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Monotone iterative technique for time-space fractional diffusion equations involving delay*

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Abstract. This paper considers the initial boundary value problem for the time-space fractional delayed diffusion equation with fractional Laplacian. By using the semigroup theory of operators and the monotone iterative technique, the existence and uniqueness of mild solutions for the abstract time-space evolution equation with delay under some quasimonotone conditions are obtained. Finally, the abstract results are applied to the time-space fractional delayed diffusion equation with fractional Laplacian operator, which improve and generalize the recent results of this issue.

Keywords: time-space fractional diffusion equations, operator semigroup, monotone iterative technique, existence and uniqueness, delay.

1 Introduction

In recent years, the research on the time-space fractional diffusion equation with fractional Laplacian has attracted wide attention of scholars. The time-space fractional diffusion equation, which is generalizations of classical diffusion equation of integer order, is one of the most commonly used models to describe several anomalous physical aspects and procedures in natural conditions, such as mechanics of materials, fluid mechanics, image processing, finance, biology, signal processing and control (see [5, 10, 21, 22, 26–28]). The initial value problems for the time-space fractional diffusion equation have been extensively studied, and many properties of their solutions have been studied because of the importance in applications (see [3, 11, 18–20] and references therein).

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Recently, the initial boundary value problems of the time-space fractional diffusion equations with fractional Laplacian have been considered by several authors (see [9, 12, 24, 29]). In [9], Chen et al. studied a homogeneous time-space diffusion equation. Combining the Mittag–Leffler function to the time fractional problem with an eigenfunction expansion of the fractional Laplacian on bounded domains, the existence of strong solutions was obtained by separation of variables. In [12], Jia and Li focused on an inhomogeneous time-space fractional diffusion equation. By utilizing properties of time fractional derivative operator and fractional Laplace operator, maximum principles for classical solution and weak solution were obtained. In [29], Toniazzi focused on an inhomogeneous time-space fractional linear diffusion equation involving time nonlocal initial condition and proved the existence and uniqueness of classical solutions along with the stochastic representation for the solution. In all these works, the existence theory of solutions for the semilinear equations is not involved.

Specially, Padgett in [24] investigated the initial boundary value problem for the time-space fractional semilinear diffusion equation with fractional Laplacian

$$\begin{aligned} &{}^{c}D_t^{\alpha}u(x,t) = -(-\Delta)^{\beta}u(x,t) + f\bigl(u(x,t)\bigr), \quad (x,t) \in \Omega \times (0,a),\\ &u|_{\partial\Omega} = 0, \qquad u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^d$ with sooth boundary $\partial\Omega$, ${}^{c}D_t^{\alpha}$ denotes the Caputo time-fractional derivative of order $\alpha \in (0,1)$, and $(-\Delta)^{\beta}$ is the fractional Laplacian with $\beta \in (0,1)$. Under the assumption that the nonlinear reaction term f satisfies a local Lipschitz condition, the author has obtained the existence and uniqueness of (1) by means of Banach fixed point theorem. In fact, in the complex reaction-diffusion processes, the nonlinear function f represents the source of material or population, which depends on time in diversified manners in many contexts. Thus, we hope that the nonlinear function f satisfies more general growth conditions than Lipschitz type conditions.

On the other hand, the monotone iteration technique for upper and lower solutions is an effective and widely used mathematical method. By using this method not only the existence theory of solutions can be obtained, but also the approximate iteration sequence of solutions can be obtained, which provides a reasonable and effective theoretical basis for using the computer to obtain the approximate solution. However, as far as we know, there are few results for the diffusion equations with delay by means of the method for the lower and upper solutions coupled with the monotone iterative technique (see [15, 16]).

Motivated by the papers mentioned above, we study the following initial boundary value problem for the fractional delayed semilinear diffusion equation with fractional Laplacian:

$$\begin{aligned} &{}^{c}D_t^{\alpha}u(x,t) + (-\Delta)^{\beta}u(x,t) = f\bigl(x,t,u(x,t),u(x,\,t+\tau)\bigr), \quad (x,t) \in \overline{\Omega} \times [0,a],\\ &u|_{\partial\Omega} = 0, \qquad u(x,\tau) = \varphi(x,\tau), \quad \tau \in [-r,0],\ x \in \overline{\Omega}, \end{aligned} \tag{2}$$

where $\overline{\Omega} \in \mathbb{R}^d$ is a bounded domain with C^2-boundary $\partial\Omega$ for $d \in \mathbb{N}$, $0 < \alpha, \beta \leqslant 1$, ${}^{c}D_t^{\alpha}$ denotes the Caputo fractional derivation of order $\alpha \in (0,1)$, $(-\Delta)^{\beta}$ is a realization of the fractional Laplace operator acting in space. $f : \overline{\Omega} \times [0,a] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous functions, $\varphi \in C(\mathbb{R} \times [-r, 0])$, $r > 0$, is a constant. In this paper, our main purpose is to establish a general principle of lower and upper solutions coupled with the monotone iterative technique to the initial boundary value problem (2) and study the existence of maximal and minimal mild solutions, which will greatly enrich and expand the results mentioned above.

As we all know, there are many different definitions of Laplacian operator and fractional Laplacian operator on a bounded domain Ω. For the properties of differential definitions and their relations, we can refer to [14, 17, 21] and references therein. Once the Laplace operator Δ is defined, according to Balakrishnan's definition, a common definition of fractional Laplacian is provided by fractional power of the nonnegative operator $-\Delta$ (see [4, 32])

$$(-\Delta)^{\beta} u = \frac{\sin \alpha\pi}{\pi} \int_0^{\infty} \mu^{\beta-1} (\mu I - \Delta)^{-1} (-\Delta) u \, \mathrm{d}\mu, \quad 0 < \beta < 1,$$

for $u \in D(-\Delta)$—the domain of the consider Laplace operator. Throughout this paper, we introduce the definition of function calculus of fractional Laplacian through Dirichlet Laplacian, which means that $-\Delta : L^2(\Omega) \to L^2(\Omega)$ is the classical Laplacian with domain $D(-\Delta) = \{u \in H_0^1(\Omega) \mid \Delta u \in L^2(\Omega)\}$. As we all know, the operator is unbounded, closed, positive define self-adjoint and has a compact inverse.

Hence, if λ_i $(i = 1, 2 \dots)$ are the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary conditions considered in $L^2(\Omega)$ and e_i as its corresponding eigenfunction, then

$$(-\Delta)^{\beta} e_i = \lambda_i^{\beta} e_i, \quad x \in \Omega, \qquad e|_{\partial\Omega} = 0.$$

Thus, we can define the fractional Laplacian to be

$$D\big((-\Delta)^{\beta}\big) := \Bigg\{ u \in L^2(\Omega)\colon\, u|_{\partial\Omega} = 0, \\ \big\|(-\Delta)^{\beta} u\big\|^2_{L^2(\Omega)} = \sum_{i=1}^{\infty} \big(\lambda^{\beta} \langle u, e_i \rangle\big)^2 < \infty \Bigg\},$$

$$(-\Delta)^{\beta} u := \sum_{i=1}^{\infty} \lambda_i^{\beta} \langle u, e_i \rangle e_i.$$

From [6, 7] it follows that the Balakrishnan definition is equivalent to the spectral definition in $L^2(\Omega)$.

