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# Monotone Quantities and Unique Limits for Evolving Convex Hypersurfaces

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## 1 Introduction

The aim of this paper is to introduce a new family of monotone integral quantities associated with certain parabolic evolution equations for hypersurfaces, and to deduce from these some results about the limiting behaviour of the evolving hypersurfaces.

A variety of parabolic equations for hypersurfaces have been considered. One of the earliest was the Gauss curvature flow, introduced in [Fi] as a model for the changing shape of a stone wearing on a beach. The stone is represented by a bounded convex region, and each point on its surface moves in the inward normal direction with speed equal to the Gauss curvature: If the surface at time $t$ is given by an embedding $x_t$, then

$$\frac{\partial x}{\partial t} = -K\mathbf{n},$$

where $K$ is the Gauss curvature, and $\mathbf{n}$ the outward unit normal. Firey showed that stones which are symmetric about the origin shrink to points in finite time, and are asymptotically spherical in shape.

Other evolution equations have been considered since then, of the form

$$\frac{\partial x}{\partial t} = -F\mathbf{n}, \tag{1}$$

where $x$ is an embedding into $\mathbb{R}^{n+1}$, and $F$ depends on the curvature and normal direction of the hypersurface. Examples include flows by mean curvature ([Hu1]) with $F = H$, the $n$th root of the Gauss curvature ([Ch1]) with $F = K^{1/n}$, and many other homogeneous degree 1 functions of the principal curvatures ([Ch2], [An1]). Flows which expand hypersurfaces

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with $F$ homogeneous of degree $-1$ have also been considered, with quite general results ([U1], [U2], [Ge], [Hu2]).

The behaviour of solutions of equations of this kind can be quite complicated, even in the case where $F$ depends only on the principal curvatures and not explicitly on the normal direction. In particular, hypersurfaces evolving by small powers of their Gauss curvature do not in general become spherical [An3], and a given equation can have several different solutions which evolve by contracting without changing shape [An7], [An8], [An9]. There are very few equations for which the behaviour of solutions is well understood, other than those mentioned above with $F$ homogeneous of degree 1 or of negative degree in the principal curvatures. A single exception is the flow with $F = K^{1/(n+2)}$, which has a remarkable invariance under the special affine group. In [An5] and [ST], it was shown that solutions become ellipsoidal in shape as they contract to points. As a guiding principle, we expect that flows in which $F$ is homogeneous of large degree in the curvatures will have solutions which are asymptotically homothetic—that is, the solution hypersurfaces can be rescaled to converge as the final time is approached, to a limit which satisfies the identity

$$F = c\langle x, \mathbf{n}\rangle \tag{2}$$

for some $c > 0$. This implies that the limit hypersurface evolves by shrinking without changing shape. On the other hand, if $F$ is homogeneous of small degree, we expect that some isoperimetric ratio will usually become unbounded as the solution shrinks to a point. In the case of curves in the plane, this picture has been confirmed in detail ([An6], [An7]). In higher dimensions, results are known only for flows involving Gauss curvature ([An8], [An9]).

**Theorem 1.** Let $\psi \in C^\infty(S^n)$ be strictly positive, and $\alpha \in (1/(n+2), 1/n]$. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open bounded convex region, and let $M_0$ be the boundary of $\Omega$. Then there exists a family of $C^\infty$ embeddings $\{x_t\colon\ S^n \to \mathbb{R}^{n+1}\}_{0<t<T}$, unique up to composition with an arbitrary time-independent smooth diffeomorphism of $S^n$, such that the hypersurfaces $M_t = x_t(S^n)$ bound strictly convex open regions $\Omega_t$ for $t > 0$, converge to $M_0$ in Hausdorff distance as $t \to 0$, and satisfy the evolution equation (1) with

$$F = \psi(\mathbf{n})K^\alpha.$$

The embeddings $x_t$ converge uniformly to a point $p \in \mathbb{R}^{n+1}$ as $t \to T$, and the rescaled embeddings

$$\tilde{x}_t = \left(\frac{\mathrm{Vol}(S^n)}{\mathrm{Vol}(\Omega_t)}\right)^{1/n+1}\left(x_t - p\right)$$

have images which converge in $C^\infty$ for a subsequence of times approaching $T$ to a hypersurface $\Sigma$ which satisfies (2). $\square$

Part of the difficulty in treating the general case of equation (1) is that there is no associated variational principle, and consequently it is difficult to find quantities which improve under the flow. The present paper concerns a family of flows of a somewhat special form, for which something more can be said. In particular, we prove the existence of many improving integral quantities for these flows, and use these to simplify the possible types of behaviour: We show that if a solution approaches a solution of (2) modulo rescaling, even in a very weak sense or on a subsequence of times, then it must converge smoothly to that solution. Hence a solution can have at most one limiting shape.

The class of flows we consider was introduced by the author in [An3], and includes the Gauss curvature flows (with $F = \psi K^\alpha$) and flows with $F = (K/E_k)^\alpha$, where $E_k$ is the $k$th elementary symmetric function of the principal curvatures. We will refer to these flows as "mixed discriminant" flows, or MDFs for brevity. The complete description of this class is given in Section 2. For each of these flows there is a family of associated integral quantities, which we introduce in Section 3. In particular, any hypersurface satisfying the identity (2) is necessarily a critical point of every one of these quantities. We show in Section 4 that some of these quantities evolve monotonically in time for MDF solutions. The main result, given in Section 6, is the following theorem.

**Theorem 2.** Let $\{x_t\}_{0\le t<T}$ be a solution of a mixed discriminant flow, converging to a point in $\mathbb{R}^{n+1}$ as $t \to T$. Suppose there exist sequences $t_k \to T$, $R_k \to \infty$, and $p_k \in \mathbb{R}^{n+1}$ such that the hypersurfaces $R_k\left(x_{t_k}(S^n) - p_k\right)$ converge in Hausdorff distance as $k \to \infty$ to a compact convex $C^2$ hypersurface $\Sigma$ with $F > 0$. Then $\Sigma$ is $C^\infty$ and satisfies the identity (2) for some choice of origin in $\mathbb{R}^{n+1}$, and there exists $p \in \mathbb{R}^{n+1}$ such that the hypersurfaces

$$\tilde{M}_t = \left(\frac{\mathrm{Vol}(\Sigma)}{\mathrm{Vol}(\Omega_t)}\right)^{1/n+1}\left(x_t(S^n) - p\right)$$

converge in $C^\infty$ to $\Sigma$ as $t \to T$, where $\Omega_t$ is the region enclosed by the hypersurface $x_t(S^n)$. $\square$

In particular, this improves the result of Theorem 1 for Gauss curvature flows: Convergence for a subsequence of times to a homothetic limit is improved to convergence in $C^\infty$ as $t \to T$. Our argument is similar to that of Simon [Si1] which applied to gradient flows of convex functionals, and to the uniqueness problem for tangent cones of minimal surfaces and harmonic maps. The present case is complicated by the fact that the flows are fully nonlinear, and are not gradient flows, so some work is required to relate the evolution equations to the gradients of appropriate functionals.

One of the main technical difficulties which arises in the proof is that of proving bounds on the radius of curvature. This difficulty stems from the explicit dependence of the speed on the normal direction, which introduces terms into the evolution equation for the radii of curvature which we can control only when the solution is close to a solution of (2). The control of such terms should be important in the study of other anisotropic equations, such as anisotropic mean curvature flows which are important in modelling interfaces ([Gu], [AG1], [AG2]).

We remark that some of the integral quantities we use have been considered before: Firey [Fi] showed that the integral $\int_{\tilde{M}_t} K \ln\langle x, \mathbf{n}\rangle \, d\mu$ decreases for solutions of the Gauss curvature flow. This was extended to flows of curves in [An6, Lemma I1.16]. A second integral quantity is the *entropy*, which was found by Hamilton for the curve shortening flow in [Ha] and extended to the higher-dimensional Gauss curvature flow (with $F = K$) by Chow in [Ch3]. It is given by $\int_{\tilde{M}_t} K \ln K \, d\mu$, and decreases for solutions of the Gauss curvature flow. This was generalised for other MDF solutions in [An3]. Both of these examples are included in the family of integral quantities we consider in this paper.

## 2 Notation and preliminary results

In this section, we review some notation and results concerning convex regions in Euclidean space, including the definitions of mixed volume and mixed discriminant. We also define the mixed discriminant flows and discuss some of their elementary properties.

### Support functions

The support function $s\colon S^n \to \mathbb{R}$ of a convex region $\Omega$ in $\mathbb{R}^{n+1}$ is defined by

$$s(z) = \sup_{y \in \Omega} \langle y, z\rangle \tag{3}$$

for each $z$ in $S^n$. This gives the distance of each supporting hyperplane of $\Omega$ from the origin. The region $\Omega$ can be recovered from $s$ as follows:

$$\Omega = \bigcap_{z \in S^n} \{y \in \mathbb{R}^{n+1}\colon \langle y, z\rangle \le s(z)\}.$$

For $\Omega$ strictly convex and smoothly bounded, there is a natural embedding $\bar{x}$ describing the boundary $\partial\Omega$, such that the Gauss map $z \to \mathbf{n}(\bar{x}(z))$ is the identity on $S^n$. This is given in terms of $s$ by the following expression:

$$\bar{x}(z) = s(z)z + \nabla s(z) \tag{4}$$

where $\nabla s$ is the gradient vector of $s$ with respect to the standard metric $g$ on $S^n$. For any $f \in C^2(S^1)$, we define an associated bilinear form $\mathfrak{r}[f]$ by

$$\mathfrak{r}_{ij}[f] = \nabla_i \nabla_j f + f g_{ij}, \tag{5}$$

where $\nabla$ is the Levi-Civita connection of $g$. Then the principal radii of curvature of $\partial\Omega$ at the point $\bar{x}(z)$ are the eigenvalues with respect to $g$ of $\mathfrak{r}[s]$ at $z$. The constant curvature of $g$ implies that $\mathfrak{r}$ has totally symmetric covariant derivative: $\nabla_i \mathfrak{r}[f]_{jk} = \nabla_j \mathfrak{r}[f]_{ik}$.

If $\Omega$ is a convex region and $\varepsilon > 0$, then $\varepsilon\Omega$ is the convex region $\{\varepsilon a\colon\ a \in \Omega\}$. For any two convex regions $\Omega_1$ and $\Omega_2$, the Minkowski sum $\Omega_1 + \Omega_2$ is the convex region $\{a + b\colon\ a \in \Omega_1, b \in \Omega_2\}$. If $\Omega_1$ and $\Omega_2$ have support functions $s_1$ and $s_2$ respectively, then $\varepsilon_1\Omega_1 + \varepsilon_2\Omega_2$ has support function $s = \varepsilon_1 s_1 + \varepsilon_2 s_2$.

## Mixed volumes and mixed discriminants

The volume $\mathrm{Vol}(\Omega)$ of a convex region $\Omega$ can be calculated in terms of its support function $s$:

$$\mathrm{Vol}(\Omega) = \frac{1}{n+1} \int_{S^n} s \det\left(\mathfrak{r}[s]\right)\, d\mu$$

where $d\mu$ is the standard measure on $S^n$, and the determinant is taken with respect to $g$.

Let $\Omega_i$, $i = 1, \ldots, N$ be convex regions with support functions $s_i$, and consider the Minkowski sum $\sum_{i=1}^N \epsilon_i \Omega_i$ for arbitrary positive $\epsilon_i$. The support function of this sum is a linear combination of the support functions $s_i$, and so the volume is a degree $n+1$ polynomial of the coefficients $\varepsilon_i$:

$$\mathrm{Vol}\left(\sum \epsilon_i \Omega_i\right) = \frac{1}{n+1} \sum_{1 \le i_0, \ldots, i_n \le N} \epsilon_{i_0} \ldots \epsilon_{i_n} V\left(\Omega_{i_0}, \ldots, \Omega_{i_n}\right).$$

The coefficient $V\left(\Omega_{i_0}, \ldots, \Omega_{i_n}\right)$ is called the *mixed volume* of the $n+1$ regions $\Omega_{i_0}, \ldots, \Omega_{i_n}$, and is given by

$$V\left(\Omega_0, \ldots, \Omega_n\right) = \int_{S^n} s_0 \mathcal{Q}\left[s_1, \ldots, s_n\right] d\mu$$

where $\mathcal{Q}$ is given in terms of the bilinear forms $\mathfrak{r}[s_i]$ by

$$\mathcal{Q}\left[s_1, \ldots, s_n\right] = \frac{1}{n!} \sum_{\sigma, \tau \in S_n} \mathrm{sgn}(\tau)\, \mathrm{sgn}(\sigma) \mathfrak{r}\left[s_1\right]_{\tau(1)}^{\sigma(1)} \ldots \mathfrak{r}\left[s_n\right]_{\tau(n)}^{\sigma(n)}, \tag{6}$$

where the sum is over all pairs of permutations on $n$ elements. The operator $\mathcal{Q}$ is called the *mixed discriminant* of $s_1, \ldots, s_n$ (see [Al2], and [Hö, Proposition 2.1.31]).

**Proposition 3.** (1) $\mathcal{Q}$ is symmetric: $\mathcal{Q}[f_1, \ldots, f_n] = \mathcal{Q}\left[f_{\sigma_1}, \ldots, f_{\sigma_n}\right]$ for any permutation $\sigma$;

(2) $\mathcal{Q}[f_1, \ldots, f_n] > 0$ for any $f_1, \ldots, f_n$ with $\mathfrak{r}[f_i]$ positive definite;

(3) If $\mathfrak{r}[f_i]$ is positive definite for $i = 2, \ldots, n$, then $\mathcal{Q}[f] := \mathcal{Q}[f, f_2, \ldots, f_n]$ is a nondegenerate second-order linear elliptic operator:

$$\mathcal{Q}[f] = \sum_{i,j} \dot{\mathcal{Q}}^{ij}\left(\nabla_i \nabla_j f + g_{ij} f\right)$$

where $\dot{\mathcal{Q}} = \dot{\mathcal{Q}}[f_2, \ldots, f_n]$ is positive definite and symmetric;

(4) For any $f_2, \ldots, f_n$, $\sum_i \nabla_i \dot{\mathcal{Q}}^{ij} = 0$.

