

MONOTONE OPERATORS AND NONLINEAR INTEGRAL EQUATIONS OF HAMMERSTEIN TYPE

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A nonlinear integral equation of Hammerstein type is one of the form

$$(1) \quad u(x) + \int_G K(x, y)f(y, u(y))dy = 0,$$

where G is a measure space with σ -finite measure dy and the unknown function $u(x)$ is defined on G . In operator-theoretic terms, the problem of determining the solutions of the equation (1) with u lying in a given Banach space Y of functions on G can be put in the form of the nonlinear functional equation

$$(2) \quad u + AN(u) = 0$$

with the linear and nonlinear mappings A and N given by

$$(3) \quad Av(x) = \int_G K(x, y)v(y)dy, \quad Nu(x) = f(x, u(x)).$$

In the present note, we establish general results on the existence and uniqueness of solutions of equation (2) for the Banach space $Y = X^*$ under appropriate assumptions of weak monotonicity type upon the mappings A and N . We note that Hammerstein equations have an extensive literature which includes Hammerstein [11], Iglisch [12], Golomb [10], Dolph [7], Rothe [18], Vainberg [19], [20], and Krasnosel'skiĭ [16]. The first application of the concept of monotone operator in this problem was made implicitly by Golomb [10] and explicitly by Vainberg [19]. More recent papers applying monotonicity concepts to Hammerstein equations include Dolph-Minty [8], Kolodner [13], Brézis [3], Kolomy [14], [15], Amann [1], [2], de Figueiredo-Gupta [9] and Vainberg [20].

We employ the following definitions: If X is a real Banach space, X^* its conjugate space, we let (w, u) denote the duality pairing between the element w of X^* and the element u of X . A mapping A of X into X^* is said to be monotone if for all u, v in X we have

$$(A(u) - A(v), u - v) \geq 0.$$

A mapping N of X^* into X is said to be hemicontinuous if it is continuous from each line segment of X^* to the weak topology of X .

DEFINITION 1. If A is a bounded monotone linear mapping of X into X^* , then A is said to be *angle-bounded* with constant $c \geq 0$ if for all u, v in X

$$|(A(u), v) - (A(v), u)| \leq 2c\{(A(u), u)\}^{1/2}\{(A(v), v)\}^{1/2}.$$

DEFINITION 2. If A is a bounded linear mapping of X into X^* , A is said to be *symmetric* if for all u and v in X ,

$$(A(u), v) = (A(v), u).$$

THEOREM 1. Let X be a general real Banach space, X^* its conjugate space. Let A be a monotone angle-bounded continuous linear mapping of X into X^* with constant of angle-boundedness $c \geq 0$. Let N be a hemicontinuous (possibly nonlinear) mapping of X^* into X such that for a given constant $k \geq 0$

$$(4) \quad (v - v_1, N(v) - N(v_1)) \geq -k\|v - v_1\|_{X^*}^2$$

for all v and v_1 in X^* . Suppose finally that there exists a constant R with $k(1+c^2)R < 1$ such that for all u in X

$$(5) \quad (A(u), u) \leq R\|u\|_X^2.$$

Then there exists exactly one solution w in X^* of the nonlinear equation

$$(6) \quad w + AN(w) = 0.$$

Some special cases of Theorem 1 are the following:

THEOREM 2. Let X be a general real Banach space, X^* its conjugate space, A a bounded linear mapping of X into X^* with A monotone and angle-bounded. Let N be a hemicontinuous (possibly nonlinear) mapping of X^* into X which is monotone, that is,

$$(v - v_1, N(v) - N(v_1)) \geq 0$$

for all v and v_1 in X^* .

