{I}^\alpha(f)(t)$

$f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is α-integrable on $[a, b]$ if and only if $t^{\alpha-1} f$ is integrable on $[a, b]$.

Lemma 2.1 ([1]). *Assume that $f : [a, \infty) \to \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$T^\alpha \mathcal{I}_a^\alpha f(t) = f(t).$$

Lemma 2.2 ([1]). *Let $f : (a, \infty) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$\mathcal{I}_a^\alpha T^\alpha(f)(t) = f(t) - f(a).$$

In the rest of this section, we assume $a, b \in \mathbb{R}, 0 < a < b$.

Theorem 2.2 (fractional Gronwall inequality [1]). *Let y be a continuous, nonnegative function on an interval $J = [a, b]$, δ and k be nonnegative constants such that*

$$y(t) \leq \delta + \int_a^t k s^{\alpha-1} y(s) ds := \delta + \int_a^t k y(s) d_\alpha s \quad (t \in J).$$

Then for all $t \in J$

$$y(t) \leq \delta e^{k\frac{t^\alpha}{\alpha}}.$$

Theorem 2.3 (Integration by parts [22]). *Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b f(x) T^\alpha(g)(x) d_\alpha x = f\, g|_a^b - \int_a^b g(x) T^\alpha(f)(x) d_\alpha x.$$

Definition 2.3 ([47]). Let $p \in [1, +\infty]$ and let $f : [a, b] \subset \mathbb{R} \to \overline{\mathbb{R}}$ be a measurable function. Say that f belongs to $L^p_\alpha([a, b])$ provided that either

$$\int_a^b |f(t)|^p d_\alpha t < +\infty \qquad \text{if } 1 \leq p < +\infty,$$

or there exists a constant $C \in \mathbb{R}$ such that

$$|f| \leq C \qquad \text{a.e. on } [a, b] \text{ if } p = +\infty.$$

Theorem 2.4 ([47]). *Let $p \in [1, +\infty]$. Then the set $L^p_\alpha([a, b])$ is a Banach space endowed with the norm defined for $f \in L^p_\alpha([a, b])$ as*

$$\|f\|_{L^p_\alpha([a,b])} = \begin{cases} \left(\int_a^b |f(t)|^p d_\alpha t\right)^{1/p} & \text{if } 1 \leq p < +\infty, \\ \inf \{C \in \mathbb{R} : |f| \leq C \text{ a.e. on } [a, b]\} & \text{if } p = \infty. \end{cases}$$

Moreover, $L^2_\alpha([a, b])$ is a Hilbert space with the inner product given for every $(f, g) \in L^2_\alpha([a, b]) \times L^2_\alpha([a, b])$ by

$$\langle f, g \rangle_{L^2_\alpha([a,b])} = \int_{[a,b]} f(t) g(t) d_\alpha t.$$

Before the statement of the properties, we denote

$$C_0([a,b]) = \{f : [a,b] \to \mathbb{R}, f \text{ is continuous on } [a,b] \text{ with compact support in}[a,b]\}$$

Theorem 2.5 ([47]). *Let $f \in L^1_\alpha([a,b])$ be such that the following equality is true:*

$$\int_{[a,b]} f(t)u(t)d_\alpha t = 0, \quad \forall u \in C_0([a,b]),$$

then

$$f \equiv 0, \quad a.e. \ on \ [a,b].$$

An interesting particular case of the Theorem 1 in [42] is the following

Corollary 2.6 ([42]). Let $\Omega \subset R^n$ be an open and connected set, assume $(0,T) \subset R^+$, and define $Q = (0,T) \times \Omega$. Given a linear system of conformable fractional ordinary differential equations of order α, $0 < \alpha \leq 1$, written in matrix form as :

$$T^\alpha_t y(t) = A(t)y(t) + B(t), \tag{2.1}$$

where the matrices $A(t)$ and $B(t)$ are assumed to be continuous in an interval $[0,T]$, then $\forall (t_0, y_0) \in Q$ there exists an unique continuous solution y in $[0,t_1] \subset [0,T]$ such that $t_0 \in [0,t_1]$ and $y(t_0) = y_0$.

Remark 2.1. The above Theorem asserts that finite time blow-up is equivalent to global nonexistence. More precisely, if $(t_0, y_0) \in Q$ there exists a solution $y(t)$ of (2.1) with $y(t_0) = y_0$, defined on a maximal interval of existence $[0, t_{\max})$, where $t_0 < t_{\max} \leq T$, and if $t_{\max} < T$ then

$$\lim_{t \to t_{\max}} |y(t)| = +\infty.$$

Lemma 2.3 (Aubin-Lions [33, 43]). *Let X_0, X, and X_1 be three Banach spaces such that*

$$X_0 \subset X \subset X_1,$$

the injection of X into X_1 being continuous, and the injection of X_0 into X is compact. Then for each $\eta > 0$, there exists some constant c_η depending on η (and on the spaces X_0, X, X_1) such that:

$$\|v\|_X \leq \eta \|v\|_{X_0} + c_\eta \|v\|_{X_1}, \forall v \in X_0$$

3. Some lemmas for time fractional PDEs

In this section, we prove some lemmas needed in the proof of our main theorems.

For $p \in [1, +\infty[$ we set the space

$$L^p_\alpha([a,b], E) = \left\{ u : [a,b] \to E : \int_{[a,b]} ||f(t)||^p_E d_\alpha t < +\infty \right\}.$$

Let $H^1(\Omega)$ denote the usual Sobolev space of square integrable functions and let $\left(H^1(\Omega)\right)'$ denote its dual space. Then by identifying $L^2(\Omega)$ with its dual space, $H^1(\Omega) \subset L^2(\Omega) \subset \left(H^1(\Omega)\right)'$ forms an evolution triple with all the embeddings being continuous, dense and compact.

We set $V = L^2_\alpha\left(0,T; H^1(\Omega)\right)$, denote its dual space by $V' = L^2_\alpha(0,T; (H^1(\Omega))')$, and define a function space W by

$$W = \left\{ w \in V | T^\alpha_t u \in V' \right\},$$

where the conformable derivative T^α_t is understood in the sense of distributions,

Theorem 3.1. *The spaces V and W endowed with norms*

$$\|u\|_V = \left(\int_Q (|u(t,x)|^p + |\nabla u(t,x)|^p)\, dx d_\alpha t \right)^{\frac{1}{p}},$$

$$\|u\|_W = \|u\|_V + \|T_t^\alpha u\|_{V'},$$

are a Banach spaces which are separable and reflexive

Proof. The proof is due to the separability and reflexivity of V and $V^{'}$, respectively. □

The following Lemma is a generalization of Lions-Magness Lemma [34].

Lemma 3.1. *Let V, H, and V' be Hilbert spaces such that $V \subset H \equiv H' \subset V'$, where V' is the dual of V and the injections are continuous. Suppose that $\mathbf{u} \in L^2_\alpha([0,T];V)$ and $T_t^\alpha \mathbf{u} \in L^2_\alpha([0,T];V')$. Then u is equal a.e. to a continuous function from $[0,T]$ into H, and the following equality holds in the distribution sense on $(0,T)$*

$$T_t^\alpha(|\mathbf{u}|_H^2) = 2\,\langle T_t^\alpha \mathbf{u}, \mathbf{u}\rangle_{V',V}. \tag{3.1}$$

Moreover, the embedding $W \subset C([0,T];H)$ is continuous.