The structure of this paper is as follows. In Section 2, we collect some known concepts and results about the operator semigroup and provide preliminary results, which can be used in the theorems stated and proved in this paper. In Section 3, we present our abstract results and apply the operator semigroup theory and monotone iterative method of the lower and upper solution to prove them. In the last section, applying our abstract results to the initial boundary value problem for the time-space fractional delayed semilinear diffusion equation with fractional Laplacian, we get the existence and uniqueness of positive solutions.

2 Preliminaries

Throughout this paper, we assume that $(E, \|\cdot\|)$ is an ordered Banach space with the partial-order "$\leqslant$" induced by the positive cone $K = \{u \in E \mid u \geqslant \theta\}$ and K is normal, θ is the zero element of E.

Denote $J := [-r, a]$, and let $C(J, E)$ be the Banach space composed of all continuous functions from J to E equipped with the norm $\|u\|_C = \max_{t\in J} \|u(t)\|$. Evidently, $C(J, E)$ is also ordered Banach space, the positive cone $K_C = \{u \in C(J, E) \mid u(t) \in K, t \in J\}$ is also normal. Similarly, $\mathcal{B}$ is ordered Banach space with norm $\|\phi\|_B = \max_{s\in[-r,0]} \|\phi(s)\|$, and the positive cone $K_B = \{\phi \in \mathcal{B} \mod \phi(s) \in K, s \in [-r, 0]\}$.

For $v, w \in C(J, E)$ with $v \leqslant w$, we denote the order interval $\{u \mid v \leqslant u \leqslant w\}$ by $[v, w]$. Moreover, we denote $\{u(t) \mid v(t) \leqslant u(t) \leqslant w(t), t \in J\}$ in E and $\{u_t \mid v_t \leqslant u_t \leqslant w_t, t \in [0, a]\}$ in $\mathcal{B}$ by $[v(t), w(t)]$ and $[v_t, w_t]$, respectively.

Next, we recall some essential properties of operator semigroup.

Let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a uniformly bounded C_0-semigroup $T(t)$ $(t \geqslant 0)$ on E. Thus, there is $M \geqslant 1$ such that

$$\sup_{t\in[0,+\infty)} \|T(t)\| \leqslant M < +\infty.$$

From [25, 32] it follows that A is a nonnegative operator and

$$\|(\lambda I + A)^{-1}\| < \frac{M}{\lambda} < \infty, \quad \lambda > 0.$$

Therefore, for any $0 < \beta < 1$, according to the Balakrishnan definition [4, 32], we can define the fractional power A^β of the nonnegative operator A by

$$A^\beta u := \frac{\sin \beta\pi}{\pi} \int_0^\infty \lambda^{\beta-1}(\lambda I + A)^{-1} Au \,\mathrm{d}\lambda, \quad u \in D(A).$$

Then, from [32] we find that $-A^\beta$ is a closed densely defined operator and generates an analytic semigroup $T_\beta(t)$ $(t \geqslant 0)$, which can be expressed as

$$T_\beta(t) = \int_0^\infty f_{\beta,t}(s) T(s) \,\mathrm{d}s, \quad t > 0, \tag{3}$$

where $f_{\beta,t}(\cdot)$ is defined by

$$f_{\beta,t}(s) = \frac{1}{2\pi\mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathrm{e}^{zs-tz^\beta} \,\mathrm{d}z, \quad \sigma > 0, \tag{4}$$

and the brach of z^β is so taken that $\mathrm{Re}(z^\beta) > 0$ for $\mathrm{Re}(z) > 0$. The convergence of integral (4) is apparent in virtue of the convergence factor e^{-tz^β}. Moreover, $f_{\beta,t}(s) \geqslant 0$

for all $s > 0$, and $\int_0^\infty f_{\beta,t}(s)\,\mathrm{d}s = 1$. For more properties of the function $f_{\beta,t}(s)$, one can refer to [32].

By the definition of the semigroup $T_\beta(t)$ and the properties of $f_{\beta,t}(s)$ one can find that $T_\beta(t)$ is continuous by operator norm and $\|T_\beta(t)\| \leqslant M$ for any $t \geqslant 0$ and $\beta \in (0,1)$. Moreover, in ordered Banach space E, if the C_0-semigroup $T(t)$ $(t \geqslant 0)$ generated by $-A$ is positive, then the semigroup $T_\beta(t)$ $(t \geqslant 0)$ generated by $-A^\beta$ is positive for any $\beta \in (0,1)$. Furthermore, we can obtain the following lemma.

Lemma 1. *If the uniformly bounded C_0-semigroup $T(t)$ $(t \geqslant 0)$ generated by $-A$ is compact, then the semigroup $T_\beta(t)$ $(t \geqslant 0)$ generated by $-A^\beta$ is compact.*

Proof. Let $\varepsilon > 0$ be arbitrary, and let

$$T_{\beta,\varepsilon}(t) := \int_\varepsilon^\infty f_{\beta,t}(s)T(s)\,\mathrm{d}s = T(\varepsilon)\int_\varepsilon^\infty f_{\beta,t}(s)T(s-\varepsilon)\,\mathrm{d}s.$$

One can easily obtain that $\int_\varepsilon^\infty f_{\beta,t}(s)T(s-\varepsilon)\,\mathrm{d}s$ is a linear bounded operator for every $t > 0$. Hence, by the compactness of the semigroup $T(t)$ $(t \geqslant 0)$, $T_{\beta,\varepsilon}(t)$ is compact for every $t > 0$. On the other hand, note that

$$\big\|T_{\beta,\varepsilon}(t) - T_\beta(t)\big\| \leqslant \left\|\int_0^\varepsilon f_{\beta,t}(s)T(s)\,\mathrm{d}s\right\| \leqslant M\int_0^\varepsilon f_{\beta,t}(s)\,\mathrm{d}s.$$

Hence, by the boundedness of $f_{\beta,t}(s)$ in s and the compactness of $T_{\beta,\varepsilon}(t)$ for $t > 0$ one can obtain that $T_\beta(t)$ $(t \geqslant 0)$ is compact. $\square$

For more details of the definitions and properties of C_0-semigroups or positive C_0-semigroups, see [23, 25, 32].

As for the definition of Caputo fractional derivation, we can refer to many references (see [8, 13, 30] and so on), which will not be repeated here. In the following, we only give some operators needed in this paper and their related properties.

