Mathematical Problems in Engineering
Volume 5, pp. 291–315
Reprints available directly from the publisher
Photocopying permitted by license only

Monotone Methods for Solving a Boundary Value Problem of Second Order Discrete System

YUAN-MING WANG [a] and RAVI P. AGARWAL [b,*]

[a] *Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China;*
[b] *Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260*

(Received 1 June 1999)

A new concept of a pair of upper and lower solutions is introduced for a boundary value problem of second order discrete system. A comparison result is given. An existence theorem for a solution is established in terms of upper and lower solutions. A monotone iterative scheme is proposed, and the monotone convergence rate of the iteration is compared and analyzed. The numerical results are given.

Keywords: Boundary value problem of second order discrete system; Monotone iterative method; Upper and lower solutions; Monotone convergence

1. INTRODUCTION

In studying some problems arising in solid state physics, chemical reaction and some other topics, we have to consider boundary value problems of discrete systems. These problems are also natural consequences of discretizations of boundary value problems of continuous systems. Thus more and more researchers are paying attention to such problems, e.g., see [1–10]. Let $N \geq 2$ be a positive integer, $I_1^{N-1} = \{1, 2, \ldots, N-1\}$ and $I_0^N = I_1^{N-1} \cup \{0, N\}$. Furthermore, let $u(t) = (u_1(t), \ldots, u_n(t))^T : I_0^N \to \mathbf{R}^n$ be a vector function of t and the given

* Corresponding author. Tel.: 65-874-2754. Fax: 65-779-5452.

function $f: I_0^N \times \mathbf{R}^n \to \mathbf{R}^n$ with the components f_i, be assumed to be continuous in its arguments. For the function $v(t) : I_0^N \to \mathbf{R}$, we define

$$\delta^2 v(t) = v(t-1) - 2v(t) + v(t+1), \quad t \in I_1^{N-1},$$

$$P_N v(t) = \frac{1}{12N^2}(v(t-1) + 10v(t) + v(t+1)), \quad t \in I_1^{N-1}.$$

Let

$$\Delta^2 = \operatorname{diag}(\delta^2, \ldots, \delta^2), \qquad \mathbf{P}_N = \operatorname{diag}(P_N, \ldots, P_N).$$

Then we consider the following boundary value problem of second order discrete system:

$$-\Delta^2 u(t) + \mathbf{P}_N f\left(\frac{t}{N}, u(t)\right) = 0, \quad t \in I_1^{N-1}, \ u(0) = \alpha, \ u(N) = \beta, \tag{1.1}$$

where $\alpha, \beta \in \mathbf{R}^n$ are known vectors. The motivation to study the problem (1.1) is due to the fact that it is the natural discrete analog of the continuous boundary value problem:

$$-y''(x) + f(x, y) = 0, \quad 0 < x < 1, \ y(0) = \alpha, \ y(1) = \beta,$$

by the fourth order Numerov's method (based on Numerov's formula $\delta^2 y_i = (1/N^2)(1 + \frac{1}{12}\delta^2)y_i''$, e.g., see [11,12]). It is well known that the nature of the solution of a continuous problem is not identical with the solution of its discrete analog (see [3,12]). It is of interest to study the problem (1.1). Recently, the author in [13] proposed a monotone iterative method for the case of a single equation. This method leads not only to the existence and uniqueness of a solution but the process of iterations gives also a computational algorithm for solutions. The properties of systems are very different from those of a single equation. This paper is devoted to extending the monotone iterative method for a single equation to the system (1.1). We also remark that only low regularity conditions are imposed on f in this paper.

The outline of this paper is as follows. In Section 2, we introduce a new concept of a pair of upper and lower solutions of (1.1), and give a comparison result. Then we study the existence of a solution in terms of upper and lower solutions. In Section 3, we propose an iterative scheme for solving (1.1). Only low regularity conditions on f are necessary to assure the monotone convergence of the iteration. Especially, we give two sufficient conditions ensuring the monotone convergence of the iteration to the unique solution of (1.1) in some sector defined by the upper and lower solutions. The convergence rate of the iteration is compared and analyzed in Section 4. In the final section, we present some numerical results which coincide with the theoretical analysis in the previous sections and illustrate this method.

2. A COMPARISON RESULT AND EXISTENCE OF A SOLUTION

Without further mention, we assume that all the inequalities involving vectors are componentwise. The ith component of a vector $u \in \mathbf{R}^n$ is denoted by u_i. For convenience, we define

$$\mathcal{S} = \{u(t) |\ u(t) : I_0^N \longrightarrow \mathbf{R}^n\}.$$

Let $u(t)$, $v(t)$ and $w(t)$ be the vector functions in $\mathcal{S}$. We say that $w \in [u, v]$ if $u(t) \leq w(t) \leq v(t)$ for all $t \in I_0^N$.

We now introduce a new concept of a pair of upper and lower solutions of (1.1).

Definition 2.1 *Let $P = (P_{i,j})$ be an $n \times n$ nonnegative matrix. A pair of vector functions $\bar{u}, \underline{u} \in \mathcal{S}$ is called a pair of upper and lower solutions of* (1.1) *with the nonnegative matrix P, if*

(i) *for all $i = 1, 2, \ldots, n$,*

$$f_i\left(\frac{t}{N}, u(t)\right) - f_i\left(\frac{t}{N}, \tilde{u}(t)\right) \leq \sum_{j=1, j\neq i}^{n} P_{i,j}(u_j(t) - \tilde{u}_j(t)), \quad t \in I_0^N, \tag{2.1}$$

whenever $\min(\bar{u}(t), \underline{u}(t)) \leq \tilde{u}(t) \leq u(t) \leq \max(\bar{u}(t), \underline{u}(t))$ *and* $u_i(t) = \tilde{u}_i(t)$;

(ii) *for all* $i = 1, 2, \ldots, n$,

$$\begin{aligned}
&-\delta^2\bar{u}_i(t) + P_N f_i\Big(\frac{t}{N}, \bar{u}(t)\Big) \\
&\quad - P_N\Bigg(\sum_{j=1, j\neq i}^{n} P_{i,j}(\bar{u}_j(t) - \underline{u}_j(t))\Bigg) \geq 0, \quad t \in I_1^{N-1}, \\
&-\delta^2\underline{u}_i(t) + P_N f_i\Big(\frac{t}{N}, \underline{u}(t)\Big) \\
&\quad + P_N\Bigg(\sum_{j=1, j\neq i}^{n} P_{i,j}(\bar{u}_j(t) - \underline{u}_j(t))\Bigg) \leq 0, \quad t \in I_1^{N-1}, \\
&\qquad \bar{u}(0) \geq \alpha \geq \underline{u}(0), \quad \bar{u}(N) \geq \beta \geq \underline{u}(N).
\end{aligned} \tag{2.2}$$

Remark 2.1 If $P_{i,j} = 0$, $j \neq i$ which implies that f_i is quasimonotone nonincreasing with respect to u in $[\min(\bar{u}, \underline{u}), \max(\bar{u}, \underline{u})]$, i.e., for fixed $t \in I_0^N$, $f_i(t/N, u(t))$ is monotone nonincreasing in $u_j(t)$ for all $j \neq i$, then (2.2) is reduced to the simple form:

$$\begin{aligned}
&-\Delta^2\bar{u}(t) + \mathbf{P}_N f\Big(\frac{t}{N}, \bar{u}(t)\Big) \geq 0, \quad t \in I_1^{N-1}, \\
&-\Delta^2\underline{u}(t) + \mathbf{P}_N f\Big(\frac{t}{N}, \underline{u}(t)\Big) \leq 0, \quad t \in I_1^{N-1}, \\
&\quad \bar{u}(0) \geq \alpha \geq \underline{u}(0), \ \bar{u}(N) \geq \beta \geq \underline{u}(N).
\end{aligned} \tag{2.3}$$

For the theoretical analysis, we first introduce some terminologies. An $n \times n$ real matrix A is called a monotone matrix, if $AZ \geq 0$ implies $Z \geq 0$ for any vector $Z \in \mathbf{R}^n$ (see [14–16]). A necessary and sufficient condition for the monotonicity of an $n \times n$ real matrix A is the existence of the inverse $A^{-1} \geq 0$ (see [14–16]). An $n \times n$ real matrix $A = (A_{i,j})$ is called an M-matrix if $A_{i,j} \leq 0$ for all $i \neq j$ and $A^{-1} \geq 0$ (see [16]). We need the following known results.

LEMMA 2.1 (See Theorem 3, p. 298 of [15]) *Let ν be the identity matrix. If a matrix $T = \nu - S$, $S \geq 0$ and for certain matrix norm $\|\cdot\|$, $\|S\| < 1$, then T is a monotone matrix.*

LEMMA 2.2 (See [17]) *Let A be an M-matrix. Then there exists a positive diagonal matrix E such that the matrix EAE^{-1} is strictly diagonally dominant.*

From now on, let ν be the identity matrix and define the symmetric tridiagonal matrices $A=(A_{i,j})$ and $B=(B_{i,j})$ as

$$\begin{array}{lll} A_{i,i}=2, & B_{i,i}=\frac{5}{6}, & i=1,2,\ldots,N-1; \\ A_{i,i-1}=-1, & B_{i,i-1}=\frac{1}{12}, & i=2,3,\ldots,N-1; \\ A_{i,i+1}=-1, & B_{i,i+1}=\frac{1}{12}, & i=1,2,\ldots,N-2. \end{array} \tag{2.4}$$

Let M be a given constant and set

$$\mathcal{N}(M)=\begin{cases} \dfrac{M}{12N^2}, & M\geq 0, \\ -\dfrac{5+\cos(\pi/N)}{24N^2\sin^2(\pi/2N)}M, & M<0. \end{cases} \tag{2.5}$$

We have the following results (see [13]).