(5) If $\mathfrak{r}[f_i] > 0$ for $i = 2, \ldots, n$, then

$$\mathcal{Q}[f_1, f_1, f_3, \ldots, f_n]\, \mathcal{Q}[f_2, f_2, f_3, \ldots, f_n] \leq \mathcal{Q}[f_1, f_2, f_3, \ldots, f_n]^2 . \qquad \square$$

Property (5) amounts to a concavity property for mixed discriminants:

$$\mathcal{Q}\left[\underbrace{s, \ldots, s}_{k \text{ times}}, s_{k+1}, \ldots, s_n\right]^{1/k}$$

is a concave function of the components of $\mathfrak{r}[s]$, provided $\mathfrak{r}[s_j] > 0$ for $j = k+1, \ldots, n$.

These properties of $\mathcal{Q}$ allow us to deduce some important properties of the mixed volumes, as shown in the following.

**Proposition 4.** For $\Omega_0, \Omega_0', \Omega_1, \ldots, \Omega_n \subset \mathbb{R}^{n+1}$ convex and $p \in \mathbb{R}^{n+1}$, $V$ is:

(1) Symmetric: $V(\Omega_0, \ldots, \Omega_n) = V(\Omega_{\sigma_0}, \ldots, \Omega_{\sigma_n})$ for any permutation $\sigma$;

(2) Translation-invariant: $V(\Omega_0 + p, \Omega_1, \ldots, \Omega_n) = V(\Omega_0, \Omega_1, \ldots, \Omega_n)$;

(3) Positive: $V(\Omega_0, \ldots, \Omega_n) \geq 0$;

(4) Monotone: If $\Omega_0 \subseteq \Omega_0'$, then $V(\Omega_0, \Omega_1, \ldots, \Omega_n) \leq V(\Omega_0', \Omega_1, \ldots, \Omega_n)$. $\square$

Property (1) follows from statement (4) of Proposition 3, which allows integration by parts. Property (2) follows because $\mathfrak{r}[\langle z, p\rangle] = 0$. Positivity follows since we can choose the origin to make $s_0$ positive, and $\mathcal{Q}[s_1, \ldots, s_n]$ is positive by part (2) of Proposition 3. Monotonicity follows since $s_0 \leq s_0'$ and $\mathcal{Q}[s_1, \ldots, s_n] \geq 0$.

## The Aleksandrov-Fenchel inequalities

The Aleksandrov-Fenchel inequalities relate the various mixed volumes which can be formed from a collection of convex regions.

**Theorem 5 ([Al1], [Al2], [Fe]).** For $\Omega_0, \ldots, \Omega_n \subset \mathbb{R}^{n+1}$ bounded and convex,

$$V\left(\Omega_0, \Omega_0, \Omega_2, \ldots, \Omega_n\right) V\left(\Omega_1, \Omega_1, \Omega_2, \ldots, \Omega_n\right) \leq V\left(\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_n\right)^2 . \qquad \square$$

Mixed discriminant flows

For convenience, we define for $k \in \{1,\ldots,n\}$ and functions $s, s_{k+1},\ldots,s_n$ the $k$th order mixed discriminant

$$Q_k[s; s_{k+1},\ldots,s_n] = Q\left[\underbrace{s,\ldots,s}_{k \text{ times}}, s_{k+1},\ldots,s_n\right]. \tag{7}$$

By a *mixed discriminant flow* we mean a flow of the form (1) where

$$F(\bar{x}(z)) = \psi(z) Q_k[s;\aleph]^{-\alpha}$$

for some $\alpha > 0$ and $k \in \{1,\ldots,n\}$, and $\psi\colon\ S^n \to \mathbb{R}$ smooth and strictly positive. Here $s$ is the support function of the evolving convex region, and $\aleph$ denotes some fixed collection of smooth functions $s_{k+1},\ldots,s_n$ such that $\mathfrak{r}[s_i]$ is positive definite for $i = k+1,\ldots,n$.

Particular examples of mixed discriminant flows are the Gauss curvature flows (with $k = n$), and the anisotropic harmonic mean curvature flows (with $k = 1$). If $\aleph$ is taken to consist of $n-k$ copies of the unit ball and $\psi \equiv 1$, the corresponding flow takes the form $F = E_k(r_1,\ldots,r_n)^{-\alpha} = K/E_{n-k}(\kappa_1,\ldots,\kappa_n)^{\alpha}$, where $r_1,\ldots,r_n$ are the principal radii of curvature, $\kappa_1,\ldots,\kappa_n$ are the principal curvatures, and $E_k$ is the $k$th elementary symmetric function.

In considering the mixed discriminant flows, it is useful to work with the induced evolution equation for the support function $s$:

$$\frac{\partial}{\partial t}s = -\psi Q_k[s;\aleph]^{-\alpha}. \tag{8}$$

Property (3) of Proposition 3 shows that this is a fully nonlinear second order scalar parabolic partial differential equation, and property (5) shows that the right-hand side of equation (8) is a concave function of the second derivatives of $s$. A smooth solution of equation (8) with $\mathfrak{r}[s] > 0$ can be used to construct a smooth, strictly convex solution of the original equation (1), and vice versa: The embeddings given by equation (4) give such a solution after suitable reparametrisation.

## 3 Integral quantities

In this section, we introduce a family of integral quantities associated with any mixed discriminant flow, and show that any homothetic solution is a critical point of every one of these functionals.

Fix a number $\alpha > 0$, a smooth, strictly positive function $\psi$ on $S^n$, an integer $k \in \{1,\ldots,n\}$, and a collection $\aleph = \{s_{k+1},\ldots,s_n\}$ of support functions of smooth, strictly convex regions (if $k < n$). We define the mixed volume $V_{k+1}[s;\aleph]$ by

$$V_{k+1}[s;\aleph] = V\left[\underbrace{s,\dots,s}_{k+1\ \text{times}}, s_{k+1},\dots,s_n\right] = \int_{S^n} s\mathcal{Q}_k[s;\aleph]\,d\mu.$$

We denote by $\tilde{s}$ the support function of the region given by rescaling to constant $V_{k+1}$:

$$\tilde{s} = s\left(\frac{|S^n|}{V_{k+1}[s;\aleph]}\right)^{\frac{1}{k+1}}.$$

Then for any function $G\colon \mathbb{R}\to\mathbb{R}$, we define

$$\mathcal{Z}_G = \int_{S^n} \tilde{s}\mathcal{Q}_k[\tilde{s},\aleph]G\left(\frac{\psi}{\tilde{s}\mathcal{Q}_k[\tilde{s};\aleph]^\alpha}\right)d\mu.$$

In particular, for each real number $\beta$, we take $\mathcal{Z}_\beta$ to be $\mathcal{Z}_G$ where $G(x) = x^\beta$:

$$\mathcal{Z}_\beta = V_{k+1}[s;\aleph]^{\beta-1+k(\alpha\beta-1)}\left(\int_{S^n} s\mathcal{Q}_k[s;\aleph]\left(\frac{\psi}{s\mathcal{Q}_k[s;\aleph]^\alpha}\right)^\beta d\mu\right)^{k+1}.$$

In the special case $\alpha = 1$, this quantity is trivial when $\beta = 1$, and we instead modify the definition to give two separate integrals:

$$\mathcal{Z}_1^+ = \exp\left\{\frac{1}{|S^n|}\int_{S^n}\psi\log\mathcal{Q}_k[s;\aleph]\,d\mu\right\}V_{k+1}[s;\aleph]^{-\frac{k}{k+1}},$$

which appears as the limit of $\mathcal{Z}_1^{1/(1-\alpha)}$ as $\alpha\to 1$, and

$$\mathcal{Z}_1^- = \exp\left\{\frac{1}{|S^n|}\int_{S^n}\psi\log s\,d\mu\right\}V_{k+1}[s;\aleph]^{-\frac{1}{k+1}},$$

which is the limit of $\mathcal{Z}_{1/\alpha}^{\alpha/(\alpha-1)}$ as $\alpha\to 1$.

The special case $\beta = 1$ (or $\mathcal{Z}_1^+$ if $\alpha = 1$) gives the *entropy*, considered before for these flows in [An3], and in special cases before that in [Ha] and [Ch3]. In the case $\alpha = 1$, $k = n$, $\psi\equiv 1$, the quantity $\mathcal{Z}_1^-$ was considered by Firey in [Fi]. The quantity $\mathcal{Z}_{1/\alpha}$ in the case $n = 1$ played a role in [An6].

**Proposition 6.** If $s$ is the support function of a solution of equation (2), then $s$ is a critical point of the quantity $\mathcal{Z}_G$ for any smooth function $G$. □

Proof. We consider a variation $\frac{\partial}{\partial t}s = \eta$. Then define

$$\begin{aligned}\tilde{\eta} &= \frac{\partial}{\partial t}\tilde{s}\\ &= \frac{\partial}{\partial t}\left(s\left(\frac{|S^n|}{V_{k+1}[s;\aleph]}\right)^{\frac{1}{k+1}}\right)\\ &= \left(\frac{|S^n|}{V_{k+1}[s;\aleph]}\right)^{\frac{1}{k+1}}\left(\eta - \frac{s}{V_{k+1}[s;\aleph]}\int_{S^n}\eta\mathcal{Q}_k[s;\aleph]\,d\mu\right),\end{aligned}$$

so that $\int_{S^n} \tilde{\eta}\mathcal{Q}[\tilde{s};\aleph]\,d\mu = 0$. Then

$$\frac{\partial}{\partial t}\mathcal{Z}_G = \int_{S^n} \tilde{\eta}\mathcal{Q}_k[\tilde{s};\aleph]G\,d\mu + k\int_{S^n} \tilde{s}\mathcal{Q}[\tilde{\eta}, s, \ldots, s;\aleph]G\,d\mu$$
$$- \int_{S^n} \tilde{s}\mathcal{Q}_k[\tilde{s};\aleph]G' \frac{\psi}{\tilde{s}\mathcal{Q}_k[\tilde{s};\aleph]^\alpha}\left(\frac{\tilde{\eta}}{\tilde{s}} + k\alpha\frac{\mathcal{Q}[\tilde{\eta}, s, \ldots, s;\aleph]}{\mathcal{Q}_k[\tilde{s};\aleph]}\right)\,d\mu.$$

In the special case where equation (2) holds, we have

$$\frac{\psi}{\tilde{s}\mathcal{Q}_k[\tilde{s};\aleph]^\alpha} = c$$

for some constant $c$, and so $G$ and $G'$ are constants, and

$$\frac{\partial}{\partial t}\mathcal{Z}_G = G(c)\int_{S^n} \tilde{\eta}\mathcal{Q}_k[\tilde{s};\aleph] + k\tilde{s}\mathcal{Q}[\tilde{\eta}, \tilde{s}, \ldots, \tilde{s};\aleph]\,d\mu$$
$$- cG'(c)\int_{S^n} \tilde{\eta}\mathcal{Q}_k[\tilde{s};\aleph] + \alpha k\tilde{s}\mathcal{Q}[\tilde{\eta}, \tilde{s}, \ldots, \tilde{s};\aleph]\,d\mu.$$

The identity (4) of Proposition 4 gives

$$\int_{S^n} \tilde{s}\mathcal{Q}[\tilde{\eta}, \tilde{s}, \ldots, \tilde{s};\aleph]\,d\mu = \int_{S^n} \tilde{\eta}\mathcal{Q}_k[\tilde{s};\aleph]\,d\mu$$

after integrating by parts twice. Therefore, for a solution of equation (2) we have

$$\frac{\partial}{\partial t}\mathcal{Z}_G = \big((1+k)G(c) - (1+\alpha k)cG'(c)\big)\int_{S^n} \tilde{\eta}\mathcal{Q}_k[\tilde{s};\aleph]\,d\mu = 0. \qquad \blacksquare$$

## 4 Monotonicity in time

In this section, we show for any $\alpha$ that there is a nontrivial range of $\beta$ for which $\mathcal{Z}_\beta$ evolves monotonically in time for a solution of an MDF.

**Theorem 7.** For a positive solution $s$ of equation (8), the quantity $\mathcal{Z}_\beta$ increases if $\alpha < 1$, and decreases if $\alpha > 1$, provided that

$$\beta \in \begin{cases} [1, \beta_-] \cup [\beta_+, 1/\alpha], \text{ if } k > 8 \text{ and } 0 < \alpha < 1 + \frac{4(1-\sqrt{k+1})}{k}, \\ [1, 1/\alpha], \text{ if } k \le 8 \text{ or } 1 + \frac{4(1-\sqrt{k+1})}{k} \le \alpha \le 1, \\ [1/\alpha, 1], \text{ if } 1 \le \alpha \le 1 + \frac{4(1+\sqrt{k+1})}{k}, \\ [1/\alpha, \beta_-] \cup [\beta_+, 1], \text{if } \alpha > 1 + \frac{4(1+\sqrt{k+1})}{k}, \end{cases}$$

where

$$\beta_\pm = \frac{k(1+\alpha) - 2 \pm \sqrt{(k(1+\alpha)-2)^2 - 4(k+1)(1+k\alpha)}}{2(1+k\alpha)}.$$

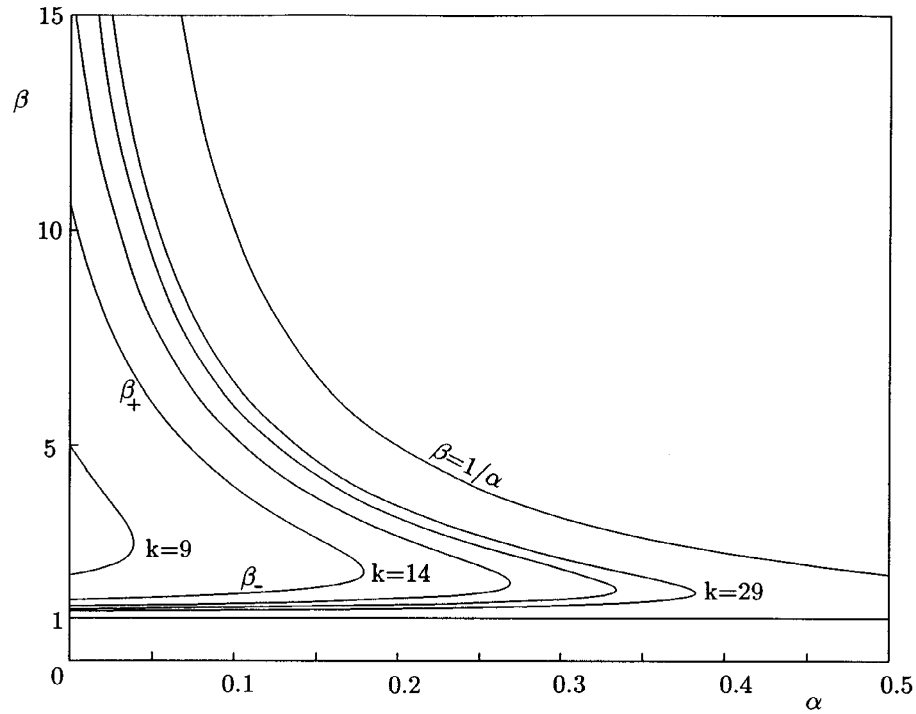


**Figure 1** Small $\alpha$: Graphs of $\beta = 1/\alpha$, and of $\beta_\pm$ for various $k$. The upper part of the curve for any given $k$ is the graph of $\beta_+$, and the lower curve gives $\beta_-$. The allowed values of $\beta$ are those which lie between 1 and $1/\alpha$ but not between $\beta_-$ and $\beta_+$.