For a given C_0-semigroup $T(t)$ $(t \geqslant 0)$, we define the family of operators $U_\alpha(t)$ $(t \geqslant 0)$ and $V_\alpha(t)$ $(t \geqslant 0)$ in E as follows:

$$U_\alpha(t) = \int_0^\infty \xi_\alpha(s)T\big(t^\alpha s\big)\,\mathrm{d}s, \qquad V_\alpha(t) = \alpha\int_0^\infty s\xi_\alpha(s)T\big(t^\alpha s\big)\,\mathrm{d}s,$$

where

$$\xi_\alpha(s) = \frac{1}{\pi\alpha}\sum_{n=1}^\infty (-s)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad s \in (0,\infty), \tag{5}$$

is a probability density function defined on $(0,\infty)$, which satisfies

$$\xi_\alpha(s) \geqslant 0, \quad s \in (0,\infty), \qquad \int_0^\infty \xi_\alpha(s)\,\mathrm{d}s = 1, \qquad \int_0^\infty s\xi_\alpha(s)\,\mathrm{d}s = \frac{1}{\Gamma(1+\alpha)}.$$

The following lemma may be find in [8, 30].

Lemma 2. *The operators $U_\alpha(t)$ $(t \geqslant 0)$ and $V_\alpha(t)$ $(t \geqslant 0)$ have the following properties*:

(i) *$U_\alpha(t)$ $(t \geqslant 0)$ and $V_\alpha(t)$ $(t \geqslant 0)$ are strongly continuous operators, i.e., for any $x \in E$ and $0 \leqslant t_1 \leqslant t_2$,*

$$\left\|U_\alpha(t_2)x - U_\alpha(t_1)x\right\| \to 0, \quad \left\|V_\alpha(t_2)x - V_\alpha(t_1)x\right\| \to 0 \quad \text{as } t_2 - t_1 \to 0.$$

(ii) *If C_0-semigroup $T(t)$ $(t \geqslant 0)$ is uniformly bounded, then $U_\alpha(t)$ and $V_\alpha(t)$ are linear bounded operators for any fixed $t \in \mathbb{R}^+$, i.e.,*

$$\left\|U_\alpha(t)x\right\| \leqslant M\|x\|, \qquad \left\|V_\alpha(t)x\right\| \leqslant \frac{M}{\Gamma(\alpha)}\|x\| \quad \forall x \in E.$$

(iii) *If C_0-semigroup $T(t)$ $(t \geqslant 0)$ is compact, then $U_\alpha(t)$ and $V_\alpha(t)$ are compact operators for every $t > 0$.*

(iv) *If C_0-semigroup $T(t)$ $(t \geqslant 0)$ is continuous by operator norm for every $t > 0$, then $U_\alpha(t)$ and $V_\alpha(t)$ are uniformly continuous for $t > 0$.*

(v) *If C_0-semigroup $T(t)$ $(t \geqslant 0)$ is positive, then $U_\alpha(t)$ and $V_\alpha(t)$ are positive operators.*

In the proof, we also need the following inequality.

Lemma 3. (*See* [31].) *Assume that $f(t)$ is a locally integrable, nonnegative function on $0 \leqslant t < \kappa$ (some $\kappa \leqslant \infty$), $g(t)$ is a nonnegative, nondecreasing, continuous bounded function on $0 \leqslant t < \kappa$, and $\alpha > 0$. Suppose that $h(t)$ is locally integrable and nonnegative on $0 \leqslant t < \Lambda$ with*

$$h(t) \leqslant f(t) + g(t) \int_0^t (t-s)^{\alpha-1} h(s)\,\mathrm{d}s.$$

Then

$$h(t) \leqslant f(t) + \int_0^t \left(\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1} f(s)\right) \mathrm{d}s.$$

3 Abstract results

In this section, we discuss the existence of the minimum and maximum mild solutions for the abstract time-space fractional evolution equation with delay

$$\begin{aligned} &{}^cD_t^\alpha u(t) + A^\beta u(t) = F\bigl(t, u(t), u_t\bigr), \quad t \in [0, a], \\ &u(t) = \varphi(t), \quad t \in [-r, 0], \end{aligned} \tag{6}$$

where $0 < \alpha, \beta \leqslant 1$, ${}^cD_t^\alpha$ is the Caputo fractional derivation of order $\alpha \in (0, 1)$; A: $D(A) \subset E \to E$ is a closed linear operator, and $-A$ generates a positive compact semigroup $T(t)$ $(t \geqslant 0)$ in E, which is uniformly bounded with $\sup_{t\geqslant 0} \|T(t)\| = M < \infty$, A^β denotes the βth fractional power operator of A according to the Blakrishman definition; $F\colon [0,a] \times E \times \mathcal{B} \to E$ is a continuous function, which will be specified later; $\mathcal{B} := C([-r,0],E)$ denotes the space of continuous functions from $[-r,0]$ into E provided with the uniform norm topology, $r > 0$ is a constant; $\varphi \in \mathcal{B}$ is given. For $t \geqslant 0$, $u_t \in \mathcal{B}$ denotes the history function defined by $u_t(s) = u(t+s)$ for $s \in [-r,0]$, where u is a continuous function from $[-r,a]$ into E.

In order to introduce the definitions of the lower or upper solution and the mild solution for the time-space fractional delayed evolution equation (6), we set

$$C^\alpha([0,a],E) = \bigl\{u \in C\bigl([0,a],E\bigr) \bigm| {}^c D_t^\alpha u \text{ exists, and } {}^c D_t^\alpha u \in C([0,a],E)\bigr\},$$

and denote by E_1 the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$.

Definition 1. A function $w \in C([-r,a],E)$ is said to be an upper solution of Eq. (6) if $w|_{[0,a]} \in C^\alpha([0,a],E) \cap C([0,a],E_1)$ and

$$\begin{aligned} &{}^c D_t^\alpha w(t) + A^\beta w(t) \geqslant F\bigl(t, w(t), w_t\bigr), \quad t \in [0,a], \\ &w(t) \geqslant \varphi(t), \quad t \in [-r,0]. \end{aligned} \tag{7}$$

If the inequality of (7) is inverse, it is said to be a lower solution.

Definition 2. A function $u \in C([-r,a],E)$ is said to be a mild solution of Eq. (6) if it satisfies

$$u(t) = \begin{cases} U_{\alpha,\beta}(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1} V_{\alpha,\beta}(t-s) F(s,u(s),u_s)\,\mathrm{d}s, & t \in [0,a], \\ \varphi(t), & t \in [-r,0]. \end{cases}$$

Next, we present and demonstrate our main results.

Theorem 1. *Assume that Eq.* (6) *has upper and lower solutions* $w^{(0)}, v^{(0)}$ *satisfying* $v^{(0)} \leqslant w^{(0)}$. *If* $F : [0,a] \times E \times \mathcal{B} \to E$ *is a continuous function satisfying*

(H1) *for any* $t \in [0,a]$, $x_1, x_2 \in E$ *and* $\phi_1, \phi_2 \in \mathcal{B}$ *with* $v^{(0)}(t) \leqslant x_1 \leqslant x_2 \leqslant w^{(0)}(t)$ *and* $v_t^{(0)} \leqslant \phi_1 \leqslant \phi_2 \leqslant w_t^{(0)}$, *there is a constant* $C \geqslant 0$ *such that*

$$F(t,x_2,\phi_2) - f(t,x_1,\phi_1) \geqslant -C(x_2 - x_1),$$

then Eq. (1) *has maximal and minimal mild solutions* $\overline{u}, \underline{u} \in [v^{(0)}, w^{(0)}]$.

