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MONOTONE META-LINDELÖF SPACES

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Abstract. In this paper, we study the monotone meta-Lindelöf property. Relationships between monotone meta-Lindelöf spaces and other spaces are investigated. Behaviors of monotone meta-Lindelöf GO-spaces in their linearly ordered extensions are revealed.

Keywords: monotonically meta-Lindelöf, compact, point-countable, order, linearly ordered extension

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1. Preliminaries

Monotone topological properties play an important role in the research of general topology (see [3]–[5], [7], [9] and [14]). In [3], the authors studied monotone Lindelöf spaces.

A space X is *monotonically Lindelöf* if for each open cover $\mathscr{U}$ of X there exists a countable open cover $r(\mathscr{U})$ of X refining $\mathscr{U}$ such that if $\mathscr{U}$ and $\mathscr{V}$ are open covers and $\mathscr{U}$ refines $\mathscr{V}$, then $r(\mathscr{U})$ refines $r(\mathscr{V})$. Monotone Lindelöf spaces are Lindelöf, however a Lindelöf space may not be monotonically Lindelöf.

In this paper, we introduce the monotone meta-Lindelöf property which is weaker than monotone Lindelöfness but stronger than meta-Lindelöfness. Properties of monotone meta-Lindelöf spaces are investigated. Behaviors of monotone meta-Lindelöf GO-spaces in their linearly ordered extensions are revealed.

Recall that a generalized ordered space (GO-space) is a Hausdorff space X equipped with a linear order and having a base of convex sets (a set A is called convex if $x \in A$ for every x lying between two points of A). If the topology of X coincides with the open interval topology of the given linear order, we say that

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X is a linearly ordered topological space ($LOTS$). Čech showed that the class of GO-spaces is the same as the class of spaces that can be topologically embedded in some $LOTS$ (see [10]).

If X is a GO-space and Y is a $LOTS$ containing X as a subspace, and the order on X is inherited from the order on Y, then Y called a linearly ordered extension of X. If the GO-space X is closed (respectively, dense) in the $LOTS$ Y, then Y is called the closed (respectively, dense) linearly ordered extension of X.

Throughout the paper, spaces are topological spaces and Hausdorff, mappings are continuous and surjective. Let $\mathscr{U}$ and $\mathscr{V}$ be open covers of the space X. If $\mathscr{U}$ refines $\mathscr{V}$, we say that $\mathscr{U}$ is a refinement of $\mathscr{V}$, denoted by $\mathscr{U} \prec \mathscr{V}$. A space X is meta-Lindelöf if every open cover $\mathscr{U}$ of X has a point-countable open refinement $\mathscr{V}$. $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{Z}$ denote the set of all real numbers, the set of all rational numbers, the set of all irrational numbers and the set of all integers respectively. The spaces $[0,\omega_1)$ and $[0,\omega_1]$ are the usual ordinal spaces unless specifically stated and the space $[0,1]$ is the subspace of the real line $\mathbb{R}$. Other terms and symbols can be found in [6] and [10].

2. The definition of monotone meta-Lindelöf spaces

Definition 1. A space X is monotonically meta-Lindelöf if each open cover $\mathscr{U}$ of X has a point-countable open refinement $r(\mathscr{U})$ such that if $\mathscr{U}$ and $\mathscr{V}$ are open covers and $\mathscr{U} \prec \mathscr{V}$, then $r(\mathscr{U}) \prec r(\mathscr{V})$. In this case, r is called a monotone meta-Lindelöf operator for the space X.

Proposition 1. *Spaces with a point-countable base are monotonically meta-Lindelöf.*

Proof. Let the space X have a point-countable base $\mathscr{B}$. For any open cover $\mathscr{U}$ of X, put $r(\mathscr{U}) = \{B \in \mathscr{B}\colon \exists U \in \mathscr{U}$ such that $B \subset U\}$, then r is a monotone meta-Lindelöf operator for X. □

Proposition 1 is not reversible (see Example 3).

Obviously,

$$(\Diamond) \qquad \text{monotone Lindelöf} \Rightarrow \text{monotone meta-Lindelöf} \Rightarrow \text{meta-Lindelöf}.$$

Examples 1, 2 and Proposition 2 show that the implications in $(\Diamond)$ are not reversible.