Remark 3.1. As a consequence of the previous identifications, the scalar product in H of $f \in H$ and $v \in V$ is the same as the scalar product of f and v in the duality between $V^{'}$ and V:

$$\langle f, v\rangle_{V',V} = (f,v)_H, \quad \forall f \in H, \forall v \in V. \tag{3.2}$$

Proof. Let $\eta < 0 < T < \beta$, and let

$$W(\eta,\beta) = \left\{ u \in L^2_\alpha([\eta,\beta];V), \text{ with } T^\alpha u \in L^2_\alpha([\eta,\beta];V') \right\}.$$

Let θ be a C^∞ function equal to 1 on $[0,T]$ and zero in neighborhood of η and β, define v on $[\eta,\beta]$ by $v(t) = u(t)$ if $t > 0$ and $v(t) = u(-t)$ if $t < 0$. We set $\omega = \theta v$ on $[0,T]$, then $\omega \in W(\eta,\beta)$.

Now by regularization on t we approximate ω in $W(\eta,\beta)$ by a sequence of C^α function ω_m vanishing in a neighborhood of η and β, we have $(\omega_m(t), T^\alpha\omega_m(t)) \in V \times V'$ and by (3.2) one can write for any $t \in (\eta,\beta)$

$$|\omega_m(t)|_H^2 = \int_\eta^t T_s^\alpha(\omega_m(s),\omega_m(s))_H d_\alpha s, \tag{3.3}$$

$$= 2\int_\eta^t \langle T_s^\alpha \omega_m(s), \omega_m(s)\rangle d_\alpha s, \tag{3.4}$$

$$\leq 2\int_\eta^t \|T_s^\alpha \omega_m(s)\|_{V'} \|\omega_m(s)\|_V d_\alpha s, \tag{3.5}$$

$$\leq \int_\eta^t \left(\|T_s^\alpha \omega_m(s)\|_{V'}^2 + \|\omega_m(s)\|_V^2 \right) d_\alpha s. \tag{3.6}$$

Hence $|\omega_m(t)|_H \leq \|\omega_m\|_{W(\eta,\beta)}$, thus the continuous ω_m converge uniformly in H to a continuous function, which means that $\omega \in C([\eta,\beta],H)$, after possible modification on a negligible set, and hence $u \in C([0,T];H)$.

Now becuse of (3.4), the equality (3.1) for w_m is obvious, therefore we are allowed to pass to the limit in the distribution sense; in the limit we find precisely (3.1). □

Let X_0, X, X_1, be three Banach spaces such that

$$X_0 \subset X \subset X_1, \tag{3.7}$$

where the injections are continuous and:

$$X_i \text{ is reflexive, } i = 0, 1. \tag{3.8}$$

$$\text{The injection } X_0 \to X \text{is compact.} \tag{3.9}$$

Let $T > 0$ be a fixed finite number, and let p_0, p_1, be two finite numbers such that $p_i > 1, i = 0, 1$. We consider the space

$$W = \left\{v \in L^{p_0}_\alpha \left(0, T; X_0\right), T^\alpha_t v \in L^{p_1}_\alpha \left(0, T; X_1\right)\right\}. \tag{3.10}$$

The space W is provided with the norm

$$\|v\|_W = \|v\|_{L^{p_0}_\alpha(0,T;X_0)} + \|T^\alpha_t v\|_{L^{p_1}_\alpha(0,T;X_1)},$$

which makes it a Banach space.

Lemma 3.2. *Let (3.7) to (3.10) be satisfied. Then the injection of W into $L^{p_0}_\alpha(0, T : X)$ is compact.*

Remark 3.2.
- When $\alpha = 1$, the result of Lemma 3.2 yield the result of Lions-Aubin which is [33, Theorem 5.1].
- For Caputo's derivative the result of above Lemma is proved in [32], for Riemann-Liouville derivative see [40].

Proof. By defintion of W we have $W \subset L^{p_0}_\alpha(0, T; X)$ with a continuous injection, we shall prove that this injection is compact.

Let u_m be some sequence which is bounded in W. We must prove that this sequence contains a sub-sequence u_μ strongly convergent in $L^{p_0}_\alpha(0, T; X)$ since the spaces X_i are reflexive spaces and $1 < p_i < +\infty$, the spaces $L^{p_i}_\alpha \left(0, T : X_i\right), i = 0, 1$, are likewise reflexive and hence W is reflexive. Therefore, there exists some u in W and some sub-sequence which for simplicity we still denote by u_m, with $u_m \to u$ in W weakly, as $m \to \infty$
which means $u_m \to u$ in $L^{p_0}\left(0, T; X_0\right)$ weakly and $T^\alpha u_m \to T^\alpha u$ in $L^{p_1}\left(0, T; X_1\right)$ weakly. It suffices to prove that

$$v_m = u_m - u \to 0 \text{ in } L^{p_0}_\alpha(0, T; X) \text{ strongly.} \tag{3.11}$$

In fact, due to Lemma 2.3, we have

$$\|v_m\|_{L^{p_0}_\alpha(0,T;X)} \leq \eta \|v_m\|_{L^{p_0}_\alpha(0,T;X_0)} + c_\eta \|v_m\|_{L^{p_0}_\alpha(0,T;X_1)},$$

and since the sequence v_m is bounded in $L^{p_0}_\alpha(0, T; X_0)$:

$$\|v_m\|_{L^{p_0}_\alpha(0,T;X)} \leq C_1\eta + c_\eta \|v_m\|_{L^{p_0}_\alpha(0,T;X_1)}. \tag{3.12}$$

Since η is arbitrary, the theorem will be proved if we show that

$$v_m \to 0 \text{ in } L^{p_0}_\alpha \left(0, T; X_1\right) \text{ strongly.} \tag{3.13}$$

To prove (3.13) we observe that

$$W \subset C\left([0, T]; X_1\right),$$

with a continuous injection; the inclusion results from Lemma 2.3, and the continuity of the injection is very easy to check. We infer from this, the majoration

$$\|v_m(t)\|_{X_1} \leq C_2, \quad \forall t \in [0, T], \forall m.$$

According to Lebesgue's theorem, (3.13) is now proved if we show that, for almost every t in $[0,T]$

$$v_m(t) \to 0 \text{ in } X_1 \text{ strongly, as } m \to \infty. \tag{3.14}$$

We shall prove (3.14) for $t=0$; the proof would be similar for any other t. We write

$$v_m(0) = v_m(t) - \int_0^t T_\tau^\alpha v_m(\tau) d_\alpha \tau,$$

and by integration between 0 and s

$$v_m(0) = \frac{\alpha}{s^\alpha}\left\{\int_0^s v_m(t)d_\alpha t - \int_0^s\int_0^t T_\tau^\alpha v_m(\tau)d_\alpha\tau d_\alpha t\right\},$$

integration by part gives

$$v_m(0) = \frac{\alpha}{s^\alpha}\left\{\int_0^s v_m(t)d_\alpha t - \int_0^s (\frac{s^\alpha}{\alpha} - \frac{t^\alpha}{\alpha})T_t^\alpha v_m(t)d_\alpha t\right\}.$$

Hence

$$v_m(0) = a_m(\alpha) + b_m(\alpha),$$

with

$$a_m(\alpha) = \frac{\alpha}{s^\alpha}\int_0^s v_m(t)\mathrm{d}t, \quad b_m(\alpha) = -\frac{1}{s^\alpha}\int_0^s (s^\alpha - t^\alpha)T_t^\alpha v_m(t)d_\alpha t,$$

hence given $\varepsilon > 0$, we can chose $s \in (0,T)$ such that

$$\|b_m(\alpha)\|_{X_1} \le \int_0^s \|T_t^\alpha v_m(t)\|_{X_1} d_\alpha t \le \frac{\varepsilon}{2}.$$

Then, for this fixed s, we observe that, as $m \to \infty, a_m(\alpha) \to 0$ in X_0 weakly and thus in X_1 strongly; for m sufficiently large

$$\|a_m(\alpha)\|_{X_1} \le \frac{\varepsilon}{2},$$

and (3.14), for $t = 0$, follows.