In the case $\alpha = 1$, $\mathcal{Z}_1^-$ decreases and $\mathcal{Z}_1^+$ increases. The time derivative is zero only when equation (2) is satisfied (possibly after translation if $\beta = 1$, or in the case of $\mathcal{Z}_1^+$ for $\alpha = 1$). □

Proof. From equation (8) and the definition of $\tilde{s}$, we have

$$\frac{\partial}{\partial t}\tilde{s} = \left(\frac{|S^n|}{V_{k+1}[s;\aleph]}\right)^{\frac{1+k\alpha}{1+k}} \left(-\psi Q_k[\tilde{s};\aleph]^{-\alpha} + \frac{\tilde{s}}{|S^n|}\int_{S^n} \psi Q_k[\tilde{s};\aleph]^{1-\alpha}\, d\mu\right).$$

It is convenient to define a new time variable $\tau$ by

$$\tau = \int_0^t \left(\frac{|S^n|}{V_{k+1}[s_u;\aleph]}\right)^{1+k\alpha/1+k} du,$$

so that

$$\frac{\partial}{\partial \tau}\tilde{s} = -\psi Q_k[\tilde{s};\aleph]^{-\alpha} + \frac{\tilde{s}}{|S^n|}\int_{S^n} \psi Q_k[\tilde{s};\aleph]^{1-\alpha}\, d\mu.$$

In the following calculation, we denote by $\rho$ the quantity $\psi/\tilde{s}Q_k[\tilde{s};\aleph]^\alpha$, and use the abbre-

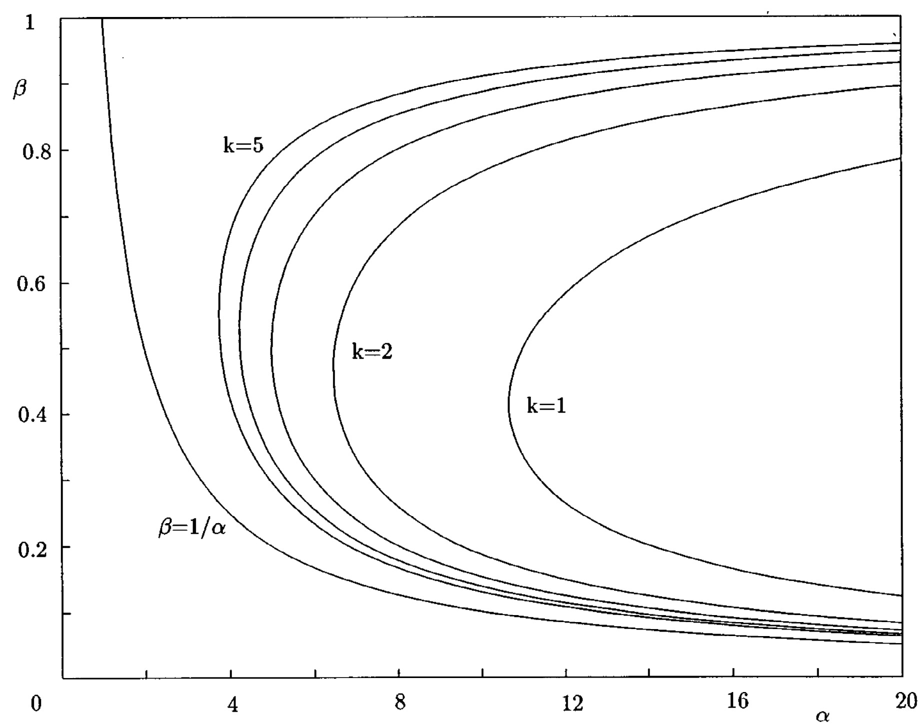


**Figure 2** Large $\alpha$: The allowed values of $\beta$ are again those which are between $1/\alpha$ and 1 but not between $\beta_-$ and $\beta_+$. $\beta_\pm$ are graphed for various values of $k$. For any $k$ and $\alpha$, there are always allowed values of $\beta$ close to 1 and to $1/\alpha$.

viations $\mathcal{Q}_k = \mathcal{Q}_k[\tilde{s};\aleph]$ and $\mathcal{Q}_k[f] = \mathcal{Q}[f,\tilde{s},\dots,\tilde{s};\aleph]$ for any function $f$:

$$\begin{aligned}\frac{\partial}{\partial\tau}\mathcal{Z}_\beta &= -(1-\beta)\left(\mathcal{Z}_{\beta+1} - \frac{1}{|S^n|}\mathcal{Z}_1\mathcal{Z}_\beta\right)\\ &\qquad - k(1-\alpha\beta)\left(\int_{S^n}\tilde{s}\rho^\beta\mathcal{Q}_k[\tilde{s}\rho]\,d\mu - \frac{1}{|S^n|}\mathcal{Z}_\beta\mathcal{Z}_1\right).\end{aligned}$$

Consider the second bracket in more detail: By property (3) of Proposition 3, we have

$$\begin{aligned}\int_{S^n}\tilde{s}\rho^\beta\mathcal{Q}_k[\tilde{s}\rho]\,d\mu &= \int_{S^n}\tilde{s}\rho^\beta\dot{\mathcal{Q}}^{ij}\left(\nabla_i\nabla_j(\tilde{s}\rho) + g_{ij}\tilde{s}\rho\right)\,d\mu\\ &= \int_{S^n}\tilde{s}\rho^\beta\dot{\mathcal{Q}}^{ij}\left(\mathfrak{r}[s]_{ij}\rho + 2\nabla_i s\nabla_j\rho + \tilde{s}\nabla_i\nabla_j\rho\right)\,d\mu\\ &= \mathcal{Z}_{1+\beta} - \beta\int_{S^n}\tilde{s}^2\dot{\mathcal{Q}}^{ij}\nabla_i\rho\nabla_j\rho^\beta\,d\mu\\ &= \mathcal{Z}_{1+\beta} - \frac{4\beta}{(1+\beta)^2}\int_{S^n}\tilde{s}^2\dot{\mathcal{Q}}^{ij}\nabla_i\left(\rho^{\frac{1+\beta}{2}}\right)\nabla_j\left(\rho^{\frac{1+\beta}{2}}\right)\,d\mu,\end{aligned}$$

where we used the identity (4) from Proposition 3 to integrate by parts.

**Lemma 8.** For any $f \in C^\infty(S^n)$,

$$\int_{S^n} \tilde{s}^2 \dot{\mathcal{Q}}^{ij} \nabla_i f \nabla_j f \, d\mu \geq \int_{S^n} \tilde{s} \mathcal{Q}_k f^2 \, d\mu - \frac{1}{|S^n|} \left( \int_{S^n} \tilde{s} \mathcal{Q}_k f \, d\mu \right)^2 ,$$

with equality if and only if $f = c + 1/\tilde{s}\langle z, p\rangle$ for some constant $c$ and some $p \in \mathbb{R}^{n+1}$. □

Proof. The Aleksandrov-Fenchel inequalities give

$$V[\Omega', \Omega'; \aleph] V[\Omega, \Omega; \aleph] \leq V[\Omega, \Omega', \aleph]^2$$

for any convex regions $\Omega$ and $\Omega'$. Furthermore, equality holds if and only if $\Omega$ and $\Omega'$ are scaled translates of each other ([Sc, Theorem 6.6.8]), since $\aleph$ consists of support functions of smooth, strictly convex regions.

Fix $f \in C^\infty(S^n)$, and let $\Omega$ have support function $\tilde{s}$. For $c$ sufficiently large, $\mathfrak{r}[(f+c)\tilde{s}]$ is positive definite, and so $(f + c)\tilde{s}$ is the support function of some convex region $\Omega'$. The Aleksandrov-Fenchel inequality then reads

$$\begin{aligned}
0 &\geq \int_{S^n} \tilde{s}(f + c)\mathcal{Q}[\tilde{s}(f + c)] \, d\mu \int_{S^n} \tilde{s}\mathcal{Q}_k \, d\mu - \left( \int_{S^n} \tilde{s}(f + c)\mathcal{Q}_k \, d\mu \right)^2 \\
&= \left( c^2 \int_{S^n} \tilde{s}\mathcal{Q}_k \, d\mu + 2c \int_{S^n} \tilde{s}\mathcal{Q}_k f \, d\mu + \int_{S^n} \tilde{s} f \mathcal{Q}_k[\tilde{s} f] \, d\mu \right) \int_{S^n} \tilde{s}\mathcal{Q}_k \, d\mu \\
&\quad - c^2 \left( \int_{S^n} \tilde{s}\mathcal{Q}_k \, d\mu \right)^2 - 2c \int_{S^n} \tilde{s}\mathcal{Q}_k \, d\mu \int_{S^n} \tilde{s}\mathcal{Q}_k f \, d\mu - \left( \int_{S^n} \tilde{s}\mathcal{Q}_k f \, d\mu \right)^2 \\
&= |S^n| \int_{S^n} \tilde{s} f \mathcal{Q}_k[\tilde{s} f] \, d\mu - \left( \int_{S^n} \tilde{s}\mathcal{Q}_k f \, d\mu \right)^2 \\
&= |S^n| \left( \int_{S^n} \tilde{s}\mathcal{Q}_k f^2 \, d\mu - \int_{S^n} \tilde{s}^2 \dot{\mathcal{Q}}^{ij} \nabla_i f \nabla_j f \, d\mu \right) - \left( \int_{S^n} \tilde{s}\mathcal{Q}_k f \, d\mu \right)^2 ,
\end{aligned}$$

where we used the identity $\int_{S^n} \tilde{s}\mathcal{Q}_k \, d\mu = |S^n|$ and integration by parts. ■

We write the evolution equation for $\mathcal{Z}_\beta$ as follows:

$$\begin{aligned}
\frac{\partial}{\partial \tau} \mathcal{Z}_\beta = (\beta - 1) \left( k(1 - \alpha\beta) \frac{1 - \beta}{(1 + \beta)^2} + 1 \right) \mathcal{Z}_{\beta+1} \\
+ \big(1 - \beta + k(1 - \alpha\beta)\big) \frac{\mathcal{Z}_\beta \mathcal{Z}_1}{|S^n|} - \frac{4\beta k(1 - \alpha\beta)}{(1 + \beta)^2} \frac{\mathcal{Z}^2_{(\beta+1)/2}}{|S^n|} \\
+ \frac{4\beta k(1 - \alpha\beta)}{(1 + \beta)^2} \left( \int_{S^n} \tilde{s}^2 \dot{\mathcal{Q}}^{ij} \nabla_i \rho^{\frac{1+\beta}{2}} \nabla_j \rho^{\frac{1+\beta}{2}} \, d\mu - \mathcal{Z}_{\beta+1} + \frac{1}{|S^n|} \mathcal{Z}^2_{\frac{\beta+1}{2}} \right).
\end{aligned}$$

Lemma 8 with $f = \rho^{(1+\beta)/2}$ shows that the quantity in the last bracket is nonnegative. If $\beta > 0$, then the Hölder inequality shows that $|S^n|\mathcal{Z}_{\beta+1}$ is larger than both $\mathcal{Z}_\beta \mathcal{Z}_1$ and $\mathcal{Z}^2_{(\beta+1)/2}$. Therefore, the entire time derivative has a sign, provided that the coefficient of $\mathcal{Z}_{\beta+1}$ has the same sign as $1 - \alpha\beta$. The coefficient of $\mathcal{Z}_{\beta+1}$ is equal to

$$\frac{(\beta - 1)}{(\beta + 1)^2} \big( (1 + k\alpha)\beta^2 + (2 - k(1 + \alpha))\beta + k + 1 \big) ,$$

which has the same sign as $\beta - 1$ unless $\beta_- < \beta < \beta_+$. So outside this range, the time derivative has a sign provided that $1 - \alpha\beta$ and $\beta - 1$ have the same sign. ■

The quantities $\mathcal{Z}_\beta$ depend on the choice of origin (unless $\beta = 1$). By choosing the origin at each time, we obtain the following corollary.

**Corollary 9.** If $s$ is a smooth solution of (8) with $\mathfrak{r}[s] > 0$, with $\Omega_t$ the convex region with support function $s_t$, then for $\beta$ as in Theorem 7, $\inf_{p\in\Omega_t} \mathcal{Z}_\beta[s_t - \langle p, z\rangle]$ increases for $\alpha < 1$, and $\sup_{p\in\Omega_t} \mathcal{Z}_\beta[s_t - \langle p, z\rangle]$ decreases for $\alpha > 1$. If $\alpha = 1$, then $\sup_{p\in\Omega_t} \mathcal{Z}_1^-[s_t - \langle p, z\rangle]$ decreases. □

## 5 Regularity estimates

The main result of this section (Proposition 11) is that a solution which is close in Hausdorff distance to a homothetic solution $\Sigma$ is subsequently smooth, strictly convex, and close to $\Sigma$ in any $C^k$ norm. For this we need to assume that the homothetic solution itself is nondegenerate, in the sense that it is a $C^2$ hypersurface, and that it has speed $F$ strictly positive. This result implies immediately that if there is a subsequence of times on which a solution converges in Hausdorff distance to a homothetic solution (after rescaling and possibly translation), then there is a subsequence of times for which the rescaled solutions converge in $C^\infty$ to $\Sigma$ (Proposition 17).