Lemma 2.3 *The matrix* $A+(M/N^2)B$ *is an M-matrix provided* $\mathcal{N}(M)<1$.

Lemma 2.4 *Let* $u(t): I_0^N \to \mathbf{R}$ *such that*

$$-\delta^2 u(t)+MP_N u(t)\geq 0, \quad t\in I_1^{N-1}, \ u(0)\geq 0, \ u(N)\geq 0,$$

where M is a given constant. If $\mathcal{N}(M)<1$, *then* $u(t)\geq 0$ *for all* $t\in I_0^N$.

Now, we give a comparison result for a pair of upper and lower solutions of (1.1).

Theorem 2.1 *Let* $\bar{u}, \underline{u}$ *be a pair of upper and lower solutions of the problem* (1.1) *with the nonnegative matrix* $P=(P_{i,j})$. *In addition, there exists a matrix* $Q=(Q_{i,j})$ *such that for all* $i=1,2,\ldots,n$,

$$\sum_{j=1}^{n} Q_{i,j}(u_j(t)-\tilde{u}_j(t))\leq f_i\left(\frac{t}{N},u(t)\right)-f_i\left(\frac{t}{N},\tilde{u}(t)\right) \leq \sum_{j=1}^{n} P_{i,j}(u_j(t)-\tilde{u}_j(t)), \quad t\in I_0^N, \tag{2.6}$$

whenever $\min(\underline{u}(t),\bar{u}(t))\leq\tilde{u}(t)\leq u(t)\leq\max(\underline{u}(t),\bar{u}(t))$. *Set* $\sigma_1=\max_{i,j}P_{i,j}$, $\sigma_2=\min_{i,j}Q_{i,j}$, $\sigma_3=\max_i P_{i,i}$ *and* $\sigma_4=\min_i P_{i,i}$. *If*

$$\max(\mathcal{N}(2\sigma_3),\mathcal{N}(2\sigma_4+n\sigma_2-2n\sigma_1))<1,$$

then $\bar{u}(t)\geq\underline{u}(t)$ *for all* $t\in I_0^N$.

Proof Let $w(t) = \bar{u}(t) - \underline{u}(t)$. We have from (2.2) that for all $i = 1, 2, \ldots, n$,

$$-\delta^2 w_i(t) + P_N\left(f_i\left(\frac{t}{N}, \bar{u}(t)\right) - f_i\left(\frac{t}{N}, \underline{u}(t)\right) - 2\sum_{j=1, j\neq i}^{n} P_{i,j} w_j(t)\right) \geq 0,$$
$$t \in I_1^{N-1},\ w(0) \geq 0,\ w(N) \geq 0.$$

Let $v(t) = \max(\underline{u}(t), \bar{u}(t))$. By (2.6),

$$\begin{aligned} & f_i\left(\frac{t}{N}, \bar{u}(t)\right) - f_i\left(\frac{t}{N}, \underline{u}(t)\right) - 2\sum_{j=1, j\neq i}^{n} P_{i,j} w_j(t) \\ & \leq \sum_{j=1}^{n} Q_{i,j}(\bar{u}_j(t) - v_j(t)) + \sum_{j=1}^{n} P_{i,j}(v_j(t) - \underline{u}_j(t)) - 2\sum_{j=1, j\neq i}^{n} P_{i,j} w_j(t) \\ & \leq \sum_{j=1}^{n} (P_{i,j} - Q_{i,j})(v_j(t) - \bar{u}_j(t)) - \sum_{j=1}^{n} P_{i,j} w_j(t) + 2P_{i,i} w_i(t). \end{aligned}$$

By introducing $w^+(t) = \max(0, w(t))$ and $w^-(t) = w(t) - w^+(t)$, we further have

$$\begin{aligned} & f_i\left(\frac{t}{N}, \bar{u}(t)\right) - f_i\left(\frac{t}{N}, \underline{u}(t)\right) - 2\sum_{j=1, j\neq i}^{n} P_{i,j} w_j(t) \\ & \leq \sum_{j=1}^{n} (Q_{i,j} - 2P_{i,j}) w_j^-(t) + 2P_{i,i} w_i(t) \\ & \leq (\sigma_2 - 2\sigma_1)\sum_{j=1}^{n} w_j^-(t) + 2P_{i,i} w_i(t). \end{aligned}$$

So we obtain

$$-\delta^2 w_i(t) + 2P_{i,i} P_N w_i(t) \geq (2\sigma_1 - \sigma_2) P_N\left(\sum_{j=1}^{n} w_j^-(t)\right),$$
$$t \in I_1^{N-1},\ w(0) \geq 0,\ w(N) \geq 0. \tag{2.7}$$

Let $W_i, G_i, W_i^-, W^- \in \mathbf{R}^{N-1}$ be defined by

$$\begin{aligned} W_i &= (w_i(1), w_i(2), \ldots, w_i(N-1))^{\mathrm{T}}, \\ G_i &= (w_i(0), 0, \ldots, 0, w_i(N))^{\mathrm{T}}, \\ W_i^- &= (w_i^-(1), w_i^-(2), \ldots, w_i^-(N-1))^{\mathrm{T}}, \quad W^- = \sum_{i=1}^{n} W_i^-. \end{aligned}$$

Then (2.7) may be written as

$$\left(A + \frac{2P_{i,i}}{N^2} B\right) W_i \geq \frac{2\sigma_1 - \sigma_2}{N^2} BW^- + \left(1 - \frac{P_{i,i}}{6N^2}\right) G_i,$$

$$w(0) \geq 0, \;\; w(N) \geq 0.$$

Since $\mathcal{N}(2\sigma_3) < 1$, we have $(1 - P_{i,i}/6N^2)G_i \geq 0$ and from Lemma 2.3,

$$\left(A + \frac{2\sigma_4}{N^2} B\right)^{-1} \geq \left(A + \frac{2P_{i,i}}{N^2} B\right)^{-1} \geq 0.$$

These facts lead to

$$W_i \geq \frac{2\sigma_1 - \sigma_2}{N^2} \left(A + \frac{2P_{i,i}}{N^2} B\right)^{-1} BW^- \geq \frac{2\sigma_1 - \sigma_2}{N^2} \left(A + \frac{2\sigma_4}{N^2} B\right)^{-1} BW^-$$

or

$$W^- \geq \frac{(2\sigma_1 - \sigma_2)n}{N^2} \left(A + \frac{2\sigma_4}{N^2} B\right)^{-1} BW^-. \tag{2.8}$$

Since $\mathcal{N}(2\sigma_4 + n\sigma_2 - 2n\sigma_1) < 1$, we have from Lemma 2.3 that the matrix

$$A + \frac{1}{N^2}(2\sigma_4 + n\sigma_2 - 2n\sigma_1)B$$

is an M-matrix. By Lemma 2.2, there exists a positive diagonal matrix E such that

$$E\left(A + \frac{1}{N^2}(2\sigma_4 + n\sigma_2 - 2n\sigma_1)B\right)E^{-1}$$

is strictly diagonally dominant. Then for $e=(1,1,\ldots,1)^{\mathrm{T}}$ we have

$$E\left(A+\frac{2\sigma_4}{N^2}B\right)E^{-1}e > \frac{2n\sigma_1-n\sigma_2}{N^2}EBE^{-1}e$$

and so

$$\left\|\frac{2n\sigma_1-n\sigma_2}{N^2}E\left(A+\frac{2\sigma_4}{N^2}B\right)^{-1}BE^{-1}\right\|_\infty < 1. \tag{2.9}$$

From (2.8),

$$EW^- \geq \frac{(2\sigma_1-\sigma_2)n}{N^2}E\left(A+\frac{2\sigma_4}{N^2}B\right)^{-1}BE^{-1}EW^-$$

or

$$\left(\nu-\frac{2n\sigma_1-n\sigma_2}{N^2}E\left(A+\frac{2\sigma_4}{N^2}B\right)^{-1}BE^{-1}\right)EW^- \geq 0.$$

By Lemma 2.1 and (2.9), the matrix

$$\nu-\frac{2n\sigma_1-n\sigma_2}{N^2}E\left(A+\frac{2\sigma_4}{N^2}B\right)^{-1}BE^{-1}$$

is monotone and so $EW^- \geq 0$. This implies $W^- \geq 0$ or $\bar{u}(t) \geq \underline{u}(t)$ for all $t \in I_0^N$. This completes the proof.

There is no definitive result for the existence of a pair of upper and lower solutions. But in practical problems, such pair can be easily constructed. We now turn to the existence of a solution of (1.1).