Proof. Obviously, Eq. (6) can be rewritten in the following form:

$$\begin{aligned} &{}^c D_t^\alpha u(t) + A^\beta u(t) + Cu(t) = F\bigl(t, u(t), u_t\bigr) + Cu(t), \quad t \geqslant 0, \\ &u(t) = \varphi(t), \quad t \in [-r,0], \end{aligned} \tag{8}$$

where constant C is decided by condition (H1).

According to the discussion in the preparatory part, one can obtain that for any $\beta \in (0,1)$, $-A^\beta$ generated a uniformly bounded, positive and compact semigroup $T_\beta(t)$ $(t \geqslant 0)$ satisfying $\|T_\beta(t)\| \leqslant M$ for every $t \geqslant 0$.

We denote by $S_\beta(t) = \mathrm{e}^{-Ct}T_\beta(t)$ $(t \geqslant 0)$ the C_0-semigroup generated by $-(CI + A^\beta)$. Obviously,

$$\|S_\beta(t)\| = \|\mathrm{e}^{-Ct}T_\beta(t)\| \leqslant M\mathrm{e}^{-Ct} \leqslant M, \quad t \geqslant 0.$$

Moreover, by the positivity and compactness of the semigroup $T_\beta(t)$ $(t \geqslant 0)$ one can see that $S_\beta(t)$ $(t \geqslant 0)$ is a positive compact semigroup. Define two operators $\mathcal{U}_{\alpha,\beta}(t)$ $(t \geqslant 0)$ and $\mathcal{V}_{\alpha,\beta}(t)$ $(t \geqslant 0)$ by

$$\mathcal{U}_{\alpha,\beta}(t)x = \int_0^\infty \xi_\alpha(s)S_\beta(t^\alpha s)x\,\mathrm{d}s, \qquad \mathcal{V}_{\alpha,\beta}(t)x = \alpha\int_0^\infty s\xi_\alpha(s)S_\beta(t^\alpha s)x\,\mathrm{d}s,$$

where $x \in E$, and $\xi_\alpha(s)$ is the function defined by (5). Thus, the operators $\mathcal{U}_{\alpha,\beta}(t)$ $(t \geqslant 0)$ and $\mathcal{V}_{\alpha,\beta}(t)$ $(t \geqslant 0)$ have properties (i)–(v) in Lemma 2, and for each $t \geqslant 0$,

$$\|\mathcal{U}_{\alpha,\beta}(t)x\| \leqslant M\|x\|, \qquad \|\mathcal{V}_{\alpha,\beta}(t)\| \leqslant \frac{M}{\Gamma(\alpha)}\|x\| \quad \forall x \in E. \tag{9}$$

For each $u \in [v^{(0)}, w^{(0)}]$ and $t \in [0,a]$, we have $u_t \in [v_t^{(0)}, w_t^{(0)}] \subset \mathcal{B}$. Now, we define operator $\mathcal{Q}$ on $[v^{(0)}, w^{(0)}]$ by

$$\mathcal{Q}u(t) = \begin{cases} \mathcal{U}_{\alpha,\beta}(t)\varphi(0) + \int_0^t (t-s)^{q-1}\mathcal{V}_{\alpha,\beta}(t-s) \\ \qquad\qquad \times \left(F(s,u(s),u_s) + Cu(s)\right)\mathrm{d}s, \quad t \in [0,a], \\ \varphi(t), \quad t \in [-r,0]. \end{cases} \tag{10}$$

From the normality of the cone K, condition (H1) and the continuity of F one can deduce that for any $u \in [v^{(0)}, w^{(0)}]$, there is a constant $M_0 > 0$ such that

$$\max_{t\in[0,a]}\left\{\left\|F\left(t,u(t),u_t\right)\right\| + C\left\|u(t)\right\|\right\} \leqslant M_0. \tag{11}$$

Hence, it is easy to show that $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to C([-r,a],E)$ is well defined. By Definition 2 and (8) it can be asserted that $u \in [v^{(0)}, w^{(0)}]$ is a mild solution of Eq. (6) if u is a fixed point of $\mathcal{Q}$.

Next, we prove it in four steps.

Step 1. $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is monotone increasing.

On the one hand, let

$${}^cD_t^\alpha v^{(0)}(t) + A^\beta v^{(0)}(t) + Cv^{(0)}(t) := h(t), \quad t \geqslant 0.$$

By the positivity of operators $\mathcal{U}_{\alpha,\beta}(t)$ and $\mathcal{V}_{\alpha,\beta}(t)$, for $t \geqslant 0$, one can obtain that

$$\begin{aligned} v^{(0)}(t) &= \mathcal{U}_{\alpha,\beta}(t)v^{(0)}(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{V}_{\alpha,\beta}(t-s)h(s)\,\mathrm{d}s \\ &\leqslant \mathcal{U}_{\alpha,\beta}(t)\varphi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{V}_{\alpha,\beta}(t-s)\bigl(F\bigl(s, v^{(0)}(s), v_s^{(0)}\bigr) + Cv^{(0)}(s)\bigr)\,\mathrm{d}s \\ &= \mathcal{Q}v^{(0)}(t), \end{aligned}$$

and $v^{(0)}(t) \leqslant \varphi(t)$ for $t \in [-r, 0]$. Thus, $v^{(0)} \leqslant \mathcal{Q}v^{(0)}$. Similarly, $\mathcal{Q}w^{(0)} \leqslant w^{(0)}$ can be obtained.

On the other hand, for any $u^{(1)}, u^{(2)} \in [v^{(0)}, w^{(0)}]$ with $u^{(1)} \leqslant u^{(2)}$ and $t \geqslant 0$, we can see $v^{(0)}(t) \leqslant u^{(1)}(t) \leqslant u^{(2)}(t) \leqslant w^{(0)}(t)$, $v_t^{(0)} \leqslant u_t^{(1)} \leqslant u_t^{(2)} \leqslant w_t^{(0)}$. Thus, by condition (H1), $F(t, u^{(2)}(t), u_t^{(2)}) + Cu^{(2)}(t) \geqslant F(t, u^{(1)}(t), u_t^{(1)}) + Cu^{(1)}(t)$. Hence, by the positivity of $\mathcal{V}_{\alpha,\beta}(t)$ ($t \geqslant 0$) one can see

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1}\mathcal{V}_{\alpha,\beta}(t-s)\bigl(F\bigl(s, u^{(2)}(s), u_s^{(2)}\bigr) + Cu^{(2)}(s)\bigr)\,\mathrm{d}s \\ &\qquad \geqslant \int_0^t (t-s)^{\alpha-1}\mathcal{V}_{\alpha,\beta}(t-s)\bigl(F\bigl(s, u^{(1)}(s), u_s^{(1)}\bigr) + Cu^{(1)}(s)\bigr)\,\mathrm{d}s. \end{aligned} \tag{12}$$

Combining with (10), (12) and the positivity of $\mathcal{U}_{\alpha,\beta}(t)$ ($t \geqslant 0$), it is easy to see $\mathcal{Q}u^{(1)} \leqslant \mathcal{Q}u^{(2)}$. Therefore, $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is monotone increasing.