The most difficult step in the proof of Proposition 11 is the proof of a $C^{1,1}$ bound for the solution $s$. Our proof works only in the case where the solutions are close to the homothetic solution, for reasons which are entirely due to the explicit anisotropy in the operator $\mathcal{Q}_k$—there is no such difficulty in isotropic cases, or in the cases $k = 1$ or $k = n$. Once this bound is established, the evolution equation remains uniformly parabolic, and further regularity follows from the results of Krylov and Safonov [KS] and Schauder estimates.

**Proposition 10.** Suppose $\Sigma$ is a $C^2$ convex hypersurface with support function $\sigma$ satisfying equation (2), where $F$ is of the form (7). If $F > 0$, then $\Sigma$ is $C^\infty$ and strictly convex. □

Proof. For convenience here and henceforward, we first arrange (by rescaling time by a constant and adjusting $\psi$ accordingly) that $c = 1$ in equation (2). The assumption that $\Sigma$ is $C^2$ implies that the principal curvatures are bounded, and hence the principal radii of curvature are bounded below: $\mathfrak{r}[\sigma] \geq C_0 g$. Since $\aleph$ consists of support functions of smooth, strictly convex regions, there exist positive constants $C_1$ and $C_2$ such that $C_1 g_{ij} \leq \mathfrak{r}[s_m]_{ij} \leq C_2 g_{ij}$ for $m = k+1, \dots, n$. Hence by Property (2) of Proposition 3 we have $C_1^{n-k}E_k[\sigma] \le \mathcal{Q}_k[\sigma;\aleph] \le C_2^{n-k}E_k[\sigma]$, where $E_k = \mathcal{Q}_k[\sigma;1,\dots,1]$ is the $k$th elementary symmetric function of the eigenvalues of $\mathfrak{r}[\sigma]$. If the eigenvalues of $\mathfrak{r}[\sigma]$ are $r_1,\dots,r_n$, then

$$E_k = \frac{k!(n-k)!}{n!}\sum_{1\le i_1<\dots<i_k\le n} r_{i_1}\dots r_{i_k} \ge \frac{k}{n} r_{\max} r_{\min}^{k-1}$$

where $r_{\max} = \max_{1\le i\le n} r_i$ and $r_{\min} = \min_{1\le i\le n} r_i$. Therefore

$$r_{\max} \le \frac{nE_k}{k r_{\min}^{k-1}} \le \frac{n\mathcal{Q}_k[\sigma;\aleph]}{kC_1^{n-k}C_0^{k-1}} \le \frac{n(\sup\psi)^{1/\alpha}}{kC_1^{n-k}C_0^{k-1}(\inf F)^{1/\alpha}},$$

so $\mathfrak{r}[\sigma]$ is bounded, and $\Sigma$ is strictly convex. By the identity (2), $\sigma$ satisfies the uniformly elliptic equation

$$\mathcal{Q}_k[\sigma;\aleph]^{1/k} = \left(\frac{\psi}{\sigma}\right)^{1/k\alpha},$$

in which $\mathcal{Q}_k^{1/k}$ is a monotone, concave function of the second derivatives of $\sigma$. By Theorem 5.5 of [K], [Ev], or Theorem 17.14 of [GT], we derive $C^{2,\alpha}$ bounds for $\sigma$. Bounds on higher derivatives follow from Schauder estimates (e.g., [GT, Theorem 6.2]). ■

**Proposition 11.** Let $\Sigma$ be a $C^2$ convex hypersurface with $F > 0$, satisfying equation (2) and having support function $\sigma$. Then for any $t_0 \in (0, 1/(1+k\alpha))$, $\varepsilon > 0$, and $k \ge 1$, there exist $\delta > 0$ such that whenever $s\colon S^n \times [0,T) \to \mathbb{R}$ is a solution of (8) (maximally extended in time) with $|s(z,0) - \sigma(z)| < \delta$ for all $z \in S^n$, then $T > t_0$,

$$\left| s(z,t) - \big(1-(1+k\alpha)t\big)^{\frac{1}{1+k\alpha}}\sigma(z)\right| < \varepsilon$$

for all $z \in S^n$ and all $t \in [0,t_0]$, and

$$\left| s_{t_0} - \big(1-(1+k\alpha)t_0\big)^{\frac{1}{1+k\alpha}}\sigma\right|_{C^k} < \varepsilon. \qquad \square$$

Proof. The Hausdorff distance between the two solutions remains small.

**Lemma 12.** If $(1-\delta)\sigma(z) \le s(z,0) \le (1+\delta)\sigma(z)$ for all $z$, and equation (8) holds, then

$$\big((1-\delta)^{1+k\alpha} - (1+k\alpha)t\big)^{\frac{1}{1+k\alpha}}\sigma \le s(.,t) \le \big((1+\delta)^{1+k\alpha} - (1+k\alpha)t\big)^{\frac{1}{1+k\alpha}}\sigma$$

for all $z \in S^n$ and $0 \le t \le \frac{1}{1+k\alpha}(1-\delta)^{1+k\alpha}$. □

Proof. By the maximum principle for equation (8), the solution $s$ remains between the solutions obtained by evolving $(1\pm\delta)\sigma$. ■

From this we also deduce bounds on the gradient of the support function.

**Lemma 13.** There exists a constant $C$ depending only on $\Sigma$ such that any convex hypersurface $M$ with support function $s$ satisfying $(1-\delta)\sigma \le s \le (1+\delta)\sigma$ necessarily satisfies

$$|D(s-\sigma)| \le C\sqrt{\delta}. \qquad \square$$

Proof. This result is purely geometric, and does not depend on the evolution equation. By Proposition 10 there exist $C_3$ and $C_4$ such that $C_3^{-1} g \le \mathfrak{r}[\sigma] \le C_3 g$ and $C_4^{-1} \le \sigma \le C_4$. By hypothesis, the point $\bar{x}(z)$ on $M$ with normal $z$ lies between the hypersurface $(1+\delta)\Sigma$ and the hyperplane $\langle y, z\rangle = (1-\delta)\sigma(z)$. Equation (4) gives $\bar{x}(z) = s(z)z + Ds(z)$, so it suffices to bound the width of this region in directions perpendicular to $z$. The radii of curvature of $\Sigma$ are bounded by $C_3$, so the region is contained inside a spherical cap of height $2\delta\sigma(z)$ and radius $C_3$, which has width bounded by $\min\{2C_3, 4\sqrt{C_3 C_4 \delta}\}$. ■

Next we control the speed $F$ above and below.

**Lemma 14.** There exist constants $\delta_0 > 0$ and $C_5$ such that if $\delta < \delta_0$ and $s$ satisfies equation (8) with $(1-\delta)\sigma(z) \le s(z,0) \le (1+\delta)\sigma(z)$, then

$$\left|F(z,\sqrt{\delta}) - \sigma(z)\right| \le C_5 \delta^{1/4}. \qquad \square$$

Proof. We use Lemma 12, together with the following (Theorem 5.6 from [An4]):

$$\frac{d}{dt}F + \frac{\alpha F}{(1+\alpha)t} \ge 0. \tag{9}$$

Equivalently, the quantity $Ft^{\alpha/(1+\alpha)}$ is nondecreasing pointwise.

The result of Lemma 12 on the time interval $I = \left[\sqrt{\delta}, \sqrt{\delta}\left(1+\delta^{1/4}\right)\right]$ gives

$$\int_I F(z,t)dt = s\left(z,\sqrt{\delta}\right) - s\left(z,\sqrt{\delta}(1+\delta^{1/4})\right) \le \left(\delta^{3/4} + C\delta\right)\sigma(z).$$

By the estimate (9), we also have

$$F(z,t) \ge F(z,\sqrt{\delta})\left(\frac{t}{\sqrt{\delta}}\right)^{-\alpha/1+\alpha} \ge F(z,\sqrt{\delta})(1 - C\delta^{1/4}),$$

and so

$$\int_I F(z,t)dt \ge F(z,\sqrt{\delta})(1-C\delta^{1/4})\delta^{3/4}.$$

Therefore we have

$$F(z,\sqrt{\delta}) - \sigma(z) \le C\delta^{1/4}.$$

The estimate on $F$ from below follows by applying the same method on the time interval $[\sqrt{\delta}(1-\delta^{1/4}), \sqrt{\delta}]$. ■

**Lemma 15.** For any $t_0 \in \big(0, 1/1 + k\alpha\big)$, there are $\delta_1 > 0$ and $C_6$ such that if $\delta < \delta_1$ and $s$ satisfies (8) with $(1-\delta)\sigma(z) \le s(z,0) \le (1+\delta)\sigma(z)$, then for all $z \in S^n$,

$$\mathfrak{r}[s]_{ij}(z, t_0) \le C_6 g_{ij}. \qquad \square$$

Proof. By Lemmas 12–14 and scaling, we can assume

$$\begin{aligned}
&\left|s(z,t) - \big(1-(1+k\alpha)t\big)^{1/(1+k\alpha)} \sigma(z)\right| \le C\delta;\\
&\left|Ds(z,t) - \big(1-(1+k\alpha)t\big)^{1/(1+k\alpha)} D\sigma(z)\right| \le C\sqrt{\delta};\\
&\left|F(z,t) - \big(1-(1+k\alpha)t\big)^{-k\alpha/(1+k\alpha)} \sigma(z)\right| \le C\delta^{1/4},
\end{aligned}$$

on the time interval $[\sqrt{\delta}, t_0]$. The evolution equation for $\mathfrak{r}[s]$ is as follows:

$$\begin{aligned}
\frac{\partial}{\partial t}\mathfrak{r}[s]_{ij} &= \mathfrak{r}[-F]_{ij}\\
&= \nabla_i\left(-\mathcal{Q}_k^{-\alpha}\nabla_j\psi + \alpha\psi\mathcal{Q}_k^{-(1+\alpha)}\nabla_j\mathcal{Q}_k\right) - g_{ij}\mathcal{Q}_k^{-\alpha}\\
&= \alpha\psi\mathcal{Q}_k^{-(1+\alpha)}\nabla_i\nabla_j\mathcal{Q}_k - \alpha(1+\alpha)\psi\mathcal{Q}_k^{-(2+\alpha)}\nabla_i\mathcal{Q}_k\nabla_j\mathcal{Q}_k\\
&\quad + \alpha\nabla_i\psi\mathcal{Q}_k^{-(1+\alpha)}\nabla_j\mathcal{Q}_k + \alpha\nabla_j\psi\mathcal{Q}_k^{-(1+\alpha)}\nabla_i\mathcal{Q}_k - \mathcal{Q}_k^{-\alpha}\mathfrak{r}[\psi]_{ij}.
\end{aligned} \tag{10}$$

The second derivatives of $\mathcal{Q}_k$ can be expanded as follows:

$$\begin{aligned}
\nabla_i\nabla_j\mathcal{Q}_k &= \nabla_i\left[k\dot{\mathcal{Q}}^{pq}\left[\underbrace{s,\dots,s}_{k-1\text{ times}};\aleph\right]\nabla_j\mathfrak{r}_{pq} + \sum_{a=k+1}^{n}\dot{\mathcal{Q}}^{pq}\left[\underbrace{s,\dots,s}_{k\text{ times}};\aleph\backslash\{s_a\}\right]\nabla_j\mathfrak{r}[s_a]_{pq}\right]\\
&= k\dot{\mathcal{Q}}^{pq}\left[\underbrace{s,\dots,s}_{k-1\text{ times}};\aleph\right]\frac{\nabla_i\nabla_j+\nabla_j\nabla_i}{2}\mathfrak{r}[s]_{pq}\\
&\quad + \sum_{a=k+1}^{n}\dot{\mathcal{Q}}^{pq}\left[\underbrace{s,\dots,s}_{k\text{ times}};\aleph\backslash\{s_a\}\right]\frac{\nabla_i\nabla_j+\nabla_j\nabla_i}{2}\mathfrak{r}[s_a]_{pq}\\
&\quad + k(k-1)\ddot{\mathcal{Q}}^{pq\,mn}\left[\underbrace{s,\dots,s}_{k-2\text{ times}};\aleph\right]\nabla_i\mathfrak{r}[s]_{mn}\nabla_j\mathfrak{r}[s]_{pq}\\
&\quad + k\sum_{a=k+1}^{n}\ddot{\mathcal{Q}}^{pq\,mn}\left[\underbrace{s,\dots,s}_{k-1\text{ times}};\aleph\backslash\{s_a\}\right]\nabla_i\mathfrak{r}[s]_{pq}\nabla_j\mathfrak{r}[s_a]_{mn}
\end{aligned}$$

$$+ k \sum_{a=k+1}^{n} \ddot{Q}^{pq\ mn} \left[ \underbrace{s, \ldots, s}_{k-1 \text{ times}}; \aleph \backslash \{s_a\} \right] \nabla_j \mathfrak{r}[s]_{pq} \nabla_i \mathfrak{r}[s_a]_{mn}$$

$$+ \sum_{\substack{k+1 \le a, b \le n \\ a \ne b}} \ddot{Q}^{pq\ mn} \left[ \underbrace{s, \ldots, s}_{k \text{ times}}; \aleph \backslash \{s_a, s_b\} \right] \nabla_i \mathfrak{r}[s_a]_{pq} \nabla_j \mathfrak{r}[s_b]_{mn}. \tag{11}$$