THEOREM 2.2 *Let $\bar{u}, \underline{u}$ be a pair of upper and lower solutions of* (1.1) *with the nonnegative matrix $P=(P_{i,j})$ such that $\bar{u}(t) \geq \underline{u}(t)$ for all $t \in I_0^N$. In addition, there exist constants M_i such that for all $i=1,2,\ldots,n$,*

$$f_i\left(\frac{t}{N},u(t)\right)-f_i\left(\frac{t}{N},\tilde{u}(t)\right) \leq M_i(u_i(t)-\tilde{u}_i(t)), \quad t \in I_0^N, \tag{2.10}$$

whenever $\underline{u}(t) \leq \tilde{u}(t) \leq u(t) \leq \bar{u}(t)$ *and* $u_j(t) = \tilde{u}_j(t)$, $j \neq i$. *If* $\max_i \mathcal{N}(M_i) < 1$, *then the problem* (1.1) *has at least one solution* $u^* \in [\underline{u}, \bar{u}]$.

Proof We consider the following uncoupled problem:

$$-\delta^2 v_i(t) + M_i P_N v_i(t) = M_i P_N u_i(t) - P_N f_i\left(\frac{t}{N}, u(t)\right),$$
$$t \in I_1^{N-1},\ v(0) = \alpha,\ v(N) = \beta,\ i = 1, 2, \ldots, n. \tag{2.11}$$

Since $\max_i \mathcal{N}(M_i) < 1$, Lemma 2.4 ensures that (2.11) has the unique solution $v(t) = (v_1(t), \ldots, v_n(t))^{\mathrm{T}}$. Now, we define the map $\mathcal{T} : [\underline{u}, \bar{u}] \to \mathcal{S}$ as

$$\mathcal{T}u(t) = v(t), \quad \forall u \in [\underline{u}, \bar{u}],\ t \in I_0^N. \tag{2.12}$$

We first show $v \in [\underline{u}, \bar{u}]$. Let $w(t) = v(t) - \underline{u}(t)$. It follows from (2.11), (2.2), (2.1) and (2.10) that for all $i = 1, 2, \ldots, n$,

$$\begin{aligned}
&-\delta^2 w_i(t) + M_i P_N w_i(t) \\
&\geq P_N\Bigg(M_i(u_i(t) - \underline{u}_i(t)) + \sum_{j=1, j\neq i}^{n} P_{i,j}(\bar{u}_j(t) - \underline{u}_j(t)) \\
&\qquad - f_i\left(\frac{t}{N}, u(t)\right) + f_i\left(\frac{t}{N}, \underline{u}(t)\right)\Bigg) \\
&\geq P_N\left(\sum_{j=1, j\neq i}^{n} P_{i,j}(\bar{u}_j(t) - \underline{u}_j(t))\right) \geq 0.
\end{aligned}$$

By Lemma 2.4, we have $w_i(t) \geq 0$ for all $i = 1, 2, \ldots, n$ and so $v(t) \geq \underline{u}(t)$ for all $t \in I_0^N$. Similarly $v(t) \leq \bar{u}(t)$ for all $t \in I_0^N$. In view of the continuity of f, it is clear that $\mathcal{T}$ is a bounded continuous map from $[\underline{u}, \bar{u}]$ into itself. Since $[\underline{u}, \bar{u}]$ is a finite dimensional space, $\mathcal{T}$ is a completely continuous map. Thus by Schauder's fixed point theorem, $\mathcal{T}$ has at least one fixed point $u^* \in [\underline{u}, \bar{u}]$. Obviously, this is a solution of (1.1) in $[\underline{u}, \bar{u}]$.

3. A MONOTONE ITERATIVE SCHEME

So far, we have shown that if (1.1) possesses a pair of upper and lower solutions $\bar{u}(t)$ and $\underline{u}(t)$ such that $\bar{u}(t) \geq \underline{u}(t)$ for all $t \in I_0^N$, then it has at least one solution. Moreover the upper and lower solutions may serve as the upper bound and the lower bound for the solution. In this section, we develop a monotone iterative scheme which yields monotone sequences improving the bounds. Besides, under certain additional conditions, the sequences of the upper and the lower bounds converge to the unique solution in some sector defined by the upper and lower solutions. For this, only low regularity conditions are imposed on f.

Let $\bar{u}, \underline{u}$ be a pair of upper and lower solutions of (1.1) with the nonnegative matrix $P=(P_{i,j})$. We consider the following iterative scheme:

$$\begin{aligned}
&\bar{u}^{(0)}(t) = \bar{u}(t), \quad \underline{u}^{(0)}(t) = \underline{u}(t), \qquad t \in I_0^N,\\
&g_i^{(m)}(t) = \sum_{j=1}^{i-1} P_{i,j}\Big(\bar{u}_j^{(m)}(t) - \underline{u}_j^{(m)}(t)\Big) + \sum_{j=i+1}^{n} P_{i,j}\Big(\bar{u}_j^{(m-1)}(t) - \underline{u}_j^{(m-1)}(t)\Big),\\
&-\delta^2 \bar{u}_i^{(m)}(t) + M_i^* P_N \bar{u}_i^{(m)}(t)\\
&\quad = P_N\Big(M_i^* \bar{u}_i^{(m-1)}(t) - f_i\Big(\frac{t}{N}, \bar{u}_1^{(m)}(t), \ldots, \bar{u}_{i-1}^{(m)}(t),\\
&\qquad \bar{u}_i^{(m-1)}(t), \ldots, \bar{u}_n^{(m-1)}(t)\Big) + g_i^{(m)}(t)\Big),\\
&-\delta^2 \underline{u}_i^{(m)}(t) + M_i^* P_N \underline{u}_i^{(m)}(t)\\
&\quad = P_N\Big(M_i^* \underline{u}_i^{(m-1)}(t) - f_i\Big(\frac{t}{N}, \underline{u}_1^{(m)}(t), \ldots, \underline{u}_{i-1}^{(m)}(t),\\
&\qquad \underline{u}_i^{(m-1)}(t), \ldots, \underline{u}_n^{(m-1)}(t)\Big) - g_i^{(m)}(t)\Big),\\
&\qquad i = 1, 2, \ldots, n, \quad t \in I_1^{N-1},\\
&\qquad \bar{u}^{(m)}(0) = \underline{u}^{(m)}(0) = \alpha, \quad \bar{u}^{(m)}(N) = \underline{u}^{(m)}(N) = \beta,
\end{aligned} \tag{3.1}$$

where M_i^* are some constants specified later. By Lemma 2.4, the above iteration (3.1) is well defined provided $\max_i \mathcal{N}(M_i^*) < 1$.

Remark 3.1 If $n=1$, the iteration (3.1) is reduced to that established in [13].

THEOREM 3.1 *Let $\bar{u}, \underline{u}$ be a pair of upper and lower solutions of* (1.1) *with the nonnegative matrix $P=(P_{i,j})$ such that $\bar{u}(t) \geq \underline{u}(t)$ for all $t \in I_0^N$. In addition, there exist constants M_i such that for all $i=1,2,\ldots,n$,*

$$f_i\left(\frac{t}{N}, u(t)\right) - f_i\left(\frac{t}{N}, \tilde{u}(t)\right) \leq M_i(u_i(t) - \tilde{u}_i(t)), \quad t \in I_0^N, \tag{3.2}$$

whenever $\underline{u}(t) \leq \tilde{u}(t) \leq u(t) \leq \bar{u}(t)$ *and* $\tilde{u}_j(t) = u_j(t)$, $j \neq i$. *If* $\max_i \mathcal{N}(M_i) < 1$, *then the sequences $\{\bar{u}^{(m)}(t)\}$ and $\{\underline{u}^{(m)}(t)\}$ defined by* (3.1) *with $M_i^* = M_i$ for all $i=1,2,\ldots,n$, converge to the limits $\bar{u}^*(t)$ and $\underline{u}^*(t)$, respectively. Moreover,*

$$\begin{aligned} \underline{u}(t) &\leq \underline{u}^{(m)}(t) \leq \underline{u}^{(m+1)}(t) \leq \underline{u}^*(t) \leq \bar{u}^*(t) \\ &\leq \bar{u}^{(m+1)}(t) \leq \bar{u}^{(m)}(t) \leq \bar{u}(t), \quad t \in I_0^N. \end{aligned} \tag{3.3}$$

Besides, for any solution u^* of (1.1) *in* $[\underline{u}, \bar{u}]$, *we have* $u^* \in [\underline{u}^*, \bar{u}^*]$.

