MONOTONE n-FRAMES ARE TAME[1]

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An *n-frame* $\mathfrak{F}_n$ is the union of n arcs which are disjoint except for a common end point called the *branch point* of $\mathfrak{F}_n$. An n-frame of E^3 is *tamely imbedded* provided there is an autohomeomorphism of E^3 which carries it onto a polygonal n-frame. We will say that an n-frame $\mathfrak{F}_n$ in E^3 is *monotone* provided each geometric 2-sphere centered at the branch point of $\mathfrak{F}_n$ intersects each of the n defining arcs of $\mathfrak{F}_n$ in at most one point. Because each monotone n-frame is of the same imbedding type as a monotone n-frame having its branch point at the origin and its end points on the unit sphere, we will assume that a monotone n-frame has these properties as well as a prescribed ordering, say $a_1, a_2, \cdots, a_n$, of its defining arcs.

Since for each $s \in (0, 1]$, the arc a_i intersects the sphere of radius s centered at the origin in a single point $p_i(s)$, the equation $\phi(s) = (p_1(s), \cdots, p_n(s))$, $0 < s \leqq 1$, defines a continuous function from $(0, 1]$ into the space $F_n(S^2)$ of all n-tuples of distinct points of S^2. In this way there is established a one-one correspondence between monotone n-frames $\mathfrak{F}_n$ and continuous maps ϕ of $(0, 1]$ into $F_n(S^2)$. We restrict our attention to frames $\mathfrak{F}_n$ for which the n-tuple of end points $(p_1(1), \cdots, p_n(1))$ is the base point $(p_1, \cdots, p_n)$ of $F_n(S^2)$.

Let $LH(B^3)$ be the space of all orientation preserving autohomeomorphisms H of the unit ball B^3 which are level preserving in the sense that $\|H(x)\| = \|x\|$ for all $x \in B^3$. To each $H \in LH(B^3)$ there corresponds a family h_s of autohomeomorphisms of the boundary sphere S^2 of B^3 defined by $h_s(x) = H(sx)/s$ for $s \in (0, 1]$, $x \in S^2$. Therefore $LH(B^3)$ is in one-one correspondence with the set of maps of $(0, 1]$ into the group $G(S^2)$ of all orientation preserving autohomeomorphisms of S^2.

In order to relate monotone n-frames $\mathfrak{F}_n$ and level preserving homeomorphisms H, consider the map $\rho_n: G(S^2) \to F_n(S^2)$ defined by $\rho_n(f) = (f(p_1), \cdots, f(p_n))$, where $(p_1, \cdots, p_n)$ is the base point of $F_n(S^2)$.

The following theorem is patterned after a result of G. S. McCarty [**3**, Lemma 4.1], but, as used here, is better viewed as a generalization of a fundamental theorem of E. Artin in braid theory [**1**, p. 104]. (See

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D. M. Dahm [2, p. 15], where this generalization is developed for braids on arbitrary manifolds.) Although the theorem is stated for S^2, the proof given is valid for any connected manifold of dimension at least 2.

THEOREM 1. *$G(S^2)$ is a fiber bundle over $F_n(S^2)$ relative to the map ρ_n.*

PROOF. $G(S^2)$ is an effective transitive group of transformations of $F_n(S^2)$. Consequently, by the bundle structure theorem, it suffices to show that ρ_n admits a local cross-section. But, since the basic open sets in $F_n(S^2)$ may be chosen to be products of n disjoint open discs, this is a consequence of the following elementary fact. Let $G(B^2, S^1)$ be the group of autohomeomorphisms of the 2-cell B^2 which are pointwise fixed on the boundary S^1, and let π be the map of $G(B^2, S^1)$ onto the interior of B^2 defined by $\pi(f)=f(0)$. Then π admits a global cross section.

As a corollary to Theorem 1, we have our main result.

THEOREM 2. *Any monotone n-frame is tame.*

PROOF. Let $\mathfrak{F}_n$ be a monotone n-frame. Then $\mathfrak{F}_n$ corresponds to a map ϕ: $(0, 1]\rightarrow F_n(S^2)$. Choosing the base point $(p_1, \cdots, p_n)$ to be $\phi(1)$, the map ϕ lifts to a map h: $(0, 1]\rightarrow G(S^2)$ such that h_1 is the identity. The corresponding map H in $LH(B^3)$ is an autohomeomorphism of B^3 carrying $\mathfrak{F}_n$ onto n radial segments in B^3. Since H is the identity on the boundary sphere, H extends to all of E^3 and hence $\mathfrak{F}_n$ is tame.

ADDED IN PROOF. The authors are grateful to Professor R. H. Bing for pointing out that Theorem 2 (generalized, as noted, to n-manifolds) is equivalent to Proposition 2.3 of: R. H. Bing and V. L. Klee, *Every simple closed curve in E^3 is unknotted in E^4*, J. London Math. Soc. **39** (1964), 86–94.

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