

MONOTONICITY OF ENTROPY FOR REAL MULTIMODAL MAPS

HENK BRUIN AND SEBASTIAN VAN STRIEN

ABSTRACT. In 1992, Milnor [28] posed the *Monotonicity Conjecture* that within a family of real multimodal polynomial interval maps with only real critical points, the *isentropes*, *i.e.*, the sets of parameters for which the topological entropy is constant, are connected. This conjecture was already proved in the mid-1980s for quadratic maps by a number of different methods, see [30, 10, 9, 26, 42]. In 2000, Milnor & Tresser [31], provided a proof for the case of cubic maps. In this paper we will prove the general case of this 20 year old conjecture.

1. INTRODUCTION AND STATEMENT OF RESULTS.

Given a family of continuous maps f from an interval I to itself, one can ask how its ‘dynamical complexity’ depends on f . Let us assume that I can be decomposed into finitely many subintervals I_0, \dots, I_b on which f is monotone. The smallest number $b+1$ of such intervals is called the *lap number* $\ell(f)$ of f . Note that b is the number of extrema of f , and is often called the *modality* of f . Maps for which b is equal to one or two are called *unimodal* and *bimodal* respectively. One natural measurement of the dynamical complexity of f is the rate of exponential growth of the lap numbers $\ell(f^n)$ where f^n denotes the n -th iterate of f . This growth rate $\lim_{n \rightarrow \infty} \frac{1}{n} \log \ell(f^n)$ exists and is equal to the usual notion of *topological entropy* $h_{top}(f)$ of f , see [33] and also [26, Section II.7]. Topological entropy can be used to classify maps with finite modality, up to semi-conjugacy, see [30] – a bit like the rotation number enables a classification up to semi-conjugacy of degree one circle maps. Conventionally, a map is called *chaotic* if and only if $h_{top}(f) > 0$.

For continuous interval maps, $h_{top}(f)$ coincides with the exponential growth rate of the number of n -periodic orbits. Therefore if we consider a family of continuous interval maps f_t , $t \in [0, 1]$ and $h_{top}(f_1) > h_{top}(f_0)$, then many periodic orbits are created as the parameter t increases from 0 to 1. However, it should be noted that entropy is only a coarse indicator of the birth of periodic orbits, because periodic

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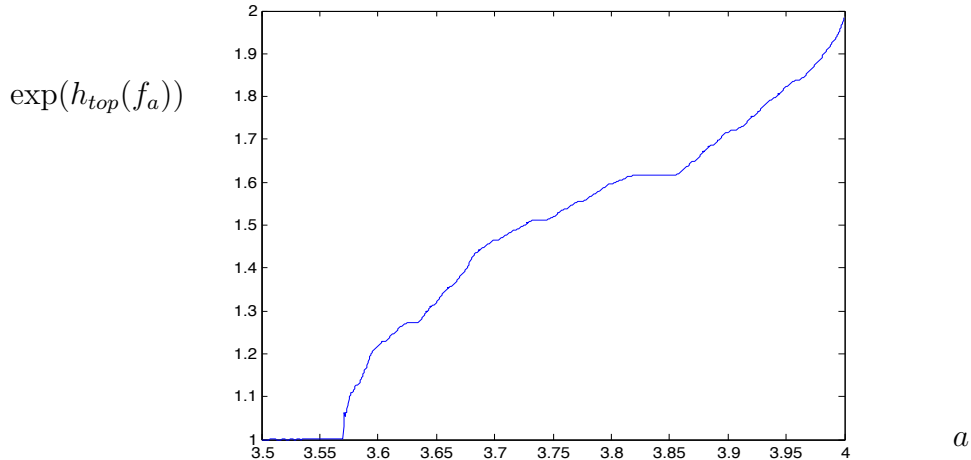


FIGURE 1. Monotonicity of entropy for the maps $f_a(x) = ax(1-x)$, $a \in [3.5, 4]$.

orbits can both appear and disappear in parameter ranges of constant entropy. This is clearly true in modality ≥ 2 (since entropy is only a one-dimensional observable in higher dimensional parameter space), but already in modality one, entropy is constant on any parameter interval within a period doubling cascade. In order to obtain a complete picture on the emergence and disappearance of periodic orbits, it is therefore necessary to combine monotonicity of entropy with so-called Thurston Rigidity, which we explain later on in this introduction.

In this paper we consider the space of b -modal continuous interval maps. Obviously, $h_{top}(f) \in [0, \log(b+1)]$ for a b -modal map. It turns out that $f \mapsto h_{top}(f)$ is not continuous on the entire space of b -modal maps, but if we restrict to C^1 -smooth maps then it is, see [30, 43, 32, 26].

The question whether $h_{top}(f)$ ‘increases’ with f goes at least back to the early 70s, see [27]. In the unimodal situation, one of the simplest ways of asking this question is as follows. Let $I = [0, 1]$ and consider a smooth unimodal map $f: [0, 1] \rightarrow [0, 1]$ with $f(0) = f(1) = 0$, $f(1/2) = 1$ and the family $f_a(x) = af(x)$, $a \in [0, 1]$.

- (1) Does the topological entropy of $f_a(x) = af(x)$ increase with $a \in [0, 1]$?

As mentioned before, entropy *cannot* be *strictly* increasing with a . It has been conjectured in the 90’s that if a C^3 unimodal convex map $f: [0, 1] \rightarrow [0, 1]$ as above has negative Schwarzian and is symmetric around the critical point, then the answer to (1) is positive. This conjecture is subtle: there are C^3 close maps $f, g: [0, 1] \rightarrow [0, 1]$ of this type for which $f \leq g$ yet $h_{top}(f) > h_{top}(g)$, see [3]. Moreover, none of the assumptions can be dropped, see the examples in [45, 19, 34] and also [26, Section II.10].

It was shown in the mid 1980's by Douady & Hubbard [9, 10] and Milnor & Thurston [30] that for the quadratic family $f_a(x) = 4ax(1-x)$ the entropy $h_{top}(f_a)$ depends monotonically on $a \in [0, 1]$. All known proofs of this use that the quadratic map can be extended to the complex plane and require tools from complex analysis, see the above references and also Tsujii's proof [42] and [26, Section II.10]. To show how subtle this question is, let us note that it was only very recently shown that the topological entropy of $f_a(x) = a \sin(\pi x)$ depends monotonically on a , see [37]. In fact, that paper shows that (1) holds for each unimodal $f: [0, 1] \rightarrow [0, 1]$ with $f(0) = f(1) = 0$ which extends to an entire transcendental map on the complex plane, with a finite number of singular values and satisfying the so-called sector condition.

The above proofs not only show that the topological entropy of $f_a(x) = 4ax(1-x)$ increases with a , but also that periodic orbits do not disappear when a increases. In other words, as a increases, only new periodic orbits are created (by period doubling and saddle-node bifurcations). That this is true, follows essentially from the following property:

Thurston Rigidity (combinatorially equivalent critically finite maps are unique): Consider f_a and $f_{a'}$ for which their critical points $c = 1/2$ have finite orbits O and O' . If there exists an order preserving homeomorphism $h: I \rightarrow I$ with $h(O) = O'$ and $h \circ f_a = f_{a'} \circ h$, then $a = a'$.

In fact, much more is known: whenever f_a and $f_{a'}$ have the same 'kneading invariant' and f_a has no periodic attractor, then $a = a'$. This result was proven in [13, 23] and is usually called the *density of hyperbolicity for real quadratic maps*. This result implies that there exists a dense set $H \subset [0, \log 2]$ so that for any $h_0 \in H$ there exists *precisely one* $a \in [0, 1]$ with $h_{top}(f_a) = h_0$. It follows that $a \mapsto h_{top}(f_a)$ is a devil's staircase, the plateaus of which correspond to intervals of parameters containing a periodic attractor and the subsequent period doubling cascade. By density of hyperbolicity, such parameters form a dense set, and so there exists no interval of parameters on which $a \mapsto h_{top}(f_a)$ is strictly increasing.

Douady & Hubbard, see [10], showed the following monotonicity result:

Bifurcations are monotone in the quadratic family: Assume that (a_-, a_+) is a parameter range so that the quadratic family $f_a(x) = 4ax(1-x)$ has a hyperbolic periodic attractor p_a of period n for each $a \in (a_-, a_+)$, then $a \mapsto Df_a^n(p_a) \in (-1, 1)$ is differentiable and strictly decreasing.

The corresponding parameter space for higher degree polynomials is parametrized by Blaschke products, see [29] and Theorems 3.4 and 3.6 below. Combining Thurston

Rigidity with the previous property shows that period doubling cascades are traversed monotonically in the quadratic family. (We should point out that there are additional results on the transversality of bifurcations in polynomial families, see [39, 11, 22].)

Let us turn to real cubic maps. Take $I = [-1, 1]$ and cubic maps $f: I \rightarrow I$ with exactly two critical points, both in the interior of I . Since this space consists of two connected components, it makes sense to separate the cases where $f(-1) = -1, f(1) = 1$ and where $f(-1) = 1, f(1) = -1$. In the former case, such cubic maps can be written in the form $f_{a,b}(x) = ax^3 + bx^2 + (1-a)x - b$ where $(a, b) \in \mathbb{R}^2$ are contained in a simply-connected region bounded four algebraic curves (this follows as in [28]). It is not hard to show that for a smooth one-parameter family f_t of such cubic maps, $t \mapsto h_{top}(f_t)$ need not be monotone, see Remark 4.2. Perhaps this is not too surprising, as the level sets of $(a, b) \mapsto h_{top}(f_{a,b})$ are very complicated fractal-like sets. Related to this is the result by Kan, Koçak & Yorke [18] that within the Hénon family $F_{a,b}(x, y) = (1 - ax^2 + by, x)$, the entropy of $F_{a,b}$ does not depend monotonically on a for fixed b .