In a $LOTS$, monotone meta-Lindelöfness does not imply monotone Lindelöfness:

Example 1. Let $X = [0,1] \times (0,1)$ be equipped with the open interval topology of the lexicographical order. Then X is monotonically meta-Lindelöf, but not monotonically Lindelöf.

Proof. For each $t \in [0,1]$, $\{t\} \times (0,1)$ has a countable base $\mathscr{B}_t$, so X has a point-countable base $\mathscr{B} = \{B \in \mathscr{B}_t \colon t \in [0,1]\}$. By Proposition 1, X is monotonically meta-Lindelöf. Since the open cover $\{\{t\} \times (0,1) \colon t \in [0,1]\}$ of X has no countable subcover X is not Lindelöf. So X is not monotonically Lindelöf. □

In *GO*-spaces, monotone meta-Lindelöfness does not imply monotone Lindelöfness:

Example 2. The Michael line M (a *GO*-space) is monotonically meta-Lindelöf, but not monotonically Lindelöf.

Proof. Note that the Michael line M (the real line with the irrationals isolated and the rationals having their usual neighborhoods) has a point-countable base $\mathscr{B} = \{(a,b) \colon a,b \in \mathbb{Q}\} \cup \{\{p\} \colon p \in \mathbb{P}\}$. By Proposition 1 M is monotonically meta-Lindelöf. However M is not monotonically Lindelöf since it is not Lindelöf ([13]). □

Example 3. The space $X = ([0,\omega_1) \times \mathbb{Z}) \cup \{\langle \omega_1, 0\rangle\}$ equipped with the lexicographical-order topology is monotonically meta-Lindelöf, but without a point-countable base.

Proof. For any open cover $\mathscr{U}$ of X, [3] noted that if

$$\alpha = \alpha(\mathscr{U}) = \min\{\alpha' \in [0,\omega_1) \colon (\langle \alpha', 0\rangle, \langle \omega_1, 0\rangle] \subset U \text{ for some } U \in \mathscr{U}\}$$

and

$$r(\mathscr{U}) = \{(\langle \alpha, 0\rangle, \langle \omega_1, 0\rangle]\} \cup \{\{\langle \beta, \kappa\rangle\} \colon \beta < \alpha \text{ and } \kappa \in \mathbb{Z} \text{ or } \beta = \alpha \text{ and } \kappa \leqslant 0\},$$

then r is a monotone Lindelöf operator. So r is also a monotone meta-Lindelöf operator. Since the point $\langle \omega_1, 0\rangle$ has no countable neighborhood base, X has no point-countable base. □

In Example 2.3 of [3], it is shown that $[0,\omega_1]$ is not monotonically Lindelöf. Note that in its proof, if r is assumed to be a monotone meta-Lindelöf operator for $[0,\omega_1]$, then $r(\mathscr{U}_\gamma)$ is a point-countable open refinement of $\mathscr{U}_\gamma$. Thus from the proof of Example 2.3 of [3], we can see that the following stronger result is true.

Proposition 2. *The compact LOTS $[0,\omega_1]$ is not a monotone meta-Lindelöf space.*

Corollary 1. *A monotonically meta-Lindelöf compact LOTS X is first countable.*

P r o o f. Let $\prec$ be the linear order on X. If the compact *LOTS* X is not first countable, then it contains a closed subspace which is homomorphic to $[0,\omega_1]$: in fact, let $p \in X$ have no countable neighborhood base. Without loss of generality, we may assume that p has no immediate predecessor and is not the minimal element and any strict increasing sequence of $\{x \in X\colon\ x \prec p\}$ cannot be convergent to p.

Take $x_0 \in X$ such that $x_0 \prec p$. Start with x_0, by transfinite induction, we can obtain a closed subset $F = \{x_\alpha \prec p\colon\ \alpha \in [0,\omega_1]\}$ of X where for each limit ordinal $\gamma \in [0,\omega_1]$, $x_\gamma = \sup\{x_\alpha\colon\ \alpha < \gamma\}$ (since X is a compact *LOTS* this can be done) and $x_\alpha \prec x_\beta$ whenever $\alpha < \beta$. Clearly F is homomorphic to $[0,\omega_1]$. By Proposition 3 (1), F (homomorphic to $[0,\omega_1]$) is monotonically meta-Lindelöf. This contradicts Proposition 2. □

The compact *LOTS* $[0,\omega_1]$ in Proposition 2 is not connected. We will see that a connected compact *LOTS* may not imply monotone meta-Lindelöfness.