Passing to the limit in (3.12) we see by (3.13) that

$$\overline{\lim_{m\to\infty}} \|v_m\|_{L_\alpha^{p_0}(0,T;X)} \le c\eta,$$

since $\eta > 0$ is arbitrarily small in Lemma 2.3, this upper limit is 0 and thus (3.11) is proved. This completes the proof. □

4. Main results

Let $H_0^1(\Omega)$ be the subspace of $H^1(\Omega)$ whose elements have generalized homogeneous boundary values, and denote by $H^{-1}(\Omega)$ its dual space. One can consult [15] and [5] for general properties of Sobolev spaces. Then obviously $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ forms also an evolution triple, and all statements made above remain true also in this situation when setting $V_0 = L_\alpha^2\left(0,T;H_0^1(\Omega)\right)$ $V_0' = L_\alpha^2\left(0,T;H^{-1}(\Omega)\right)$ and $W_0 = \left\{w \in V_0 | T_t^\alpha w \in V_0'\right\}$.

Furthermore, by Lemma 3.1 the embedding $W \subset C([0,T];L^2(\Omega))$ is continuous. Finally, because $H^1(\Omega) \subset L^2(\Omega)$ is compactly embedded, by Lemma 3.2 we have a compact embedding of $W \subset L_\alpha^2\left(0,T;L^2(\Omega)\right)$

We denote the duality pairing between the elements of V_0' and V_0 by $\langle\cdot,\cdot\rangle$, and define the bi-linear form B associated with the operator $(-\Delta)$ by

$$\langle -\Delta u, \varphi\rangle = B[u,\varphi] \equiv \sum_{i,j=1}^{N} \int_{[0,T]} \int_{\Omega} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} \mathrm{d}x\mathrm{d}_\alpha t, \quad \varphi \in V_0.$$

We set $L_\alpha = L^2_\alpha[0,T;L^2(\Omega)]$ the cone L^+_α of all nonnegative elements of L_α. This induces a corresponding partial ordering also in the subspace W of $L^2(Q)$, and if $u,\overline{u} \in W$ with $\underline{u} \leq \overline{u}$ then $[\underline{u},\overline{u}] := \{u \in W | \underline{u} \leq u \leq \overline{u}\}$ denotes the order interval formed by $\underline{u}$ and $\overline{u}$. Let $(\cdot,\cdot)$ denote the inner product in L_α, define by

$$(u,v)_{L_\alpha\times L_\alpha} := \int_0^T \int_\Omega u(t,x)v(t,x)dxd_\alpha t = \int_\Omega \int_0^T u(t,x)v(t,x)d_\alpha tdx \quad \forall u,v \in L_\alpha,$$

and denote by F the Nemytskij operator related with the function f by $Fu(t,x) = f(t,x,u(t,x))$

Definition 4.1. A function $u \in W_0$ is called a weak periodic solution of (1.1) if

1. $u(0,x) = u(T,x)$ in Ω,
2. $\langle T^\alpha_t(u),\varphi\rangle + B[u,\varphi] = (Fu,\varphi)$ for all $\varphi \in V_0$,

where

$$\langle T^\alpha_t(u),\varphi\rangle = \int_0^T \int_\Omega T^\alpha_t(u)(t,x)\varphi(t,x)dxd_\alpha t,$$

$$(Fu,\varphi) = \int_0^T \int_\Omega F(u)(t,x)\varphi(t,x)dxd_\alpha t = \int_0^T \int_\Omega f(t,x,u)\varphi(t,x)dxd_\alpha t.$$

Definition 4.2. A function $\underline{u} \in W$ is said to be a weak lower solution of (1.1) if

1. $\underline{u}(t,x) \leq 0$ on Γ, $\underline{u}(0,x) \leq \underline{u}(T,x)$ in Ω,
2. $\langle T^\alpha_t(\underline{u}),\varphi\rangle + B[\underline{u},\varphi] \leq (F\underline{u},\varphi)$ for all $\varphi \in V_0 \cap L^+_\alpha$.

A weak upper solution of (1.1) is defined similarly by reversing the inequalities.

Let us make the following assumptions on the nonlinear right-hand side f.
$(A1)$ $f : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is f is measurable in $(t,x) \in Q$ for each $u \in \mathbb{R}$ and continuous in u for a.e. $(t,x) \in Q$.
$(A2)$ $f(t,x,r) : Q \times \mathbf{R} \to \mathbf{R}$ is nondecreasing with respect to r for a.e. $(t,x) \in Q$.
$(A3)$ that there is a nonnegative function $L \in L^\infty(Q)$ such that

$$f(t,x,u_1) - f(t,x,u_2) \leq L(t,x)(u_1 - u_2), \tag{4.1}$$

whenever $u_1 \geq u_2$ and a.e. $(t,x) \in Q$.

4.1. Comparison results

We can now prove the following comparison result.

Theorem 4.1. *Let (A3) be satisfied and let $\underline{u}$ and $\overline{u}$ be lower and upper solutions of* (1.1), *respectively such that*

$$\underline{u}(0,x) \leq \overline{u}(0,x) \text{ on } \Omega. \tag{4.2}$$

Then we have $\underline{u}(t,.) \leq \overline{u}(t,.)$ a.e in Ω, for all $t \in [0,T]$.