Yet a compelling question is whether, within the space of all real cubic maps, those with a given topological entropy form a connected set. In the early 1990's Milnor made this question precise, by defining the following space. Consider the space P_ϵ^b of real polynomials f with

- (1) precisely b distinct critical points, *all of which* are real, non-degenerate and contained in the interior of I ;
- (2) $f(\partial I) \subset \partial I$;
- (3) with *shape* $\epsilon = \epsilon(f)$, where

$$\epsilon(f) = \begin{cases} +1 & \text{if } f \text{ is increasing at the left endpoint of } I, \\ -1 & \text{otherwise.} \end{cases}$$

Note that P_ϵ^b consists of polynomials of degree $d = b + 1$.

Milnor's conjecture essentially asserts that within this space, bifurcations are 'as efficient as possible':

Milnor's Monotonicity of Entropy Conjecture [28]: For each $\epsilon \in \{-, +\}$, $b \in \mathbb{N}$ and $h_0 \geq 0$, the isentrope

$$\{f \in P_\epsilon^b \ ; \ h_{top}(f) = h_0\}$$

is connected.

This conjecture was motivated by numerical experiments, made for the family of real cubic maps, of the isentropes and also by considering the '*bones*' for this family. These bones are curves within the parameter space such that one critical point is periodic with a specified order type, and were introduced by MacKay and Tresser in

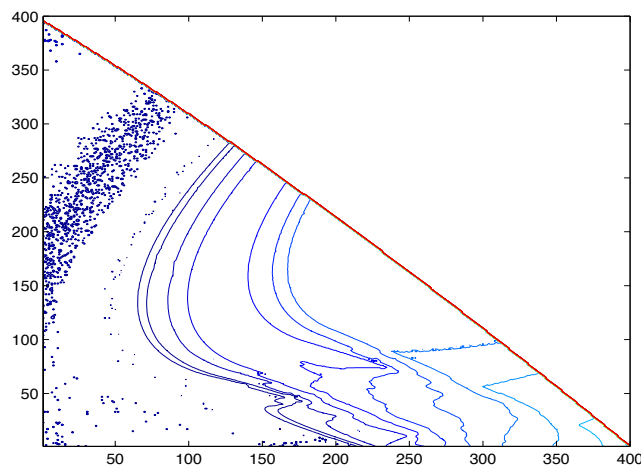


FIGURE 2. Isentropes for cubic maps $f_{a,b}(x) = ax^3 + bx^2 + (1-a)x - b$. The horizontal axis gives $a \in [2.5, 4]$ and the vertical axis $b \in [0, \sqrt{4a-a}]$. The maps $f_{a,-b}$ and $f_{a,b}$ are conjugate, and $f_{a,b}([-1, 1]) \not\subset [-1, 1]$ for $|b| > \sqrt{4a-a}$.

a study of the boundary of chaos for bimodal maps of the interval [24]. A few years later it was shown in [8] that in the cubic case $b = 2$, this conjecture follows from another conjecture (density of hyperbolicity for cubic maps). We should emphasize that although Thurston Rigidity holds for polynomials of any degree, this is *not* sufficient for proving the monotonicity conjecture for degree $d > 2$. In 2000, Milnor & Tresser [31] showed that one does not quite need density of hyperbolicity for real cubic maps. More precisely, they showed that on some curves in the parameter space, the bimodal family behaves essentially like a one-parameter family of unimodal maps. Combining this with an extension of density of hyperbolicity in the quadratic case, due to Heckman [16] and using the Jordan theorem, they were able to conclude the cubic case. Using similar techniques as Milnor & Tresser, Radulescu [36] proved monotonicity of entropy for a two-parameter family of quartic polynomials made up of the composition of two quadratic maps.

In this paper we solve this conjecture in full generality:

Main Theorem (Milnor's Monotonicity of Entropy Conjecture). *For each $\epsilon \in \{+, -\}$, $b \in \mathbb{N}$ and $h_0 \geq 0$, the isentrope*

$$I(h_0) = \{f \in P_\epsilon^b ; h_{top}(f) = h_0\}$$

and the set

$$I(h_0^+) := I(h_0) \cap \text{closure}(\{f \in P_\epsilon^b ; h_{top}(f) > h_0\})$$

are both connected.

In particular, the boundary of ‘chaos’, *i.e.*, the boundary of the set of maps in P_ϵ^b with positive entropy is connected. In fact, the proof of the theorem also shows that for each $h_1 \leq h_0$, the set

$$I = \{f \in P_\epsilon^b \ ; \ h_1 \leq h_{top}(f) \leq h_0\}$$

is connected.

The set of maps $P_\epsilon^b \subset P^b$ with shape ϵ can be parametrized by the coefficients of the polynomial, or more suitably by the critical values of f , see [26, Section II.4] or [30]. As mentioned, one should not expect that the entropy depends monotonically on any of these parameters.

Remark 1.1. *We should emphasise that we prove that the isentrope $I(h_0) = \{f \in P_\epsilon^b \ ; \ h_{top}(f) = h_0\}$, rather than the weaker statement that the closure of this space is connected. (So we prove connectedness within the space of maps with non-degenerate critical points, rather than merely in the closure of this space.)*

As in Milnor & Tresser’s paper [31], our proof relies on stunted sawtooth maps. The other important ingredient is density of hyperbolicity, but now for real polynomials of arbitrary degree, see [20] and [21]. More precisely, we use an analogue of Thurston Rigidity proved in [20] which holds for all real polynomials with real critical points *regardless* of whether the orbits of the critical points are finite or not.

On the way to proving the Main Theorem, we will also prove the following result (see Theorem 5.12 and Theorem 6.1):

Theorem A. *Fix $\epsilon \in \{+, -\}$, $b \in \mathbb{N}$, let $f \in P_\epsilon^b$ and define*

$$\mathcal{H}(f) = \{g \in P_\epsilon^b \ ; \ g \text{ has the same kneading invariants as } f\}.$$

Then $\mathcal{H}(f)$ is connected.

1.1. Related results and some conjectures. In this paper we will also consider the space of so-called admissible stunted sawtooth maps \mathcal{S}_*^b , and show that isentropes within this space are contractible, see Theorem 7.1. In view of this, we would like to propose the following

Conjecture. *Any isentrope $\{f \in P_\epsilon^b \ ; \ h_{top}(f) = s\}$ is contractible.*

Isentropes in P_ϵ^b are extremely complicated. Indeed, we prove in [4] the following

Theorem. *When $b \geq 4$, there are infinitely many values for $s \geq 0$ for which $\{f \in P_\epsilon^b \ ; \ h_{top}(f) = s\}$ is not locally connected.*

In fact, it is not known whether there exists *any* value $s \in [0, \log(b+1)]$ so that the corresponding isentrope $\{f \in P_\epsilon^b \ ; \ h_{top}(f) = s\}$ is locally connected. The methods used in the proof of the previous theorem rely on $b \geq 4$, and it is possible that each isentrope is connected in the cubic case.

In the survey [41] a number of related questions and conjectures are discussed. In particular, the following question due to Tresser: Consider the space Pol_ϵ^d of real polynomials f of degree d , *not necessarily with all critical points on the real line*, but still with $f(\{\pm 1\}) \subset \{\pm 1\}$ and $\epsilon(f) = \epsilon$ as in the definition of P^b .

Conjecture 1.2 (Tresser). *Fix $\epsilon \in \{-1, 1\}$. Isentropes in Pol_ϵ^d are connected.*

Davoud Cheraghi and the second author have made progress towards this conjecture in the context of real unimodal polynomials of degree 4 with at most one real critical point, see [7].

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2. STRATEGY OF THE PROOF, ORGANIZATION OF THE PAPER AND NOTATION

It is well-known that any multimodal map (with positive topological entropy) is entropy-preservingly semi-conjugate to a piecewise monotone map of constant slope, [35, 30]. However, such piecewise affine maps do not exhibit all possible combinatorial types which exist for polynomials maps. Instead, one of the ingredients in Milnor & Tresser's proof is to consider the space of stunted sawtooth maps, all obtained from a single sawtooth map S_0 as in Figure 3. For example, for each cubic map, there exists a stunted sawtooth map T as in this figure with the same combinatorics. This map T is obtained by moving the two plateaus up or down as appropriate - in a way which is made precise in Section 5. In our paper, we will use the space \mathcal{S}_ϵ^b of b -modal stunted sawtooth maps to “parametrize” the space P_ϵ^b of b -modal polynomials. Indeed, we introduce a map

$$\Psi: P_\epsilon^b \rightarrow \mathcal{S}_\epsilon^b$$

which assigns to $f \in P_\epsilon^b$ the unique map $T \in \mathcal{S}_\epsilon^b$ which has the same ‘kneading invariant’ (*i.e.*, symbolic itineraries of critical points) as f . We discuss the definition of Ψ in detail in Section 5. An important property of Ψ is that $\Psi(f)$ and f have the same topological entropy.

