MONOTONE NONCOMPACT MAPPINGS OF E^r ONTO E^k FOR $r \geqq 4$ AND $k \geqq 3$

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Introduction. Using results of [1] and [4], we give counterexamples to the so-called monotone mapping problem for $r \geqq 4$. That is, if f is a monotone map of E^r onto E^r ($f^{-1}(y)$ is a compact connected set), then is f a compact map (inverse image of compact sets compact) [7]? Whyburn [6] has shown that for $r=2$ any monotone mapping of E^2 onto E^2 is necessarily a compact mapping. E. H. Connell [2] has shown that if f is a monotone mapping of E^r onto E^r such that for $p \in E^r$, $H_k[f^{-1}(p)]=0$ for $k=1, 2, \cdots$, then f is compact. If G is a domain in E^r and $f: G \to E^r$ is $(r-2)$-acyclic (counter-images of points are compact and have trivial cohomology groups in dimensions $\leqq r-2$), then it is shown in [5] that a number of given conditions are equivalent and that each of the conditions implies that f is compact. In particular, if f is a $(r-2)$-acyclic map of E^r onto itself, then f is compact. In [3], before they had discovered the reference [5], the authors obtained another proof of this latter result. Here, the proof is quite direct and is obtained by showing that if f is a monotone map of E^r onto E^r such that for some flat $(r-1)$-sphere $S \subset E^r$, the inverse image of a neighborhood of S is compact and $f_*: H_{r-1}(f^{-1}(S)) \to H_{r-1}(S)$ is nontrivial (e.g. if f is a homeomorphism on some open set), then f is compact. Besides being quite direct, the proofs given in [3] are noteworthy in that they are of general interest and easy to follow, particularly for someone unfamiliar with the theory in this area.

Definitions and notation. E^n will denote Euclidean n-space. The unit n-sphere in E^{n+1} will be denoted by S^n and the unit n-ball in E^n by I^n. Let S_t^{n-1} denote the $(n-1)$-sphere in E^n of radius t (thus $S_1^{n-1}=S^{n-1}$). Let $E_+^n = E^{n-1} \times [0, \infty) \subset E^n$. For convenience in notation we will denote the origin of E^n by p_n.

If for $t \in (0, \infty)$ and $x \in S^{n-1}$ we identify the point $(x, t) \in S^{n-1} \times (0, \infty)$ with the point $tx \in S_t^{n-1}$, then we can consider E^n as $p_n \cup (S^{n-1} \times (0, \infty))$. In a similar fashion, we can consider E_+^n as $p_n \cup (I^{n-1} \times (0, \infty))$, where $I^{n-1} \times t$ is identified, in a natural manner, with $S_t^{n-1} \cap E_+^n$.

For $1 \leqq r \leqq n$, thinking of E^n as $E^{r-1} \times E^{n-r+1}$, if for each $t \in (0, \infty)$ and $y \in E^{r-1}$ we identify $y \times t \times S^{n-r}$ with $y \times S_t^{n-r} \subset y \times E^{n-r+1}$, then we can express E^n as $(E^{r-1} \times p_{n-r+1}) \cup (E^{r-1} \times (0, \infty) \times S^{n-r})$. Thus

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given $1 \leqq r \leqq n$ we can define a natural map $h_{n,r}$ taking E^n onto $E^r_+ = E^{r-1} \times [0, \infty)$ by defining

$$h_{n,r}(y, p_{n-r+1}) = (y, 0) \in E^{r-1} \times 0 \subset E^r_+$$

and

$$h_{n,r}(y, t, x) = (y, t) \in E^{r-1} \times t \subset E^r_+.$$

We note that the preimage of a point is either a point or an $(n-r)$-sphere. Thus, if $1 \leqq r < n$, $h_{n,r}: E^n \to E^r_+$ is a monotone compact map.

In [4], for each $n > m \geqq 3$, the author described a monotone noncompact map of E^n onto E^m. Since we will need to make use of one of the maps defined there in our main result, it will be convenient to indicate briefly here how the above maps are obtained. By [4, Lemma 2], there exists a 1-1 continuous noncompact map, which we will denote here by h, taking E^3_+ onto E^3. Hence for $n \geqq 3$, there exists a 1-1 continuous noncompact map H_n taking E^n_+ onto E^n. That is, considering E^n_+ as $E^3_+ \times E^{n-3}$, $H_n(x, y) = (h(x), y)$, where $x \in E^3_+$ and $y \in E^{n-3}$. The monotone noncompact map of E^n onto E^m $(n > m \geqq 3)$ mentioned above is simply the composition

$$E^n \xrightarrow{h_{n,m}} E^m_+ \xrightarrow{H_m} E^m.$$

By [1, Theorem III], for $n \geqq 3$ and $m \geqq 2$, there exist monotone maps

(1) $f^1_{n,m}$ of I^n onto I^m,
(2) $f^2_{n,m}$ of I^n onto S^m,
(3) $f^3_{n,m}$ of S^n onto S^m and
(4) $f^4_{n,m}$ of S^n onto I^m.

(Since this difficult result seemed to be an interesting pathological result with no immediate applications, due to the fact that the proof itself was so difficult and involved and required extremely complicated notation, a proof of this theorem has never been published. However, a proof of this result was presented at the Summer Institute on Set Theoretic Topology held at the University of Wisconsin in 1955.)

Main results. We now can easily describe the counterexamples to the so-called monotone mapping problem for $r \geqq 4$ along with a number of other related results.

THEOREM 1. *For $r \geqq 4$ and $k \geqq 3$, there exist monotone compact maps*
(1) $F^1_{r,k}$ *of* E^r_+ *onto* E^k_+,

(2) $F^2_{r,k}$ *of* E^r_+ *onto* E^k,
(3) $F^3_{r,k}$ *of* E^r *onto* E^k, *and*
(4) $F^4_{r,k}$ *of* E^r *onto* E^k_+.