In order to produce an elliptic operator as the leading term in the evolution equation, we note the following identity for the first term above:

$$\begin{aligned}
(\nabla_p \nabla_q + \nabla_q \nabla_p) \mathfrak{r}[s]_{ij} &= \frac{1}{2} (\nabla_p \nabla_i \mathfrak{r}[s]_{qj} + \nabla_p \nabla_j \mathfrak{r}[s]_{qi} + \nabla_q \nabla_i \mathfrak{r}[s]_{pj} + \nabla_q \nabla_j \mathfrak{r}[s]_{pi}) \\
&= \frac{1}{2} (\nabla_i \nabla_p \mathfrak{r}[s]_{qj} + \nabla_j \nabla_p \mathfrak{r}[s]_{qi} + \nabla_i \nabla_q \mathfrak{r}[s]_{pj} + \nabla_j \nabla_q \mathfrak{r}[s]_{pi}) \\
&\quad + \frac{1}{2} (g_{pq} \mathfrak{r}[s]_{ij} - g_{iq} \mathfrak{r}[s]_{jp} + g_{pj} \mathfrak{r}[s]_{qi} - g_{ij} \mathfrak{r}[s]_{pq} \\
&\qquad + g_{pq} \mathfrak{r}[s]_{ij} - g_{jq} \mathfrak{r}[s]_{ip} + g_{pi} \mathfrak{r}[s]_{qj} - g_{ij} \mathfrak{r}[s]_{pq} \\
&\qquad + g_{pq} \mathfrak{r}[s]_{ij} - g_{ip} \mathfrak{r}[s]_{jq} + g_{qj} \mathfrak{r}[s]_{pi} - g_{ij} \mathfrak{r}[s]_{pq} \\
&\qquad + g_{pq} \mathfrak{r}[s]_{ij} - g_{jp} \mathfrak{r}[s]_{iq} + g_{qi} \mathfrak{r}[s]_{pj} - g_{ij} \mathfrak{r}[s]_{pq}) \\
&= (\nabla_i \nabla_j + \nabla_j \nabla_i) \mathfrak{r}[s]_{pq} + g_{pq} \mathfrak{r}[s]_{ij} - g_{ij} \mathfrak{r}[s]_{pq}.
\end{aligned} \tag{12}$$

The second and last terms in (11) we estimate from above: There exists some constant $C$ such that

$$\nabla_i \mathfrak{r}[s_a]_{pq} \nabla_j \mathfrak{r}[s_b]_{mn} + \nabla_j \mathfrak{r}[s_a]_{pq} \nabla_i \mathfrak{r}[s_b]_{mn} \le C g_{ij} \mathfrak{r}[s_a]_{pq} \mathfrak{r}[s_b]_{mn}$$

and

$$\frac{1}{2} (\nabla_i \nabla_j + \nabla_j \nabla_i) \mathfrak{r}[s_a]_{pq} \le C g_{ij} \mathfrak{r}[s_a]_{pq}$$

for $a, b = k+1, \ldots, n$.

We bound the third term in equation (11) using the concavity property of the mixed discriminants: By item (5) of Proposition 3, we have for each $i$,

$$Q_k \ddot{Q}^{pq\ mn} \left[ \underbrace{s, \ldots, s}_{k-2 \text{ times}}; \aleph \right] \nabla_i \mathfrak{r}[s]_{pq} \nabla_i \mathfrak{r}[s]_{mn} \le \left( \dot{Q}^{pq} \left[ \underbrace{s, \ldots, s}_{k-1 \text{ times}}; \aleph \right] \nabla_i \mathfrak{r}[s]_{pq} \right)^2. \tag{13}$$

The last term here will be controlled in terms of gradients of $Q_k$:

$$k \dot{Q}^{pq} \left[ \underbrace{s, \ldots, s}_{k-1 \text{ times}}; \aleph \right] \nabla_i \mathfrak{r}[s]_{pq} = \nabla_i Q_k - \sum_{a=k+1}^{n} \dot{Q}^{pq} \left[ \underbrace{s, \ldots, s}_{k \text{ times}}; \aleph \backslash \{s_a\} \right] \nabla_i \mathfrak{r}[s_a]_{pq} \tag{14}$$

where we have

$$\left|\dot{\mathcal{Q}}^{pq}\left[\underbrace{s,\ldots,s}_{k\text{ times}};\aleph\backslash\{s_a\}\right]\nabla_i\mathfrak{r}[s_a]_{pq}\right|\le C\mathcal{Q}_k.$$

Combining estimates (10–14), we obtain the following inequality:

$$\begin{aligned}\frac{\partial\mathfrak{r}_{ij}}{\partial t}&\le k\alpha\psi\mathcal{Q}_k^{-(1+\alpha)}\dot{\mathcal{Q}}^{pq}\nabla_p\nabla_q\mathfrak{r}_{ij}-k\alpha\psi\mathcal{Q}_k^{-(1+\alpha)}\mathcal{Q}\left[\underbrace{s,\ldots,s}_{k-1\text{ times}},1;\aleph\right]\mathfrak{r}_{ij}+C\mathcal{Q}_k^{-\alpha}g_{ij}\\&\quad+k\alpha\psi\mathcal{Q}_k^{-(1+\alpha)}\sum_{\substack{k+1\le a,b\le n\\ a\ne b}}\ddot{\mathcal{Q}}^{pq\ mn}\left[\underbrace{s,\ldots,s}_{k-1\text{ times}};\aleph\backslash\{s_a\}\right]\nabla_i\mathfrak{r}[s_a]_{pq}\nabla_j\mathfrak{r}[s]_{mn}\\&\quad+k\alpha\psi\mathcal{Q}_k^{-(1+\alpha)}\sum_{\substack{k+1\le a,b\le n\\ a\ne b}}\ddot{\mathcal{Q}}^{pq\ mn}\left[\underbrace{s,\ldots,s}_{k-1\text{ times}};\aleph\backslash\{s_a\}\right]\nabla_j\mathfrak{r}[s_a]_{pq}\nabla_i\mathfrak{r}[s]_{mn}.\end{aligned}$$

The last two terms in this expression are yet to be controlled. These are the most troublesome terms in the entire estimate, and it is only in controlling these that we require the oscillation of $s/\sigma$ to be small. As will be shown below, these terms can be controlled using the leading elliptic term, at the expense of terms of the form

$$\mathcal{Q}\left[\underbrace{s,\ldots,s}_{k-1\text{ times}},1;\aleph\right]\mathfrak{r}_{ij}.$$

These are in turn controlled using good terms in the evolution equation for $s$, as long as the oscillation of $s/\sigma$ is sufficiently small (note that in the case $k=n$, these terms do not arise). The evolution equation for $s$ can be written as follows:

$$\begin{aligned}\frac{\partial}{\partial t}s&=-\psi\mathcal{Q}_k^{-\alpha}\\&=k\alpha\psi\mathcal{Q}_k^{-1-\alpha}\dot{\mathcal{Q}}^{pq}\nabla_p\nabla_q s-(1+k\alpha)\psi\mathcal{Q}_k^{-\alpha}\\&\quad+k\alpha s\psi\mathcal{Q}_k^{-1-\alpha}\mathcal{Q}\left[\underbrace{s,\ldots,s}_{k-1\text{ times}},1;\aleph\right].\end{aligned}$$

The first term here is elliptic, and the last is bounded above and below, but the second term becomes large if some eigenvalue of $\mathfrak{r}$ is large.

By choosing $\delta_1$ sufficiently small, we can ensure that $s-\rho(t)\sigma$ remains strictly positive up to time $t_0$, where $\rho(t)=(1-\gamma)\int_{S^n}s\,d\mu/\int_{S^n}\sigma\,d\mu$ for some $\gamma>0$ to be chosen later. Consider the function $w$ on $\{(z,v)\in TS^n\colon |v|\ne 0\}$ defined by

$$w(z,v)=\frac{\mathfrak{r}[s](v,v)}{g(v,v)(s-\rho(t)\sigma)}.$$

If a maximum of $w$ occurs at time $t$ at a point $(z,\nu)$, then we assume without loss of generality that $|\nu| = 1$ and that we have normal coordinates $\{z_1,\dots,z_n\}$ about $z$ such that $\nu = \partial_{z_1}$. Then we have local coordinates for $TS^n$ given by $\{z_1,\dots,z_n,\nu_1,\dots,\nu_n\}$. The criticality conditions at the point $(z,\nu)$ are then

$$0 = \partial_{z_i} w = \frac{1}{s-\rho\sigma}\left(\frac{\nabla_i \mathfrak{r}[s]_{11}}{g_{11}} - (\nabla_i s - \rho\nabla_i\sigma)\frac{\mathfrak{r}[s]_{11}}{g_{11}(s-\rho\sigma)}\right)$$

and

$$0 = \partial_{\nu_i} w = \frac{2}{s-\rho\sigma}\left(\frac{\mathfrak{r}[s]_{1i}}{g_{11}} - \frac{\mathfrak{r}[s]_{11}}{g_{11}^2}g_{1i}\right)$$

for $i = 1,\dots,n$. Since the point $(z,\nu)$ is a maximum of $w$, we also have the extremality conditions

$$\begin{aligned}
0 &\geq (\partial_{z_i} + \Lambda_i^k\partial_{\nu_k})(\partial_{z_j} + \Lambda_j^\ell\partial_{\nu_\ell})w \\
&= \frac{1}{s-\rho\sigma}\big(\nabla_i\nabla_j\mathfrak{r}[s]_{11} - w\nabla_i\nabla_j(s-\rho\sigma)\big) + 2\sum_{k,\ell\neq 1}\frac{\Lambda_i^k\Lambda_j^\ell}{s-\rho\sigma}(\mathfrak{r}[s]_{k\ell} - \mathfrak{r}[s]_{11}g_{k\ell}) \\
&\quad + 2\sum_{k\neq 1}\frac{\Lambda_i^k}{s-\rho\sigma}\nabla_1\mathfrak{r}[s]_{kj} + 2\sum_{\ell\neq 1}\frac{\Lambda_j^\ell}{s-\rho\sigma}\nabla_1\mathfrak{r}[s]_{\ell i}
\end{aligned} \tag{15}$$

in the sense that this matrix is positive definite for arbitrary $\Lambda_i^j$. Now consider the evolution equation for $w$ at this maximum point:

$$\begin{aligned}
\frac{\partial}{\partial t}w(z,\nu) &= \frac{1}{s-\rho\sigma}\left(\frac{\partial}{\partial t}\mathfrak{r}[s]_{11} - w\frac{\partial}{\partial t}(s-\rho\sigma)\right) \\
&\leq k\alpha\psi Q_k^{-(1+\alpha)}\dot{Q}^{pq}\Bigg(\frac{1}{s-\rho\sigma}\big(\nabla_p\nabla_q\mathfrak{r}[s]_{11} - w\nabla_p\nabla_q(s-\rho\sigma)\big) \\
&\qquad\qquad + 2\frac{\Lambda_p^m}{s-\rho\sigma}\nabla_1\mathfrak{r}_{qm} + 2\frac{\Lambda_q^m}{s-\rho\sigma}\nabla_1\mathfrak{r}_{pm}\Bigg) \\
&\quad + \frac{1}{s-\rho\sigma}\left(CQ_k^{-\alpha} - 2k\alpha\psi Q_k^{-(1+\alpha)}Q\left[\underbrace{s,\dots,s}_{k-1\text{ times}},1;\aleph\right]\mathfrak{r}_{11}\right) \\
&\quad - \frac{w}{s-\rho\sigma}\left(k\alpha\rho Q\left[\underbrace{s,\dots,s}_{k-1\text{ times}},\sigma;\aleph\right] - (1+k\alpha)\psi Q_k^{-\alpha} + \frac{\int_{S^n}F\,d\mu}{\int_{S^n}\sigma\,d\mu}\sigma\right)
\end{aligned} \tag{16}$$

where

$$\Lambda_i^j = \left(\dot{Q}^{-1}\right)_{ip}\sum_{a=k+1}^{n}\ddot{Q}^{pj\,mn}\left[\underbrace{s,\dots,s}_{k-1\text{ times}};\aleph\backslash\{s_a\}\right]\nabla_1\mathfrak{r}[s_a]_{mn}.$$

The first bracket can be estimated using the inequality (15), provided we have an estimate on the matrices $\Lambda_i^j$. More specifically, we need to estimate the terms produced by (15), which are

$$\frac{2}{s-\rho\sigma}k\alpha\psi\mathcal{Q}_k^{-(1+\alpha)}\sum_{m,n\neq 1}\dot{\mathcal{Q}}^{pq}\Lambda_p^m\Lambda_q^n\left(\mathfrak{r}_{11}g_{mn}-\mathfrak{r}_{mn}\right)$$
$$+\frac{4k\alpha\psi}{s-\rho\sigma}\mathcal{Q}_k^{-(1+\alpha)}\sum_{a=k+1}^{n}\ddot{\mathcal{Q}}^{p1\ mn}\left[\underbrace{s,\dots,s}_{k-1\text{ times}};\aleph\backslash\{s_a\}\right]\nabla_1\mathfrak{r}[s_a]_{mn}\nabla_p\mathfrak{r}_{11}. \tag{17}$$

Consider the first term: There is some constant $C$ such that $-C\mathfrak{r}[s_a]\leq\nabla_\nu\mathfrak{r}[s_a]\leq C\mathfrak{r}[s_a]$ for every unit vector $\nu$ and each $a=k+1,\dots,n$. By the monotonicity of the mixed discriminants (property (2) from Proposition 3), this implies

$$-C\dot{\mathcal{Q}}^{pq}\leq\ddot{\mathcal{Q}}^{pq\ mn}\left[\underbrace{s,\dots,s}_{k-1\text{ times}};\aleph\backslash\{s_a\}\right]\nabla_1\mathfrak{r}[s_a]_{mn}\leq C\dot{\mathcal{Q}}^{pq}.$$