Then there exist exactly one solution w in X^* of the equation

$$w + AN(w) = 0.$$

The result of Theorem 2 was obtained by Amann [1] under an assumption that X^* has a dense continuous linear imbedding in a Hilbert space. As we show below, such assumptions are not needed in the proof of Theorem 1, nor in the proof of Theorem 2 which is a special case of Theorem 1 for $k=0$. Another special case is the following

THEOREM 3. *Let X be a general real Banach space, X^* its conjugate space, A a bounded linear mapping of X into X^* which is monotone and symmetric. Suppose that N is a hemicontinuous (possibly nonlinear) mapping of X^* into X such that for a given $k \geq 0$ and all v, v_1 in X^* ,*

$$(v - v_1, N(v) - N(v_1)) \geq -k\|v - v_1\|_{X^*}^2.$$

Suppose that $k\|A\| < 1$.

Then the equation $w + AN(w) = 0$ has exactly one solution w in X^ .*

We note that when A is symmetric then A is angle-bounded with constant of angle-boundedness $c = 0$.

The result of Theorem 3 was obtained by Golomb [10] for $X = L^2(G)$ and by Vainberg [20] for $X = L^p(G)$, using variational methods. Our method, on the other hand, consists in splitting the linear operator A via a Hilbert space H (Theorem 4) and reducing the equation (6) to an equivalent equation in H , which is then solved by using the results of Browder [5] and Minty [17] for monotone operator equations.

We may add that the proof of Theorems 1, 2, 3 and 4 can be adapted in a straightforward manner to the case of a real locally-convex space.

We turn now to the proof of Theorem 1. An essential tool in that proof is the following auxiliary theorem.

THEOREM 4. *Let X be a Banach space, X^* its conjugate space, A a bounded linear mapping of X into X^* which is monotone and angle-bounded. Then there exists a Hilbert space H , a continuous linear mapping S of X into H with S^* injective and a bounded skew-symmetric linear mapping B of H into H such that $A = S^*(I + B)S$, and the following two inequalities hold:*

- (i) $\|B\| \leq c$, with c the constant of angle-boundedness of A .
- (ii) $\|S\|^2 \leq R$ if and only if for all u in X , $(A(u), u) \leq R\|u\|_X^2$.

PROOF. We introduce a symmetric bilinear form on X by setting

$$[u, u_1] = \frac{1}{2}\{(A(u), u_1) + (A(u_1), u)\}.$$

Since $[u, u] = (A(u), u) \geq 0$ by the monotonicity of A , we have the Cauchy-Schwarz inequality

$$(7) \quad |[u, u_1]| \leq \{[u, u]\}^{1/2}\{[u_1, u_1]\}^{1/2}.$$

Let N be the subset of those u in X for which $[u, u] = 0$. It follows from (7) that $N = \{u | u \text{ in } X, [u, v] = 0 \text{ for all } v \text{ in } X\}$ and hence, N is a closed vector-subspace of X . Let H_0 denote the quotient vector-space

X/N and endow H_0 with the inner product $[u + N, u_1 + N] = [u, u_1]$. Under this inner product, H_0 is a pre-Hilbert space. We denote by H the Hilbert space obtained from H_0 by completing it with respect to this inner product. We let S denote the linear mapping of X into H obtained from the natural projection of X onto X/N . Since A is a bounded linear mapping of X into X^* , there exists a least constant $R \geq 0$ such that $(A(u), u) \leq R\|u\|_X^2$ for all u in X . Hence, for all u in X we have

$$\|S(u)\|_H^2 = [S(u), S(u)] = (A(u), u) \leq R\|u\|_X^2$$

that is, S is a bounded linear mapping of X into H with $\|S\|^2 \leq R$. Since the range of S coincides with H_0 , S has a dense range in H . Hence the adjoint mapping S^* of H into X^* is injective.

Since A is angle-bounded with constant of angle-boundedness $c \geq 0$, it follows that for all u, v in X

$$|(A(u), v) - (A(v), u)| \leq 2c\|S(u)\|_H\|S(v)\|_H.$$

Hence the function

$$h(S(u), S(v)) = \frac{1}{2}\{(A(u), v) - (A(v), u)\}$$

is well defined on H_0 and is a bounded bilinear form on H_0 . Let h also denote the unique extension of this bounded bilinear form on H_0 to H . It follows that there exists a well-defined and unique bounded linear mapping B of H into H such that for all u, v in X

$$h(S(u), S(v)) = [B(S(u)), S(v)].$$

Since $h(S(u), S(v)) = -h(S(v), S(u))$ for all u, v in X , it follows that B is skew-symmetric on H_0 and hence on all of H by continuity. We note that

$$|[B(S(u)), S(v)]| \leq c\|S(u)\|_H\|S(v)\|_H$$

for all u, v in X and so $\|B\| \leq c$.