Proof. The definition of lower and upper weak solutions of (1.1) yields

$$\underline{u}(t,x) - \overline{u}(t,x) \leq 0 \text{ on } \Gamma \text{ and } \underline{u}(0,x) - \overline{u}(0,x) \leq \underline{u}(T,x) - \overline{u}(T,x) \text{ in } \Omega,$$

and

$$\langle T^\alpha_t(\underline{u} - \overline{u}),\varphi\rangle + B[\underline{u} - \overline{u},\varphi] \leq (F\underline{u} - F\overline{u},\varphi), \tag{4.3}$$

for all $\varphi \in V_0 \cap L^+_\alpha$. Taking the test function

$$\varphi = (\underline{u} - \overline{u})^+ := \max\{(\underline{u} - \overline{u}), 0\} \in V_0 \cap L^+_\alpha,$$

then by (4.2) $(\underline{u} - \overline{u})(0, x) \leq 0$ it follows that $(\underline{u} - \overline{u})^+(0, x) = 0$ in Ω and by using Lemma 2.2 we get for any $\tau \in (0, T]$ the following equality

$$\begin{aligned} \int_\Omega \int_0^\tau T^\alpha_t (\underline{u} - \overline{u})(\underline{u} - \overline{u})^+ d_\alpha t \; dx &= \frac{1}{2} \int_\Omega \int_0^\tau T^\alpha_t \left[((\underline{u} - \overline{u})^+)^2\right] d_\alpha t \; dx, \\ &= \frac{1}{2} \left\|(\underline{u} - \overline{u})^+(\tau, \cdot)\right\|^2_{L^2(\Omega)}, \end{aligned}$$

and by using $(A3)$ we get from the weak formulation (4.3) for any $\tau \in (0, T]$ the following inequality:

$$\frac{1}{2} \left\|(\underline{u} - \overline{u})^+(\cdot, \tau)\right\|^2_{L^2(\Omega)} + \left\|\nabla(\underline{u} - \overline{u})^+\right\|^2_{L^2_\alpha(0,\tau,L^2(\Omega))} \leq \|L\|_{L^\infty(Q_\tau)} \int_{Q_\tau} t^{1-\alpha} \left((\underline{u} - \overline{u})^+\right)^2 \mathrm{d}x\mathrm{d}t, \tag{4.4}$$

where $Q_\tau := (0, \tau) \times \Omega \subset Q$, and $\|\nabla(\underline{u} - \overline{u})^+\|^2_{L^2_\alpha(0,\tau,L^2(\Omega))} \geq 0$
Thus, by setting

$$y(\tau) = \left\|(\underline{u} - \overline{u})^+(\cdot, \tau)\right\|^2_{L^2(\Omega)},$$

from (4.4) we obtain the inequality

$$y(\tau) \leq 2\|L\|_{L^\infty(Q)} \int_0^\tau t^{1-\alpha} y(t)\mathrm{d}t \quad \text{for all } \tau \in [0, T],$$

since due to Lemma 3.1 $(\underline{u} - \overline{u}) \in C\left([0, T]; L^2(\Omega)\right)$ and $y(\tau) \geq 0$, by applying conformable Gronwall's Lemma $y(\tau) = 0$ for any $\tau \in [0, T]$, which implies

$(\underline{u} - \overline{u})^+(t, x) = 0$ for a.e. $x \in \Omega$, for all $t \in [0, T]$. i.e., $\underline{u}(t, x) \leq \overline{u}(t, x)$ for a.e. $x \in \Omega$ and for all $t \in [0, T]$, proving the claim. □

The following Lemma is useful in our discussion.

Lemma 4.1. *For any $p \in W$ satisfying $p(t, x) \leq 0$ on Γ $p(0, x) \leq 0$ in Ω and*

$$\langle T^\alpha_t(p), \varphi\rangle + B[p, \varphi] \leq 0 \textit{ for all } \varphi \in V_0 \cap L^+_\alpha,$$

we have $p(t, .) \leq 0$ a.e. in Ω, for all $t \in [0, T]$.

Proof. Let $\varphi(t, x) = p^+(t, x) = \sup\{p(t, x), 0\}$ then $\varphi \in V_0 \cap L^2_+(\Omega)$. Hence we have $p^+(0, x) = 0$ in Ω and

$$\left(T^\alpha_t(p), p^+\right) + B[p, p^+] \leq 0,$$

we have $B[p, p^+] \geq 0$, we get for any $\tau \in (0, T]$

$$\begin{aligned} 0 \geq \left\langle T^\alpha_t(p), p^+\right\rangle &= \int_\Omega \int_0^\tau T^\alpha_t(p) p^+ dt \; dx, \\ &= \frac{1}{2} \left\|p^+(\tau, \cdot)\right\|^2_{L^2(\Omega)} - \frac{1}{2} \left\|p^+(0, \cdot)\right\|^2_{L^2(\Omega)}, \\ &= \frac{1}{2} \left\|p^+(\tau, \cdot)\right\|^2_{L^2(\Omega)} \geq 0, \end{aligned}$$

which implies $p^+(t, x) = 0$ a.e. $x \in Q$, for all $t \in [0, T]$. i.e. $p(t, .) \leq 0$ a.e. in Ω, for all $t \in [0, T]$. □

4.2. Existence and uniqueness theorem for linear equations

The following theorem proves uniqueness of solution of the linear fractional initial boundary value problem (IBVP).

Theorem 4.2. *Given $h \in L_\alpha$ and $g \in L^2(\Omega)$, then, the following linear parabolic conformable fractional IBVP has one and only one weak solution $u \in W_0$,*

$$\begin{cases} T_t^\alpha(u(t,x)) - \Delta u(t,x) = h(t,x), & \text{in } Q \\ u(t,x) = 0, & \text{on } \Gamma \\ u(0,x) = g(x), & \text{in } \Omega \end{cases} \tag{4.5}$$

Moreover, the energy estimate

$$\|u(t,.)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2_\alpha(0,t,H^1_0(\Omega))} \le \|g\|^2_{L^2(\Omega)} + C^2\|h\|^2_{L^2_\alpha(0,t,L^2(\Omega))},$$

holds for each $t \in [0,T]$

4.2.1. Proof of existence

The weak formulation of (4.5) for the homogeneous Dirichlet boundary condition reads as follows, given $h \in L_\alpha$ and $g \in L^2(\Omega)$, find $u \in W_0$ such that

$$\begin{cases} T_t^\alpha(u(t),v) + (\nabla u(t),\nabla) = (h(t),v) \quad \forall v \in H^1_0(\Omega), \quad a.e \quad t \in (0,T) \\ u(0) = g \end{cases} \tag{4.6}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$, $u(t) = u(t,.)$ and $h(t) = h(t,.)$.

Remark 4.1. Equation (4.6) may be interpreted in the sense of distributions. To see this, observe that, for every $v \in V$, the real function

$$w(t) = \langle T_t^\alpha u(t), v\rangle,$$

is a distribution in $\mathfrak{D}'(0,T)$ and

$$\langle T_t^\alpha u(t), v\rangle = T_t^\alpha(u(t),v) \in \mathfrak{D}'(0,T),$$

This means that, for every $\psi \in \mathfrak{D}(0,T)$, we have

$$\int_0^T \langle T_t^\alpha u(t), v\rangle \psi(t) d_\alpha t = -\int_0^T (u(t),v) T_t^\alpha \psi(t) d_\alpha t. \tag{4.7}$$

We divide the proof into three steps.