Proof We use induction to assert that for all $m=0,1,2,\ldots,$

$$\underline{u}(t) \leq \underline{u}^{(m)}(t) \leq \underline{u}^{(m+1)}(t) \leq \bar{u}^{(m+1)}(t) \leq \bar{u}^{(m)}(t) \leq \bar{u}(t), \quad t \in I_0^N. \tag{3.4}$$

Firstly, we have from (3.1), (2.2), (2.1) and (3.2) that

$$\begin{aligned} &-\delta^2(\underline{u}_1^{(1)}(t) - \underline{u}_1^{(0)}(t)) + M_1 P_N(\underline{u}_1^{(1)}(t) - \underline{u}_1^{(0)}(t)) \\ &\quad = \delta^2 \underline{u}_1^{(0)}(t) - P_N\left(f_1\left(\frac{t}{N}, \underline{u}^{(0)}(t)\right) + \sum_{j=1, j\neq i}^{n} P_{i,j}(\bar{u}_j^{(0)}(t) - \underline{u}_j^{(0)}(t))\right) \geq 0, \\ &-\delta^2(\bar{u}_1^{(0)}(t) - \bar{u}_1^{(1)}(t)) + M_1 P_N(\bar{u}_1^{(0)}(t) - \bar{u}_1^{(1)}(t)) \\ &\quad = -\delta^2 \bar{u}_1^{(0)}(t) + P_N\left(f_1\left(\frac{t}{N}, \bar{u}^{(0)}(t)\right) + \sum_{j=1, j\neq i}^{n} P_{i,j}(\bar{u}_j^{(0)}(t) - \underline{u}_j^{(0)}(t))\right) \geq 0, \\ &-\delta^2(\bar{u}_1^{(1)}(t) - \underline{u}_1^{(1)}(t)) + M_1 P_N(\bar{u}_1^{(1)}(t) - \underline{u}_1^{(1)}(t)) \\ &\quad = P_N\Big(M_1(\bar{u}_1^{(0)}(t) - \underline{u}_1^{(0)}(t)) + f_1\left(\frac{t}{N}, \underline{u}^{(0)}(t)\right) \\ &\qquad - f_1\left(\frac{t}{N}, \bar{u}^{(0)}(t)\right) + 2g_1^{(1)}(t)\Big) \geq 0. \end{aligned}$$

By Lemma 2.4, the above estimates imply that $\underline{u}_1^{(0)}(t) \leq \underline{u}_1^{(1)}(t) \leq \bar{u}_1^{(1)}(t) \leq \bar{u}_1^{(0)}(t)$ for all $t \in I_0^N$. For convenience, let

$$\tilde{f}_i\Big(\frac{t}{N}, u(t)\Big) = f_i\Big(\frac{t}{N}, u(t)\Big) - \sum_{j=1, j\neq i}^{n} P_{i,j}\, u_j(t). \tag{3.5}$$

Then we know from (2.1) that $\tilde{f}_i$ is quasimonotone nonincreasing in $[\underline{u}, \bar{u}]$. By (3.1) and (2.2),

$$\begin{aligned}
&-\delta^2\Big(\underline{u}_i^{(1)}(t) - \underline{u}_i^{(0)}(t)\Big) + M_i P_N\Big(\underline{u}_i^{(1)}(t) - \underline{u}_i^{(0)}(t)\Big) \\
&\quad = \delta^2 \underline{u}_i^{(0)}(t) - P_N\Big(f_i\Big(\frac{t}{N}, \underline{u}_1^{(1)}(t), \ldots, \underline{u}_{i-1}^{(1)}(t), \\
&\qquad\qquad \underline{u}_i^{(0)}(t), \ldots, \underline{u}_n^{(0)}(t)\Big) + g_i^{(1)}(t)\Big) \\
&\quad \geq P_N\Big(\tilde{f}_i\Big(\frac{t}{N}, \underline{u}^{(0)}(t)\Big) - \tilde{f}_i\Big(\frac{t}{N}, \underline{u}_1^{(1)}(t), \ldots, \underline{u}_{i-1}^{(1)}(t), \underline{u}_i^{(0)}(t), \ldots, \underline{u}_n^{(0)}(t)\Big)\Big) \\
&\qquad + P_N\left(\sum_{j=1}^{i-1} P_{i,j}\Big(\bar{u}_j^{(0)}(t) - \bar{u}_j^{(1)}(t)\Big)\right), \\
&-\delta^2\Big(\bar{u}_i^{(0)}(t) - \bar{u}_i^{(1)}(t)\Big) + M_i P_N\Big(\bar{u}_i^{(0)}(t) - \bar{u}_i^{(1)}(t)\Big) \\
&\quad = -\delta^2 \bar{u}_i^{(0)}(t) + P_N\Big(f_i\Big(\frac{t}{N}, \bar{u}_1^{(1)}(t), \ldots, \bar{u}_{i-1}^{(1)}(t), \\
&\qquad\qquad \bar{u}_i^{(0)}(t), \ldots, \bar{u}_n^{(0)}(t)\Big) - g_i^{(1)}(t)\Big) \\
&\quad \geq P_N\Big(\tilde{f}_i\Big(\frac{t}{N}, \bar{u}_1^{(1)}(t), \ldots, \bar{u}_{i-1}^{(1)}(t), \bar{u}_i^{(0)}(t), \ldots, \bar{u}_n^{(0)}(t)\Big) - \tilde{f}_i\Big(\frac{t}{N}, \bar{u}^{(0)}(t)\Big)\Big) \\
&\qquad + P_N\left(\sum_{j=1}^{i-1} P_{i,j}\Big(\underline{u}_j^{(1)}(t) - \underline{u}_j^{(0)}(t)\Big)\right), \\
&-\delta^2\Big(\bar{u}_i^{(1)}(t) - \underline{u}_i^{(1)}(t)\Big) + M_i P_N\Big(\bar{u}_i^{(1)}(t) - \underline{u}_i^{(1)}(t)\Big) \\
&\quad = P_N\Big(M_i\Big(\bar{u}_i^{(0)}(t) - \underline{u}_i^{(0)}(t)\Big) \\
&\qquad\qquad - \tilde{f}_i\Big(\frac{t}{N}, \bar{u}_1^{(1)}(t), \ldots, \bar{u}_{i-1}^{(1)}(t), \bar{u}_i^{(0)}(t), \ldots, \bar{u}_n^{(0)}(t)\Big)\Big) \\
&\qquad + P_N\Big(\tilde{f}_i\Big(\frac{t}{N}, \underline{u}_1^{(1)}(t), \ldots, \underline{u}_{i-1}^{(1)}(t), \underline{u}_i^{(0)}(t), \ldots, \underline{u}_n^{(0)}(t)\Big) + g_i^{(1)}(t)\Big).
\end{aligned}$$

By the quasimonotonicity of $\tilde{f}_i$, (3.2) and Lemma 2.4, an induction argument for i shows that

$$\underline{u}^{(0)}(t) \leq \underline{u}^{(1)}(t) \leq \bar{u}^{(1)}(t) \leq \bar{u}^{(0)}(t), \quad t \in I_0^N.$$

Assume that (3.4) holds for some $m \geq 0$. We have from (3.1) that for all $i = 1, 2, \ldots, n$,

$$
\begin{aligned}
&-\delta^2(\underline{u}_i^{(m+2)}(t) - \underline{u}_i^{(m+1)}(t)) + M_i P_N(\underline{u}_i^{(m+2)}(t) - \underline{u}_i^{(m+1)}(t)) \\
&\quad = P_N(M_i(\underline{u}_i^{(m+1)}(t) - \underline{u}_i^{(m)}(t))) \\
&\qquad + P_N \tilde{f}_i\Big(\frac{t}{N}, \underline{u}_1^{(m+1)}(t), \ldots, \underline{u}_{i-1}^{(m+1)}(t), \underline{u}_i^{(m)}(t), \ldots, \underline{u}_n^{(m)}(t)\Big) \\
&\qquad - P_N \tilde{f}_i\Big(\frac{t}{N}, \underline{u}_1^{(m+2)}(t), \ldots, \underline{u}_{i-1}^{(m+2)}(t), \underline{u}_i^{(m+1)}(t), \ldots, \underline{u}_n^{(m+1)}(t)\Big) \\
&\qquad + P_N\Bigg(\sum_{j=1}^{i-1} P_{i,j}(\bar{u}_j^{(m+1)}(t) - \bar{u}_j^{(m+2)}(t)) \\
&\qquad\qquad + \sum_{j=i+1}^{n} P_{i,j}(\bar{u}_j^{(m)}(t) - \bar{u}_j^{(m+1)}(t))\Bigg), \\
&-\delta^2(\bar{u}_i^{(m+1)}(t) - \bar{u}_i^{(m+2)}(t)) + M_i P_N(\bar{u}_i^{(m+1)}(t) - \bar{u}_i^{(m+2)}(t)) \\
&\quad = P_N(M_i(\bar{u}_i^{(m)}(t) - \bar{u}_i^{(m+1)}(t))) \\
&\qquad + P_N \tilde{f}_i\Big(\frac{t}{N}, \bar{u}_1^{(m+2)}(t), \ldots, \bar{u}_{i-1}^{(m+2)}(t), \bar{u}_i^{(m+1)}(t), \ldots, \bar{u}_n^{(m+1)}(t)\Big) \\
&\qquad - P_N \tilde{f}_i\Big(\frac{t}{N}, \bar{u}_1^{(m+1)}(t), \ldots, \bar{u}_{i-1}^{(m+1)}(t), \bar{u}_i^{(m)}(t), \ldots, \bar{u}_n^{(m)}(t)\Big) \\
&\qquad + P_N\Bigg(\sum_{j=1}^{i-1} P_{i,j}(\underline{u}_j^{(m+2)}(t) - \underline{u}_j^{(m+1)}(t)) \\
&\qquad\qquad + \sum_{j=i+1}^{n} P_{i,j}(\underline{u}_j^{(m+1)}(t) - \underline{u}_j^{(m)}(t))\Bigg), \\
&-\delta^2(\bar{u}_i^{(m+2)}(t) - \underline{u}_i^{(m+2)}(t)) + M_i P_N(\bar{u}_i^{(m+2)}(t) - \underline{u}_i^{(m+2)}(t)) \\
&\quad = P_N(M_i(\bar{u}_i^{(m+1)}(t) - \underline{u}_i^{(m+1)}(t))) \\
&\qquad + P_N \tilde{f}_i\Big(\frac{t}{N}, \underline{u}_1^{(m+2)}(t), \ldots, \underline{u}_{i-1}^{(m+2)}(t), \underline{u}_i^{(m+1)}(t), \ldots, \underline{u}_n^{(m+1)}(t)\Big) \\
&\qquad - P_N\Big(\tilde{f}_i\Big(\frac{t}{N}, \bar{u}_1^{(m+2)}(t), \ldots, \bar{u}_{i-1}^{(m+2)}(t), \bar{u}_i^{(m+1)}(t), \ldots, \bar{u}_n^{(m+1)}(t)\Big) \\
&\qquad\qquad + g_i^{(m+2)}(t)\Big).
\end{aligned}
$$

Again by the quasimonotonicity of $\tilde{f}_i$, (3.2) and Lemma 2.4, an induction argument for i show that

$$\underline{u}(t) \leq \underline{u}^{(m+1)}(t) \leq \underline{u}^{(m+2)}(t) \leq \bar{u}^{(m+2)}(t) \leq \bar{u}^{(m+1)}(t) \leq \bar{u}(t), \quad t \in I_0^N,$$

and so (3.4) holds for $m+1$. The induction for (3.4) is completed.