Recall that *the long line* Z is the space $Z = [0,\omega_1) \times [0,1)$ with the open interval topology generated by the lexicographical order. Obviously, Z is countably compact but not compact. By Theorem 9.2 of [1] Z is not meta-Lindelöf. The space $Z^* = Z \cup \{\omega_1\}$ is called *the extended long line* (that is, for any $z \in Z$, $z < \omega_1$ and Z^* is equipped with the open interval topology, equivalently, Z^* is the one-point compactification of Z) (see [13]).

Corollary 2. *The connected compact LOTS Z^* is not monotonically meta-Lindelöf.*

P r o o f. Note that Z^* is not first countable since the point ω_1 has no countable neighborhood base. So by Corollary 1, Z^* is not monotonically meta-Lindelöf. □

To be clear at a glance, we give the following diagram, the implications are not reversible.

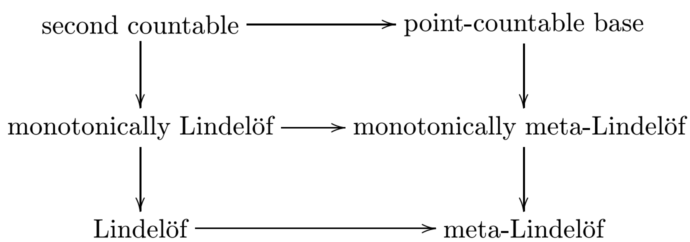

Diagram (∗).

Recall that a space X is said to have calibre ω_1 if a point-countable family of non-empty open subsets is countable [11].

Remark 1. If X has calibre ω_1 and each $x \in X$ has an open neighborhood U_x with a point-countable base, then the properties in Diagram $(*)$ are equivalent.

In fact, let X be meta-Lindelöf and $\mathscr{W}$ be a point-countable open refinement of $\mathscr{U} = \{U_x \colon x \in X\}$. For each $W \in \mathscr{W}$, take a $U \in \mathscr{U}$ such that $W \subset U$. Put $\mathscr{B}_W = \{W \cap B \colon B \in \mathscr{B}_U\}$, where $\mathscr{B}_U$ is a point-countable base of U. Then $\mathscr{B} = \bigcup\{\mathscr{B}_W \colon W \in \mathscr{W}\}$ is a point-countable base for X. Since X has calibre ω_1, $\mathscr{B}$ is a countable base for X.

Recall that a mapping $f \colon X \to Y$ is an s-mapping if for every $y \in Y$, $f^{-1}(y)$ is separable.

Proposition 3.

(1) *Monotone meta-Lindelöfness is hereditary for closed subspaces;*

(2) *monotone meta-Lindelöfness is preserved by open s-mappings.*

Proof. (1) Let the space X be monotonically meta-Lindelöf and r be a monotone meta-Lindelöf operator for X. Suppose that $F \subset X$ is closed. For any open cover $\mathscr{U}_F$ of F, there exists a family $\mathscr{U}$ of open subsets of X such that $\mathscr{U}_F = \{U \cap F \colon U \in \mathscr{U}\}$. Put $\mathscr{U}' = \{U \cup (X - F) \colon U \in \mathscr{U}\}$ and $r_F(\mathscr{U}_F) = \{W \cap F \colon W \in r(\mathscr{U}')\}$, then r_F is a monotone meta-Lindelöf operator for F.

(2) Let $f \colon X \to Y$ be an open s-mapping, X be monotonically meta-Lindelöf and r_X be a monotone meta-Lindelöf operator for X. For any open cover $\mathscr{U}$ of Y, put $r_Y(\mathscr{U}) = \{f(W) \colon W \in r_X(f^{-1}(\mathscr{U}))\}$. For any $y \in Y$, since $f^{-1}(y)$ is separable and $r_X(f^{-1}(\mathscr{U}))$ is point-countable, $\{G \in r_X(f^{-1}(\mathscr{U})) \colon G \cap f^{-1}(y) \neq \emptyset\}$ is countable. So $r_Y(\mathscr{U})$ is a point-countable open refinement of $\mathscr{U}$. Clearly r_Y is a monotone meta-Lindelöf operator for the space Y. $\square$

Remark 2.