Step 1: Solution of the approximate problem

We employ the so-called Faedo-Galerkin method, and construct an approximate sequence solving suitable finite dimensional problems. Since $H^1_0(\Omega)$ is a closed subspace of $H^1(\Omega)$, it is a separable Hilbert space. Let $\{\phi_j\}_{j\ge1}$ be a complete orthonormal basis in $H^1_0(\Omega)$ and define $V^n :=$ span $\{\phi_1,\dots,\phi_n\}$ (cf. Brézis [15]). Consider the approximate problem: for each $t \in [0,T]$ find $u^n(t) \in V^n$ such that:

$$\begin{cases} T_t^\alpha\left(u^n(t),\phi_j\right) + (\nabla u^n(t),\nabla\phi_j) = (h(t),\phi_j), \forall j = 1,\dots,n, t \in (0,T), \\ u^n(0) = g^n := P_n(g) = \displaystyle\sum_{s=1}^N \rho_s\phi_s, \end{cases} \tag{4.8}$$

where P_n is the orthogonal projection in $L^2(\Omega)$ on V^n, hence the vector ρ is the solution of the linear system $(M\rho)_j = (u_0,\phi_j)$, M being the mass matrix $M_{js} := (\phi_j,\phi_s)$. Properties of projection operators imply

$$\|g^{(n)}\|_{L^2(\Omega)} \le \|g\|_{L^2(\Omega)}, \tag{4.9}$$

since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$ and $\{\phi_i\}$ is a basis for $H_0^1(\Omega)$, we easily deduce that

$$g^{(n)} \to g \quad \text{in } L^2(\Omega) \quad \text{as } n \to \infty,$$

since $\{\phi_j\}, j = 1, \dots, n$, is a basis for V^n, the equation in (4.8) is indeed satisfied for each $v^n \in V^n$. Writing

$$u^n(t) = \sum_{s=1}^{n} c_s^n(t)\phi_s.$$

The system (4.8) can be rewritten as

$$\begin{cases} M(T_t^\alpha \mathbf{c}^n(t)) + A\mathbf{c}^n(t) = \widetilde{h}(t), \\ M\mathbf{c}^n(0) = \mathbf{c}_0, \end{cases} \tag{4.10}$$

where for $i, j = 1, \dots, n$

$$M_{ij} := (\phi_i, \phi_j), A_{ij} := (\nabla\phi_i, \nabla\phi_j), \widetilde{h}_i(t) := (h(t), \phi_i), c_{0,i} := (g, \phi_i),$$

since M is positive definite, then by Corollary 2.6 the linear differential system (4.10) has a maximal solution defined on some interval $[0, t_n]$. If $t_n < T$, then $|\boldsymbol{c}_n(t)|$ must tend to $+\infty$ as $t \to t_n$; the priori estimates we shall prove later show that this does not happen and therefore $t_n = T$.

As $\widetilde{h} \in L^2_\alpha(0,T)$, it follows $c_n \in L^2_\alpha(0,T)$, and $T_t^\alpha c_n \in L^2_\alpha(0,T)$ i.e. $u_n \in W_0$.

Step 2: a priori estimates for u_n and $T^\alpha u_n$.

We will obtain a priori estimates independent of n for the functions u_n , $T^\alpha u_n$ and then pass to the limit. The proofs are given later.

Lemma 4.2 (Estimate of u_n)**.** *For every $t \in (0,T]$, the following estimate holds:*

$$\|u_n(t)\|^2_{L^2(\Omega)} + \|u_n\|^2_{L^2_\alpha(0,t,H_0^1(\Omega))} \le \|g\|^2_{L^2(\Omega)} + C^2\|h\|^2_{L^2_\alpha(0,t,L^2(\Omega))}. \tag{4.11}$$

Lemma 4.3 (Estimate of $T^\alpha u_n$)**.** *For every $t \in [0,T]$, the following estimate holds:*

$$\|T_\tau^\alpha u_n\|^2_{L^2_\alpha(0,t,H^{-1}(\Omega))} \le 2\|g\|^2_{L^2(\Omega)} + 4C^2\|h\|^2_{L^2_\alpha(0,t,L^2(\Omega))}. \tag{4.12}$$

Step 3: Passage to limits

Lemmas 4.2 and 4.3 show that the sequence of Galerkin's approximations u_n is bounded in $L^\infty(0,T;H_0^1(\Omega))$, hence in V_0 while $T_t^\alpha u_n$ is bounded in V_0'. We now use the known weak compactness theorem and deduce that there exists a sub-sequence, which for simplicity we still denote by u_n, such that, as $n \to \infty$

$$u_n \rightharpoonup u \quad \text{weakly in } V_0,$$

$$T_t^\alpha u_n \rightharpoonup T_t^\alpha u \quad \text{weakly in } V_0'.$$

This means that there exists $u \in V_0$ such that

$$\int_0^T (\nabla u_n(t), \nabla v(t))\, d_\alpha t \to \int_0^T (\nabla u(t), \nabla v(t)) d_\alpha t,$$

and

$$\int_0^T \langle T_t^\alpha u_n(t), v(t)\rangle d_\alpha t \to \int_0^T \langle T_t^\alpha u(t), v(t)\rangle d_\alpha t,$$

for all $v \in L^2(0,T;V_n)$

We want to use these properties to pass to the limit as $n \to +\infty$ in problem (4.8) , keeping in mind that the test functions $v(t)$ have to be chosen in V_n. Fix $v \in V_0$; we may write

$v(t) = \sum_{k=1}^{\infty} v_k(t)\phi_k$ which is convergent in $H_0^1(\Omega)$, for a.e. $t \in [0,T]$.

Let

$$v_j(t) = \sum_{k=1}^{j} v_k(t)\phi_k,$$

and keep j fixed, for the time being. If $n \geq j$, then $v_j \in L^2(0,T;V_n)$. Multiplying equation (4.8) by $v_k(t)$ and summing for $k = 1,\ldots,j$, we get

$$\langle T_t^\alpha u_n(t), v_j(t)\rangle + (\nabla u_n(t), \nabla v_j(t)) = (h(t), v_j(t)).$$

An integration over $(0,T)$ yields

$$\int_0^T (T_t^\alpha u_n(t), v_j(t))\, d_\alpha t + \int_0^T (\nabla u_n(t), \nabla v_j(t))\, d_\alpha t = \int_0^T (h(t), v_j(t))\, d_\alpha t. \tag{4.13}$$

Thanks to the weak convergence of u_n and $T_t^\alpha u_n$ in their respective spaces, we can let $n \to +\infty$, since

$$\int_0^T \langle T_t^\alpha u_n(t), v_j(t)\rangle\, d_\alpha t \to \int_0^T \langle T_t^\alpha u(t), v_j(t)\rangle\, d_\alpha t.$$

We obtain

$$\int_0^T [\langle T_t^\alpha u(t), v_j(t)\rangle + (\nabla u(t), \nabla v_j(t))]\, d_\alpha t = \int_0^T (h(t), v_j(t))\, d_\alpha t.$$

Now, let $j \to +\infty$ observing that $v_j \to v$ in V and in particular weakly in this space as well. We obtain

$$\int_0^T [\langle T_t^\alpha u, v\rangle + (\nabla u, \nabla v)]\, d_\alpha t = \int_0^T (h, v) d_\alpha t, \tag{4.14}$$

for all $v \in L^2_\alpha(0,T;V)$. This entails

$$\langle T_t^\alpha u(t), v\rangle + (\nabla u(t), \nabla v) = (h(t), v),$$

for all $v \in V_0$ and a.e. $t \in [0,T]$.