In view of (3.4), there exist limits $\bar{u}^*(t)$ and $\underline{u}^*(t)$ such that

$$\lim_{m\to\infty} \bar{u}^{(m)}(t) = \bar{u}^*(t), \quad \lim_{m\to\infty} \underline{u}^{(m)}(t) = \underline{u}^*(t)$$

and (3.3) holds.

Now, let $u^*(t)$ be any possible solution of (1.1) in $[\underline{u}, \bar{u}]$. Suppose that $u^* \in [\underline{u}^{(m)}, \bar{u}^{(m)}]$ for some $m \geq 0$. By the similar argument as that for (3.4), we get

$$\underline{u}^{(m+1)}(t) \leq u^*(t) \leq \bar{u}^{(m+1)}(t), \quad t \in I_0^N.$$

This proves

$$\underline{u}^{(m)}(t) \leq u^*(t) \leq \bar{u}^{(m)}(t), \quad t \in I_0^N, \ m = 0, 1, 2, \ldots .$$

Letting $m \to \infty$, we see that $u^* \in [\underline{u}^*, \bar{u}^*]$. This completes the proof.

If $P_{i,j} = 0, j \neq i$ which implies that f_i is quasimonotone nonincreasing in $[\underline{u}, \bar{u}]$, then limits $\underline{u}^*(t)$ and $\bar{u}^*(t)$ in Theorem 3.1 are the maximal and minimal solutions of (1.1) in $[\underline{u}, \bar{u}]$, respectively. Here, the maximal and minimal property of the solutions $\bar{u}^*(t)$ and $\underline{u}^*(t)$ is in the sense that if $u^*(t)$ is a solution of (1.1) in $[\underline{u}, \bar{u}]$, then $u^* \in [\underline{u}^*, \bar{u}^*]$. In the general case, if the limits $\underline{u}^*(t)$ and $\bar{u}^*(t)$ coincide, then their common value is the unique solution of (1.1) in $[\underline{u}, \bar{u}]$.

THEOREM 3.2 *Assume that the hypothesis in Theorem* 3.1 *hold, and let* $\bar{u}^*(t)$ *and* $\underline{u}^*(t)$ *be the limits obtained from the corresponding monotone sequences. Besides, there exists a matrix* $Q = (Q_{i,j})$ *such that for all* $i = 1, 2, \ldots, n$,

$$f_i\left(\frac{t}{N}, u(t)\right) - f_i\left(\frac{t}{N}, \tilde{u}(t)\right) \geq \sum_{j=1}^{n} Q_{i,j}(u_j(t) - \tilde{u}_j(t)), \quad t \in I_0^N, \quad (3.6)$$

whenever $\underline{u}(t) \le \tilde{u}(t) \le u(t) \le \bar{u}(t)$. *Set* $\gamma_1 = \max_{i \ne j} P_{i,j}$ *and* $\gamma_2 = \min_{i \ne j} Q_{i,j}$. *If*

$$\max_i(\mathcal{N}(M_i), \mathcal{N}(n\gamma_2 - 2(n-1)\gamma_1)) < 1,$$

then for all $t \in I_0^N$, $\underline{u}^*(t) = \bar{u}^*(t)$ *and is the unique solution of* (1.1) *in* $[\underline{u}, \bar{u}]$.

Proof Let $w^*(t) = \bar{u}^*(t) - \underline{u}^*(t)$. Obviously, $w^*(t) \ge 0$ for all $t \in I_0^N$ and

$$\begin{aligned}
-\delta^2 w_i^*(t) &= P_N\left(f_i\left(\frac{t}{N}, \underline{u}^*(t)\right) - f_i\left(\frac{t}{N}, \bar{u}^*(t)\right)\right) + 2P_N\left(\sum_{j=1, j\ne i}^{n} P_{i,j} w_j^*(t)\right) \\
&\le P_N\left(\sum_{j=1}^{n} Q_{i,j}(\underline{u}_j^*(t) - \bar{u}_j^*(t))\right) + 2P_N\left(\sum_{j=1, j\ne i}^{n} P_{i,j} w_j^*(t)\right) \\
&\le -\gamma_2 P_N\left(\sum_{j=1}^{n} w_j^*(t)\right) + 2\gamma_1 P_N\left(\sum_{j=1, j\ne i}^{n} w_j^*(t)\right).
\end{aligned}$$

Summing the above result over all i, we have

$$-\delta^2\left(\sum_{i=1}^{n} w_i^*(t)\right) \le -n\gamma_2 P_N\left(\sum_{j=1}^{n} w_j^*(t)\right) + 2(n-1)\gamma_1 P_N\left(\sum_{j=1}^{n} w_j^*(t)\right)$$

or

$$-\delta^2\left(\sum_{i=1}^{n} w_i^*(t)\right) + (n\gamma_2 - 2(n-1)\gamma_1) P_N\left(\sum_{i=1}^{n} w_i^*(t)\right) \le 0.$$

By the boundary conditions and Lemma 2.4, we get $\sum_{i=1}^{n} w_i^*(t) \le 0$ which leads to $w^*(t) = 0$ for all $t \in I_0^N$. This proves $\underline{u}^*(t) = \bar{u}^*(t)$ for all $t \in I_0^N$.

THEOREM 3.3 *Assume that the hypothesis in Theorem* 3.2 *hold. If* $\max_i \mathcal{N}(M_i) < 1$ *and* $2(n-1)\gamma_1 - n\gamma_2^- < 1$ *where* $\gamma_2^- = \min(0, \gamma_2)$, *then for all* $t \in I_0^N$, $\underline{u}^*(t) = \bar{u}^*(t)$ *and is the unique solution of* (1.1) *in* $[\underline{u}, \bar{u}]$.

Proof Let $w^*(t) = \bar{u}^*(t) - \underline{u}^*(t)$. Then $w^*(0) = w^*(N) = 0$ and $w^*(t) \geq 0$ for all $t \in I_1^{N-1}$. Moreover by the proof of Theorem 3.2,

$$-\delta^2 w_i^*(t) \leq -\gamma_2 P_N\left(\sum_{j=1}^{n} w_j^*(t)\right) + 2\gamma_1 P_N\left(\sum_{j=1,j\neq i}^{n} w_j^*(t)\right),$$
$$i = 1, 2, \ldots, n. \qquad (3.7)$$

Let

$$|W^*|_1^2 = \max_i \sum_{t\in I_1^N} (w_i^*(t) - w_i^*(t-1))^2, \qquad \|W^*\|^2 = \max_i \sum_{t\in I_1^{N-1}} w_i^{*2}(t).$$

We have

$$\begin{aligned}\|W^*\|^2 &= \max_i \sum_{t\in I_1^{N-1}} w_i^{*2}(t) = \max_i \sum_{t\in I_1^N}\left(\sum_{\tilde{t}\in I_1^t}(w_i^*(\tilde{t}) - w_i^*(\tilde{t}-1))\right)^2 \\ &\leq \max_i \sum_{t\in I_1^N}\left(N\sum_{\tilde{t}\in I_1^N}(w_i^*(\tilde{t}) - w_i^*(\tilde{t}-1))^2\right) \\ &= N^2 |W^*|_1^2.\end{aligned}$$

Now, multiplying the inequality (3.7) by $w_i^*(t)$ and summing the result over all t, we obtain

$$\begin{aligned}&-\sum_{t\in I_1^{N-1}} (\delta^2 w_i^*(t)) w_i^*(t) \\ &\leq -\gamma_2^- \sum_{j=1}^{n}\left(\frac{1}{12N^2}\sum_{t\in I_1^{N-1}} (w_j^*(t-1) + 10 w_j^*(t) + w_j^*(t+1)) w_i^*(t)\right) \\ &\quad + 2\gamma_1 \sum_{j=1,j\neq i}^{n}\left(\frac{1}{12N^2}\sum_{t\in I_1^{N-1}} (w_j^*(t-1) + 10 w_j^*(t) + w_j^*(t+1)) w_i^*(t)\right) \\ &\leq \frac{-\gamma_2^-}{2N^2}\sum_{j=1}^{n}\left(\sum_{t\in I_1^{N-1}} (w_j^{*2}(t) + w_i^{*2}(t))\right) \\ &\quad + \frac{\gamma_1}{N^2}\sum_{j=1,j\neq i}^{n}\left(\sum_{t\in I_1^{N-1}} (w_j^{*2}(t) + w_i^{*2}(t))\right) \\ &\leq \frac{2\gamma_1(n-1) - n\gamma_2^-}{N^2}\|W^*\|^2 \leq (2\gamma_1(n-1) - n\gamma_2^-)|W^*|_1^2.\end{aligned}$$

On the other hand,

$$-\sum_{t\in I_1^{N-1}}(\delta^2 w_i^*(t))w_i^*(t)=\sum_{t\in I_1^N}(w_i^*(t)-w_i^*(t-1))^2.$$

Finally, we get

$$\sum_{t\in I_1^N}(w_i^*(t)-w_i^*(t-1))^2\leq(2\gamma_1(n-1)-n\gamma_2^-)|W^*|_1^2$$

or

$$|W^*|_1^2\leq(2\gamma_1(n-1)-n\gamma_2^-)|W^*|_1^2$$

from which, and the boundary conditions, the conclusion follows.