(1) *Monotone meta-Lindelöfness is not hereditary for open subspaces*: the space X in Example 3 has an open subspace $[0, \omega_1) \times \{0\}$ homomorphic to the space $[0, \omega_1)$ which is countably compact but not compact. By Theorem 9.2 of [1], $[0, \omega_1) \times \{0\}$ is not meta-Lindelöf and thus not monotonically meta-Lindelöf.

(2) *Monotone meta-Lindelöfness is not preserved by open mappings*: the first countable T_0-space $[0, \omega_1)$ is an image of a metric space X under an open mapping (see 4.2.D of [6]). Since the metric space X has a point-countable base, X is monotonically meta-Lindelöf, but $[0, \omega_1)$ is not.

(3) *Separable (hence countable) monotone meta-Lindelöf spaces are monotone Lindelöf*: this follows the fact that in separable spaces, point-countable family of open sets is countable.

(4) *Monotone meta-Lindelöfness is not productive*: the Sorgenfrey line S (the real line with half-open intervals of the form $[a, b)$ as a basis for the topology) is a

separable GO-space. By Proposition 3.1 of [3], S is monotonically Lindelöf (so monotonically meta-Lindelöf). However, $S\times S$ is not Lindelöf since it has a closed non-Lindelöf subspace $\{\langle x,-x\rangle\colon x\in S\}$. Since $S\times S$ is separable it is not meta-Lindelöf and thus not monotonically meta-Lindelöf.

Example 4. The preimage of a monotone meta-Lindelöf space under a perfect mapping need not to be monotonically meta-Lindelöf.

Proof. Let $X=[0,\omega_1]\times[0,1]$ and $p\colon X\to[0,1]$ be the projection onto the second coordinate. Clearly f is perfect. By Proposition 1, $[0,1]$ is monotonically meta-Lindelöf. Since X has a closed subspace $[0,\omega_1]\times\{0\}$ homomorphic to $[0,\omega_1]$ which is not monotonically meta-Lindelöf (see Proposition 2), X is not monotonically meta-Lindelöf. □

By Proposition 2, the compact $LOTS$ $[0,\omega_1]$ is not monotonically meta-Lindelöf. We will show that there exists a compact space Y which is neither monotonically meta-Lindelöf nor a GO-space (so not a $LOTS$).

Proposition 4. *Let $Y=X\cup\{p\}$ $(p\notin X)$ be the one-point compactification of the discrete space X of cardinality of ω_1. Then*
(1) *Y is not monotonically meta-Lindelöf;*
(2) *Y is not a GO-space.*

Proof. (1) Assume that Y is monotonically meta-Lindelöf and r is a monotone meta-Lindelöf operator. Then the open cover $\mathscr{U}_0=\{Y\setminus\{x\}\colon x\in X\}$ of Y has its point-countable refinement $r(\mathscr{U}_0)$.

Put $\mathscr{U}_0'=\{U\in\mathscr{U}_0\colon \exists V\in r(\mathscr{U}_0)$ such that $p\in V\subset U\}$, then $\mathscr{U}_0'$ is countable since $Y\setminus V$ is finite with $p\in V\in r(\mathscr{U}_0)$. Obviously $\mathscr{U}_1=\mathscr{U}_0\setminus\mathscr{U}_0'$ is still an open cover of Y.

Put $\mathscr{U}_1'=\{U\in\mathscr{U}_1\colon \exists V\in r(\mathscr{U}_1)$ such that $p\in V\subset U\}$, then $\mathscr{U}_1'$ is countable and $\mathscr{U}_2=\mathscr{U}_1\setminus\mathscr{U}_1'$ is an open cover of Y.

Suppose that for each $i<\omega$ we have obtained an open cover $\mathscr{U}_i$ of Y and countable $\mathscr{U}_i'\subset\mathscr{U}_i$ with $\mathscr{U}_{i+1}\prec\mathscr{U}_i$ and $\mathscr{U}_i'\cap\mathscr{U}_j'=\emptyset$ for $i\neq j$. Put $\mathscr{U}_\omega=\mathscr{U}_0\setminus\bigcup\{\mathscr{U}_i'\colon i<\omega\}$, then for each $i<\omega$ the open cover $\mathscr{U}_\omega$ of Y refines $\mathscr{U}_i$. Thus $r(\mathscr{U}_\omega)\prec r(\mathscr{U}_i)$. So for each $i<\omega$, we can take $V\in r(\mathscr{U}_\omega)$, $V_i\in r(\mathscr{U}_i)$ and $U_i\in\mathscr{U}_i'$ such that $p\in V\subset V_i\subset U_i$. This contradicts the finiteness of $Y\setminus V$.