Since we shall fix the shape ϵ during the proof, we shall write from now on mostly P^b and \mathcal{S}^b instead of P_ϵ^b and \mathcal{S}_ϵ^b . One of the crucial benefits of the space \mathcal{S}^b is that outside its plateaus, any map $T \in \mathcal{S}^b$ agrees with the same map S_0 . This means that all orbits of two stunted sawtooth maps T and \tilde{T} agree except on the preimages of their plateaus. By decreasing the width of a plateau Z_i (*i.e.*, by moving the image $T(Z_i)$ up or down depending on whether T assumes a local maximum or minimum at Z_i), we create new orbits without destroying any others, and hence the entropy can only increase. This means that within the space \mathcal{S}^b , entropy is a monotone

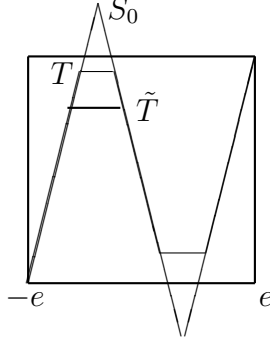


FIGURE 3. Two bimodal stunted sawtooth maps T and \tilde{T} (drawn in bold lines) constructed from the same sawtooth map S_0 (drawn in thin lines).

function of each critical value separately, a property which fails for P^b , see Remark 4.2 and [4]. Using this idea, Milnor & Tresser showed the following:

Theorem 2.1 ([31]). *Isentropes in \mathcal{S}_ϵ^b are connected and contractible.*

This result would imply the main theorem if $\Psi: P_\epsilon^b \rightarrow \mathcal{S}_\epsilon^b$ was a homeomorphism, but as we shall see that is unfortunately not the case.

2.1. Non-bijectivity of $\Psi: P^b \rightarrow \mathcal{S}^b$. One of the reasons the map $\Psi: P^b \rightarrow \mathcal{S}^b$ is non-bijective is simple to see: if $f \in P^b$ is hyperbolic (*i.e.*, if each critical point is in the basin of a hyperbolic periodic attractor), then the itinerary of each critical point of f is eventually periodic. From the definition of Ψ it then follows that the endpoints of each plateau of $T = \Psi(f)$ are also eventually periodic. Since there are uncountably many hyperbolic maps in P^b and only countably many such maps T , the map Ψ is obviously not injective. Neither is Ψ surjective, see Example 5.2. To overcome this we consider equivalence classes in P^b and \mathcal{S}^b .

2.2. Equivalence classes in P^b : partial conjugacy. The first ingredient aimed at overcoming the fact that $\Psi: P^b \rightarrow \mathcal{S}^b$ is neither injective nor surjective is to define a notion of equivalence classes within these spaces, corresponding to sets of maps which have the same dynamics except inside the basins of their attractors. It turns out that we will need slightly different notions within the spaces P^b and \mathcal{S}^b . For P^b , two maps f and \tilde{f} will be taken to be equivalent if they are *partially conjugate*. For this to hold, we require that they are conjugate (on the real line) outside the basins $B(f)$ and $B(\tilde{f})$ of their periodic attractors and that they have the same number of critical points in corresponding components of $B(f)$ and $B(\tilde{f})$. In other words, two interval maps $f, \tilde{f}: I \rightarrow I$ are partially conjugate, if there exists a homeomorphism $h: I \rightarrow I$ so that $h \circ f = \tilde{f} \circ h$ holds outside $B(f)$, so that $h(B(f)) = B(\tilde{f})$ and so that h maps critical points of f to critical points of \tilde{f} . Given $f \in P^b$, we define

$\mathcal{PH}(f)$ to be the set of polynomials $\tilde{f} \in P^b$ which are partially conjugate to f .

For more precise definitions see Section 3. Although the space $\mathcal{H}(f)$ (of maps with the same kneading as f) and $\mathcal{PH}(f)$ are closely related, neither is a subset of the other one, see Example 5.2. Extending the rigidity theorems from [21] we obtain

Theorem 2.2 (See Theorem 3.4). *For any $f \in P_\epsilon^b$, the set $\mathcal{PH}(f)$ is connected.*

This result is a non-trivial extension of Douady & Hubbard's result that hyperbolic components with the space of (complex) quadratic polynomials are topologically discs. We emphasize that an important ingredient in the proof of this theorem is that all critical points of maps in P^b are real. The situation when two real polynomials are conjugate on the real line, but have critical points which are outside the real line, is more subtle and the subject of ongoing research, see [7].

2.3. Preplateau equivalence in \mathcal{S}^b . The same definition can also be used to say when $T, \tilde{T} \in \mathcal{S}^b$ are partially conjugate. In Lemma 5.1 we will see that if $f, \tilde{f} \in P^b$ are partially conjugate then $T = \Psi(f), \tilde{T} = \Psi(\tilde{f})$ are partially conjugate, but unfortunately the reverse implication does not hold. This is why we also introduce another equivalence class within the space \mathcal{S}^b . Indeed, in \mathcal{S}^b we will define a set $W(T)$ which deviates slightly from $B(T)$ and which is based on the preimages of plateaus, see Section 4.2. We then say that $T, \tilde{T} \in \mathcal{S}^b$ are equivalent if $W(\tilde{T}) = W(T)$ and define

$$\langle T \rangle = \{ \tilde{T} ; W(\tilde{T}) = W(T) \} \quad \text{and} \quad [T] = \text{closure}(\langle T \rangle).$$

In Theorem 4.17 we show that $\langle T \rangle$ and therefore its closure $[T]$ is connected (in fact, it is a cell). From the definitions it follows that all maps within $\mathcal{PH}(f)$ (respectively within $[T]$) have the same topological entropy.

2.4. The set $\mathcal{A}_b \subset P^b$ and a useful property of the map Ψ . Unfortunately it is not true that $\Psi(\mathcal{PH}(f)) \subset [\Psi(f)]$ for any $f \in P^b$. To address this issue and in order to relate $\mathcal{PH}(f)$ and $[T]$, we introduce a subset $\mathcal{A}_b \subset P^b$, see Definition 5.3, which enables us to obtain the following property:

Proposition 2.3 (See Proposition 5.4). *If $f \in \mathcal{A}_b$ then $\Psi(\mathcal{PH}(f)) \subset [\Psi(f)]$. Within any $\mathcal{PH}(f)$ we can find special maps $f_0 \in \mathcal{PH}(f) \cap \mathcal{A}_b$.*

Because of this proposition we can morally view Ψ as a map which sends equivalence classes (consisting of partially conjugate maps) in P^b to equivalence classes (consisting of preplateau equivalent maps) in \mathcal{S}^b .

2.5. Non-surjectivity of $\Psi: P^b \rightarrow \mathcal{S}^b$ because of wandering pairs. There is an additional, more serious way, in which $\Psi: P^b \rightarrow \mathcal{S}^b$ is not surjective, and this has to do with wandering intervals. An interval $J \subset [-1, 1]$ is called *wandering* for f if all its iterates are disjoint and $f^n(J)$ does not converge to an attracting periodic orbit as $n \rightarrow \infty$. It is well-known that polynomial interval maps (indeed C^2 interval map with non-flat critical points) have no wandering intervals. This implies that, when

$b \geq 3$, there are many stunted sawtooth maps T in \mathcal{S}^b for which there is no $f \in P^b$ with $T = \Psi(f)$. Indeed, take T with the property that there exists an interval containing two adjacent plateaus which is eventually mapped into a third plateau which is not eventually periodic. Then for any interval map f with $\Psi(f) = T$, the interval connecting the corresponding adjacent critical points would be wandering. Since a polynomial f does not have wandering intervals, these adjacent critical points coincide and so that f has only two critical points. This in turn implies that $T \neq \Psi(f)$. Therefore by using merely the space \mathcal{S}^b we could at best prove that isentropes within P^b are connected within the larger space of polynomial maps with $\leq b$ critical points.

2.6. The space of non-degenerate stunted sawtooth maps \mathcal{S}_*^b . To overcome this problem we define the notion of *wandering pairs* of plateaus, in Definition 4.12, and introduce the space \mathcal{S}_*^b of stunted sawtooth maps without wandering pairs. It turns out that Ψ maps P^b into \mathcal{S}_*^b . The topology of \mathcal{S}_*^b is much more complicated than that of \mathcal{S}^b , which makes it tricky to show that isentropes in \mathcal{S}_*^b are contractible. In \mathcal{S}^b , this is much easier: within \mathcal{S}^b one can construct a retract of an isentrope by moving plateaus with relatively great liberty. To construct a retract for isentropes in \mathcal{S}_*^b we are forced to move plateaus in exactly the right order and with exactly the right speed. The description of this occupies most of Section 7, and leads to

Theorem 2.4 (Connectivity of isentropes in $\mathcal{S}_{\epsilon,*}^b$, see Theorem 7.1). *Isentropes in $\mathcal{S}_{\epsilon,*}^b$ are connected and even contractible.*

Remark 2.5. *The construction in Theorem 7.1 of this contraction is quite subtle. Nevertheless it would be interesting to explore whether one can use the same methodology to construct deformations within the space P^b (and show that isentropes within that space are contractible).*

2.7. The main steps in the proof. With these notions in place, we will obtain that Ψ is ‘almost’ surjective and injective:

Proposition 2.6 (Ψ is ‘almost’ surjective, see Proposition 5.9). *Take $T \in \mathcal{S}_*^b$ and let $\tilde{T} \in \langle T \rangle_{\mathfrak{q}}$. Then there exists a polynomial $f \in P^b \cap \mathcal{A}_b$ such that $\tilde{T} = \Psi(f)$ and $T \in [\Psi(f)]$.*

Proposition 2.7 (Ψ is almost injective, see Proposition 5.10). *The map $\Psi: P^b \rightarrow \mathcal{S}_*^b$ is ‘almost injective’ in the sense that if $f_1, f_2 \in \mathcal{A}_b$ and $[\Psi(f_1)] \cap [\Psi(f_2)] \neq \emptyset$, then $\overline{\mathcal{PH}(f_1)} \cap \overline{\mathcal{PH}(f_2)} \neq \emptyset$.*

To prove the latter proposition, it is important to analyze how two subsets $\mathcal{PH}(f)$ and $\mathcal{PH}(\tilde{f})$ of the space of polynomials P_ϵ^b can intersect. It is for this reason that we require a description of what bifurcations occur at the boundary of these sets, see Theorem 3.6. A corresponding description for the boundary of $\langle T \rangle$ within the space \mathcal{S}_ϵ^b is also needed, and is given in Theorem 4.17.