Remark 3.2 In the determination of the monotone sequence from the iteration (3.1) it is only needed to solve an uncoupled linear two-point discrete boundary value problem in each iteration. We may use any one method established in [9], which is efficient in constructing the solutions of such problems.

Remark 3.3 The crucial point for ensuring the monotone convergence of the iteration (3.1) is to find a pair of vector functions $\bar{u},\underline{u}\in\mathcal{S}$ such that $\bar{u}(t)\geq\underline{u}(t)$ for all $t\in I_0^N$ as well as (2.1) and (2.2) hold. In the final section, we give an example where such a pair of $\bar{u}(t)$ and $\underline{u}(t)$ can be easily constructed.

4. THE CONVERGENCE RATE OF THE ITERATION

In this section, we compare and analyze the convergence rate of the iteration (3.1). We begin with the following comparison result.

THEOREM 4.1 *Let $\bar{u}$, $\underline{u}$ be a pair of upper and lower solutions of the problem (1.1) with the nonnegative matrix $P=(P_{i,j})$ such that $\bar{u}(t)\geq\underline{u}(t)$ for all $t\in I_0^N$. In addition, there exist constants M_i such that* *for all* $i=1,2,\ldots,n$,

$$f_i\left(\frac{t}{N},u(t)\right)-f_i\left(\frac{t}{N},\tilde{u}(t)\right)\leq M_i(u_i(t)-\tilde{u}_i(t)),\quad t\in I_0^N, \tag{4.1}$$

whenever $\underline{u}(t)\leq\tilde{u}(t)\leq u(t)\leq\bar{u}(t)$ *and* $\tilde{u}_j(t)=u_j(t)$, $j\neq i$. *Set* $\bar{M}=\max_i M_i$. *Let* $\{\bar{u}^{(m)}(t)\}$ *and* $\{\underline{u}^{(m)}(t)\}$ *denote the sequences from the iteration* (3.1) *with* $M_i^*=\bar{M}$ *for all* $i=1,2,\ldots,n$. *Also, let* $\{\bar{u}'^{(m)}(t)\}$ *and* $\{\underline{u}'^{(m)}(t)\}$ *denote the sequences from the iteration* (3.1) *with* $M_i^*=M_i$ *for all* $i=1,2,\ldots,n$. *If* $\max_i\mathcal{N}(M_i)<1$, *then all above sequences have the monotone convergence described in Theorem* 3.1 *and*

$$\bar{u}^{(m)}(t)\geq\bar{u}'^{(m)}(t),\quad \underline{u}^{(m)}(t)\leq\underline{u}'^{(m)}(t),\quad t\in I_0^N,\qquad m=0,1,2,\ldots. \tag{4.2}$$

Proof We only prove (4.2). From Theorem 3.1, we have that for all $t\in I_0^N$ and $m=0,1,2,\ldots$,

$$\underline{u}(t)\leq\underline{u}^{(m)}(t)\leq\underline{u}^{(m+1)}(t)\leq\bar{u}^{(m+1)}(t)\leq\bar{u}^{(m)}(t)\leq\bar{u}(t),$$

$$\underline{u}(t)\leq\underline{u}'^{(m)}(t)\leq\underline{u}'^{(m+1)}(t)\leq\bar{u}'^{(m+1)}(t)\leq\bar{u}'^{(m)}(t)\leq\bar{u}(t).$$

Clearly (4.2) is true from $m=0$. Suppose that (4.2) holds for some $m\geq 0$. Let $\bar{w}^{(m+1)}(t)=\bar{u}^{(m+1)}(t)-\bar{u}'^{(m+1)}(t)$ and $\underline{w}^{(m+1)}(t)=\underline{u}'^{(m+1)}(t)-\underline{u}^{(m+1)}(t)$. Then we have $\bar{w}^{(m+1)}(t)=\underline{w}^{(m+1)}(t)=0$ for $t=0,N$ and for all $t\in I_1^{N-1}$ and $i=1,2,\ldots,n$,

$$\begin{aligned}
&-\delta^2\bar{w}_i^{(m+1)}(t)+\bar{M}P_N\bar{w}_i^{(m+1)}(t)\\
&=P_N\Big(\bar{M}(\bar{u}_i^{(m)}(t)-\bar{u}_i'^{(m)}(t))\\
&\qquad+\tilde{f}_i\Big(\frac{t}{N},\bar{u}_1'^{(m+1)}(t),\ldots,\bar{u}_{i-1}'^{(m+1)}(t),\bar{u}_i'^{(m)}(t),\ldots,\bar{u}_n'^{(m)}(t)\Big)\Big)\\
&\quad-P_N\tilde{f}_i\Big(\frac{t}{N},\bar{u}_1^{(m+1)}(t),\ldots,\bar{u}_{i-1}^{(m+1)}(t),\bar{u}_i^{(m)}(t),\ldots,\bar{u}_n^{(m)}(t)\Big)\\
&\quad+P_N\left(\sum_{j=1}^{i-1}P_{i,j}(\underline{u}_j'^{(m+1)}(t)-\underline{u}_j^{(m+1)}(t))\right.\\
&\qquad\left.+\sum_{j=i+1}^{n}P_{i,j}(\underline{u}_j'^{(m)}(t)-\underline{u}_j^{(m)}(t))\right)\\
&\quad+(\bar{M}-M_i)P_N(\bar{u}_i'^{(m)}(t)-\bar{u}_i'^{(m+1)}(t)),
\end{aligned}$$

$$
\begin{aligned}
&-\delta^2 \underline{w}_i^{(m+1)}(t) + M_i P_N \underline{w}_i^{(m+1)}(t) \\
&\quad = P_N\Big(M_i(\underline{u}_i^{\prime(m)}(t) - \underline{u}_i^{(m)}(t)) \\
&\qquad + \tilde{f}_i\Big(\frac{t}{N}, \underline{u}_1^{(m+1)}(t), \dots, \underline{u}_{i-1}^{(m+1)}(t), \underline{u}_i^{(m)}(t), \dots, \underline{u}_n^{(m)}(t)\Big)\Big) \\
&\quad - P_N \tilde{f}_i\Big(\frac{t}{N}, \underline{u}_1^{\prime(m+1)}(t), \dots, \underline{u}_{i-1}^{\prime(m+1)}(t), \underline{u}_i^{\prime(m)}(t), \dots, \underline{u}_n^{\prime(m)}(t)\Big) \\
&\quad + P_N\Bigg(\sum_{j=1}^{i-1} P_{i,j}(\bar{u}_j^{(m+1)}(t) - \bar{u}_j^{\prime(m+1)}(t)) \\
&\qquad + \sum_{j=i+1}^{n} P_{i,j}(\bar{u}_j^{(m)}(t) - \bar{u}_j^{\prime(m)}(t))\Bigg) \\
&\quad + (M_i - \bar{M}) P_N(\underline{u}_i^{(m)}(t) - \underline{u}_i^{(m+1)}(t)),
\end{aligned}
$$

where as before, $\tilde{f}_i$ is defined by (3.5). By the quasimonotonicity of $\tilde{f}_i$ and (4.1), we use induction for i and Lemma 2.4 to get

$$
\bar{u}^{(m+1)}(t) \geq \bar{u}^{\prime(m+1)}(t), \quad \underline{u}^{(m+1)}(t) \leq \underline{u}^{\prime(m+1)}(t), \qquad t \in I_0^N.
$$

This shows that (4.2) holds for $m+1$. The induction for (4.2) is completed.

Next, we estimate the convergence rate of the iteration (3.1).