Now $\dot{\mathcal{Q}}^{pq}$ is a positive definite symmetric matrix, and so has a well-defined positive definite square root $\dot{\mathcal{Q}}^{1/2}$. Multiplying by the inverse of this matrix on the left and the right gives

$$-Cg^{pq}\leq\left(\dot{\mathcal{Q}}^{1/2}\right)^{pr}\Lambda_r^s\left(\dot{\mathcal{Q}}^{-1/2}\right)^{sq}\leq Cg^{pq},$$

and so the symmetric matrix $\tilde{\Lambda}$ obtained by conjugating $\Lambda$ by the square root of $\dot{\mathcal{Q}}$ has bounded eigenvalues and hence bounded norm. The terms we must control have the form

$$\frac{2k\alpha\psi}{s-\rho\sigma}\mathcal{Q}_k^{-(1+\alpha)}g^{pq}\tilde{\Lambda}_p^r\tilde{\Lambda}_q^s\left(\dot{\mathcal{Q}}^{1/2}\right)^{rm}\left(\dot{\mathcal{Q}}^{1/2}\right)^{sn}\left(\mathfrak{r}_{11}g_{mn}-\mathfrak{r}_{mn}\right),$$

and are therefore bounded by $C\mathcal{Q}_k^{-(1+\alpha)}\left(\mathfrak{r}_{11}\mathcal{Q}_{k-1}-\mathcal{Q}_k\right)/(s-\rho\sigma)$. Finally, to control the remaining terms in equation (17), we use the criticality condition $\nabla_p\mathfrak{r}_{11}=w\nabla_p(s-\rho\sigma)$. Then we have

$$\begin{aligned}\frac{\partial}{\partial t}w(z,\nu)&\leq C\mathcal{Q}_k^{-(1+\alpha)}\mathcal{Q}_{k-1}w+\frac{Cw}{s-\rho\sigma}\mathcal{Q}_k^{-(1+\alpha)}\mathcal{Q}_{k-1}|\nabla(s-\rho\sigma)|\\&\quad+\frac{C\mathcal{Q}_k^{-\alpha}}{s-\rho\sigma}-C\frac{w\mathcal{Q}_k^{-(1+\alpha)}\mathcal{Q}_{k-1}}{s-\rho\sigma}\\&\leq -Cw\mathcal{Q}_{k-1}\mathcal{Q}_k^{-(1+\alpha)}\frac{1-C|\nabla(s-\rho\sigma)|-C(s-\rho\sigma)}{s-\rho\sigma}+\frac{C\mathcal{Q}_k^{-\alpha}}{s-\rho\sigma}.\end{aligned}$$

The first term here is the most important: We have

$$\mathcal{Q}_{k-1}\geq cE_{k-1}\geq cE_1^{1/(k-1)}E_k^{(k-2)/(k-1)}\geq c\mathfrak{r}_{11}^{1/(k-1)}\mathcal{Q}_k^{1-1/(k-1)}$$

and so

$$\frac{\mathcal{Q}_{k-1}w}{s-\rho\sigma}\geq cw^{1+1/(k-1)}\frac{\mathcal{Q}_k^{1-1/(k-1)}}{(s-\rho\sigma)^{1-1/(k-1)}},$$

and so, provided that $s - \rho\sigma$ and $|\nabla(s - \rho\sigma)|$ are sufficiently small, we have

$$\frac{\partial}{\partial t} w \leq -Cw^{1+1/(k-1)} \frac{\mathcal{Q}_k^{-\alpha-1/(k-1)}}{(s-\rho\sigma)^{1-1/(k-1)}} + C\frac{\mathcal{Q}_k^{-\alpha}}{s-\rho\sigma}.$$

Finally, since the exponent of $w$ is greater than 1 and the remaining terms and coefficients are bounded, we obtain by the maximum principle a bound on $w$, independent of initial data. This completes the proof of Lemma 15, since a bound on $w$ implies a bound on $\mathfrak{r}$.

■

**Lemma 16.** If $C_7^{-1} \leq \mathcal{Q}_k \leq C_7$ and $\mathfrak{r}[s] \leq C_8 g$, then there exists a constant $C_9$ such that $C_9^{-1} g \leq \dot{\mathcal{Q}} \leq C_9 g$. □

Proof. The upper bound on $\dot{\mathcal{Q}}$ follows immediately from the bound on $\mathfrak{r}$. The lower bound is proved as follows: By monotonicity of the mixed discriminants,

$$\dot{\mathcal{Q}}^{ij} \geq c\dot{E}_k^{ij}$$

for some constant c depending only on $\aleph$. The matrix $\dot{E}_k^{ij}$ is diagonal when $\mathfrak{r}[s]$ is diagonal: If $\{e_1, \ldots, e_n\}$ is a basis for which $\mathfrak{r}[s] = \operatorname{diag}(r_1, \ldots, r_n)$, then $\dot{E}_k = \operatorname{diag}(q_1, \ldots, q_n)$, where

$$\begin{aligned}
q_i &= c(k, n) \sum_{\substack{1\leq i_1<\ldots<i_{k-1}\leq n\\ i_j\neq i}} r_{i_1}\ldots r_{i_{k-1}}\\
&\geq c \max_{\substack{1\leq i_1<\ldots<i_{k-1}\leq n\\ i_j\neq i}} r_{i_1}\ldots r_{i_{k-1}}\\
&\geq \frac{c}{\max_{1\leq j\leq n} r_j} \max_{1\leq i_1<\ldots<i_k\leq n} r_{i_1}\ldots r_{i_k}\\
&\geq c' \frac{E_k}{r_{\max}}\\
&\geq c'' \frac{\mathcal{Q}_k}{r_{\max}}\\
&\geq C.
\end{aligned}$$

Hence we have a bound below on the eigenvalues of $\dot{\mathcal{Q}}$, as required. ■

The proof of Proposition 11 now follows from the result of Lemmas 15 and 16: $s$ satisfies a fully nonlinear uniformly parabolic equation (by Lemma 16), which is concave in the second derivatives of $s$. Hence the second derivatives of $s$ are uniformly Hölder continuous by Theorem 5.5 of [K] (see also Corollary 14.9 of [Lm]), and all higher derivatives are uniformly bounded by parabolic Schauder estimates (see for example [Si2] or Theorem 4.9 in [Lm]). Finally, interpolation inequalities (see Theorem 7.28 of [GT]) show that any $C^k$ norm of $s/\sigma$ can be made arbitrarily small by taking the oscillation of $s/\sigma$ sufficiently small, since the $C^{2k}$ norm is bounded. ■

An immediate corollary of Proposition 11 follows.

**Proposition 17.** Suppose $\sigma$ is as in Proposition 10, and $s$ is a solution of (8). If there exist sequences $t_i \to T$, $R_i \to \infty$, and $p_i \in \mathbb{R}^{n+1}$ such that $R_i(s_{t_i} - \langle z, p_i\rangle) \to \sigma$ uniformly, then there exist $t_i'$, $R_i'$, and $p_i'$ such that $R_i'\left(s_{t_i'} - \langle z, p_i'\rangle\right) \to \sigma$ in $C^\infty(S^n)$. □

## 6 The convergence argument

In this section, we adapt a method of [Si1] to complete the proof of Theorem 2. This method involves bounding the distance that the solution can move away from the limit in terms of the change in $\mathcal{Z}_\beta$ for some $\beta$. The proof of this depends crucially on a bound below for the norm of the gradient of $\mathcal{Z}_\beta$ in $L^2$ near its critical point $\sigma$. This bound is proved by reducing to an inequality for real-analytic functions on finite-dimensional spaces proved by Łojasiewicz [L]. For a recent exposition of the Łojasiewicz inequality and related topics, see [MV], especially Theorem 4.14.

The Łojasiewicz estimate has been used to prove convergence for gradient flows of real-analytic functions. In our case, our evolution equations are not gradient flows, but we do have monotone quantities. We will show that the angle between the direction of motion and the gradient of one of these functionals remains acute, with cosine bounded away from zero. This weaker condition suffices for the Łojasiewicz argument.

To illustrate the argument, we first describe an analogous situation for ordinary differential equations: Suppose $E\colon M \to \mathbb{R}$ is a real-analytic function on a (real-analytic) finite-dimensional Riemannian manifold $M$, and $V$ is a vector field on $M$ which satisfies the angle condition $\langle V, \nabla E\rangle \geq c_0|V|\,|\nabla E|$ for some constant $c_0 > 0$. Suppose $x\colon [0,\infty) \to M$ satisfies the ordinary differential equation $\dot{x} = -V(x)$, where $\dot{x} = dx/dt$.

Suppose there is a subsequence of times $\{t_k\}$ approaching infinity such that $x(t_k)$ approaches a limit $x_\infty \in M$. Then $x_\infty$ is necessarily a critical point of $E$, and since $E(x(t))$ is a decreasing function of $t$, we have $\lim_{t\to\infty} E(x(t)) = E(x_\infty)$. We show that $x(t)$ approaches $x_\infty$ as $t \to \infty$. The result of Łojasiewicz [L] is that there exists a neighbourhood $U$ of $x_\infty$ in $M$ and a constant $\theta \in (0, 1/2]$ such that $|\nabla E(\xi)| \geq |E(\xi) - E(x_\infty)|^{1-\theta}$ for all $\xi \in U$.

Given $\varepsilon > 0$, we must show that there exists $T(\varepsilon)$ such that for every $t \geq T(\varepsilon)$ we have $|x(t) - x_\infty| < \varepsilon$. First, decrease $\varepsilon$ if necessary to ensure that $B_\varepsilon(x_\infty) \subset U$. Then choose $k$ sufficiently large to satisfy $|x(t_k) - x_\infty| < 1/2\varepsilon$ and $|E(x(t_k)) - E(x_\infty)| \leq (c_0\theta\varepsilon/2)^{1/\theta}$. Then we have

$$
\begin{aligned}
|\dot{x}| &= |V(x)| \\
&\leq \frac{\langle V, \nabla E\rangle}{c_0|\nabla E|}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{|\dot{E}|}{c_0|\nabla E|} \\
&\leq \frac{1}{c_0}|\dot{E}||E - E(x_\infty)|^{\theta-1} \\
&= \frac{1}{c_0\theta}\left|\frac{d}{dt}\left((E - E(x_\infty))^\theta\right)\right|.
\end{aligned}
$$

Integrating from $t_k$ to any $t > t_k$, we obtain

$$
|x(t) - x(t_k)| \leq \frac{1}{c_0\theta}|E(x(t_k)) - E(x_\infty)|^\theta < \frac{1}{2}\varepsilon,
$$

and so

$$
|x(t) - x_\infty| \leq |x(t) - x(t_k)| + |x_{t_k} + x_\infty| < \varepsilon
$$

for all $t > t_k$. Thus $T(\varepsilon) = t_k$ suffices.

To apply this argument to the present situation, we take the real-analytic function $\mathcal{Z}_{1/\alpha}^{\alpha/(\alpha-1)}$ if $\alpha \neq 1$, and $\mathcal{Z}_-$ if $\alpha = 1$.

As before, we denote by $\tilde{s}$ the rescaled support function

$$
\left(\frac{V_{k+1}[\sigma]}{V_{k+1}[s]}\right)^{1/(k+1)} s,
$$

and define a new time parameter $\tau$ by

$$
\frac{\partial}{\partial\tau} = \left(\frac{V_{k+1}[s]}{V_{k+1}[\sigma]}\right)^{(1+k\alpha)/(k+1)} \frac{\partial}{\partial t}.
$$

**Proposition 18.** There exists a neighbourhood in $C^2$ about $\sigma$ in which

$$
\frac{d}{d\tau}\left(\mathcal{Z}_{1/\alpha}^{\alpha/(\alpha-1)}\right) \leq -C\left\|\frac{\partial}{\partial\tau}\tilde{s}\right\|_{L^2(S^n)}\left\|\nabla^{L^2(S^n)} Z_{1/\alpha}^{\alpha/(\alpha-1)}\right\|_{L^2(S^n)}
$$

for some $C > 0$. □

Proof. Direct calculation gives expressions for $\frac{\partial}{\partial\tau}\tilde{s}$ and $\nabla^{L^2(S^n)} Z_{1/\alpha}^{\alpha/(\alpha-1)}$:

$$
\begin{aligned}
&\frac{\partial}{\partial\tau}\tilde{s} = -\psi\mathcal{Q}_k[\tilde{s}]^{-\alpha} + \mathcal{Z}_1\tilde{s}, \\
&\nabla^{L^2(S^n)} Z_{1/\alpha}^{\alpha/(\alpha-1)} = \mathcal{Z}_{1/\alpha}^{1/(\alpha-1)}\left(\left(\frac{\psi}{\tilde{s}}\right)^{1/\alpha} - \mathcal{Q}_k[\tilde{s}]\mathcal{Z}_{1/\alpha}\right).
\end{aligned}
$$

Let

$$
\phi = \frac{\psi}{\tilde{s}\mathcal{Q}_k[\tilde{s}]^\alpha}.
$$

Then we have

$$
\frac{\partial}{\partial\tau}\tilde{s} = -\tilde{s}\left(\phi - \mathcal{Z}_1\right) \tag{18}
$$

and

$$\nabla^{L^2(S^n)} Z_{1/\alpha}^{\alpha/(\alpha-1)} = \mathcal{Z}_{1/\alpha}^{\alpha/(\alpha-1)} \mathcal{Q}_k[\tilde{s}] \left( \frac{\phi^{1/\alpha}}{\mathcal{Z}_{1/\alpha}} - 1 \right). \tag{19}$$

Taking the $L^2$ norm of each of these, we obtain

$$\begin{aligned} \left\| \frac{\partial}{\partial \tau} \tilde{s} \right\|_{L^2(S^n)}^2 &= \int_{S^n} \tilde{s}^2 \left( \phi - \mathcal{Z}_1 \right)^2 \, d\mu \\ &\le \sup_{S^n} \frac{\tilde{s}}{\mathcal{Q}_k[\tilde{s}]} \left( \mathcal{Z}_2 - \mathcal{Z}_1^2 \right) \end{aligned} \tag{20}$$

and

$$\begin{aligned} \left\| \nabla^{L^2(S^n)} Z_{1/\alpha}^{\alpha/(\alpha-1)} \right\|_{L^2(S^n)}^2 &= \mathcal{Z}_{1/\alpha}^2 \int_{S^n} \mathcal{Q}_k[\tilde{s}]^2 \left( \phi^{1/\alpha} - \mathcal{Z}_{1/\alpha} \right)^2 \, d\mu \\ &\le \mathcal{Z}_{1/\alpha}^2 \sup_{S^n} \frac{\mathcal{Q}_k[\tilde{s}]}{\tilde{s}} \left( \mathcal{Z}_{2/\alpha} - \mathcal{Z}_{1/\alpha}^2 \right). \end{aligned} \tag{21}$$

We can also write the time derivative of $\mathcal{Z}_{1/\alpha}^{\alpha/(\alpha-1)}$ in terms of $\phi$:

$$\frac{d}{d\tau} \mathcal{Z}_{1/\alpha}^{\alpha/(\alpha-1)} = -\mathcal{Z}_{1/\alpha}^{1/(\alpha-1)} \left( \mathcal{Z}_{1+1/\alpha} - \mathcal{Z}_1 \mathcal{Z}_{1/\alpha} \right). \tag{22}$$

We consider any $C^2$ neighbourhood of $\sigma$ consisting of functions $s$ satisfying $\mathfrak{r}[s] > 0$ and the conditions

$$\sup_{S^n} \phi \le e^C \inf_{S^n} \phi \quad \text{and} \quad \left( \sup_{S^n} \frac{\mathcal{Q}_k[\tilde{s}]}{\tilde{s}} \right) \left( \sup_{S^n} \frac{\tilde{s}}{\mathcal{Q}_k[\tilde{s}]} \right) \le e^C$$

for some constant $C > 0$.