Next, let

$$v^{(i)} = \mathcal{Q}v^{(i-1)}, \quad w^{(i)} = \mathcal{Q}w^{(i-1)}, \quad i = 1, 2, \dots, \tag{13}$$

then we can obtain two sequences $\{v^{(i)}\}$ and $\{w^{(i)}\}$ in $[v^{(0)}, w^{(0)}]$. By the monotonicity of the operator $\mathcal{Q}$ one can see

$$v^{(0)} \leqslant v^{(1)} \leqslant v^{(2)} \leqslant \cdots \leqslant v^{(i)} \leqslant \cdots \leqslant w^{(i)} \leqslant \cdots \leqslant w^{(2)} \leqslant w^{(1)} \leqslant w^{(0)}.$$

Step 2. $\{v^{(i)}\}$ and $\{w^{(i)}\}$ are equicontinuous in $[-r, a]$.

In fact, for each $u \in [v^{(0)}, w^{(0)}]$, by (10) we only consider it on $[0, a]$. Without loss of generality, let $0 \leqslant t_1 < t_2 \leqslant a$. By (10) one can see

$$\begin{aligned} &\bigl\|\mathcal{Q}u(t_2) - \mathcal{Q}u(t_1)\bigr\| \\ &\quad = \Biggl\|\mathcal{U}_{\alpha,\beta}(t_2)u(0) + \int_0^{t_2} (t_2-s)^{\alpha-1}\mathcal{V}_{\alpha,\beta}(t_2-s)\bigl(F\bigl(s, u(s), u_s\bigr) + Cu(s)\bigr)\,\mathrm{d}s \\ &\qquad - \mathcal{U}_{\alpha,\beta}(t_1)u(0) - \int_0^{t_1} (t_1-s)^{\alpha-1}\mathcal{V}_{\alpha,\beta}(t_1-s)\bigl(F\bigl(s, u(s), u_s\bigr) + Cu(s)\bigr)\,\mathrm{d}s\Biggr\| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \big\|\mathcal{U}_{\alpha,\beta}(t_2)u(0)-\mathcal{U}_{\alpha,\beta}(t_1)u(0)\big\| \\
&\quad+\int_0^{t_1}\big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big)\big\|\mathcal{V}_{\alpha,\beta}(t_2-s)\big\|\big\|F\big(s,u(s),u_s\big)+Cu(s)\big\|\,\mathrm{d}s \\
&\quad+\int_0^{t_1}(t_1-s)^{\alpha-1}\big\|\mathcal{V}_{\alpha,\beta}(t_2-s)-\mathcal{V}_{\alpha,\beta}(t_1-s)\big\|\big\|F\big(s,u(s),u_s\big)+Cu(s)\big\|\,\mathrm{d}s \\
&\quad+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\big\|\mathcal{V}_{\alpha,\beta}(t_2-s)\big\|\big\|F\big(s,u(s),u_s\big)+Cu(s)\big\|\,\mathrm{d}s \\
&:= J_1+J_2+J_3+J_4.
\end{aligned}
$$

Next, we check if $\|J_i\|$ tend to 0 as $t_2-t_1\to 0$ ($i=1,2,3,4$), which are not dependent on $u\in[v^{(0)},w^{(0)}]$. It is easy to see that $J_1\to 0$ as $t_2-t_1\to 0$ by Lemma 2(i). By (9) and (11) we can obtain

$$
\begin{aligned}
J_2&=\int_0^{t_1}\big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big)\big\|\mathcal{V}_{\alpha,\beta}(t_2-s)\big\|\big\|F\big(s,u(s),u_s\big)+Cu(s)\big\|\,\mathrm{d}s \\
&\leqslant \frac{MM_0}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\,\mathrm{d}s \leqslant \frac{MM_0}{\Gamma(\alpha)}\big(t_1^\alpha-t_2^\alpha+(t_2-t_1)^\alpha\big) \\
&\leqslant \frac{2MM_0}{\Gamma(\alpha)}(t_2-t_1)^\alpha\to 0 \quad \text{as } t_2-t_1\to 0.
\end{aligned}
$$

If $t_1=0$ and $0<t_2\leqslant a$, then it easy to see that $J_3=0$. For $t_1>0$ and $\epsilon>0$ small enough, by (9), (11) and Lemma 2(iv) we get that

$$
\begin{aligned}
J_3&=\int_0^{t_1}(t_1-s)^{\alpha-1}\big\|\mathcal{V}_{\alpha,\beta}(t_2-s)-\mathcal{V}_{\alpha,\beta}(t_1-s)\big\|\big\|F\big(s,u(s),u_s\big)+Cu(s)\big\|\,\mathrm{d}s \\
&\leqslant M_0\int_0^{t_1-\epsilon}(t_1-s)^{\alpha-1}\big\|\mathcal{V}_{\alpha,\beta}(t_2-s)-\mathcal{V}_{\alpha,\beta}(t_1-s)\big\|\,\mathrm{d}s \\
&\quad+M_0\int_{t_1-\epsilon}^{t_1}(t_1-s)^{\alpha-1}\big\|\mathcal{V}_{\alpha,\beta}(t_2-s)-\mathcal{V}_{\alpha,\beta}(t_1-s)\big\|\,\mathrm{d}s \\
&\leqslant \sup_{s\in[0,t_1-\epsilon]}\big\|\mathcal{V}_{\alpha,\beta}(t_2-s)-\mathcal{V}_{\alpha,\beta}(t_1-s)\big\|M_0\int_0^{t_1-\epsilon}(t_1-s)^{\alpha-1}\,\mathrm{d}s \\
&\quad+\frac{2MM_0}{\Gamma(\alpha)}\int_{t_1-\epsilon}^{t_1}(t_1-s)^{\alpha-1}\,\mathrm{d}s
\end{aligned}
$$

$$\leqslant \sup_{s\in[0,t_1-\epsilon]} \left\|\mathcal{V}_{\alpha,\beta}(t_2-s)-\mathcal{V}_{\alpha,\beta}(t_1-s)\right\| \frac{M_0(t_1^\alpha-\epsilon^\alpha)}{\alpha} + \frac{2MM_0\epsilon^\alpha}{\Gamma(\alpha)}$$
$$\to 0 \quad \text{as } t_2-t_1\to 0.$$

Finally, by (9) and (11) we have

$$J_4 \leqslant \frac{MM_0}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\,\mathrm{d}s = \frac{M_0}{\Gamma(\alpha)}(t_2-t_1)^\alpha \to 0 \quad \text{as } t_2-t_1\to 0.$$

Therefore, $\|\mathcal{Q}u(t_2)-\mathcal{Q}u(t_1)\|$ tends to 0 independently of $u\in[v^{(0)},w^{(0)}]$ as $t_2-t_1\to 0$, which implies that $\mathcal{Q}:[v^{(0)},w^{(0)}]\to[v^{(0)},w^{(0)}]$ is equicontinuous.