THEOREM 4.2 *Assume that all conditions of Theorems* 3.1 *and* 3.2 *hold. Let* $\{\bar{u}^{(m)}(t)\}$ *and* $\{\underline{u}^{(m)}(t)\}$ *be the sequences given in Theorem* 3.1, *and* $\bar{u}^*(t)$ *and* $\underline{u}^{(*)}(t)$ *be their limits respectively. Set* $\bar{M} = \max_i M_i$. *Then there exists a positive diagonal matrix* $E = \mathrm{diag}(E_1, \dots, E_{N-1})$ *such that for all* $t \in I_0^N$ *and* $m = 0, 1, 2, \dots,$

$$
\begin{aligned}
&\sum_{i=1}^{n}(|\bar{u}_i^{(m)}(t) - \bar{u}_i^*(t)| + |\underline{u}_i^{(m)}(t) - \underline{u}_i^*(t)|) \\
&\quad \leq \frac{\max_i E_i}{\min_i E_i} \rho^m \Bigg(\max_{t \in I_1^{N-1}} \sum_{i=1}^{n}(|\bar{u}_i^{(0)}(t) - \bar{u}_i^*(t)| + |\underline{u}_i^{(0)}(t) - \underline{u}_i^*(t)|) \\
&\qquad + \frac{1}{10} \max_{t=0,N} \sum_{i=1}^{n}(|\bar{u}_i^{(0)}(t) - \bar{u}_i^*(t)| + |\underline{u}_i^{(0)}(t) - \underline{u}_i^*(t)|)\Bigg)
\end{aligned}
$$

where

$$\rho = \left\| \frac{\bar{M} + 2(n-1)\gamma_1 - n\gamma_2}{N^2} E\left(A + \frac{\bar{M}}{N^2} B\right)^{-1} BE^{-1} \right\|_{\infty} < 1.$$

Proof By the monotone convergence of the sequences, the comparison result in Theorem 4.1 and the uniqueness of the solutions, we only need to consider the case of $M_i^* = \bar{M}$ for all $i = 1, 2, \ldots, n$. Let $\bar{w}^{(m)}(t) = \bar{u}^{(m)}(t) - \bar{u}^*(t)$ and $\underline{w}^{(m)}(t) = \underline{u}^*(t) - \underline{u}^{(m)}(t)$. Then $\bar{w}^{(m)}(t) \geq 0$ and $\underline{w}^{(m)}(t) \geq 0$ for all $t \in I_0^N$ and $m = 0, 1, 2, \ldots$ Moreover for all $i = 1, 2, \ldots, n$, $t \in I_1^N$ and $m \geq 1$,

$$\begin{aligned}
&-\delta^2 \bar{w}_i^{(m)}(t) + \bar{M} P_N \bar{w}_i^{(m)}(t) \\
&\quad = P_N\Big(\bar{M}\bar{w}_i^{(m-1)}(t) - \tilde{f}_i\Big(\frac{t}{N}, \bar{u}_1^{(m)}(t), \ldots, \bar{u}_{i-1}^{(m)}(t), \\
&\qquad \bar{u}_i^{(m-1)}(t), \ldots, \bar{u}_n^{(m-1)}(t)\Big)\Big) + P_N \tilde{f}_i\Big(\frac{t}{N}, u^*(t)\Big) \\
&\qquad + P_N\left(\sum_{j=1}^{i-1} P_{i,j} \underline{w}_j^{(m)}(t) + \sum_{j=i+1}^{n} P_{i,j} \underline{w}_j^{(m-1)}(t)\right) \\
&\quad \leq P_N\left((\bar{M} - \gamma_2)\bar{w}_i^{(m-1)}(t) + (\gamma_1 - \gamma_2)\left(\sum_{j=1}^{i-1} \bar{w}_j^{(m)}(t) + \sum_{j=i+1}^{n} \bar{w}_j^{(m-1)}(t)\right)\right) \\
&\qquad + \gamma_1 P_N\left(\sum_{j=1}^{i-1} \underline{w}_j^{(m)}(t) + \sum_{j=i+1}^{n} \underline{w}_j^{(m-1)}(t)\right) \\
&\quad \leq P_N\Bigg((\bar{M} - \gamma_2)\bar{w}_i^{(m-1)}(t) + (\gamma_1 - \gamma_2) \sum_{j=1, j\neq i}^{n} \bar{w}_j^{(m-1)}(t) \\
&\qquad + \gamma_1 \sum_{j=1, j\neq 1}^{n} \underline{w}_j^{(m-1)}(t)\Bigg),
\end{aligned}$$

where $\tilde{f}_i$ is defined by (3.5). Similarly for all $i = 1, 2, \ldots, n$, $t \in I_1^{N-1}$ and $m \geq 1$,

$$\begin{aligned}
&-\delta^2 \underline{w}_i^{(m)}(t) + \bar{M} P_N \underline{w}_i^{(m)}(t) \\
&\quad \leq P_N\Bigg((\bar{M} - \gamma_2)\underline{w}_i^{(m-1)}(t) + (\gamma_1 - \gamma_2) \\
&\qquad \times \sum_{j=1, j\neq i}^{n} \underline{w}_j^{(m-1)}(t) + \gamma_1 \sum_{j=1, j\neq i}^{n} \bar{w}_j^{(m-1)}(t)\Bigg).
\end{aligned}$$

Then we get

$$-\delta^2\left(\bar{w}_i^{(m)}(t)+\underline{w}_i^{(m)}(t)\right)+\bar{M}P_N\left(\bar{w}_i^{(m)}(t)+\underline{w}_i^{(m)}(t)\right)$$
$$\leq(\bar{M}-\gamma_2)P_N\left(\bar{w}_i^{(m-1)}(t)+\underline{w}_i^{(m-1)}(t)\right)$$
$$+(2\gamma_1-\gamma_2)P_N\left(\sum_{j=1,j\neq i}^{n}\left(\bar{w}_j^{(m-1)}(t)+\underline{w}_j^{(m-1)}(t)\right)\right).$$

Summing the above result over all i leads to

$$-\delta^2\left(\sum_{i=1}^{n}\left(\bar{w}_i^{(m)}(t)+\underline{w}_i^{(m)}(t)\right)\right)+\bar{M}P_N\left(\sum_{i=1}^{n}\left(\bar{w}_i^{(m)}(t)+\underline{w}_i^{(m)}(t)\right)\right)$$
$$\leq(\bar{M}+2(n-1)\gamma_1-n\gamma_2)P_N\left(\sum_{i=1}^{n}\left(\bar{w}_i^{(m)}(t)+\underline{w}_i^{(m)}(t)\right)\right). \tag{4.3}$$

Let $W^{(m)}, G^{(m)}\in\mathbf{R}^{N-1}$ be defined by

$$W^{(m)}=\left(\sum_{i=1}^{n}\left(\bar{w}_i^{(m)}(1)+\underline{w}_i^{(m)}(1)\right),\ldots,\sum_{i=1}^{n}\left(\bar{w}_i^{(m)}(N-1)+\underline{w}_i^{(m)}(N-1)\right)\right)^T,$$
$$G^{(m)}=\left(\sum_{i=1}^{n}\left(\bar{w}_i^{(m)}(0)+\underline{w}_i^{(m)}(0)\right),0,\ldots,0,\sum_{i=1}^{n}\left(\bar{w}_i^{(m)}(N)+\underline{w}_i^{(m)}(N)\right)\right)^T.$$

Then (4.3) may be written as

$$\left(A+\frac{\bar{M}}{N^2}B\right)W^{(m)}\leq\frac{\bar{M}+2(n-1)\gamma_1-n\gamma_2}{N^2}\left(BW^{(m-1)}+\frac{1}{12}G^{(m-1)}\right).$$

Since $\left(A+(\bar{M}/N^2)B\right)^{-1}\geq 0$ due to $\max_i\mathcal{N}(M_i)<1$ and $\frac{1}{12}G^{(m-1)}\leq\frac{1}{10}BG^{(m-1)}$, we have

$$0\leq W^{(m)}\leq\frac{\bar{M}+2(n-1)\gamma_1-n\gamma_2}{N^2}\left(A+\frac{\bar{M}}{N^2}B\right)^{-1}$$
$$\times B\left(W^{(m-1)}+\frac{1}{10}G^{(m-1)}\right). \tag{4.4}$$

Since $\mathcal{N}(n\gamma_2-2(n-1)\gamma_1)<1$, we have from Lemma 2.3 that the matrix $A+(1/N^2)(n\gamma_2-2(n-1)\gamma_1)B$ is an M-matrix. By Lemma 2.2, there exists a positive diagonal matrix $E = \mathrm{diag}(E_1, \ldots, E_{N-1})$ such that the matrix

$$E\left(A + \frac{1}{N^2}(n\gamma_2 - 2(n-1)\gamma_1\right)E^{-1}$$

is strictly diagonally dominant. Then we have

$$\frac{\bar{M} + 2(n-1)\gamma_1 - n\gamma_2}{N^2} EBE^{-1}e < E\left(A + \frac{\bar{M}}{N^2}B\right)E^{-1}e$$

where $e = (1, 1, \ldots, 1)^{\mathrm{T}}$, and so

$$\rho = \left\| \frac{\bar{M} + 2(n-1)\gamma_1 - n\gamma_2}{N^2} E\left(A + \frac{\bar{M}}{N^2}B\right)^{-1} BE^{-1} \right\|_\infty < 1.$$

Further by (4.4),

$$\|EW^{(m)}\|_\infty \leq \rho \|E(W^{(m-1)} + \tfrac{1}{10}G^{(m-1)})\|_\infty.$$

Since $G^{(m)} = 0$ for all $m \geq 1$, we have that for all $m \geq 1$,

$$\|EW^{(m)}\|_\infty \leq \rho^m \|E(W^{(0)} + \tfrac{1}{10}G^{(0)})\|_\infty.$$

Consequently for all $t \in I_0^N$ and $m \geq 1$,

$$\begin{aligned}
&\sum_{i=1}^{n}(\bar{w}_i^{(m)}(t) + \underline{w}_i^{(m)}(t)) \\
&\quad \leq \frac{\max_i E_i}{\min_i E_i}\rho^m \left(\max_{t \in I_1^{N-1}} \sum_{i=1}^{n}(\bar{w}_i^{(0)}(t) + \underline{w}_i^{(0)}(t)) \right. \\
&\qquad\qquad \left. + \frac{1}{10}\max_{t=0,N}\sum_{i=1}^{n}(\bar{w}_i^{(0)}(t) + \underline{w}_i^{(0)}(t)) \right)
\end{aligned}$$

and the conclusion follows.