PROOF. Since $r\geqq 4$ and $k\geqq 3$, the result follows by making use of the maps $f^i_{r-1,k-1}$ ($i=1, 2, 3, 4$) given above. That is, in each case for $s=r$ or k, we express E^s as $p_s\cup(S^{s-1}\times(0, \infty))$ and E^s_+ as $p_s\cup(I^{s-1}\times(0, \infty))$. Then for $i=1, 2, 3$, or 4, $F^i_{r,k}$ is defined as follows:

$$F^i_{r,k}(p_r) = p_k$$

and

$$F^i_{r,k}((x, t)) = (f^i_{r-1,k-1}(x), t),$$

where the ranges and domains are appropriately expressed in each case.

Remarks. We note here that using the maps $h_{s,k}$ taking E^s onto E^k_+ ($s>k\geqq 1$) composed with the maps $F^2_{r,s}$ and $F^3_{r,s}$ we can get additional monotone compact maps of E^r_+ onto E^k_+ and E^r onto E^k_+ having preimages of $(s-k)$-spheres as preimages of points. That is, if $r\geqq 4$, $k\geqq 1$, $s>k$, and $s\geqq 3$, let $F^{1,s}_{r,k}$ be the monotone compact map of E^r_+ onto E^k_+ given by the composition

$$E^r_+ \xrightarrow{F^2_{r,s}} E^s \xrightarrow{h_{s,k}} E^k_+,$$

and let $F^{4,s}_{r,k}$ be the monotone compact map of E^r onto E^k_+ given by the composition

$$E^r \xrightarrow{F^3_{r,s}} E^s \xrightarrow{h_{s,k}} E^k_+.$$

Here one uses the facts that the inverse image of a compact connected set under a monotone compact map is compact and connected, and the composition of compact maps is compact.

Also for $r\geqq 6$ and $k\geqq 4$, there exist monotone compact maps of E^r onto E^k_+ having preimages of points homeomorphic to continua $\times(r-s)$-spheres. That is, for $r>s\geqq 5$ and $k\geqq 4$, let $\tilde{F}^{4,s}_{r,k}$ be the monotone compact map of E^r onto E^k_+ given by the composition

$$E^r \xrightarrow{h_{r,s}} E^s_+ = E^{s-1}\times[0, \infty) \xrightarrow{F^3_{s-1,k-1}\times \text{id.}} E^{k-1}\times[0, \infty) = E^k_+.$$

THEOREM 2. *For* $r\geqq 4$ *and* $k\geqq 3$, *there exist monotone noncompact maps* $G^1_{r,k}$ *and* $G^{1,s}_{r,k}$ *of* E^r_+ *onto* E^k *and* $G^4_{r,k}$, $G^{4,s}_{r,k}$, *and* $\tilde{G}^{4,s}_{r,k}$ *of* E^r *onto* E^k.

PROOF. The maps are obtained by following the appropriate F maps by the 1-1 noncompact map H_k: $E^k_+ \to E^k$. We note the map $\tilde{G}^{4,s}_{r,k}$ is only defined for $r \geqq 6$ and $k \geqq 4$.

Questions.

Question 1. Does there exist a monotone compact map $F_{3,n}$ of E^3 onto E^n for some $n \geqq 4$?

The results of [1] do not give an answer to this question directly and it does not appear that the methods used in [1] can be modified so as to produce the desired result (i.e. one would like to have a monotone map $f_{3,n}$ of S^3 onto S^n (for some $n \geqq 4$) so that for some $y \in S^n$, $S^3 - f_{3,n}^{-1}(y) = E^3$ (or for some compact connected set $Z \subset S^n$ with $S^n - Z = E^n$, $S^3 - f_{3,n}^{-1}(Z) = E^3$)).

Question 2. Does there exist a monotone map k of E^3 onto E^3_+? We note, if Question 1 can be answered in the affirmative for some $n \geqq 4$, then the result follows for all $m \geqq 4$, Question 2 is true, and there would exist a monotone noncompact map of E^3 onto E^3. That is,

(i) the composition $E^3 \xrightarrow{F_{3,n}} E^n \xrightarrow{F^3_{n,m}} E^m$ is a monotone compact map of E^3 onto E^m;

(ii) the composition $E^3 \xrightarrow{F_{3,n}} E^n \xrightarrow{h_{n,3}} E^3_+$ is a monotone compact map of E^3 onto E^3_+; and

(iii) a map given by an affirmative answer to Question 2 or the map given by (ii) followed by the map H_3: $E^3_+ \to E^3$ gives a monotone noncompact map of E^3 onto E^3.

We also note that since the maps defined by (i) taking E^3 onto E^m (for $m \geqq 4$) are monotone and compact, we could then obtain monotone noncompact maps of E^3 onto E^m ($m \geqq 3$). That is, the compositions

$$E^3 \xrightarrow{F_{3,k}} E^k \xrightarrow{h_{k,m}} E^m_+ \xrightarrow{H_m} E^m \qquad (k > m \geqq 3)$$

or

$$E^3 \xrightarrow{F_{3,k}} E^k \xrightarrow{F^{-1}_{k,m}} E^m_+ \xrightarrow{H_m} E^m \qquad (k \geqq 4, m \geqq 3)$$

would be examples of such maps.

Finally, we note that from the above discussion an affirmative answer to Question 2 at least allows us to obtain a monotone noncompact map of E^3 onto E^3 and if Question 2 is false, then Question 1 is false for all $n \geqq 4$.

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