Step 3. $\{v^{(i)}(t)\}$ and $\{w^{(i)}(t)\}$ are relatively compact on E for each $t\in J$.

Let $\Lambda=\{v^{(i)}\}$, $\Pi=\{w^{(i)}\}$ and $\Lambda_0=\Lambda\cup\{v^{(0)}\}$, $\Pi_0=\Pi\cup\{w^{(0)}\}$. Obviously, $\Lambda(t)=(\mathcal{Q}\Lambda_0)(t)$ and $\Pi(t)=(\mathcal{Q}\Pi_0)(t)$ for $t\in J$. In view of the fact that $v^{(i)}(t)=w^{(i)}(t)=\varphi(t)$ for $t\in[-r,0]$, $\{v^{(i)}(t)\}$ and $\{w^{(i)}(t)\}$ are relatively compact on E for $t\in[-r,0]$.

Let $0<t\leqslant a$ be fixed. For any $\varepsilon\in(0,t)$ and $\delta>0$, define a set $\mathcal{Q}_{\varepsilon,\delta}\Lambda_0(t)$ by

$$\mathcal{Q}_{\varepsilon,\delta}\Lambda_0(t) := \left\{\mathcal{Q}_{\varepsilon,\delta}v^{(i)}(t) \;\middle|\; v^{(i)}\in\Lambda_0\right\},$$

$$\begin{aligned}\mathcal{Q}_{\varepsilon,\delta}v^{(i)}(t) &= \mathcal{U}_{\alpha,\beta}(t)v^{(i-1)}(0)\\ &\quad + \alpha\int_0^{t-\varepsilon}\int_\delta^\infty \tau(t-s)^{\alpha-1}\xi_\alpha(\tau)S_\beta\left((t-s)^\alpha\tau\right)\\ &\qquad\times\left(F\left(s,v^{(i-1)}(s),v_s^{(i-1)}\right)+Cv^{(i-1)}(s)\right)\mathrm{d}\tau\,\mathrm{d}s\\ &= \mathcal{U}_{\alpha,\beta}(t)v^{(i-1)}(0)\\ &\quad + \alpha S_\beta(\varepsilon^\alpha\delta)\int_0^{t-\varepsilon}\int_\delta^\infty \tau(t-s)^{\alpha-1}\xi_\alpha(\tau)S_\beta\left((t-s)^\alpha\tau-\varepsilon^\alpha\delta\right)\\ &\qquad\times\left(F\left(s,v^{(i-1)}(s),v_s^{(i-1)}\right)+Cv^{(i-1)}(s)\right)\mathrm{d}\tau\,\mathrm{d}s.\end{aligned}$$

Hence, from the compactness of $\mathcal{U}_{\alpha,\beta}(t)$ and $S_\alpha(\varepsilon^\alpha\delta)$ it follows that $\mathcal{Q}_{\varepsilon,\delta}\Lambda_0(t)$ is relatively compact in E for each $\delta>0$ and $\varepsilon\in(0,t)$. Moreover, for every $v^{(i)}\in\Lambda_0$ and $0<t\leqslant a$, one can find

$$\begin{aligned}&\left\|\mathcal{Q}v^{(i)}(t)-\mathcal{Q}_{\varepsilon,\delta}v^{(i)}(t)\right\|\\ &\quad\leqslant \left\|\alpha\int_0^t\int_0^\delta \tau(t-s)^{\alpha-1}\xi_\alpha(\tau)S_\beta\left((t-s)^\alpha\tau\right)\right.\\ &\qquad\left.\times\left(F\left(s,v^{(i-1)}(s),v_s^{(i-1)}\right)+Cv^{(i-1)}(s)\right)\mathrm{d}\tau\,\mathrm{d}s\right\|\end{aligned}$$

$$+\left\|\alpha\int_{t-\varepsilon}^{t}\int_{\delta}^{\infty}\tau(t-s)^{\alpha-1}\xi_\alpha(\tau)S_\beta\big((t-s)^\alpha\tau\big)\right.$$
$$\left.\times\big(F\big(s,v^{(i-1)}(s),v_s^{(i-1)}\big)+Cv^{(i-1)}(s)\big)\,\mathrm{d}\tau\,\mathrm{d}s\right\|$$
$$\leqslant \alpha MM_0\int_0^t(t-s)^{\alpha-1}\,\mathrm{d}s\int_0^\delta\tau\xi_\alpha(\tau)\,\mathrm{d}\tau$$
$$+\,\alpha MM_0\int_{t-\varepsilon}^{t}(t-s)^{\alpha-1}\,\mathrm{d}s\int_\delta^\infty\tau\xi_\alpha(\tau)\,\mathrm{d}\tau$$
$$\leqslant MM_0a^\alpha\int_0^\delta\tau\xi_\alpha(\tau)\,\mathrm{d}\tau+\frac{MM_0}{\Gamma(1+\alpha)}\varepsilon^\alpha\to 0\quad\text{as }\varepsilon\to 0,\ \delta\to 0.$$

Thus, the set $(\mathcal{Q}\Lambda_0)(t)$ is relatively compact, which implies that $\{v^{(i)}(t)\}$ is relatively compact on E for $0<t\leqslant a$. Thus, $\{v^{(i)}(t)\}$ is relatively compact on E for $t\in J$. Similarly, one can obtain that $\{w^{(i)}(t)\}$ is relatively compact on E for $t\in J$.

As we all know, the above argument and the Arzela–Ascoli theorem guarantee that $\{v^{(i)}\}$ and $\{w^{(i)}\}$ are relatively compact in $C(J,E)$. Hence, there are convergent subsequences in $\{v^{(i)}\}$ and $\{w^{(i)}\}$, respectively. Combining the normality of the cone K_C and the monotonicity, we obtain that $\{v^{(i)}\}$ and $\{w^{(i)}\}$ themselves are convergent, namely, there exist $\underline{u},\overline{u}\in C(J,E)$ such that $\lim_{i\to\infty}v^{(i)}=\underline{u}$ and $\lim_{i\to\infty}w^{(i)}=\overline{u}$.

Hence, taking $i\to\infty$ in (13), we have

$$\underline{u}=\mathcal{Q}\underline{u},\qquad \overline{u}=\mathcal{Q}\overline{u}.$$

Therefore, $\underline{u},\overline{u}\in\Omega$ are fixed points of $\mathcal{Q}$, which are mild solutions of Eq. (6).

Step 4. Maximal and minimal properties of $\overline{u},\underline{u}$.