Theorem 4.2 shows that the iteration (3.1) has geometric convergence rate.

5. NUMERICAL RESULTS

This section is devoted to numerical results. We consider the following problem:

$$\begin{aligned}
&-\delta^2 u_1(t) + P_N f_1\left(\frac{t}{N}, u(t)\right) = 0, \quad t \in I_1^{N-1},\\
&-\delta^2 u_2(t) + P_N f_2\left(\frac{t}{N}, u(t)\right) = 0, \quad t \in I_1^{N-1},\\
&\quad u_1(t) = u_2(t) = 0,\ t = 0, N,
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
f_1\left(\frac{t}{N}, u(t)\right) &= \frac{t}{N}\cos\left(u_1(t) - p\left(\frac{t}{N}\right)\right)\sin\left(u_2(t) - q\left(\frac{t}{N}\right)\right),\\
f_2\left(\frac{t}{N}, u(t)\right) &= \frac{t}{N}\cos\left(u_2(t) - p\left(\frac{t}{N}\right)\right)\sin\left(u_1(t) - q\left(\frac{t}{N}\right)\right),
\end{aligned}$$

and the functions p and q are assumed to be continuous in their arguments. It is clear that the problem (5.1) is of form (1.1). To use the iteration (3.1) we have to find the vector functions $\bar{u}(t)$ and $\underline{u}(t)$ such that $\bar{u}(t) \geq \underline{u}(t)$ for all $t \in I_0^N$ and for $i = 1, 2$,

$$f_i\left(\frac{t}{N}, u(t)\right) - f_i\left(\frac{t}{N}, \tilde{u}(t)\right) \leq \sum_{j=1, j\neq i}^{2} P_{i,j}(u_j(t) - \tilde{u}_j(t)), \quad t \in I_0^N, \tag{5.2}$$

whenever $\underline{u}(t) \leq \tilde{u}(t) \leq u(t) \leq \bar{u}(t)$ and $u_i(t) = \tilde{u}_i(t)$, and

$$\begin{aligned}
&-\delta^2 \bar{u}_i(t) + P_N f_i\left(\frac{t}{N}, \bar{u}(t)\right) - P_N\left(\sum_{j=1, j\neq i}^{2} P_{i,j}(\bar{u}_j(t) - \underline{u}_j(t))\right) \geq 0,\\
&\qquad\qquad t \in I_1^{N-1},\\
&-\delta^2 \underline{u}_i(t) + P_N f_i\left(\frac{t}{N}, \underline{u}(t)\right) - P_N\left(\sum_{j=1, j\neq i}^{2} P_{i,j}(\bar{u}_j(t) - \underline{u}_j(t))\right) \leq 0,\\
&\qquad\qquad t \in I_1^{N-1},\\
&\quad \bar{u}(0) \geq 0 \geq \underline{u}(0),\ \bar{u}(N) \geq 0 \geq \underline{u}(N),
\end{aligned} \tag{5.3}$$

where $P=(P_{i,j})$ is a nonnegative matrix. We take

$$P_{1,2} = P_{2,1} = 1$$

and set $\bar{u}_i(t) = -\underline{u}_i(t) = (t/N)(1 - t/N)$, $i=1,2$. By an elementary calculation, we find that the pair of $\bar{u}$ and $\underline{u}$ is a solution of the above inequalities (5.2) and (5.3). Moreover for $i=1,2$,

$$f_i\left(\frac{t}{N}, u(t)\right) - f_i\left(\frac{t}{N}, \tilde{u}(t)\right) \leq u_i(t) - \tilde{u}_i(t), \quad t \in I_0^N,$$

whenever $\underline{u}(t) \leq \tilde{u}(t) \leq u(t) \leq \bar{u}(t)$ and $u_j(t)=\tilde{u}_j(t), j \neq i$. Since all the assumptions of Theorem 3.1 are satisfied, we have a monotone iterative procedure of the form (3.1). In addition,

$$f_i\left(\frac{t}{N}, u(t)\right) - f_i\left(\frac{t}{N}, \tilde{u}(t)\right) \geq -\sum_{j=1}^{2}(u_j(t) - \tilde{u}_j(t)), \quad t \in I_0^N,\ i = 1,2,$$

whenever $\underline{u}(t) \leq \tilde{u}(t) \leq u(t) \leq \bar{u}(t)$. Thus Theorem 3.2 may be applied.

In practical computations, we specify this example with $p(t/N) = q(t/N) = 1$, $t \in I_1^{N-1}$ and $N=20$. We take $M_i^* = 1$, $i=1,2$ in the iteration (3.1) and denote by $\{\bar{u}^{(m)}(t)\}$ and $\{\underline{u}^{(m)}(t)\}$ the mth value of the iteration. Numerical results show that $\{\bar{u}^{(m)}(t)\}$ is a monotone nonincreasing sequence (see Table I), while $\{\underline{u}^{(m)}(t)\}$ is a monotone nondecreasing sequence (see Table II). The monotonicity in Tables I and II

TABLE I

m	$\bar{u}_1^{(m)}(2)$	$\bar{u}_1^{(m)}(4)$	$\bar{u}_1^{(m)}(6)$	$\bar{u}_2^{(m)}(2)$	$\bar{u}_2^{(m)}(4)$	$\bar{u}_2^{(m)}(6)$
1	0.029634	0.056425	0.078033	0.019925	0.038191	0.053290
3	0.008687	0.016824	0.023859	0.008123	0.015746	0.022367
5	0.007696	0.014930	0.021235	0.007672	0.014885	0.021173
9	0.007654	0.014849	0.021123	0.007654	0.014849	0.021123

TABLE II

m	$\underline{u}_1^{(m)}(2)$	$\underline{u}_1^{(m)}(4)$	$\underline{u}_1^{(m)}(6)$	$\underline{u}_2^{(m)}(2)$	$\underline{u}_2^{(m)}(4)$	$\underline{u}_{(2)}^{(m)}(6)$
1	−0.017180	−0.032378	−0.043975	−0.007151	−0.013410	−0.018018
3	0.006574	0.012782	0.018253	0.007141	0.013869	0.019763
5	0.007611	0.014767	0.021008	0.007634	0.014811	0.021071
9	0.007653	0.014849	0.021122	0.007654	0.014849	0.021122

agrees with the one described by Theorem 3.1. We also find that the above two sequences tend to same limit and so it is the unique solution of (5.1) in the sector $[\underline{u}, \bar{u}]$. This coincides with the uniqueness result in Theorem 3.2, because the uniqueness condition of the solution is satisfied in this example.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker Inc., New York (1992).
[2] B. Zimmer, Stability of traveling wavefronts for the discrete Nagumo equation, *SIAM J. Math. Anal.* **22**, 1016–1020 (1991).
[3] R.P. Agarwal, On boundary value problems for second order discrete systems, *Applicable Analysis* **20**, 1–17 (1985).
[4] P.Y.H. Pang and R.P. Agarwal, Periodic boundary value problems for first and second order discrete systems, *Mathl. Comput. Modelling* **16**(10), 101–112 (1992).
[5] P.Y.H. Pang and R.P. Agarwal, On discrete boundary value problems arising transport phenomena, *Appl. Math. Comp.* **60**, 193–213 (1994).
[6] P.Y.H. Pang and R.P. Agarwal, Monotone iterative methods for a general class of discrete boundary value problems, *Computers Math. Applic.* **28**(1–3), 243–254 (1994).
[7] Q. Sheng and R.P. Agarwal, Monotone methods for higher order partial difference equations, *Computers Math. Applic.* **28**(1–3), 291–307 (1994).
[8] W. Zhuang, Y. Cheng and S.S. Cheng, Monotone methods for a discrete boundary problem, *Computers Math. Applic.* **32**(12), 41–49 (1996).
[9] R.A. Usmani and R.P. Agarwal, On the numerical solution of two point boundary value problems, *Appl. Math Comp.* **25**, 247–264 (1988).
[10] Y.-M. Wang, Monotone enclosure for a class of discrete boundary value problems without monotone nonlinearities, *Computers Math. Applic.* **35**(6), 51–60 (1998).
[11] R.P. Agarwal and R.C. Gupta, On the solution of Holt's problem, *BIT* **24**, 342–346 (1984).
[12] R.P. Agarwal, On Numerov's method for solving two point boundary value problems, *Utilitas Math.* **28**, 159–174 (1985).
[13] Y.-M. Wang, Monotone methods for a boundary value problem of second order discrete equation, *Computers Math. Applic.* **36**(6), 77–92 (1998).
[14] L. Collatz, *The Numerical Treatment of Differential Equations*, 3rd edn, Springer-Verlag, Berlin (1964).
[15] L. Collatz, *Funkionalanalysis und Numerisch Mathematics*, Springer-Verlag, Berlin (1964).
[16] R.S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Inc. (1962).
[17] A. Berman and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York (1976).