From Lemma 3.1, we know that $u \in C([0,T]; L^2(\Omega))$. It remains to check that $u(t)$ satisfies the initial condition $u(0) = g$. Let v be any continuously α-differentiable function on $[0,T]$ with $v(T) = 0$. Integrating by parts , we obtain

$$\int_0^T \langle T_t^\alpha u_n, v_j\rangle d_\alpha t = (g_n, v_j(0)) - \int_0^T (u_n, T_t^\alpha v_j)\, d_\alpha t,$$

so that, from (4.13) we find

$$-\int_0^T [(u_n, T_t^\alpha v_j) + (\nabla u_n, \nabla v_j)]\, d_\alpha t = -(g_n, v_j(0)) + \int_0^T (h, v_j)\, d_\alpha t.$$

Let first $n \to +\infty$ and then $j \to +\infty$ we get

$$-\int_0^T [(u, T_t^\alpha v) + (\nabla u, \nabla v)]\, d_\alpha t = -(g, v(0)) + \int_0^T (h, v) d_\alpha t. \tag{4.15}$$

On the other hand, integrating by parts in formula (4.14) we find

$$\int_0^T [(u, T_t^\alpha v)_0 + (\nabla u, \nabla v)]\, d_\alpha t = -(u(0), v(0)) + \int_0^T (h, v) d_\alpha t. \tag{4.16}$$

Subtracting (4.16) from (4.15), we deduce

$$(u(0) - g, v(0)) = 0,$$

and the arbitrariness of $v(0)$ forces $u(0) = g$. Therefore u satisfies (4.6).

4.2.2. Proof of uniqueness

Let u_1 and u_2 be weak solutions of the same problem. Then, $w = u_1 - u_2$ is a weak solution of

$$\langle T_t^\alpha w(t), v\rangle + (\nabla w(t), \nabla v) = 0,$$

for all $v \in V$ and a.e. $t \in [0, T]$, with initial data $w(0) = 0$. Choosing $v = w(t)$ we have

$$\langle T_t^\alpha w(t), w(t)\rangle + (\nabla w(t), \nabla w(t)) = 0,$$

then by Lemma 4.1 we have $w(t) = 0$ for all $t \in [0, T]$. This gives uniqueness of the weak solution.

4.2.3. Proof of Lemma 4.2

Choosing $u^n(t)$ in (4.8) as a test function, we have

$$(T_t^\alpha u^n(t), u^n(t)) + (\nabla u^n(t), \nabla u^n(t)) = (h(t), u^n(t)), \tag{4.17}$$

for a.e. $t \in [0, T]$. Now, note that

$$(T_t^\alpha u_n(t), u_n(t)) = \frac{1}{2} T_t^\alpha(\|u_n(t)\|^2_{L^2(\Omega)}), \text{ a.e } t \in (0, T),$$

$$\text{and } \ (u_n(t), u_n(t)) = \|\nabla u_n(t)\|^2_{L^2(\Omega)} = \|u_n(t)\|^2_{H_0^1(\Omega)}.$$

From Schwarz and Poincaré inequalities and the elementary inequality

$$\begin{aligned}(h(t), u_n(t)) &\le \|h(t)\|_{L^2(\Omega)} \|u_n(t)\|_{L^2(\Omega)} \le C\|h(t)\|_{L^2(\Omega)} \|u_n(t)\|_{H_0^1(\Omega)}, \\ &\le \frac{C^2}{2}\|h(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_n(t)\|^2_{H_0^1(\Omega)}.\end{aligned}$$

Thus, from (4.17) we obtain

$$T_t^\alpha\left(\|u^n(t)\|^2_{L^2(\Omega)}\right) + \frac{1}{2}\|u^n(t)\|^2_{H_0^1(\Omega)} \le \frac{C^2}{2}\|h(t)\|^2_{L^2(\Omega)}. \tag{4.18}$$

Integrating over $(0, t), t \in (0, T]$, since $u_n(0) = g_n$ and by (4.9) we obtain

$$2\int_0^t T_\tau^\alpha\left(\|u_n(\tau)\|^2_{L^2(\Omega)}\right) d_\alpha\tau + \int_0^t \|u_n(\tau)\|^2_{H_0^1(\Omega)}\, d_\alpha\tau \le C^2 \int_0^t \|h(\tau)\|^2_{L^2(\Omega)} d_\alpha\tau,$$

$$\|u_n(t)\|^2_{L^2(\Omega)} - \|u_n(0)\|^2 + \int_0^t \|u_n(\tau)\|^2_{H_0^1(\Omega)}\, d_\alpha\tau \le C^2 \int_0^t \|h(\tau)\|^2_{L^2(\Omega)} d_\alpha\tau,$$

we may write

$$\|u_n(t)\|^2_{L^2(\Omega)} + \int_0^t \|u_n(\tau)\|^2_{H_0^1(\Omega)}\, d_\alpha\tau \le \|G_n\|^2_{L^2(\Omega)} + C^2 \int_0^t \|h(\tau)\|^2_{L^2(\Omega)} d_\alpha\tau,$$

$$\|u_n(t)\|^2_{L^2(\Omega)} + \|u_n\|^2_{L^2_\alpha(0,t,V)} \le \|g\|^2_{L^2(\Omega)} + C^2\|h\|^2_{L^2_\alpha(0,t,L^2(\Omega))}.$$

4.2.4. Proof of Lemma 4.3

We recall that

$$\|T_t^\alpha u_n(t)\|_{H^{-1}(\Omega)} = \sup\{|\langle T_t^\alpha u_n(t), \phi\rangle| \quad / \quad \|\phi\|_{H_0^1(\Omega)} = 1\}.$$

By (4.8) we have

$$|\langle T_t^\alpha u_n(t), \phi\rangle| \leq |(\nabla u_n(t), \nabla\phi)| + |(h(t), \phi)|,$$

using the Schwartz and Poincaré inequalities we obtain,

$$\begin{aligned} |\langle T_t^\alpha u_n(t), \phi\rangle| &\leq \|\nabla u_n(t)\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} + \|h(t)\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)}, \\ &\leq \left(\|\nabla u_n(t)\|_{L^2(\Omega)} + C\|h(t)\|_{L^2(\Omega)}\right)\|\nabla\phi\|_{L^2(\Omega)}, \\ &\leq \left(\|u_n(t)\|_{H^{-1}(\Omega)} + C\|h(t)\|_{L^2(\Omega)}\right)\|\phi\|_{H^{-1}(\Omega)}. \end{aligned}$$

By definition of the norm in $H^{-1}(\Omega) := (H_0^1(\Omega))'$, we may write

$$\|T_t^\alpha u_n(t)\|_{H^{-1}(\Omega)} \leq \|_n(t)\|_{H_0^1(\Omega)} + C\|h(t)\|_{L^2(\Omega)},$$

squaring both sides and integrating over (0, t) we get

$$\int_0^t \|T_\tau^\alpha u_n(\tau)\|^2_{H^{-1}(\Omega)} d_\alpha\tau \leq 2\int_0^t \|u_n(\tau)\|^2_{H_0^1(\Omega)} d_\alpha\tau + 2C^2\int_0^t \|h(\tau)\|^2_{L^2(\Omega)} d_\alpha\tau.$$

using (4.11) we get

$$\|T_\tau^\alpha u_n\|^2_{L^2_\alpha(0,t,H^{-1}(\Omega))} \leq 2\|g\|^2_{L^2(\Omega)} + 4C^2\|h\|^2_{L^2_\alpha(0,t,L^2(\Omega))}.$$

4.3. Existence theorem for nonlinear equation

The existence of a weak extremal solutions to (1.1) is given by the result below.