We estimate each of the expressions (20–22):

$$\begin{aligned} \mathcal{Z}_{1+\alpha/\alpha} - \mathcal{Z}_1 \mathcal{Z}_{1/\alpha} &= \int_{S^n} \phi^{1+\alpha/\alpha} \, d\tilde{\mu} \int_{S^n} d\tilde{\mu} - \int_{S^n} \phi \, d\tilde{\mu} \int_{S^n} \phi^{1/\alpha} \, d\tilde{\mu} \\ &= \frac{1}{2} \int \phi(x)^{\alpha+1/\alpha} + \phi(y)^{\alpha+1/\alpha} - \phi(x)\phi(y)^{1/\alpha} - \phi(y)\phi(x)^{1/\alpha} \, d\tilde{\mu}(x) \, d\tilde{\mu}(y) \\ &= \frac{1}{2} \int \left[ \phi(x)\phi(y) \right]^{\alpha+1/2\alpha} \left[ \rho^{\alpha+1/\alpha} + \rho^{-\alpha+1/\alpha} - \rho^{\alpha-1/\alpha} - \rho^{1-\alpha/\alpha} \right] d\tilde{\mu}(x) d\tilde{\mu}(y) \\ &\ge \frac{1}{2} \left( \inf_{S^n} \phi \right)^{\alpha+1/\alpha} \int \left( \rho^{\alpha+1/\alpha} + \rho^{-\alpha+1/\alpha} - \rho^{\alpha-1/\alpha} - \rho^{1-\alpha/\alpha} \right) \, d\tilde{\mu}(x) \, d\tilde{\mu}(y) \end{aligned}$$

where

$$\rho(x, y) = \sqrt{\frac{\phi(x)}{\phi(y)}}$$

and $d\tilde{\mu} = \tilde{s}\mathcal{Q}_k[\tilde{s}]d\mu$. By similar arguments, we have

$$\mathcal{Z}_2 - \mathcal{Z}_1^2 \leq \frac{1}{2}\left(\sup_{S^n} \phi\right)^2 \int_{S^n \times S^n} \left(\rho - \rho^{-1}\right)^2 \, d\tilde{\mu}(x)\, d\tilde{\mu}(y)$$

and

$$\mathcal{Z}_{2/\alpha} - \mathcal{Z}_{1/\alpha}^2 \leq \frac{1}{2}\left(\sup_{S^n} \phi\right)^{2/\alpha} \int_{S^n \times S^n} \left(\rho^{1/\alpha} - \rho^{-1/\alpha}\right)^2 \, d\tilde{\mu}(x)\, d\tilde{\mu}(y).$$

**Lemma 19.** If $e^{-C/2} \leq \rho \leq e^{C/2}$, then for $\alpha \geq 1$ we have

$$\rho^{\alpha+1/\alpha} + \rho^{-\alpha+1/\alpha} - \rho^{\alpha-1/\alpha} - \rho^{1-\alpha/\alpha} \geq \alpha\left(\rho^{1/\alpha} - \rho^{-1/\alpha}\right)^2,$$

and

$$\rho^{\alpha+1/\alpha} + \rho^{-\alpha+1/\alpha} - \rho^{\alpha-1/\alpha} - \rho^{1-\alpha/\alpha} \geq \frac{\sinh\left(\frac{C}{2\alpha}\right)}{\sinh\left(\frac{C}{2}\right)}\left(\rho - \rho^{-1}\right)^2.$$

For $0 < \alpha \leq 1$, we have

$$\rho^{\alpha+1/\alpha} + \rho^{-\alpha+1/\alpha} - \rho^{\alpha-1/\alpha} - \rho^{1-\alpha/\alpha} \geq \frac{\sinh\left(\frac{C}{2}\right)}{\sinh\left(\frac{C}{2\alpha}\right)}\left(\rho^{1/\alpha} - \rho^{-1/\alpha}\right)^2,$$

and

$$\rho^{\alpha+1/\alpha} + \rho^{-\alpha+1/\alpha} - \rho^{\alpha-1/\alpha} - \rho^{1-\alpha/\alpha} \geq \frac{1}{\alpha}\left(\rho - \rho^{-1}\right)^2. \qquad \square$$

Proof of Lemma 19. We note that

$$\rho^{\alpha+1/\alpha} + \rho^{-\alpha+1/\alpha} - \rho^{\alpha-1/\alpha} - \rho^{1-\alpha/\alpha} = \left(\rho^{1/\alpha} - \rho^{-1/\alpha}\right)\left(\rho - \rho^{-1}\right),$$

and so

$$\frac{\rho^{\alpha+1/\alpha} + \rho^{-\alpha+1/\alpha} - \rho^{\alpha-1/\alpha} - \rho^{1-\alpha/\alpha}}{\left(\rho - \rho^{-1}\right)^2} = \frac{\rho^{1/\alpha} - \rho^{-1/\alpha}}{\rho - \rho^{-1}},$$

and

$$\frac{\rho^{\alpha+1/\alpha} + \rho^{-\alpha+1/\alpha} - \rho^{\alpha-1/\alpha} - \rho^{1-\alpha/\alpha}}{\left(\rho^{1/\alpha} - \rho^{-1/\alpha}\right)^2} = \frac{\rho - \rho^{-1}}{\rho^{1/\alpha} - \rho^{-1/\alpha}}.$$

Hence it suffices to bound the ratio of $\rho - \rho^{-1}$ and $\rho^{1/\alpha} - \rho^{-1/\alpha}$ from above and below. Let $r = \ln \rho$, so that the quantity we must bound above and below is $\sinh r/\sinh(r/\alpha)$. This has limit $\alpha$ as $r$ approaches zero, and

$$\begin{aligned}\frac{d}{dr}\frac{\sinh r}{\sinh(r/\alpha)} &= \frac{\sinh r}{\sinh(r/\alpha)}\left(\frac{\cosh r}{\sinh r} - \frac{\cosh(r/\alpha)}{\alpha \sinh(r/\alpha)}\right) \\ &= \frac{\sinh r}{\sinh(r/\alpha)}\frac{\left(K(r) - K(r/\alpha)\right)}{r},\end{aligned}$$

where

$$K(a) = \frac{a\cosh(a)}{\sinh(a)},$$

which is increasing in $a$ (with derivative equal to $\sinh(2a) - 2a/2\sinh^2 a > 0$). Hence

$$\frac{\sinh r}{\sinh(r/\alpha)}$$

is increasing for $\alpha \geq 1$ and decreasing for $0 < \alpha \leq 1$. The result follows. ■

Applying the lemma directly, and using the estimate $\sup_{S^n} \phi \leq e^C \inf_{S^n} \phi$, we have

$$\mathcal{Z}_{\alpha+1/\alpha} \geq \sqrt{\min\left\{\frac{\alpha\sinh\left(\frac{C}{2\alpha}\right)}{\sinh\left(\frac{C}{2}\right)}, \frac{\sinh\left(\frac{C}{2}\right)}{\alpha\sinh\left(\frac{C}{2\alpha}\right)}\right\}} e^{-C(\alpha+1)/\alpha}\sqrt{\mathcal{Z}_2 - \mathcal{Z}_1^2}\sqrt{\mathcal{Z}_{2/\alpha} - \mathcal{Z}_{1/\alpha}^2}.$$

From the expression (20–22), this implies

$$\frac{d}{d\tau}\mathcal{Z}_{1/\alpha}^{\alpha-1/\alpha} \leq -\sqrt{\frac{\min\left\{\frac{\alpha\sinh\left(\frac{C}{2\alpha}\right)}{\sinh\left(\frac{C}{2}\right)}, \frac{\sinh\left(\frac{C}{2}\right)}{\alpha\sinh\left(\frac{C}{2\alpha}\right)}\right\}}{\sup_{S^n}\frac{\tilde{s}}{\mathcal{Q}_k[\tilde{s}]}\sup_{S^n}\frac{\mathcal{Q}_k[\tilde{s}]}{\tilde{s}}}} e^{-C(\alpha+1)/\alpha}\left\|\frac{\partial}{\partial\tau}\tilde{s}\right\|\left\|\nabla^{L^2(S^n)}Z_{1/\alpha}^{\alpha/\alpha-1}\right\|.$$

This completes the proof of Proposition 18. ■

**Proposition 20.** There exist $\theta \in (0, 1/2]$ and a neighbourhood in $C^{2,\mu}$ about $\sigma$ in which

$$\left\|\nabla^{L^2(S^n)}Z_{1/\alpha}^{\alpha/\alpha-1}[s]\right\|_{L^2(S^n)} \geq \left\|Z_{1/\alpha}^{\alpha/\alpha-1}[s] - Z_{1/\alpha}^{\alpha/\alpha-1}[\sigma]\right\|_{L^2(S^n)}^{1-\theta}. \qquad \square$$

Proof. Simon [Si1] showed that such an inequality can be deduced for gradients of functionals of a slightly different form. The present case differs in the fact that the gradient is a fully nonlinear elliptic operator rather than a quasilinear one, but the details are otherwise identical. We refer the reader to the proof of Theorem 3 in [Si1]. ■

**Proposition 21.** For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\tilde{s}_\tau$ is rescaled from a solution of equation (8) with $\lim_{t\to\infty}\mathcal{Z}_{1/\alpha}[\tilde{s}_t] = \mathcal{Z}_{1/\alpha}[\sigma]$, and $|\tilde{s}_0 - \sigma|_{L^2} < \delta$, then $\sup_{t\geq 0}|\tilde{s}_t - \sigma|_{C^{2,\mu}} < \varepsilon$. □

Proof. The argument is similar to that described above for ordinary differential equations. Some complications arise because the Łojasiewicz inequality (Proposition 20) and the angle condition (Proposition 18) have been established in a $C^{2,\mu}$ neighbourhood about $\sigma$; but the argument applied directly gives bounds only on the distance travelled by the solution in $L^2$.

The results of Section 5 can be interpreted in terms of the rescaled solution $\tilde{s}$ to give the following, where we fix a positive number $\tau_0$: For every $\varepsilon > 0$, there exists a $\delta_1(\varepsilon) > 0$ such that if $\|\tilde{s}_0 - \sigma\|_{L^2(S^n)} < \delta_1$, then

$$\|\tilde{s}_\tau - \sigma\|_{L^2(S^n)} < \varepsilon$$

for $0 \le \tau \le \tau_0$. Similarly, there exists $\delta_2(\varepsilon)$ such that $\|\tilde{s}_0 - \sigma\|_{L^2(S^n)} < \delta_1$ implies

$$|\tilde{s}_{\tau_0} - \sigma|_{C^{2,\mu}} < \varepsilon.$$

Proposition 18 gives the existence of constants $\varepsilon_1 > 0$ and $c_0 > 0$ such that

$$|\tilde{s} - \sigma|_{C^2} < \varepsilon_1 \quad \Longrightarrow \quad \frac{\partial}{\partial\tau}\mathcal{Z}_{1/\alpha}^{\alpha/(\alpha-1)} \le -c_0 \left\|\nabla\mathcal{Z}_{1/\alpha}^{\alpha/(\alpha-1)}\right\|_{L^2} \left\|\frac{\partial}{\partial\tau}\tilde{s}\right\|_{L^2},$$

and Proposition 20 gives $\varepsilon_2 > 0$ and $\theta \in (0, 1/2]$ such that

$$|\tilde{s} - \sigma|_{C^{2,\mu}} < \varepsilon_2 \implies \left\|\nabla\mathcal{Z}_{1/\alpha}^{\alpha/(\alpha-1)}\right\|_{L^2} \ge \left|\mathcal{Z}_{1/\alpha}[\tilde{s}]^{\alpha/(\alpha-1)} - \mathcal{Z}_{1/\alpha}[\sigma]^{\alpha/(\alpha-1)}\right|^{1-\theta}.$$

Finally, by the continuity of $\mathcal{Z}_{1/\alpha}$ as a functional on $C^2$, for every $\varepsilon > 0$ there exists $\delta_3(\varepsilon) > 0$ such that

$$|\tilde{s} - \sigma|_{C^2} < \delta_3 \quad \Longrightarrow \quad \left|\mathcal{Z}_{1/\alpha}[\tilde{s}]^{\alpha/(\alpha-1)} - \mathcal{Z}_{1/\alpha}[\sigma]^{\alpha/(\alpha-1)}\right| < \varepsilon.$$

Given $\varepsilon > 0$, we let $\varepsilon_* = \min\{\varepsilon, \varepsilon_1, \varepsilon_2\}$, and choose

$$\delta = \min\left\{\delta_1\left(\frac{1}{2}\delta_2\left(\varepsilon_*\right)\right), \delta_2\left(\delta_3\left(\left(\frac{c_0\theta\delta_2(\varepsilon_*)}{2}\right)^{1/\theta}\right)\right)\right\}.$$