Let $\widetilde{u}\in[v_0,w_0]$ be a fixed point of $\mathcal{Q}$, then $v^{(0)}(t)\leqslant\widetilde{u}(t)\leqslant w^{(0)}(t)$ for each $t\in J$, and

$$v^{(1)}(t)=(\mathcal{Q}v^{(0)})(t)\leqslant(\mathcal{Q}\widetilde{u})(t)=\widetilde{u}(t)\leqslant(\mathcal{Q}w^{(0)})(t)=w^{(1)}(t),$$

namely, $v^{(1)}\leqslant\widetilde{u}\leqslant w^{(1)}$. In general,

$$v^{(i)}\leqslant\widetilde{u}\leqslant w^{(i)},\quad i=1,2,\dots. \tag{14}$$

Taking $i\to\infty$ in (14), we can obtain $\underline{u}\leqslant\widetilde{u}\leqslant\overline{u}$, which implies that $\overline{u},\underline{u}$ are maximal and minimal mild solutions of Eq. (6). □

In the above works, the key assumption (H1) (the monotone on the history function of the nonlinear function) is employed. However, we hope that the nonlinear function is quasimonotonicity. In this case, the results have more extensive application background.

In fact, we find that if Eq. (6) has the upper and lower solutions $w^{(0)}$, $v^{(0)}$ with $v^{(0)}\leqslant w^{(0)}$ and

(H2) for any $u^{(1)}, u^{(2)} \in [v^{(0)}, w^{(0)}]$ with $u^{(2)} \geqslant u^{(1)}$, there exists a sufficiently small constant $C_0 > 0$ such that

$$u^{(2)}(t) - u^{(1)}(t) \geqslant C_0\big(u_t^{(2)}(\cdot) - u_t^{(1)}(\cdot)\big), \quad t \in [0,a],$$

then condition (H1) can be replaced by

(H3) there are nonnegative constants C_1, C_2 such that

$$F(t, x_2, \phi_2) - f(t, x_1, \phi_1) \geqslant -C_1(x_2 - x_1) - C_2\big(\phi_2(\cdot) - \phi_1(\cdot)\big)$$

for any $t \in [0,a]$, $x_1, x_2 \in E$ and $\phi_1, \phi_2 \in \mathcal{B}$ with $v^{(0)}(t) \leqslant x_1 \leqslant x_2 \leqslant w^{(0)}(t)$ and $v_t^{(0)} \leqslant \phi_1 \leqslant \phi_2 \leqslant w_t^{(0)}$.

In fact, for every $t \in [0,a]$ and $u^{(1)}, u^{(2)} \in [v^{(0)}, w^{(0)}]$ satisfying $u^{(1)} \leqslant u^{(2)}$, one can obtain that $v^{(0)}(t) \leqslant u^{(1)}(t) \leqslant u^{(2)}(t) \leqslant w^{(0)}(t)$, $v_t^{(0)} \leqslant u_t^{(1)} \leqslant u_t^{(2)} \leqslant w_t^{(0)}$. From conditions (H2) and (H3) it follows that

$$\begin{aligned}
&F\big(t, u^{(2)}(t), u_t^{(2)}\big) - F\big(t, u^{(1)}(t), u_t^{(2)}\big) \\
&\quad \geqslant -C_1\big(u^{(2)}(t) - u^{(1)}(t)\big) - C_2\big(u_t^{(2)}(\cdot) - u_t^{(1)}(\cdot)\big) \\
&\quad \geqslant -C_1\big(u^{(2)}(t) - u^{(1)}(t)\big) - \frac{C_2}{C_0}\big(u^{(2)}(t) - u^{(1)}(t)\big) \\
&\quad = -\left(C_1 + \frac{C_2}{C_0}\right)\big(u^{(2)}(t) - u^{(1)}(t)\big) \\
&\quad := -C\big(u^{(2)}(t) - u^{(1)}(t)\big).
\end{aligned}$$

Hence, we can obtain the following result from Theorem 1.

Theorem 2. *Assume that Eq.* (6) *has upper and lower solutions $w^{(0)}$, $v^{(0)}$ with $v^{(0)} \leqslant w^{(0)}$. If $F : [0,a] \times E \times \mathcal{B} \to E$ is continuous and satisfies* (H2) *and* (H3)*, then Eq.* (6) *has maximal and minimal mild solutions $\overline{u}, \underline{u} \in [v^{(0)}, w^{(0)}]$.*

Remark. Obviously, condition (H2) is easy to satisfy, and condition (H3) weakens condition (H1). Thus, Theorem 2 partially improve Theorem 1.

In the end of this section, we study the uniqueness of the mild solution for Eq. (6).

Theorem 3. *Assume that Eq.* (6) *has upper and lower solutions $w^{(0)}$, $v^{(0)}$ with $v^{(0)} \leqslant w^{(0)}$. If $F : [0,a] \times E \times \mathcal{B} \to E$ is continuous and satisfies* (H2), (H3) *and*

(H4) *there exist positive constants L_1, L_2 such that, for any $x_1, x_2 \in E$ and $\phi_1, \phi_2 \in \mathcal{B}$ with $v^{(0)}(t) \leqslant x_1 \leqslant x_2 \leqslant w^{(0)}(t)$, $v_t^{(0)} \leqslant \phi_1 \leqslant \phi_2 \leqslant w_t^{(0)}$,*

$$F(t, x_2, \phi_2) - F(t, x_1, \phi_1) \leqslant L_1(x_2 - x_1) + L_2\big(\phi_2(\cdot) - \phi_1(\cdot)\big), \quad t \in [0,a],$$

then Eq. (6) *has a unique mild solution in $[v^{(0)}, w^{(0)}]$.*

Proof. From Theorem 2 we can assert that Eq. (6) has maximal and minimal mild solutions $\overline{u}, \underline{u} \in [v^{(0)}, w^{(0)}]$, which are fixed points of $\mathcal{Q}$ defined by (10).

By (10) one can find that $\overline{u} - \underline{u} \equiv \theta$ for $t \in [-r, 0]$. On the other hand, for $t \in [0, a]$,

$$\begin{aligned}
\theta \leqslant \overline{u}(t) - \underline{u}(t) &= \mathcal{Q}\overline{u}(t) - \mathcal{Q}\underline{u}(t) \\
&= \int_0^t (t-s)^{\alpha-1} \mathcal{V}_{\alpha,\beta}(t-s)\big(F\big(s, \overline{u}(s), \overline{u}_s\big) + C\overline{u}(s)\big)\,\mathrm{d}s \\
&\quad - \int_0^t (t-s)^{\alpha-1} \mathcal{V}_{\alpha,\beta}(t-s)\big(F\big(s, \underline{u}(s), \underline{u}_s\big) + C\underline{u}(s)\big)\,\mathrm{d}s \\
&\leqslant \int_0^t (t-s)^{\alpha-1} \mathcal{V}_{\alpha,\beta}(t-s)\big((L_1 + C)\big(\overline{u}(s) - \underline{u}(s)\big) + L_2\big(\overline{u}_s(\cdot) - \underline{u}_s(\cdot)\big)\big)\,\mathrm{d}s \\
&\leqslant \left(L_1 + C + \frac{L_2}{C_0}\right) \int_0^t (t-s)^{\alpha-1} \mathcal{V}_{\alpha,\beta}(t-s)\big(\overline{u}(s) - \underline{u}(s)\big)\,\mathrm{d}s,
\end{aligned}$$

where $C = C_1 + C_2/C_0$. Hence, from the normality of the cone K it follows that for any $t \in [0, a]$,

$$\big\|\overline{u}(t) - \underline{u}(t)\big\| \leqslant \frac{NM}{\Gamma(\alpha)}\left(C_1 + L + \frac{C_2}{L_0}\right) \int_0^t (t-s)^{\alpha-1} \big\|\overline{u}(s) - \underline{u}(s)\big\|\,\mathrm{d}s.$$