(2) Assume that Y is a GO-space. Then it is easy to see the compact GO-space Y is a $LOTS$. Let $\prec$ be the linear order on Y. Note that p has no countable neighborhood base.

Similar to Corollary 1, we can take a closed subspace $F=\{y_\alpha\prec p\colon \alpha\in[0,\omega_1]\}$ of Y, where for each limit ordinal $\gamma\in[0,\omega_1]$, $y_\gamma=\sup\{y_\alpha\colon \alpha<\gamma\}$, and $y_\alpha\prec y_\beta$

whenever $\alpha < \beta$, such that F is homomorphic to $[0, \omega_1]$. Obviously $y_{\omega_1} \preceq p$. If $y_{\omega_1} \prec p$, then $U = \{y \in Y\colon\ y_{\omega_1} \prec y\} \ni p$ is open and $Y \setminus U$ is infinite, a contradiction. If $y_{\omega_1} = p$, take a limit ordinal α with $0 < \alpha < \omega_1$. Then $U = \{y \in Y\colon\ y_\alpha \prec y\} \ni p$ is open and $Y \setminus U$ is infinite, a contradiction. □

3. Linearly ordered extensions of monotone meta-Lindelöf GO-spaces

Lemma 1. *For a GO-space X, the following are equivalent:*

(1) *X is monotonically meta-Lindelöf;*

(2) *each open cover $\mathscr{U}$ of X by convex sets has a point-countable open refinement $r(\mathscr{U})$ such that if $\mathscr{U}$ and $\mathscr{V}$ are open covers of X by convex sets and $\mathscr{U} \prec \mathscr{V}$, then $r(\mathscr{U}) \prec r(\mathscr{V})$;*

(3) *same as* (2), *but each member of $r(\mathscr{U})$ is a convex set.*

Proof. Note that any non-empty subset G of the GO-space X can be uniquely represented as $G = \bigcup\{S_i\colon\ i \in I\}$, where each S_i is a convex component of G and if the set G is open, then each S_i is open. Moreover, if $G \subset G'$, where $G' = \bigcup\{S'_i\colon\ i \in I'\}$ and $\{S'_i\colon\ i \in I'\}$ is the set of all convex components of G', then $\{S_i\colon\ i \in I\} \prec \{S'_i\colon\ i \in I'\}$. Then the proof is obvious. □

Let X be a GO-space with the topology τ and λ be the usual open interval topology on X. Put

$$(\dagger)\qquad R = \{x \in X\colon\ [x, \rightarrow) \in \tau \setminus \lambda\} \quad \text{and} \quad L = \{x \in X\colon\ (\leftarrow, x] \in \tau \setminus \lambda\}.$$

Define $X^* \subset X \times \mathbb{Z}$ as follows:

$$X^* = (X \times \{0\}) \cup (R \times \{k \in \mathbb{Z}\colon\ k < 0\}) \cup (L \times \{k \in \mathbb{Z}\colon\ k > 0\}).$$

Let X^* have the open interval topology generated by the lexicographical order. Then $e\colon\ X \to X^*$ defined by $e(x) = \langle x, 0\rangle$ is an order-preserving homeomorphism from X onto the closed subspace $X \times \{0\}$ of X^*. So the space X^* is a closed linearly ordered extension of X.

It is well known that if $\mathscr{P}$ is paracompactness (respectively, metrizability, Lindelöfness or quasi-developability), then a GO-space X has $\mathscr{P}$ if and only if its closed linearly ordered extension X^* has $\mathscr{P}$. Now we have

Proposition 5. *For a GO-space X, the following are equivalent:*
(1) *X is monotonically meta-Lindelöf;*
(2) *the closed linearly ordered extension X^* of X is monotonically meta-Lindelöf.*

Proof. (2) $\Rightarrow$ (1). By Proposition 3 the closed subspace $X \times \{0\}$ of X^* is monotonically meta-Lindelöf. So X is monotonically meta-Lindelöf.