Theorem 4.3. *Let (A1)-(A3) be satisfied. Assume in addition that*
(A4) $\underline{u}_0, \overline{u}_0 \in W_0$ *are respectively lower and upper solutions of* (1.1) *with* $\underline{u}_0(0,x) \leq \overline{u}_0(0,x)$ *in* Ω
(A5) For any fixed function $\eta \in [\underline{u}_0, \overline{u}_0]$*, the function* $F\eta \in L_\alpha$*.*

Then there exist monotone sequences $\{\underline{u}_n\}$ *and* $\{\overline{u}_n\}$ *which converge weakly in* W_0 *to* u_* *and* u^* *and strongly in* L_α *respectively. Moreover* u_* *and* u^* *are extremal solutions of* (1.1) *in the sense that they are solutions of* (1.1) *themselves and that if* u *is any solution of* (1.1) *satisfying* $\underline{u}(0,.) \leq u(0,.) \leq \overline{u}(0,.)$ *a.e. in* Ω*, then* $u_* \leq u \leq u^*$*.*

Proof. We prove our result in several steps.
(a) Iterative schemes.

For each $n \geq 0$ consider the following linear IBVPs

$$\begin{cases} T_t^\alpha \underline{u}_{n+1}(t,x) - \Delta\underline{u}_{n+1}(t,x) = f(t,x,\underline{u}_n(t,x)) & \text{in } Q \\ \underline{u}_{n+1}(t,x) = 0, & \text{on } \Gamma \\ \underline{u}_{n+1}(0,x) = \underline{u}_n(T,x), & \text{in } \Omega \end{cases} \tag{4.19}$$

$$\begin{cases} T_t^\alpha \overline{u}_{n+1}(t,x) - \Delta\overline{u}_{n+1}(t,x) = f(t,x,\overline{u}_n(t,x)) & \text{in } Q \\ \overline{u}_{n+1}(t,x) = 0, & \text{on } \Gamma \\ \overline{u}_{n+1}(0,x) = \overline{u}_n(T,x), & \text{in } \Omega \end{cases} \tag{4.20}$$

Our claim is to prove for any $t \in [0,T]$ the following inequality

$$\underline{u}_0(t) \leq \underline{u}_1(t) \leq \ldots \leq \underline{u}_n(t) \leq \overline{u}_n(t) \leq \ldots \leq \overline{u}_1(t) \leq \overline{u}_0(t) \qquad \text{a.e. in } \Omega. \tag{4.21}$$

We will prove it by induction.

For each $n \geq 0$, if we have $\underline{u}_0 \leq \underline{u}_n \leq \overline{u}_n \leq \overline{u}_0$, then by assumptions (A4), $F\overline{u}_n, F\underline{u}_n \in L_\alpha$, then by Theorem 4.2, the linear parabolic IBVPs (4.19) and (4.20) have one and only one weak solution $\underline{u}_{n+1}$ and $\overline{u}_{n+1}$ respectively in W_0.

For $n = 0$, we have $\underline{u}_1, \overline{u}_1 \in W_0$ as the unique solutions of (4.19) and (4.20) respectively.

Set $p = \underline{u}_0 - \underline{u}_1$ so that $p \leq 0$ on Γ and $p(0, x) \leq 0$ a.e. in Ω

$$\langle T_t^\alpha(p), \varphi\rangle + B[p, \varphi] \leq (F\underline{u}_0, \varphi) - (F\underline{u}_0, \varphi) = 0 \text{ for all } \varphi \in V_0 \cap L_\alpha^+.$$

Thus we get by Lemma 4.1 that $\underline{u}_0 \leq \underline{u}_1$ in Q and a similar argument shows $\overline{u}_1 \leq \overline{u}_0$ in Q.

In fact, by Theorem 4.1 we have $\underline{u}_0 \leq \overline{u}_0$.

We next prove that $\underline{u}_1 \leq \overline{u}_1$ in Q.

Consider $p = \underline{u}_1 - \overline{u}_1$ so that $p \leq 0$ on $\Gamma, p(0, x) \leq 0$ in Ω and

$$\langle T_t^\alpha(p), \varphi\rangle + B[p, \varphi] = (F\underline{u}_0, \varphi) - (F\overline{u}_0, \varphi) \text{ for all } \varphi \in V_0 \cap L_\alpha^+,$$

using (A2) we have $f(t, x, \underline{u}_0) \leq f(t, x, \overline{u}_0)$ then

$$\langle T_t^\alpha(p), \varphi\rangle + B[p, \varphi] \leq 0.$$

Thus we get by Lemma 4.1, that $\underline{u}_1 \leq \overline{u}_1$ in Q. As a result, it follows that

$$\underline{u}_0 \leq \underline{u}_1 \leq \overline{u}_0 \leq \overline{u}_1 \text{ in } Q.$$

Assume that for some $n \geq 1$,

$$\underline{u}_{n-1} \leq \underline{u}_n \leq \overline{u}_n \leq \overline{u}_{n-1} \text{ in } Q. \tag{4.22}$$

Then we wish to show that

$$\underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u}_{n+1} \leq \overline{u}_n \text{ in } Q. \tag{4.23}$$

To do this, let $p_n = \underline{u}_n - \underline{u}_{n+1}$ so that $p_n \leq 0$ on $\Gamma, p_n(0, x) \leq 0$ in Ω and

$$\langle T_t^\alpha(p_n), \varphi\rangle + B[p_n, \varphi] = (F\underline{u}_{n-1}, \varphi) - (F\underline{u}_n, \varphi) \text{ for all } \varphi \in V_0 \cap L_\alpha^+,$$

by (4.22) and $(A2)$ that $\langle T_t^\alpha(p_n), \varphi\rangle + B[p_n, \varphi] \leq 0$ and by Lemma 4.1 that $\underline{u}_n \leq \underline{u}_{n+1}$ and a similar argument shows $\overline{u}_{n+1} \leq \overline{u}_n$.

We next prove that $\underline{u}_{n+1} \leq \overline{u}_{n+1}$ in Q.

Consider $q_n = \underline{u}_{n+1} - \overline{u}_{n+1}$ so that $q_n \leq 0$ on $\Gamma, q_n(0, x) \leq 0$ in Ω and

$$\langle T_t^\alpha(q_n), \varphi\rangle + B[q_n, \varphi] = (F\underline{u}_n, \varphi) - (F\overline{u}_n, \varphi) \leq 0 \text{ for all } \varphi \in V_0 \cap L_\alpha^+,$$

in view of (4.22) and $(A2)$, thus we get by Lemma 4.1 $\underline{u}_{n+1} \leq \overline{u}_{n+1}$ in Q. Hence by induction (4.21) holds for all $n \geq 0$

(b) Convergence of $(\underline{u}_n), (\overline{u}_n)$ to the solution in L_α.