From Lemma 3 it follows that $\underline{u}(t) = \overline{u}(t)$ for $t \in [0, a]$. Therefore, $\underline{u} = \overline{u}$ is the unique mild solution of Eq. (6) in $[v^{(0)}, w^{(0)}]$. □

4 Results for the time-space fractional diffusion equation

In the following, we will apply our abstract results to prove the existence and uniqueness of the mild solutions for the time-space fractional delayed diffusion equation with fractional Laplacian (2).

Theorem 4. *Let $f(x, t, 0, 0) \geqslant 0$ for any $(x, t) \in \overline{\Omega} \times [0, a]$, and let $w = w(x, t) \in C(\overline{\Omega} \times [-r, a]) \cap C^{2,\alpha}(\Omega \times [0, a])$ be a nonnegative function satisfying*

$$\begin{aligned}
&{}^cD_t^\alpha w(x,t) + (-\Delta)^\beta w(x,t) \geqslant f\big(x, t, w(x,t), w(x, t+\tau)\big), \quad (x,t) \in \overline{\Omega} \times [0, a], \\
&w|_{\partial\Omega} = 0, \qquad w(x, \tau) \geqslant \varphi(\tau)(x), \quad x \in \overline{\Omega},\ \tau \in [-r, 0].
\end{aligned}$$

If for any $x \in \overline{\Omega}$, $t \in [0, a]$, $\tau \in [-r, 0]$, and $0 \leqslant u_1(x,t) \leqslant u_2(x,t) \leqslant w(x,t)$,

(A1) *there is a constant $c_0 > 0$ such that*

$$u_2(x,t) - u_1(x,t) \geqslant c_0\big(u_2(x,\ t+\tau) - u_1(x, t+\tau)\big);$$

(A2) *there are constant* $c_1, c_2 > 0$ *such that*

$$f\big(x,t,u_2(x,t),u_2(x,\,t+\tau)\big) - f\big(x,t,u_1(x,t),u_1(x,\,t+\tau)\big)$$
$$\geqslant -c_1\big(u_2(x,t)-u_1(x,t)\big) - c_2\big(u_2(x,\,t+\tau)-u_1(x,\,t+\tau)\big),$$

then semilinear time-space fractional diffusion equation boundary value problem with delay (2) *has maximal and minimal mild solutions* $\overline{u}, \underline{u} \in C([-r,a], L^2(\Omega))$ *between* 0 *and* w.

Proof. Let $E = L^2(\overline{\Omega})$ with the L^2-norm $\|\cdot\|$, $K = \{u \in E \mid u(x) \geqslant 0,\ \text{a.e. } x \in \overline{\Omega}\}$, which defines a partial ordering "$\leqslant$" on E. Thus, E is an ordered Banach space, and the positive cone K is a normal regeneration cone.

Define operator $A: D(A) \subset E \to E$ as follows:

$$D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad Au = -\Delta u.$$

From [2] it follows that $-A$ is a selfadjoint operator, which generates a uniformly bounded analytic semigroup $T(t)$ $(t \geqslant 0)$ in E. Specially, $T(t)$ $(t \geqslant 0)$ is contractive in E, hence, $\|T(t)\| \leqslant 1$ for every $t \geqslant 0$. Furthermore, we assume that λ_1 is the first eigenvalue of operator A, then $\lambda_1 > 0$ from [1, Thm. 1.16]. On the other hand, $\lambda I + A$ has a positive bounded inverse operator $(\lambda I + A)^{-1}$ for $\lambda > 0$, it follows that $T(t)$ $(t \geqslant 0)$ is a positive C_0-semigroup. Since the operator A has compact resolvent in $L^2(\Omega)$, thus $T(t)$ $(t \geqslant 0)$ is a compact semigroup (see [25]).

Obviously, the fractional power A^β of the nonnegative operator A is well defined by (3). Therefore, based on the argument in Section 2 and the properties of $T(t)$ $(t \geqslant 0)$ generated by $-A$, one can deduce that $-A^\beta$ generates a positive, compact, uniformly bounded and analytic semigroup $T_\beta(t)$ $(t \geqslant 0)$ on E, and $\|T_\beta(t)\| \leqslant 1$ for all $t \geqslant 0$.

Set $u(t)(\xi) = u(\xi,t)$, $u(t+\tau)(\xi) = u(\xi,\,t+\tau)$, and

$$F\big(t,u(t),u(t+\tau)\big)(\xi) = f\big(\xi,t,u(\xi,t),u(\xi,\,t+\tau)\big), \tag{15}$$

thus, the boundary value problem (2) can be rewritten as follows:

$$\begin{aligned} &{}^cD_t^q u(t) + A^\beta u(t) = F\big(t,u(t),u(t+\tau)\big), \quad t \in [0,a],\\ &u(\tau) = \varphi(\tau), \quad \tau \in [-r,0]. \end{aligned} \tag{16}$$

From the assumptions of the function f we can deduce that the function $F\colon [0,a] \times E \times E \to E$ defined by (15) is continuous and satisfies condition (H1). And from the assumptions one can find that $v_0 \equiv 0$, and $w_0 = w(\xi,t) \geqslant 0$ are lower and upper solutions of problem (16), respectively. By conditions (A1) and (A2) we can deduce that conditions (H2) and (H3) hold. Therefore, by Theorem 2 one can find that Eq. (16) has minimal and maximal mild solutions $\underline{u}, \overline{u} \in C([-r,a],E)$. □

According to the proof of the above theorem, it is not difficult to obtain the following uniqueness results.

Theorem 5. *Under the assumptions of Theorem* 4*, if the following condition holds*:

(A3) *there are positive constants* c_1, c_2 *such that*

$$\begin{aligned} &f\big(\xi, t, u_2(\xi, t), u_2(\xi,\, t+\tau)\big) - f\big(\xi, t, u_1(\xi, t), u_1(\xi,\, t+\tau)\big) \\ &\quad \leqslant c_1\big(u_2(\xi, t) - u_1(\xi, t)\big) + c_2\big(u_2(\xi,\, t+\tau) - u_1(\xi,\, t+\tau)\big), \end{aligned}$$

then the semilinear time-space fractional diffusion equation boundary value problem with delay (2) *has a unique mild solution* $u^* \in C([-r, a], E)$ *between* 0 *and* w.

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