(1) $\Rightarrow$ (2). We will identify X with the subspace $X \times \{0\}$ of X^*.

Let $\mathscr{U}$ be an open cover of X^* by convex sets. Then $\mathscr{U}_X = \{U \cap X\colon U \in \mathscr{U}\}$ is an open cover of X by convex sets. By Lemma 1, $\mathscr{U}_X$ has point-countable open refinement $r_X(\mathscr{U}_X)$ consisting of convex sets of X, where r_X is a monotone meta-Lindelöf operator for X. For a convex set S of X, put

$$I(S) = \{x \in S\colon \exists\, a, b \in S \text{ with } a < x < b\},$$
$$S^{\sim} = \{\langle x, k\rangle \in X^*\colon x \in I(S)\} \cup \{\langle x, 0\rangle\colon x \in S \setminus I(S)\}$$

and

$$\mathscr{S}^{\sim} = \{S^{\sim}\colon S \in r_X(\mathscr{U}_X)\}.$$

For any $S^{\sim} \in \mathscr{S}^{\sim}$ with $S \in r_X(\mathscr{U}_X)$, there exists a $U \in \mathscr{U}$ such that $S \subset U$. Since S is an open convex set and $U \subset X^*$ is convex, $S^{\sim}$ is open and $S^{\sim} \subset U$ (see Lemma 3.2 (b), (c) of [10].

Let $r(\mathscr{U}) = \mathscr{S}^{\sim} \cup \{\{\langle x, k\rangle\}\colon \langle x, k\rangle \in X^* \setminus X\}$. Since $r_X(\mathscr{U}_X)$ is point-countable and each $\{\langle x, k\rangle\}$ with $k \neq 0$ is open, $r(\mathscr{U})$ is a point-countable open cover of X^* refining $\mathscr{U}$. If $\mathscr{U}$ and $\mathscr{V}$ are open covers of X^* by convex sets and $\mathscr{U} \prec \mathscr{V}$, then $r_X(\mathscr{U}_X) \prec r_X(\mathscr{V}_X)$. For any $S \in r_X(\mathscr{U}_X)$, there exists a $T \in r_X(\mathscr{V}_X)$ such that the convex sets S and T satisfy $S \subset T$ and thus by Lemma 3.2 (a) of [10] $S^{\sim} \subset T^{\sim}$. So $r(\mathscr{U}) \prec r(\mathscr{V})$. By Lemma 1, X^* is monotonically meta-Lindelöf. $\square$

If $\mathscr{P}$ is "a continuous separating family", then the Michael line M and the Sorgenfrey line S have $\mathscr{P}$ ([8]), M^* has $\mathscr{P}$ ([2]), but S^* does not have $\mathscr{P}$ ([2], [12]). For comparison we have

Corollary 3. *For the Sorgenfrey line S and the Michael line M, their closed linearly ordered extensions S^* and M^* are monotonically meta-Lindelöf.*

For a *GO*-space X, let R and L be defined as in (†). Put

$$\ell(X) = (X \times \{0\}) \cup (R \times \{-1\}) \cup (L \times \{1\}).$$

Equip $\ell(X)$ with the open interval topology generated by the lexicographical order. Then the space $\ell(X)$ has a dense subspace $X \times \{0\}$ which is homeomorphic to the space X. So the space $\ell(X)$ is a dense linearly ordered extension of X.

Example 5. There exists a monotone meta-Lindelöf *GO*-space X for which its dense linearly ordered extension $\ell(X)$ is not monotonically meta-Lindelöf.

P r o o f. Define a topology on the linearly ordered set $X = [0, \omega_1]$ with a base as follows: points of $[0, \omega_1)$ are isolated and ω_1 has the neighborhoods of the form $[\alpha, \omega_1]$, $\alpha < \omega_1$. For an open cover $\mathscr{U}$ of X, put $\alpha_{\mathscr{U}} = \min\{\alpha\colon\ [\alpha, \omega_1] \subset U$ for some $U \in \mathscr{U}\}$ and $r(\mathscr{U}) = \{[\alpha_{\mathscr{U}}, \omega_1]\} \cup \{\{\beta\}\colon\ \beta < \alpha\}$. Then r is a monotone meta-Lindelöf operator for the *GO*-space X.