This choice guarantees that $\|\tilde{s}_\tau - \sigma\|_{L^2} < 1/2\delta_2(\varepsilon_*)$ for $0 \le \tau \le \tau_0$, and hence $|\tilde{s}_\tau - \sigma|_{C^{2,\mu}} < \varepsilon_*$ for $\tau_0 \le \tau \le 2\tau_0$. Let

$$T = \sup\{\tau_1\colon\ |\tilde{s}_\tau - \sigma|_{C^{2,\mu}} < \varepsilon_* \quad \text{for } \tau_0 \le \tau \le \tau_1\},$$

which is well-defined and greater than or equal to $2\tau_0$ in view of the previous sentence. For $\tau_0 \le \tau \le T$, we have $|\tilde{s}_\tau - \sigma|_{C^{2,\mu}} < \varepsilon_* \le \min\{\varepsilon_1, \varepsilon_2\}$, and so both the Łojasiewicz inequality and the angle condition hold. Therefore

$$\|\tilde{s}_\tau - \sigma\|_{L^2} \le \|\tilde{s}_{\tau_0} - \sigma\|_{L^2} + \frac{1}{c_0\theta}\left|\mathcal{Z}_{1/\alpha}[\tilde{s}_{\tau_0}]^{\alpha/\alpha-1} - \mathcal{Z}_{1/\alpha}[\sigma]^{\alpha/\alpha-1}\right|^{\theta}$$

for $\tau_0 \le \tau \le T$, by the argument described before Proposition 18. But we have

$$\|\tilde{s}_{\tau_0} - \sigma\|_{L^2} < \frac{1}{2}\delta_2(\varepsilon_*),$$

and

$$\|\tilde{s}_{\tau_0} - \sigma\|_{C^{2,\mu}} < \delta_3\left(\left(\frac{c_0\theta\delta_2(\varepsilon_*)}{2}\right)^{1/\theta}\right),$$

and so by the definition of $\delta_3$ we have

$$\frac{1}{c_0\theta}\left|\mathcal{Z}_{1/\alpha}[\tilde{s}_{\tau_0}]^{\alpha/\alpha-1}-\mathcal{Z}_{1/\alpha}[\sigma]^{\alpha/\alpha-1}\right|^{\theta}<\frac{1}{2}\delta_2(\varepsilon_*).$$

Therefore for $\tau_0\le\tau\le T$ (and, as we already know, for $0\le\tau\le\tau_0$) we have

$$\|\tilde{s}_\tau-\sigma\|_{L^2}<\delta_2(\varepsilon_*),$$

so by the definition of $\delta_2$, we have for $\tau_0\le\tau\le T+\tau_0$,

$$|\tilde{s}_\tau-\sigma|_{C^{2,\mu}}<\varepsilon_*.$$

This contradicts the maximality of $T$ if $T<\infty$. Therefore $T=\infty$, and we have $\tilde{s}_\tau$ within distance $\varepsilon_*$ of $\sigma$ in $C^{2,\mu}$ for all positive $\tau$, as required. ■

**Theorem 2**, **restated.** Suppose $s\colon S^n\times[0,T)\to\mathbb{R}$ is a smooth solution of equation (8), and there exist $t_i\to T$, $R_i\to\infty$, and $p_i\in\mathbb{R}^{n+1}$ such that $R_i\left(s_{t_i}-\langle p_i,z\rangle\right)$ converges in $C^0$ to the support function $\sigma$ of a $C^2$ convex hypersurface $\Sigma$ with $F>0$. Then $\Sigma$ satisfies equation (2), and is $C^\infty$ and strictly convex, and there exists $p\in\mathbb{R}^{n+1}$ such that for all $k\ge1$,

$$\lim_{t\to T}\left|\left(\frac{V_{k+1}[\sigma]}{V_{k+1}[s_t]}\right)^{1/k+1}\left(s_t-\langle z,p\rangle\right)-\sigma\right|_{C^k}=0. \qquad \square$$

Proof. First, note that the hypersurfaces $M_t$ defined by the support functions $s_t$ converge to a point $p\in\mathbb{R}^{n+1}$ as $t\to T$: If we denote by $\Omega_t$ the region enclosed by $M_t$, then $\Omega_{t_2}\subset\Omega_{t_1}$ for $t_2>t_1$ by the comparison principle, and since $R_i\to\infty$, we have $\operatorname{diam}M_{t_i}\le 1/R_i\operatorname{diam}\Sigma\to0$ as $i\to\infty$, and so $\operatorname{diam}M_t\to0$ as $t\to T$ since $\operatorname{diam}M_t$ is decreasing in $t$. Then $\bigcap_{0\le t<T}\Omega_t=\{p\}$ for some $p\in\mathbb{R}^{n+1}$.

Next we note that $R_i(p_i-p)$ approaches zero as $i\to\infty$: If not, then there exists some $\varepsilon_0>0$ and a sequence $i_j\to\infty$ such that $|R_{i_j}(p_{i_j}-p)|\ge\varepsilon_0$. Define $\tilde{\varepsilon}=\min\{\varepsilon_0,\sup\sigma\}$. We know that $|R_is_{t_i}-\langle R_ip_i,z\rangle-\sigma|_{C^0}\to0$ as $i\to\infty$. But now choose $I$ sufficiently large to ensure that $|R_is_{t_i}-\langle R_ip_i,z\rangle-\sigma|_{C^0}\le\varepsilon\sup\sigma$ for $i\ge I$, where

$$(1+\varepsilon)^{1+k\alpha}-1<\left(\frac{\tilde{\varepsilon}}{\sup\sigma}\right)^{1+k\alpha}\left(\left(\frac{3}{4}\right)^{1+k\alpha}-\left(\frac{1}{2}\right)^{1+k\alpha}\right)$$

and

$$(1-\varepsilon)^{1+k\alpha}-1>-\left(\frac{\tilde{\varepsilon}}{\sup\sigma}\right)^{1+k\alpha}\left(\left(\frac{1}{2}\right)^{1+k\alpha}-\left(\frac{1}{4}\right)^{1+k\alpha}\right).$$

Then, taking

$$\tau_i=\frac{1-\left(\frac{\tilde{\varepsilon}}{2\sup\sigma}\right)^{1+k\alpha}}{(1+k\alpha)R_i^{1+k\alpha}},$$

we have by the comparison principle (as in Lemma 12) for $i \geq I$,

$$\left((1-\varepsilon)^{1+k\alpha} - (1+k\alpha)R_i^{1+k\alpha}\tau_i\right)\sigma \leq R_i\left(s_{t_i+\tau_i} - \langle p_i, z\rangle\right)$$

and

$$R_i\left(s_{t_i+\tau_i} - \langle p_i, z\rangle\right) \leq \left((1+\varepsilon)^{1+k\alpha} - (1+k\alpha)R_i^{1+k\alpha}\tau_i\right)\sigma;$$

the choices of $\tau_i$ and $\varepsilon$ then imply that

$$\frac{\tilde{\varepsilon}\inf\sigma}{4\sup\sigma} \leq R_i s_{t_i+\tau_i} - \langle R_i p_i, z\rangle \leq \frac{3\tilde{\varepsilon}}{4}.$$

For $j$ sufficiently large, we have $i_j \geq I$, and so the last inequality holds; but also

$$|R_{i_j}(p_{i_j} - p)| \geq \varepsilon_0 \geq \tilde{\varepsilon} > \frac{3\tilde{\varepsilon}}{4} > \sup_{z\in S^n}\left(R_{i_j} s_{t_{i_j}+\tau_{i_j}} - \langle R_{i_j} p_{i_j}, z\rangle\right),$$

and so at time $t_{i_j} + \tau_{i_j}$ the point $p$ is no longer in the enclosed region $\Omega_{t_{i_j}+\tau_{i_j}}$. This is a contradiction, since $\{p\} = \bigcap \Omega_t$. Therefore $|R_i(p_i - p)| \to 0$ as $i \to \infty$.

As a consequence, we can replace the sequence $p_i$ with a constant sequence, since

$$|R_i(s_{t_i} - \langle p, z\rangle) - \sigma| \leq |R_i(s_{t_i} - \langle p_i, z\rangle - \sigma| + |R_i\langle p_i - p, z\rangle|,$$

and both terms on the right approach zero.

By Theorem 7, $\mathcal{Z}_{1/\alpha}[s_t - \langle p, z\rangle]^{\alpha/(\alpha-1)}$ is decreasing in time, and so has limit equal to $\mathcal{Z}_{1/\alpha}[\sigma]^{\alpha/(\alpha-1)}$ since $\mathcal{Z}$ converges on the sequence of times $t_i$. Also, $\tilde{s}_{t_i}$ converges in $C^0$ (and hence in $L^2$) to $\sigma$, and so for any $\varepsilon > 0$ there exists $i$ sufficiently large that $s_{t_i}$ satisfies the conditions of Proposition 21. Therefore, $\tilde{s}_t$ remains within distance $\varepsilon$ of $\sigma$ in $C^{2,\mu}$ for all $t \geq t_i$. This gives convergence in $C^{2,\mu}$. Convergence in all higher $C^k$ norms follows from Proposition 11. ■

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## References

[Al1] A. D. Aleksandrov, *Zur theorie gemischter volumina von konvexen körpern II: Neue ungleichungen zwischen den gemischten volumina und ihre andwendungen*, Mat. Sb. SSSR **2** (1937), 1205–1238.

[Al2] ———, *Zur theorie gemischter volumina von konvexen körpern IV: Die gemischten diskriminanten und die gemischten volumina*, Mat. Sb. SSSR **3** (1938), 227–251.

[An1] B. Andrews, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. Partial Differential Equations **2** (1994), 151–171.

[An2] ———, *Contraction of convex hypersurfaces in Riemannian spaces*, J. Differential Geom. **39** (1994), 407–431.

[An3] ———, *Entropy estimates for evolving hypersurfaces*, Comm. Anal. Geom. **2** (1994), 53–64.

[An4] ———, *Harnack inequalities for evolving hypersurfaces*, Math. Z. **217** (1994), 179–197.

[An5] ———, *Contraction of convex hypersurfaces by their affine normal*, J. Differential Geom. **43** (1996), 207–230.

[An6] ———, *Evolving convex curves*, preprint.

[An7] ———, *Instability of limiting shapes for curves evolving by curvature*, preprint.

[An8] ———, *Evolution of hypersurfaces by Gauss curvature*, preprint.

[An9] ———, *Eigenvalue estimates of Hersch type with applications to geometric evolution equations*, preprint.

[An10] ———, *Aleksandrov-Fenchel inequalities and curvature flows*, preprint.

[AG1] S. Angenent and M. E. Gurtin, *Multiphase thermodynamics with interfacial structure*, *2. Evolution of an isothermal interface*, Arch. Rational Mech. Anal. **108** (1989), 323–391.

[AG2] ———, *Anisotropic motion of a phase interface: Well-posedness of the initial value problem and qualitative properties of the interface*, J. Reine Angew. Math. **446** (1994), 1–47.

[Ch1] B. Chow, *Deforming convex hypersurfaces by the* $n$*th root of the Gaussian curvature*, J. Differential Geom. **22** (1985), 117–138.

[Ch2] ———, *Deforming hypersurfaces by the square root of the scalar curvature*, Invent. Math. **87** (1987), 63–82.

[Ch3] ———, *On Harnack's inequality and entropy for the Gaussian curvature flow*, Comm. Pure Appl. Math. **44** (1991), 469–483.

[Ev] L. C. Evans, *Classical solutions of fully nonlinear*, *convex*, *second order elliptic equations*, Comm. Pure Appl. Math. **35** (1982), 333–363.

[Fe] W. Fenchel, *Inégalités quadratiques entre les volumes mixtes des corps convexes*, C. R. Acad. Sci. Paris Sér. I Math. **203** (1936), 647–650.

[Fi] W. J. Firey, *Shapes of worn stones*, Mathematika **21** (1974), 1–11.

[GH] M. E. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom. **23** (1986), 69–96.

[Ge] C. Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom. **32** (1990), 199–314.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, *2d ed.*, Grundlehren Math. Wiss. **224**, Springer-Verlag, Berlin, 1983.

[Gu] M. E. Gurtin, *Multiphase thermodynamics with interfacial structure*, *I. Heat construction and the capillary balance law*, Arch. Rational Mech. Anal. **104** (1988), 195–221.

[Ha] R. S. Hamilton, *Heat Equations in Geometry*, lecture notes, University of Hawaii, 1989.

[Hö] L. Hörmander, *Notions of Convexity*, Progr. Math. **127**, Birkhäuser, Boston, 1994.

[Hu1] G. Huisken, *Flow by mean curvature of convex hypersurfaces into spheres*, J. Differential Geom. **20** (1984), 237–268.

[Hu2] ———, *On the expansion of convex hypersurfaces by the inverse of symmetric curvatures functions*, to appear.

[K] N. V. Krylov, *Nonlinear Elliptic and Parabolic Equations of the Second Order*, Math. Appl.

(Soviet Ser.) **7**, D. Reidel Publ. Co., Dordrecht, 1987.

[KS] N. V. Krylov and M. V. Safonov, *A property of the solutions of parabolic equations with measureable coefficients*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), 161–175.

[Lm] G. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, Singapore, 1997.

[L] S. Łojasiewicz, *Ensembles semi-analytiques*, IHES notes, 1965.

[MV] C. Miller and L. van den Dries, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–540.

[ST] G. Sapiro and A. Tannenbaum, *On affine plane curve evolution*, J. Funct. Anal. **119** (1994), 79–120.

[Sc] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia Math. Appl. **44**, Cambridge University Press, Cambridge, 1993.

[Si1] L. Simon, *Asymptotics for a class of nonlinear evolution equations*, *with applications to geometric problems*, Ann. of Math. **118** (1983), 525–571.

[Si2] ———, *Schauder estimates by scaling*, preprint.

[T] K. Tso, *Deforming a hypersurface by its Gauss-Kronecker curvature*, Comm. Pure Appl. Math. **38** (1985), 867–882.

[U1] J. I. E. Urbas, *An expansion of convex surfaces*, J. Differential Geom. **33** (1991), 91–125.

[U2] ———, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. **205** (1990), 355–372.

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