Proposition 2.8 (Ψ is almost continuous, see Proposition 5.13). $\Psi: P^b \rightarrow \mathcal{S}_*^b$ is ‘almost continuous’ in the following sense. Assume that $f_n \rightarrow f$ where $f_n, f \in P^b$ and $f_n \in \mathcal{A}_b$. Then there exists $T \in \mathcal{S}_*^b$ so that $\Psi(f_n) \rightarrow [T]$ and $\Psi(f) \in [T]$.

Combined this gives the following:

Theorem 2.9. *There exists a map $\Psi: P_\epsilon^b \rightarrow \mathcal{S}_{\epsilon,*}^b$ such that*

- Ψ is ‘almost continuous’, ‘almost surjective’ and ‘almost injective’ (as described in the previous three propositions);
- There exists a connected set $[\Psi(f)] \ni \Psi(f)$ such that the topological entropy of any map $T \in [\Psi(f)]$ is equal to the topological entropy of f ;
- If K is closed and connected then $\Psi^{-1}(K) = \{f \ ; \ [\Psi(f)] \cap K \neq \emptyset\}$ is connected.

Isentropes in $\mathcal{S}_{\epsilon,*}^b$ are contractible (and therefore connected), so Theorem 2.9 implies that isentropes in $P_{\epsilon,*}^b$ are connected, proving the Main Theorem.

2.8. Organization of the paper. Section 3 discusses the notion of partial conjugacy and shows that partial conjugacy classes $\mathcal{PH}(f)$ within the space P^b of polynomials are connected. It also describes when different sets $\mathcal{PH}(f)$ and $\mathcal{PH}(\tilde{f})$ have common boundary points. This section relies on methods which use complex analysis and results from the theory of holomorphic dynamics. In particular, this section it relies on a theorem on rigidity of real polynomials. The remainder of the paper only uses real methods. Section 4 discusses the space \mathcal{S}^b of stunted sawtooth maps and properties of the equivalence classes $\langle T \rangle$. Section 5 discusses the map $\Psi: P^b \rightarrow \mathcal{S}^b$ and its properties. In Section 6 the proof of the Main Theorem is provided. The technical result that isentropes in \mathcal{S}_*^b are connected (and even contractible) is proved in Section 7.

2.9. Notation used in this paper.

- $B(f)$ is the union of the basin of periodic attractors, see Definition 3.1.
- \mathbb{D} denotes the open unit disc in \mathbb{C} .
- I is an interval in \mathbb{R} .
- Z_i are (closed) plateaus of the stunted sawtooth map, see Section 4.1.
- P^b is the space of real polynomials of degree $b + 1$ mapping I (and ∂I) into itself, with precisely b non-degenerate critical points each of which is contained in the interior of I .
- $P_\epsilon^b \subset P^b$ is the space of maps which is increasing (respectively decreasing) at the left endpoint of I when $\epsilon = 1$ (respectively $\epsilon = -1$).
- $\mathcal{H}(f)$ is the space of maps $g \in P^b$ with the same kneading invariant as f , see Theorem A in the introduction.
- $\mathcal{PH}(f)$ is the space of maps $g \in P_\epsilon^b$ which are partially conjugate to f , see Definition 3.2.
- $\mathcal{KS}(f)$ is the space of Kupka-Smale maps, see Definition 3.3.

- S_0 is a sawtooth map of modality b , see Section 4.1.
- \mathcal{S}^b is the space of stunted sawtooth maps, see Section 4.1.
- $W(T)$ is the set of points which are eventually mapped into the interior of a block of plateaus, see equation (5).
- $\langle T \rangle$ is the set of maps with the same $W(T)$, see equation (6).
- $[T]$ is the closure of $\langle T \rangle$, see equation (6).
- $\min[T]$ and $[T]_b$ are certain subsets of the boundary of $\langle T \rangle$, see Definition 4.7.
- $\mathcal{J} := [Z_i, Z_j]$ is the convex hull of plateaus Z_i and Z_j , see Definition 4.12.
- \mathcal{S}_*^b is the space of non-degenerate maps in \mathcal{S}^b , see Definition 4.12.
- $\Psi: P^b \rightarrow \mathcal{S}^b$ is the map which assigns to a polynomial a stunted sawtooth map, see equation (8) in Section 5.
- \mathcal{A}_b is a subset of polynomials with parabolic periodic points, see Definition 5.3.
- $M_{n,\epsilon}, M_{n,\epsilon}^o, M_{n,\epsilon}^\Sigma$ are spaces of Blaschke products, see Definition 3.9.
- $\Gamma_t, \gamma_t, \hat{\Gamma}_t$ are entropy increasing deformations, see Section 7.2 and 7.8.
- $\delta_t, \hat{\delta}_t$ and Δ_t are entropy decreasing deformation, see Sections 7.3 and 7.9.
- K_i and \hat{K}_i are periodic intervals related to the i -th plateau of T , see Section 7.9.
- β_t is an entropy preserving deformation, see Section 7.7
- R_t and r_t are retracts, see Sections 7.5, 7.6 and 7.3.

3. THE PARTIAL CONJUGACY CLASS OF MAPS IN P^b IS CONNECTED

As usual, we say that a polynomial f is *hyperbolic* if each of its periodic orbits is hyperbolic and each of its critical points lies in the basin of a periodic attractor. A well-known result due to Douady & Hubbard asserts that each connected component of the set $\{c \in \mathbb{C}; q_c(z) = z^2 + c \text{ is hyperbolic}\}$ is topologically an open disc parametrised by the multiplier of the periodic attractor. The corresponding case for polynomials of higher degree was considered in [29]. In [11, 22] it is shown that the multipliers of non-repelling periodic points are independent parameters. In this section we will generalize these results to polynomials of higher degree with the crucial difference that we no longer assume that each critical point is in the basin of hyperbolic periodic attractors and restrict to partial conjugacy classes (defined below). We shall only prove this generalization for real polynomials, because one of the main ingredients we need is a rigidity result which is only known in that context.

Before stating this generalization we will introduce some terminology.

Definition 3.1 (Basin of an interval map). *We say that a periodic orbit O of an interval map $f: I \rightarrow I$ is attracting if its basin $B^s(O) = \{x; f^n(x) \rightarrow O \text{ as } n \rightarrow \infty\}$ contains a (possibly one-sided) neighborhood of O . Let $B(f)$ be the union of the basins of periodic attractors of f , i.e., $B(f)$ consists of all points x so that $f^n(x)$ tends to a (possibly one-sided) periodic attractor. Note that if f has a neutral periodic point (i.e., a periodic point which is non-hyperbolic), $B(f)$ need not be open. When f is a polynomial we also will consider f as acting on the complex plane, and in order to emphasize this we sometimes write $B_{\mathbb{C}}(f)$ to denote the basin of f as a subset of \mathbb{C} .*

Definition 3.2 (Partially conjugate). *We say that two b -modal maps $f, g: I \rightarrow I$ are partially conjugate if there is an orientation preserving homeomorphism $h: I \rightarrow I$ such that*

- h maps $B(f)$ onto $B(g)$;
- h maps the i -th critical point of f to the i -th critical point of g ;
- $h \circ f(x) = g \circ h(x)$ for all $x \notin B(f)$.

We denote by $\mathcal{PH}(f)$ the set of maps $g \in P_\epsilon^b$ which are partially conjugate to f .

Note that when a critical point is eventually mapped to the boundary of a component of $B(f)$ that this property persists within $\mathcal{PH}(f)$.

Definition 3.3 (Kupka-Smale maps). *Let \mathcal{KS} be the set of $g \in P_\epsilon^b$ which are Kupka-Smale in the sense that*

- g has only hyperbolic periodic points and
- g has no homoclinic orbits, i.e., no critical point of g is mapped to the boundary of a component of $B(g)$.

The set

$$\mathcal{PH}^o(f) := \mathcal{PH}(f) \cap \mathcal{KS}.$$

generalizes the notion of hyperbolic component for quadratic maps allowing, for the situation that some critical points are not attracted to periodic attractors provided the dynamics of such critical points agrees for all maps within $\mathcal{PH}^o(f)$. Note that $f \in \mathcal{KS}$ does **not** imply $\mathcal{PH}^o(f) = \mathcal{PH}(f)$ because even in this case $\mathcal{PH}(f)$ can contain maps with neutral periodic orbits.

The main result in this section is the following theorem and its more detailed version Theorem 3.6.

Theorem 3.4 (Connectedness of $\mathcal{PH}(f)$). *Let $f \in P^b$.*

- *If $f \in \mathcal{KS}$ then $\mathcal{PH}^o(f)$ is homeomorphic to an open ball of dimension equal to the number of critical points in $B(f)$.*
- *$\mathcal{PH}(f) \subset \overline{\mathcal{PH}^o(f)}$ and therefore the set $\mathcal{PH}(f)$ is connected.*

In fact, we shall also need Theorem 3.6 which states that for any $f \in P^b$ one can find a continuous family of maps f_μ , $\mu \in [0, 1]$ with $f_\mu \in \mathcal{KS}$ for $\mu \in (0, 1]$ and $f_0 = f$ with the crucial additional property that f_μ has the same dynamics as f_0 outside the basins. Before stating that theorem more formally, let us clarify what the types of non-hyperbolic periodic points can occur for maps within the space P^b .