Note that $\ell(X)$ can actually be constructed from $[0, \omega_1]$ by inserting a predecessor $\langle\alpha, -1\rangle$ at each limit ordinal α less than ω_1. These inserted predecessors in $\ell(X)$ play the role of the limit ordinals in $[0, \omega_1]$. So it is clear that $\ell(X)$ is homeomorphic to the space $[0, \omega_1]$ which is not monotonically meta-Lindelöf by Proposition 2. Hence $\ell(X)$ is not monotonically meta-Lindelöf. □

Note that for the Michael line M, the space $\ell(M) = (\mathbb{R} \times \{0\}) \cup (\mathbb{P} \times \{-1, 1\})$ with the open interval topology generated by the lexicographical order. Let

$$(\ddagger) \qquad M_1 = (\mathbb{Q} \times \{0\}) \cup (\mathbb{P} \times \{1\}) \quad \text{and} \quad M_{-1} = (\mathbb{Q} \times \{0\}) \cup (\mathbb{P} \times \{-1\})$$

be subspaces of $\ell(M)$.

Lemma 2. *Let M_1 and M_{-1} be the subspaces of $\ell(M)$ defined in ($\ddagger$). Then for any open convex set S of M_1, there exists a minimal interval I_S of $\ell(M)$ such that $S = I_S \cap M_1$. For any open convex set S of M_{-1}, an analogous conclusion holds.*

P r o o f. For an open convex set S of M_1, S must be one of the following six intervals of M_1 (for $x, y \in M_1$, by $[x, y)_{M_1}$ or $(x, y)_{M_1}$ we mean an interval of M_1 with endpoints x and y).

(1) $S = [\langle p, 1\rangle, \langle p', 1\rangle)_{M_1}$, $p, p' \in \mathbb{P}$;
(2) $S = [\langle p, 1\rangle, \langle q', 0\rangle)_{M_1}$, $p \in \mathbb{P}$, $q' \in \mathbb{Q}$;
(3) $S = (\langle p, 1\rangle, \langle p', 1\rangle)_{M_1}$, $p, p' \in \mathbb{P}$;
(4) $S = (\langle p, 1\rangle, \langle q', 0\rangle)_{M_1}$, $p \in \mathbb{P}$, $q' \in \mathbb{Q}$;
(5) $S = (\langle q, 0\rangle, \langle p', 1\rangle)_{M_1}$, $q \in \mathbb{Q}$, $p' \in \mathbb{P}$;
(6) $S = (\langle q, 0\rangle, \langle q', 0\rangle)_{M_1}$, $q, q' \in \mathbb{Q}$.

Correspondingly, take the minimal interval I_S of $\ell(M)$ such that $S = I_S \cap M_1$ as follows.

($\dot{1}$) $I_S = [\langle p, 1\rangle, \langle p', -1\rangle)$ for case (1);
($\dot{2}$) $I_S = [\langle p, 1\rangle, \langle q', 0\rangle)$ for case (2);
($\dot{3}$) $I_S = (\langle p, 1\rangle, \langle p', -1\rangle)$ for case (3);
($\dot{4}$) $I_S = (\langle p, 1\rangle, \langle q', 0\rangle)$ for case (4);

$(\dot{5})$ $I_S = (\langle p, 1\rangle, \langle p', -1\rangle)$ for case (5);

$(\dot{6})$ $I_S = (\langle q, 0\rangle, \langle q', 0\rangle)$ for case (6).

Obviously, for any open convex set S of M_{-1}, an analogous conclusion holds. □

Proposition 6. *For the Sorgenfrey line S and the Michael line M, their dense linearly ordered extensions $\ell(S)$ and $\ell(M)$ are monotonically meta-Lindelöf.*

Proof. Note that the space $\ell(S)$ is the set $\mathbb{R} \times \{-1, 0\}$ equipped with the open interval topology generated by the lexicographical order. Clearly $\ell(S)$ has a countable dense subset $\mathbb{Q} \times \{0\}$. So by Proposition 3.1 of [3] the separable space $\ell(S)$ is monotonically Lindelöf and thus monotonically meta-Lindelöf.