Finally for all u, v in X

$$\begin{aligned} (A(u), v) &= [u, v] + h(S(u), S(v)) = [S(u), S(v)] + [B(S(u)), S(v)] \\ &= [(I + B)S(u), S(v)] = (S^*(I + B)S(u), v). \end{aligned}$$

Thus $A = S^*(I + B)S$. This completes the proof of the theorem.

We also need the following elementary lemma in the proof of Theorem 1.

LEMMA. *Let H be a given Hilbert space, B a skew-symmetric bounded linear mapping of H into H . Then the bounded linear mapping $I + B$ is*

a monotone bijective mapping of H onto H . Further, for any u in H we have

$$[(I + B)^{-1}(u), u] \geq \frac{1}{1 + \|B\|^2} \|u\|_H^2.$$

PROOF OF THEOREM 1. Suppose w in X^* is a solution of the equation (6). By Theorem 4, $A = S^*(I+B)S$ and equation (6) becomes

$$(8) \quad w + S^*(I + B)SN(w) = 0.$$

Since S^* is injective, there is a unique u in H such that $w = S^*(u)$ and equation (8) becomes

$$(9) \quad S^*(u) + S^*(I + B)SNS^*(u) = 0,$$

i.e.,

$$(10) \quad S^*(u + (I + B)SNS^*(u)) = 0.$$

Since S^* is injective, equation (10) is equivalent to

$$(11) \quad u + (I + B)SNS^*(u) = 0.$$

Hence equation (6) has exactly one solution in X^* if and only if equation (11) has exactly one solution in H . Now, by the lemma, equation (11) is equivalent to the equation

$$(12) \quad (I + B)^{-1}(u) + SNS^*(u) = 0.$$

Let $T = (I+B)^{-1} + SNS^*$. For u, v in H ,

$$[T(u) - T(v), u - v] = [(I + B)^{-1}(u - v), u - v] + [SNS^*(u) - SNS^*(v), u - v].$$

By the lemma and Theorem 4(i)

$$[(I + B)^{-1}(u - v), u - v] \geq \frac{1}{1 + \|B\|^2} \|u - v\|_H^2 \geq \frac{1}{1 + c^2} \|u - v\|_H^2.$$

On the other hand,

$$[SNS^*(u) - SNS^*(v), u - v] = (S^*(u) - S^*(v), NS^*(u) - NS^*(v)) \geq -k \|S^*(u) - S^*(v)\|_{X^*}^2 \geq -kR \|u - v\|_H^2$$

by Theorem 4(ii). Combining these inequalities, we see that

$$[T(u) - T(v), u - v] \geq (1/(1 + c^2) - kR) \|u - v\|_H^2 \geq c_1 \|u - v\|_H^2$$

where $c_1 = 1/(1 + c^2) - kR > 0$ since $k(1 + c^2)R < 1$ by hypothesis.

Thus T is a monotone hemicontinuous mapping of H into H and T is injective. Moreover, for u in H we have

$$\begin{aligned} [T(u), u] &= [T(u) - T(0), u - 0] + [T(0), u] \\ &\geq c_1 \|u\|_H^2 - \|T(0)\|_H \|u\|_H. \end{aligned}$$

Since $[T(u), u]/\|u\|_H \rightarrow \infty$ as $\|u\|_H \rightarrow \infty$, T is coercive. It then follows from the results of Browder [5] and Minty [17] that T maps H onto H injectively. Hence (12) has exactly one solution in H and so by our preceding discussion, equation (6) has exactly one solution in X^* . q.e.d.

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