Lemma 3.5. *Let $f \in P^b$. Then each attracting or neutral periodic point of f is real and contains a critical point in its basin. Moreover, if p is a neutral point (say of minimal period n) then it attracts at least from one side and is of one of the following types:*

- (pd) p is attracting from both sides with multiplier -1 and up to a change of coordinates f^{2n} has the form $x \mapsto x - x^3 + O(x^4)$ near p .
- (pf) p is attracting from both sides with multiplier 1 and up to a change of coordinates f^n has the form $x \mapsto x - x^3 + O(x^4)$ near p .
- (sn) p is one-sided attracting with multiplier 1 and up to a change of coordinates f^n has the form $x \mapsto x - x^2 + O(x^3)$ near p .

Proof. Here we use that maps in P^b are real and only have real critical points. This condition implies that if p is a periodic orbit of period n with $Df^n(p) = 1$ then it must be attracting one from side (otherwise the attracting petals will not intersect the real line, but this is impossible since all critical points lie in the real line). Using that each attracting forward invariant petal of a neutral periodic point contains a critical point, the result follows. Alternatively, one can use the fact that maps $f \in P_\epsilon^b$ have negative Schwarzian derivative, i.e., $Sf = [f' \cdot f''' - (3/2)(f'')^2]/(f')^2 < 0$. Since this implies that the Schwarzian derivative of f^n and f^{2n} are negative, see Exercise IV.1.7 in [26], the required statement follows. \square

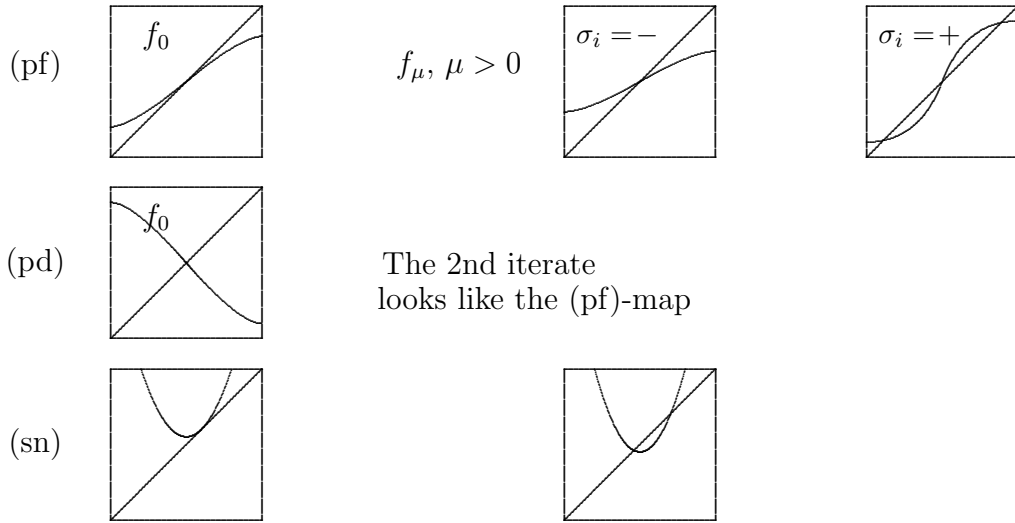


FIGURE 4. A map with a pf (pitchfork), pd (period doubling) respectively sn (saddle-node) fixed point and the corresponding unfoldings constructed in Theorem 3.6.

These case are described in Figure 4. One of the main ingredients in this paper is the following theorem which shows that one can find a deformation $f_\mu, \mu \in [0, 1]$ of a any map $f \in P^b \setminus \mathcal{KS}$ so that $f_\mu \in \mathcal{KS}$ for $\mu \in (0, 1]$ and so that $f_\mu \in \mathcal{PH}(f_1)$ for each $\mu \in (0, 1]$. Therefore the only bifurcations of f_μ are that basins can merge or split and that neutral orbits become hyperbolic.

Theorem 3.6 (Bifurcations of maps which are not in \mathcal{KS}). *Assume that $f \notin \mathcal{KS}$ and let O_1, \dots, O_k be the non-hyperbolic periodic orbits of f and pick a periodic point $p_i \in O_i$ for each $i = 1, \dots, k$. For each periodic orbit O_i , choose a sign $\sigma_i \in \{-, +\}$.*

Then there exists a family $f_\mu \in P_\epsilon^b$ and periodic points $p_{i,\mu}$ all depending continuously on $\mu \in [0, 1]$ so that $f_0 = f, p_{i,0} = p_i$ and

$$f_\mu \in \mathcal{PH}^o(f_1) \text{ for each } \mu \in (0, 1].$$

If p_i has (minimal) period n_i and

- (1) p_i is attracting from both sides and has multiplier -1 (the (pd)-case), then depending on the sign of σ_i a period doubling or a period halving bifurcation occurs as μ becomes positive; i.e., up to a parameter dependent coordinate change $f_\mu^{2n_i}$ has for each $\mu \in [0, 1]$ near $p_{i,\mu}$ the form $x \mapsto (1 + \sigma_i \mu)x - x^3 + h.o.t.$;
- (2) p_i is attracting from both sides with multiplier 1 (the (pf)-case), then depending on the sign of σ_i a pitch-fork or a reverse pitch-form bifurcation occurs as μ becomes positive; i.e., up to a parameter dependent coordinate change $f_\mu^{n_i}$ has for each $\mu \in [0, 1]$ near $p_{i,\mu}$ the form $x \mapsto (1 + \sigma_i \mu)x - x^3 + h.o.t.$;
- (3) p_i is one-sided attracting and has multiplier 1 (the (sn)-case), then one can create a saddle-node pair for f_μ as μ becomes positive; i.e., up to a parameter dependent coordinate change $f_\mu^{n_i}$ has for each $\mu \in [0, 1]$ near $p_{i,\mu}$ the form $x \mapsto (1 + \mu)x - x^2 + h.o.t.$
- (4) If p_i is one-sided attracting and has multiplier 1 and there exists a neighborhood U of p_i so that $U \setminus \{p_i\}$ is contained in $B(f)$, then one can create or destroy a saddle-node pair for f_μ as μ becomes positive; i.e., up to a parameter dependent coordinate change $f_\mu^{n_i}$ has for each $\mu \in [0, 1]$ near $p_{i,\mu}$ the form $x \mapsto x - x^2 + \sigma_i \mu + h.o.t.$
- (5) Moreover, if a critical point c is eventually mapped into the boundary of a component of $B(f)$, then $f_\mu(c)$ is contained in the interior of $B(f_\mu)$ for each $\mu \in (0, 1]$.

This theorem asserts that one can find a family of maps $f_\mu \in \mathcal{KS}$ when $\mu \in (0, 1]$ so that the two adjacent parabolic petals of $f = f_0$ at a neutral periodic point p_i of type (pf) and (pd) as in case (1) and (2) correspond for $\mu > 0$ to two adjacent hyperbolic basins when $\sigma_i = +$ or to one hyperbolic basin when $\sigma_i = -$ (so the sign of σ_i determines whether or not one has a reverse period doubling of pitchfork bifurcation). In case (4) two basins which touch at a saddle-node orbit are deformed into a map where the basins touch at a repelling orbit (when $\sigma_i = +$) or merge into one hyperbolic basin (when $\sigma_i = -$). This situation corresponds to the fixed point with multiplier $= 1$ in Figure 5.

In order to clarify case (5) in the theorem, consider the situation that $f(c)$ is contained in the boundary of a component B of $B(f)$. Then the basin $B_C(f)$ has two components B_1, B_2 which touch at c . These components B_1, B_2 can lie symmetrically in the upper and lower half plane or to the left and right of c . Which case occurs depends on whether $f(c)$ is a left or right end point of $B \cap \mathbb{R}$ and whether f has a maximum or a minimum at c . A situation with the latter case is shown in Figure 5, see also Figure 6 where U plays the role of $B \cap \mathbb{R}$.

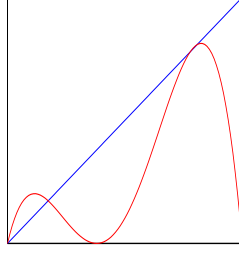


FIGURE 5. For this map, there are three degeneracies: two parabolic fixed points (one with multiplier -1 and another one with multiplier 1) and a homoclinic orbit, *i.e.*, a critical point which is mapped to the boundary of $B(f)$. In this case, both sides of the one-sided attracting fixed point with multiplier 1 are contained in $B(f)$. Theorem 3.6 shows that such a map can be embedded in a family of maps f_μ , $\mu \in [0, 1]$ so that for $\mu \in (0, 1]$ the map f_μ only has hyperbolic periodic points (in the situation shown in the figure, a unique hyperbolic fixed point) and so that the two basins are merged.

3.1. Underlying rigidity theorems. Before proving Theorem 3.4, we should remark that it is related to (and extends) the following result:

Theorem 3.7 (Rigidity Theorem, see [20]). *Let $f \in P_\epsilon^b$. Assume that f has no periodic attractors. Then $\mathcal{PH}(f) = \{f\}$.*

For the quadratic case, Theorem 3.7 was proved independently by Lyubich [23] and Graczyk & Świątek [13]. Milnor & Tresser used this result, or rather a version which applies to certain cubic maps due to Heckman, a PhD student of Świątek (see [16]), in their proof of the Main Theorem for the cubic case, see [31].