To show that the space $\ell(M) = (\mathbb{R} \times \{0\}) \cup (\mathbb{P} \times \{-1, 1\})$ is monotonically meta-Lindelöf, let the space M_r be $\mathbb{R}$ with the topology defined as follows:

Each $q \in \mathbb{Q}$ has a neighborhood base consisting of the usual open intervals;

each $p \in \mathbb{P}$ has a neighborhood base consisting of the sets $[p, x)$, $x \in \mathbb{R}$.

Clearly the GO-space M_r is separable. So by Proposition 3.1 of [3], M_r is monotonically meta-Lindelöf. It is obvious that the subspace $M_1 = (\mathbb{Q} \times \{0\}) \cup (\mathbb{P} \times \{1\})$ of $\ell(M)$ is homeomorphic to M_r. So the space M_1 is monotonically meta-Lindelöf.

Similarly, let the space M_l be $\mathbb{R}$ equipped with the topology: each $q \in \mathbb{Q}$ has a neighborhood base consisting of the usual open intervals and each $p \in \mathbb{P}$ has a neighborhood base consisting of the sets $(x, p]$, $x \in \mathbb{R}$. Then the GO-space M_l is monotonically meta-Lindelöf and M_l is homeomorphic to the subspace $M_{-1} = (\mathbb{Q} \times \{0\}) \cup (\mathbb{P} \times \{-1\})$ of $\ell(M)$.

Let $\mathscr{U}$ be an open cover of $\ell(M)$ by convex sets. Then $\mathscr{U}_1 = \{U \cap M_1 \colon U \in \mathscr{U}\}$ is an open cover of M_1 by convex sets. By Lemma 1, $\mathscr{U}_1$ has a point-countable open refinement $r_1(\mathscr{U}_1)$ by convex sets, where r_1 is a monotone meta-Lindelöf operator for M_1.

Similarly the open cover $\mathscr{U}_{-1} = \{U \cap M_{-1} \colon U \in \mathscr{U}\}$ of M_{-1} by convex sets has a point-countable open refinement $r_{-1}(\mathscr{U}_{-1})$ by convex sets, where r_{-1} is a monotone meta-Lindelöf operator for M_{-1}.

For any $S \in r_1(\mathscr{U}_1)$, there exists a $U \in \mathscr{U}$ such that $S \subset U \cap M_1 \in \mathscr{U}_1$. Since S is an open convex set of M_1, by Lemma 2 there exists a minimal interval I_S of $\ell(M)$ such that $S = I_S \cap M_1$.

Claim. *$I_S \subset U$.*

Proof. Let $x \in I_S$. If $x \in I_S \cap M_1$, then $x \in U \cap M_1 \subset U$; if $x \in I_S \setminus M_1$, then $x = \langle p_0, 0\rangle$ or $x = \langle p_0, -1\rangle$, where $p_0 \in \mathbb{P}$, and there exist $q_1, q_2 \in \mathbb{Q}$ with $q_1 < q_2$ such that $x \in (\langle q_1, 0\rangle, \langle q_2, 0\rangle)$ and $\langle q_1, 0\rangle, \langle q_2, 0\rangle \in I_S \cap M_1$. So the points $\langle q_1, 0\rangle$ and

$\langle q_2, 0\rangle$ belong to U. Since U is a convex set of $\ell(M)$ we know that $x \in U$. Thus $I_S \subset U$.

Put $\mathscr{S}_1 = \{I_S\colon S \in r_1(\mathscr{U}_1)\}$. Then the cover $\mathscr{S}_1$ of M_1 by open convex sets of $\ell(M)$ refines $\mathscr{U}$. By the point-countability of $r_1(\mathscr{U}_1)$, $\mathscr{S}_1$ is point-countable.

Similarly, we can obtain a point-countable open cover $\mathscr{S}_{-1}$ of M_{-1} by convex sets of $\ell(M)$ refining $\mathscr{U}$. Put

$$r(\mathscr{U}) = \mathscr{S}_1 \cup \mathscr{S}_{-1} \cup \{\{\langle p, 0\rangle\}\colon p \in \mathbb{P}\}.$$

Then $r(\mathscr{U})$ is a point-countable open refinement of $\mathscr{U}$ by convex sets and r is a monotone meta-Lindelöf operator for $\ell(M)$. By Lemma 1 $\ell(M)$ is monotonically meta-Lindelöf. □

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