By the monotonicity of the iterates $(\underline{u}_n)$ and $(\overline{u}_n)$ according to (3.9) there exist for all $t \in [0, T]$ the pointwise limits

$$u_*(t, x) = \lim_{n\to\infty} \underline{u}_n(t, x), \quad u^*(t, x) = \lim_{n\to\infty} \overline{u}_n(t, x) \text{ for a.e. } x \in \Omega\ .$$

Moreover, since $\underline{u}_n, \overline{u}_n \in [\underline{u}_0, \overline{u}_0]$, then

$$|\underline{u}_n| \leq \gamma = |\underline{u}_0| + |\overline{u}_0| \quad \text{ and } \quad |\overline{u}_n| \leq \gamma \text{ a.e. in } Q,$$

since $\gamma \in L_\alpha^2(0, T, L^2(\Omega))$ it follows by Lebesgue's dominated convergence theorem that

$$\underline{u}_n \to u_* \text{ and } \overline{u}_n \to u^* \quad \text{ in } L_\alpha.$$

(c) Convergence of $(\underline{u}_n), (\overline{u}_n)$ to the solution in W_0.

The proof will only be given for the convergence of $\underline{u}_n$ since the convergence of $\overline{u}_n$ can be proved quite similarly by dual reason.

The first step, $\underline{u}_n$ is bounded in V_0.

By definitions of the iteration scheme, $\bar{u}_n, \bar{u}_{n-1} \in W_0$ and satisfy

$$\langle T_t^\alpha(\underline{u}_n), \varphi\rangle + B[\underline{u}_n, \varphi] = (F\underline{u}_{n-1}, \varphi) \text{ for all } \varphi \in V_0, \quad \underline{u}_n(0,x) = \underline{u}_{n-1}(T,x), \tag{4.24}$$

taking the test function $\varphi = \underline{u}_n$ using integration by parts and assumption $(A5)$, we have

$$\frac{1}{2}\left(||\underline{u}_n(T,.)||^2_{L^2(\Omega)} - ||\underline{u}_n(0,.)||^2_{L^2(\Omega)}\right) + ||\underline{u}_n||_{V_0} = \int_0^T \int_\Omega f(t,x,\underline{u}_{n-1})\underline{u}_n dx d_\alpha t,$$

we obtain the estimate

$$||\underline{u}_n||_{V_0} \leq \frac{1}{2}||\underline{u}_n(0,.)||^2_{L^2(\Omega)} + ||F\underline{u}_{n-1}||_{L_\alpha}||\underline{u}_n||_{L_\alpha},$$

then by (A5) and (4.21) $\underline{u}_n$ is bounded in V_0.

The second step, $\underline{u}_n$ is bounded in W_0.

We recall that $\|u\|_{W_0} = \|u\|_{V_0} + \|T_t^\alpha u\|_{V_0}$, then we need to estimate $\left\|T_t^\alpha \underline{u_n}\right\|_{V_0}$

By (4.24) and $(A4)$ we have

$$|\langle T_t^\alpha \underline{u}_n, \varphi\rangle| \leq |(\nabla \underline{u}_n, \nabla\varphi)| + ||F\underline{u}_{n-1}||_{L_\alpha}||\varphi||_{L_\alpha} \leq ||\underline{u}_n||_{V_0} + C||\varphi||_{V_0},$$

hence

$$||T_t^\alpha \underline{u}_n||_{W_0} = \sup\{|\langle T_t^\alpha \underline{u}_n, \varphi\rangle|\,;\|\varphi\|_{V_0} = 1\} \leq C,$$

where $C > 0$ is a constant. Now because W_0 is reflexive and by Lemma 3.2 the embedding $W_0 \subset L_\alpha$ is compact , hence by the known weak compactness theorem, that there exists a sub-sequence, which for simplicity we still denote by $\underline{u}_n$, such that, as $n \to \infty$,

$$\underline{u}_n \rightharpoonup \underline{u}_* \text{ weakly in } W_0 \text{ and } \underline{u}_n \to \underline{u}_* \text{ strongly in } L_\alpha,$$

a similar argument shows

$$\overline{u}_n \rightharpoonup u^* \text{ weakly in } W_0 \text{ and } \overline{u}_n \to u^* \text{ strongly in } L_\alpha,$$

since B and f are continuous, allow to pass to the limit in the weak formulation of (4.19) and (4.20) as $n \to \infty$ which yield

$$\langle T_t^\alpha(u_*), \varphi\rangle + B[u_*, \varphi] = (Fu_*, \varphi) \text{ for all } \varphi \in V_0 \text{ and } u_*(0,x) = u_*(T,x) \text{ in } \Omega.$$
$$\langle T_t^\alpha(u^*), \varphi\rangle + B[u^*, \varphi] = (Fu^*, \varphi) \text{ for all } \varphi \in V_0 \text{ and } u^*(0,x) = u^*(T,x) \text{ in } \Omega.$$

(d) u_* and u^* are extremal solutions of (1.1)

Let us suppose that u is any solution of (1.1) with $\underline{u}(0,.) \leq u(0,.) \leq \overline{u}(0,.)$ a.e. in Ω, then by Lemma 4.1 we have $\underline{u}_0 \leq u \leq \overline{u}_0$ on Q . Assume that for some $n \geq 0$, we have

$$\underline{u}_n \leq u \leq \overline{u}_n \text{ on } Q\ .$$

Set $p = \underline{u}_{n+1} - u$ so that $p = 0$ on Γ, and

$$p(0,x) = \underline{u}_{n+1}(0,x) - u(0,x) = \underline{u}_n(T,x) - u(T,x) \leq 0 \text{ on } \Omega,$$

and

$$\langle T_t^\alpha(p), \varphi\rangle + B[p, \varphi] = (F\underline{u}_{n+1}, \varphi) - (Fu, \varphi) \text{ for all } \varphi \in V_0 \cap L_\alpha^+,$$

by $(A3)$ that $\langle T_t^\alpha(p), \varphi\rangle + B[p,\varphi] \leq 0$ and by Lemma 4.1 that $\underline{u}_{n+1} \leq u$ and a similar argument shows $\overline{u}_{n+1} \geq u$, so we have $\underline{u}_{n+1} \leq u \leq \overline{u}_{n+1}$ on Q. It then follows by induction that $\underline{u}_n \leq u \leq \overline{u}_n$ on Q for all $n \geq 0$. and this implies that $\underline{u}_0 \leq u_* \leq u \leq u^* \leq \overline{u}_0$ on Q. proving that u_* and u^* are extremal solutions of (1.1) .

□

Remark 4.2. The periodic solutions given by Theorem 4.3 are, in general, not unique. However, if $\underline{u}(0,.) = \overline{u}(0,.)$ a.e. in Ω or $\underline{u}_n(0,.) = \overline{u}_n(0,.)$ a.e. in Ω for some $n \geq 1$, then we have $u = u^* = u_*$ is the unique periodic solution of (1.1) in $[\underline{u}, \overline{u}]$. However, this uniqueness result is ensured only with respect to the given upper and lower solutions, and it does not rule out the possibility of other solutions outside the sector $[\underline{u}, \overline{u}]$.

5. Conclusion

In this paper, a nonlinear conformable fractional parabolic differential equation has been investigated, the existence of solutions is established using a generalized monotone iterative method combined with the method of upper and lower solutions. The existence of extremal solutions is proved. It is clear that the method of upper and lower solutions is a very effective method for studying the nonlinear fractional differential equations. However, all the results derived in this paper are more or less direct extensions of well-known results of the theory of first-order differential equations, since the conformal fractional derivative is the generalization of the first-order derivative.

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