Let us also note that Theorem 3.7 is related to *density of hyperbolicity*. We say that an interval map f is *hyperbolic* if each critical point of f is in the basin of a hyperbolic periodic attractor. Building on Theorem 3.7 it was shown in [21] that one has density of hyperbolicity: each real polynomial can be approximated by a hyperbolic real polynomial of the same degree. In fact, each C^∞ interval map can be approximated in the C^∞ topology by a hyperbolic C^∞ map, see [21]; it follows for example that within generic one-parameter families of interval maps hyperbolic maps are dense, see [40].

The main ingredient in the proof of Theorems 3.4 and 3.6 is the following:

Theorem 3.8 (Generalized Rigidity Theorem, see [20]). *Let $f, g \in P_\epsilon^b$ and assume that f, g are partially conjugate on the real line. Moreover, assume that for each periodic attractor (or parabolic point) of f there exists a conformal map $h: B_\mathbb{C}^f \rightarrow B_\mathbb{C}^g$ from the basin $B_\mathbb{C}^f$ of this periodic attractor to the basin $B_\mathbb{C}^g$ of the corresponding periodic attractor of g so that $h \circ f = g \circ h$ on $B_\mathbb{C}^f$. Then $f = g$.*

Proof. This follows from the ‘Rigidity Theorem’ on page 751 of [20] which states that f and g are quasiconformally conjugate (in this theorem parabolic periodic points are allowed). By the assumption, we can modify this conjugacy away from the Julia set to a quasiconformal homeomorphism which agrees with the conformal conjugacy outside a small neighborhood of the Julia set and with the original conjugacy on the forward orbits of the critical points (and which preserves the real line). Using the usual pullback argument, one then obtains a quasiconformally conjugacy which is conformal on the Fatou set. Since the Julia set of a map in P_ϵ^b does not carry an invariant line field, see Theorem 1 in [38], it follows that the quasiconformal conjugacy must be conformal. By the normalization imposed on maps in P_ϵ^b it follows that $f = g$. \square

3.2. The set of Blaschke products with real critical points forms a ball.

To show that $\mathcal{PH}(f)$ is connected, our strategy is to prove first that $\mathcal{PH}^o(f)$ is connected. To this end we shall identify $\mathcal{PH}^o(f)$ with a space of Blaschke products by means of quasi-conformal surgery.

Definition 3.9. For any $n \geq 1$ and $\epsilon \in \{+, -\}$, let $M_{n,\epsilon}$ denote the set of all proper (i.e., the inverse of a compact set is compact) holomorphic maps $A : \mathbb{D} \rightarrow \mathbb{D}$ of degree n of the open unit disc \mathbb{D} , preserving the real axis, such that A has $n-1$ distinct real critical points in $(-1, 1)$, and such that the sign of $A'(-1)$ is ϵ . Since $A \in M_{n,\epsilon}$ maps \mathbb{R} into itself, it can be written as $z \mapsto \sigma \prod_{i=1}^n \frac{z-a_i}{1-\bar{a}_i z}$, where $\sigma \in \{-1, 1\}$ depending on ϵ and on the parity of n , and where $\{a_i, 1 \leq i \leq n\}$ is a subset of \mathbb{D} which is symmetric with respect to the real axis. Therefore $M_{n,\epsilon}$ can be considered as a subset of \mathbb{D}^n and is thus supplied with the induced topology.

Not every map $A \in M_{n,\epsilon}$ has a fixed point, but by the Schwarz-Pick lemma any holomorphic map $A : \mathbb{D} \rightarrow \mathbb{D}$ has at most one fixed point. If A has finite degree then it extends to maps $\partial\mathbb{D}$ to itself. If $A : \mathbb{D} \rightarrow \mathbb{D}$ does not have a fixed point in \mathbb{D} then it follows from Denjoy-Wolff that there exists a unique fixed point on $\partial\mathbb{D}$ which attracts all points in \mathbb{D} .

Definition 3.10. Let $M_{n,\epsilon}^o$ (respectively $M_{n,\epsilon}^\Sigma$) be the set of maps $A \in M_{n,\epsilon}$ with the additional property that $A(0) = 0$ (respectively so that $c_1 = 0$) where $-1 < c_1 < \dots < c_{n-1} < 1$ are the critical points of A .

When $n \geq 3$, the assumption that $\{a_i, 1 \leq i \leq n\}$ is a subset of \mathbb{D} which is symmetric with respect to the real axis does *not* imply that all critical points of A are real.

Lemma 3.11. $M_{n,\epsilon}^o$ and $M_{n,\epsilon}^\Sigma$ are homeomorphic to an open Euclidean ball with (real) dimension equal to the number of critical points of maps in $M_{n,\epsilon}$, i.e., equal to $n-1$. Moreover, $M_{n,\epsilon}$ is homeomorphic to an open Euclidean ball of (real) dimension n .

Proof. Using the same argument as the proof of Lemma 3.1 in [30] or of Corollary II.4.1 in [26], one sees that maps in $M_{n,\epsilon}^o$ can be reparametrized by their critical values. Since for any map $A \in M_{n,\epsilon}^o$ there exists a unique Möbius transformation M so that $A \circ M \in M_{n,\epsilon}^\Sigma$ it follows that the space $M_{n,\epsilon}^\Sigma$ also has dimension $n - 1$. Since any map $A \in M_{n,\epsilon}$ can be written in the form $B \circ M$ where $B \in M_{n,\epsilon}^o$ and M is a Möbius transformation, this implies that the space $M_{n,\epsilon}$ is homeomorphic to a Euclidean ball of dimension n . Alternatively, this follows from the fact that $M_{n,\epsilon}^o$ can be parametrized by their critical points, see [44] and that any $A \in M_{n,\epsilon}$ can be written in the form $M \circ B$ where $B \in M_{n,\epsilon}^o$ and M is a Möbius transformation. \square

3.3. The set $\mathcal{PH}^o(f)$ is homeomorphic to a ball. Take $f \in \mathcal{KS}$ and let us associate spaces $\mathcal{M}(f)$ and $\mathcal{M}^o(f)$ to $\mathcal{PH}^o(f)$. For this we will consider $f \in P^b \cap \mathcal{KS}$ as a map acting on the complex plane, and define $B_{\mathbb{C}}(f)$ as the set of points in the complex plane whose iterates converge to periodic attractors (or parabolic points) of f . Let U_1, U_2, \dots, U_m be the components of $B_{\mathbb{C}}(f) \subset \mathbb{C}$ which contain critical points, and let n_i be the number of critical points in U_i . For each i let s_i be the minimal positive integer such that $f^{s_i}(U_i) = U_{i'}$ for some $1 \leq i' \leq m$. Note that it is conceivable that some components U_i are backward iterates of the immediate basin of the periodic attractor. Let us consider the space $\mathcal{M}(f) = \prod_{i=1}^m M_{n_i, \epsilon_i}$, where ϵ_i denotes the sign of $(f^{s_i})'$ at the left endpoint of $B_i := U_i \cap \mathbb{R}$. (If this endpoint is a critical point, then ϵ_i is the sign of the second derivative at this point.) An element $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{M}(f)$ will be viewed as a dynamical system on the disjoint union of m copies of the unit disk,

$$\mathbf{A} : \bigcup_{i=1}^m \mathbb{D}_i \rightarrow \bigcup_{i=1}^m \mathbb{D}_i, \quad \text{where } \mathbb{D}_i = \mathbb{D} \times \{i\}$$

such that $\mathbf{A}(z, i) = (A_i(z), i')$, where i' is as above. Let us say that $\mathbf{A} \sim \tilde{\mathbf{A}}$ if they are conjugate to each other via a component-preserving conformal map $\varphi : \bigcup \mathbb{D}_i \rightarrow \bigcup \mathbb{D}_i$ such that for each $1 \leq i \leq m$, $\varphi|_{\mathbb{D}_i}$ is a real symmetric (*i.e.*, $\varphi(z) = \varphi(\bar{z})$) map whose restriction to the real line preserves the orientation.

Let $\mathcal{M}^o(f)$ denote the subset of $\mathcal{M}(f)$ consisting of maps $\mathbf{A} = (A_1, A_2, \dots, A_m)$ with the following property: if \mathbf{A}^k maps \mathbb{D}_i onto itself, then \mathbf{A}^k has a fixed point in \mathbb{D}_i . In other words, if $U_{i_1}, U_{i_2}, \dots, U_{i_k}$ is a cycle of attracting basins of f , then we require that $A_{i_k} \circ \dots \circ A_{i_1}$ has a fixed point. This means that up to a Möbius transformation we can assume that the periodic points in U_i correspond to 0. It follows that $\mathcal{M}^o(f)/\sim = \prod_{i=1}^m M_{n_i, \epsilon_i}^{\delta_i}$ where $M_{n_i, \epsilon_i}^{\delta_i} = M_{n_i, \epsilon_i}^o$ when U_i contains a periodic attractor and $M_{n_i, \epsilon_i}^{\delta_i} = M_{n_i, \epsilon_i}^o$ otherwise.

Let us define a map

$$(2) \quad \Theta : \mathcal{PH}^o(f) \rightarrow \mathcal{M}^o(f)/\sim$$

as follows. For $g \in \mathcal{PH}^o(f)$, let $U_i(g) \subset \mathbb{C}$, $i = 1, 2, \dots, m$, be the components of $B_{\mathbb{C}}(g)$ containing critical points corresponding to the sets U_i from above. For each

i , let $\varphi_i : U_i(g) \rightarrow \mathbb{D}$ be some real-symmetric conformal map whose restriction to the real axis is orientation-preserving, and let

$$(3) \quad A_i(g) = \varphi_{i'} \circ g^{s_i} \circ \varphi_i^{-1}.$$

Then define

$$(4) \quad \Theta(g) = [(A_1(g), A_2(g), \dots, A_m(g))],$$

where $[A]$ denotes the equivalence class of A . Note that the space $\mathcal{M}^o(g)$ associated to any map in $g \in \mathcal{PH}^o(f)$ is the same and so this definition makes sense.

Lemma 3.12. *The map Θ defines a homeomorphism between $\mathcal{PH}^o(f)$ and $\mathcal{M}^o(f)/\sim$. In particular, $\mathcal{PH}^o(f)$ is homeomorphic to an open ball of dimension equal to the number of critical points in $B(f)$.*

Proof of Lemma 3.12. Since the sets $U_i(g)$ move continuously (in the Carathéodory topology) with respect to $g \in \mathcal{PH}^o(f)$ the map Θ is continuous. Here we use the continuous dependence of the Riemann mapping from $U_i(g)$ to \mathbb{D} as the simply connected domain $U_i(g)$ moves continuously with g , see the discussion in Section 5.1 in [25]. By the Rigidity Theorem 3.7, Θ is injective. Indeed, if $\Theta(g) = \Theta(\tilde{g})$ then g and \tilde{g} are topologically conjugate on \mathbb{R} , and moreover they are conformally conjugate near the corresponding periodic attractors. Therefore g and \tilde{g} are affinely conjugate.

Because a continuous bijective map between open subsets of Euclidean spaces is a homeomorphism (due to Brouwer's invariance of domain theorem), it remains to prove that Θ is surjective. Let $\mathbf{A} = (A_1, A_2, \dots, A_m)$ be an element in $\mathcal{M}^o(f)$. Our aim is to construct a map $g \in \mathcal{PH}^o(f)$ so that $\Theta(g) = [\mathbf{A}]$. To do this, one applies quasi-conformal surgery techniques in a standard fashion. Let us therefore be brief, and refer to the exposition given in Theorem VIII.2.1 of [6] for details. Choose $f_0 \in \mathcal{PH}^o(f)$ and let $U_i, s_i, n_i, \varepsilon_i$ be the objects associated to f_0 as above. Let $\varphi_i : U_i \rightarrow \mathbb{D}$ be a real-symmetric conformal map sending the periodic attractor in U_i to 0. Then $\varphi_{i'} \circ f_0^{s_i} \circ \varphi_i^{-1} : \mathbb{D} \rightarrow \mathbb{D}$ is a map A_i^o in M_{n_i, ε_i}^o and $\mathbf{A}^o = (A_1^o, A_2^o, \dots, A_m^o) \in \mathcal{M}^o(f)$. Define a new smooth covering map $\tilde{A}_i : \mathbb{D} \rightarrow \mathbb{D}$ as follows. Take discs $\Delta(r_i) \subset \mathbb{D}$ with $r_i < 1$ sufficiently close to 1 so that $\cup_i \Delta(r_i) \times \{i\}$ is mapped into itself by \mathbf{A} and also by \mathbf{A}^o . Let $\text{Ann}_i = ((A_i^o)^{-1} \Delta(r_i)) \setminus \Delta(r_i)$ so that Ann_i is a fundamental annulus of A_i^o . Choose $\tilde{A}_i : \mathbb{D} \rightarrow \mathbb{D}$ so that it agrees with A_i on $\Delta(r_i)$ and with A_i^o on $\mathbb{D} \setminus (\Delta(r_i) \cup \text{Ann}_i)$, and so that it is a smooth covering map on the fundamental annulus Ann_i . Next define a smooth map \tilde{f} which agrees with f outside $\cup U_i$ and which is equal to $f^{-(s-1)} \circ \varphi_i^{-1} \circ \tilde{A}_i \circ \varphi_i$ on U_i , where $f^{-(s-1)}$ stands for inverse of the conformal map $f^{s-1} : f(U_i) \rightarrow U_i'$. The smooth map \tilde{f} agrees with f outside $\cup U_i$ and is conformal outside the annuli $\varphi_i^{-1}(\text{Ann}_i)$. Since the \tilde{f} orbit of each point only hits at most once the fundamental annuli we can choose an invariant ellipse field which agrees with the standard linefield in $\varphi_i^{-1}(\Delta(r_i))$ and on the complement of $B(f)$. Using the Measurable Riemann Mapping Theorem, we obtain a K -quasiconformal

homeomorphism h so that $g := \tilde{h} \circ \tilde{f} \circ \tilde{h}^{-1}$ is again holomorphic and therefore the required polynomial of the same degree as f . Since g and f are conjugate outside $B(f)$, we have $g \in \mathcal{PH}^o(f)$.

Let us now show that $\Theta(g) = [A]$. Let $U_i^g := h(U_i)$ be the components of $B(g)$. Writing

$$H_i := \varphi_i \circ h^{-1}: h(U_i) \rightarrow \mathbb{D} \text{ and } H_{i'} := \varphi_{i'} \circ h^{-1}: h(U_{i'}) \rightarrow \mathbb{D},$$

we have that $H_{i'} \circ g^{s_i} \circ H_i^{-1}: \mathbb{D} \rightarrow \mathbb{D}$ agrees with the Blaschke product A_i on $H_i(\Delta(r_i))$. Moreover, H_i^{-1} is conformal on this set. Since the forwards iterates of critical points A are contained in this set, by pulling back via the dynamics, one obtains a sequence of K -quasiconformal maps $H_{i,n}: h(U_i) \rightarrow \mathbb{D}$, $H_{i',n}: h(U_{i'}) \rightarrow \mathbb{D}$ which are conformal on larger and larger subsets of $h(U_i)$ and $h(U_{i'})$ respectively, and so that $H_{i',n} \circ g^{s_i} \circ H_{i,n}^{-1}: \mathbb{D} \rightarrow \mathbb{D}$ agrees with A on corresponding increasing subsets of \mathbb{D} . By taking limits, one obtains conformal maps $\hat{H}_i: h(U_i) \rightarrow \mathbb{D}$, $\hat{H}_{i'}: h(U_{i'}) \rightarrow \mathbb{D}$ so that $\hat{H}_{i'} \circ g^{s_i} \circ \hat{H}_i^{-1} = A_i$. \square

3.4. The proof of Theorems 3.4 and 3.6. Take $f \in P_\epsilon^b$. Assertion (i) in Theorem 3.4 was proved in the previous lemma. So let us prove that $\mathcal{PH}(f) \subset \overline{\mathcal{PH}^o(f)}$ and that there exists a family of maps as in Theorem 3.6.

In Steps 1-4 we find a polynomial map $P \in P_\epsilon^b$ which will essentially play the role of f_1 . To find such a polynomial, we will first approximate f by a suitable continuous map g whose dynamics is the same as that of f except ‘on the basins of periodic attractors’.

Step 1. In this step we find a family of piecewise smooth interval maps g_t , $t \in [0, 1]$ with $g_0 = f$ which undergoes the required bifurcations at $t = 0$, as t becomes positive, for each of the periodic attractors as in case (1)-(4) of the assumption of Theorem 3.6. Here we ensure that g_t agrees with f outside a small neighborhood in $B(f) \cap \mathbb{R}$ of the neutral periodic points p_1, \dots, p_k . Depending on the sign of σ_i in the assumption of Theorem 3.6, we choose g_t so that as t increases, the neutral point p_i undergoes a period-doubling or period-halving bifurcation in case (1), a pitchfork or a reverse pitchfork bifurcation in case (2) and a saddle-node or a reverse saddle-node bifurcation in case (4). If p_i is as in case (3), we choose g_t so that it is merely piecewise smooth at p_i , and $g_t^{n_i}(x) = p_i + (1+t)(x - p_i) - (x - p_i)^2 + O(x - p_i)^3$ for x in a one-sided attracting neighborhood of p_i .

Step 2. First assume that c is a critical point as in case (5) so that c is mapped to the boundary of a component U of $B(f)$ and $f^{-1}(U) \cup \{c\}$ contains a (real) neighborhood of c , see Figure 6 on the right. Then we can choose the family g_t so that $g_t(c)$ is in the interior of $B(f)$ for each $t \in (0, 1]$. Let us denote $\hat{B}(f) = \cup \bar{J}$ where the union runs over all connected components J of $B(f)$ and \bar{J} is the closure



FIGURE 6. A critical point which is mapped to the boundary of a component of the basin of a periodic attractor. The cases discussed in step 2 and 3 of the proof of Theorem 3.6 are shown on the left respectively right.

of J . Note that the only difference between g_t and f is that some basins are merged or split in two and therefore $\hat{B}(g_t) = \hat{B}(f)$ for each $t \in [0, 1]$.

Step 3. Pick $t_0 > 0$ small and let $\hat{g} = g_{t_0}$. Next we need to take special care of case (5) in the situation when there exists a critical point c so that $f(c)$ is mapped into the boundary of a component U of $B(f)$ so that $f^{-1}(U) \cup \{c\}$ contains no (real) neighborhood of c , as is drawn in Figure 6 on the right. In this case choose a semi-conjugacy $h: I \rightarrow I$ (i.e., h is continuous, monotone and surjective) and a continuous b -modal interval map g so that $h \circ g = \hat{g} \circ h$ where $h^{-1}(x)$ is a point, except if $x \in \cup_{n \geq 0} \hat{g}^{-n}(c)$ for any c as in the previous sentence. Therefore we ‘glue-in’ intervals in the backward orbit of c allowing us to ‘move’ $\hat{g}(c)$ into the interior of $B(\hat{g})$. We can do this so that $g(c)$ now is mapped into the interior of the component of $B(g)$ corresponding to U . It is possible that a critical point c' is eventually mapped to c . In this case $\cup_{n \geq 0} \hat{g}^{-n}(c)$ contains c' , and we proceed in the same way. Also note that there exists an order preserving homeomorphism h_0 of $I \setminus \hat{B}(f)$ to $I \setminus \hat{B}(g)$ so that $h_0 \circ f = g \circ h_0$ on $I \setminus \hat{B}(f)$.

Step 4. Note that g has no wandering intervals and also no inessential periodic attractors. Hence, by the Fullness Theorem II.4.1 in [26], there exists a polynomial $P \in P_\epsilon^b$ which is topologically conjugate to g and so that each of its periodic orbits is hyperbolic and so that none of its critical point is mapped to the boundary of a component of the basin of P . Note that f and P are conjugate outside their basins: there exists an order preserving homeomorphism h_1 of $I \setminus \hat{B}(f)$ to $I \setminus \hat{B}(P)$ so that $h_1 \circ f = P \circ h_1$ on $I \setminus \hat{B}(f)$. In other words, f and P are conjugate outside their basin of attractors and components of $B_{\mathbb{C}}(f)$ and $B_{\mathbb{C}}(P)$ correspond to each other in the following manner:

- (1) Each component of $B_{\mathbb{C}}(f)$ containing a hyperbolic periodic point corresponds to a unique component of $B_{\mathbb{C}}(P)$.
- (2) Each component of $B_{\mathbb{C}}(f)$ of the basin of a neutral periodic point p_i of f (i.e., of a petal) corresponds to a component of the basin of a hyperbolic periodic point for P . However, two touching adjacent basins (petals) of $B_{\mathbb{C}}(f)$ may

correspond to one component of $B_{\mathbb{C}}(P)$ (so these petals are merged). This happens in case (1), (2) and (4) when the corresponding sign σ_i is negative.

- (3) Each critical point of f which is eventually mapped into the boundary of the basin of f , corresponds to a critical point of P which is contained in the interior of $B_{\mathbb{C}}(P) \cap \mathbb{R}$.

We will use P to construct the required family f_{μ} , but we do not claim that the polynomial P is close to f , even in the C^0 topology.

Let U_1, \dots, U_m be the components of $B_{\mathbb{C}}(P)$ which contain critical points. We will use the homeomorphism $\Theta: \mathcal{PH}^o(P) \rightarrow \mathcal{M}^o(P)/\sim$ defined in Subsection 3.3 to construct a family of polynomials through f . To do this, we construct in Step 5 a family of polynomials F_{μ} which may have a much higher degree than P . This family F_{μ} will be used to obtain a one-parameter family of Blaschke products, *i.e.*, a one-parameter family in $\mathcal{M}^o(P)/\sim$.

Step 5. Let U_1^f, \dots, U_k^f be the components of $B_{\mathbb{C}}(f)$ which contain critical points and let $U_{k+1}^f, \dots, U_{k'}^f$ be the components U of $B_{\mathbb{C}}(f)$ for which there exists a critical point c so that $f(c) \in \partial(U \cap \mathbb{R})$. Note that $k' = m$ by the construction of P . Let V_1^f, \dots, V_n^f be the components of $B_{\mathbb{C}}(f)$ containing the forward iterates of the sets U_1^f, \dots, U_m^f . Let $X = \cup_i \partial(V_i^f \cap \mathbb{R})$. Moreover, for each critical point c of f , let $n(c)$ be the smallest integer for which there exists a component U of $B_{\mathbb{C}}(f)$ so that $f^{n(c)}(c) \in \partial(U \cap \mathbb{R})$ and if there exists no such integer let $n(c) = 0$. Let $Y = \cup_c \{f^i(c); 0 \leq i \leq n(c) - 2\}$ where the union is taken over all critical points c of f .

Next consider a real polynomial map Q which is zero on the set $X \cup Y$, so that Q' is zero on $\text{Crit}(f)$ and let $F_{\mu} = f + \mu Q$. Note that the degree of Q and therefore of F_{μ} might be much larger than that of f . By the choice of Q each periodic point of f on the boundary of $B_{\mathbb{C}}(f)$ is still a periodic point for F_{μ} and each critical point of f is still a critical point of F_{μ} . By Theorem VI.1.2 in [6] one can choose Q so that as $\mu \in [0, 1]$ becomes positive, F_{μ} undergoes all the bifurcations of neutral periodic points and homoclinic orbits required in cases (1)-(5) of the theorem. In particular, each of the components U_1, \dots, U_m of $B_{\mathbb{C}}(P)$ corresponds in a unique way to a component $U_{1,\mu}, \dots, U_{m,\mu}$ of $B_{\mathbb{C}}(F_{\mu})$ when $\mu \in (0, 1]$ is small. Note that F_{μ} can have many more attractors than P and F_{μ} may not necessarily be conjugate to f and we do not claim that $U_{i,0} = U_i$ either. Each of the attractors in $U_{1,\mu}, \dots, U_{m,\mu}$ is hyperbolic and $U_{i,\mu}$ depend continuously on $\mu \in (0, \mu_0]$ for $\mu_0 > 0$ small. As below Lemma 3.11, we have that $F_{\mu}^{s_i}(U_{i,\mu}) = U_{i',\mu}$ and we can associate to each of the maps $F_{\mu}^{s_i}: U_{i,\mu} \rightarrow U_{i',\mu}$ a Blaschke product A_i as in (3). In this way we obtain

$$[(A_{1,\mu}, \dots, A_{2,\mu})] \in \mathcal{M}^o(P)/\sim$$

as in Lemma 3.11. Note that $[(A_{1,\mu}, \dots, A_{2,\mu})]$ depends continuously on $\mu \in (0, \mu_0]$, because the domains $U_{i,\mu}$ vary continuously.

Step 6. Next consider the map

$$\Theta : \mathcal{PH}^o(P) \rightarrow \mathcal{M}^o(P) / \sim$$

from equation (2) and (4) and define

$$f_\mu = \Theta^{-1}[(A_{1,\mu}, \dots, A_{m,\mu})].$$

Therefore f_μ is obtained from P by gluing in the components U_i Blaschke products which are obtained from the family F_μ . By definition $f_\mu \in P_\epsilon^b$ and by construction, when $\mu > 0$, all periodic orbits of f_μ are hyperbolic and no critical point of f_μ is eventually mapped in the boundary of a component of f_μ . Moreover, $f_\mu \in \mathcal{PH}^o(f_{\mu_0}) = \mathcal{PH}^o(P)$ for each $\mu \in (0, \mu_0]$.

Step 7. Let us now show that f_μ tends to f as $\mu \rightarrow 0$ (in the sense that the coefficients of f_μ converge to those of f). To see this, let $V_{1,f_\mu}, \dots, V_{n,f_\mu}$ be the components of $B(f_\mu)$ which contain forward iterates of critical points of f_μ and let $V_{1,\mu}, \dots, V_{n,\mu}$ be the corresponding components of $B(F_\mu)$. By construction there exists a family of conformal homeomorphisms

$$h_\mu : V_{1,f_\mu} \cup \dots \cup V_{n,f_\mu} \rightarrow V_{1,\mu} \cup \dots \cup V_{n,\mu}$$

so that $h_\mu \circ f_\mu = F_\mu \circ h_\mu$ on this set. Moreover, h_μ depends continuously on $\mu \in (0, \mu_0]$. Moreover, even though some of these components pinch (at places where f has a parabolic periodic point), the conformal homeomorphisms h_μ have a well-defined conformal ‘limit’

$$h : V_{1,f_0} \cup \dots \cup V_{n,f_0} \rightarrow V_1 \cup \dots \cup V_n$$

so that $h \circ f_0 = F_0 \circ h$. It follows that f_0 and $F_0 = f$ are conformally conjugate on the basin of periodic attractors. Hence by the Generalized Rigidity Theorem 3.8 we obtain that $f_0 = f$ and that $f_\mu \rightarrow f$ as $\mu \rightarrow 0$.

This completes the proof of Theorems 3.4 and 3.6.

Note that we do not state that the Julia set of F_μ is related to that of f . This enables us to avoid using the techniques employed in [14, 15]. Moreover, in general, it is not obvious how to deform a map with an attracting and a repelling orbit to one with a parabolic orbit, or vice versa to deform a map with a parabolic point to a ‘subhyperbolic’ map *in such a way that the Julia set remains topologically the same*, see [14, 15]. We are not concerned with this question.

4. PARTIAL CONJUGACY CLASSES WITHIN THE SPACE \mathcal{S}^b OF STUNTED SAWTOOTH MAPS

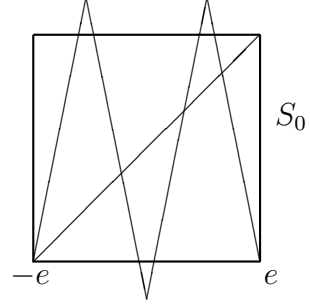
4.1. Definition of the space of stunted sawtooth maps \mathcal{S}^b . Fix the number of turning points b and the shape ϵ of the polynomials we will consider. From now on we will drop the symbol ϵ . Following [31], it will be useful to introduce a space of piecewise linear maps with b (possibly touching) plateaus. Fix the slope $\lambda = b + 2$

and let $e = b\lambda/(\lambda - 1)$. An elementary calculation shows that there exists a unique b -modal piecewise linear map S_0 (a ‘sawtooth map’) with shape ϵ and b turning points c_1, \dots, c_b at $-b + 1, -b + 3, \dots, b - 3, b - 1$ with

- $b + 1$ intervals of monotonicity

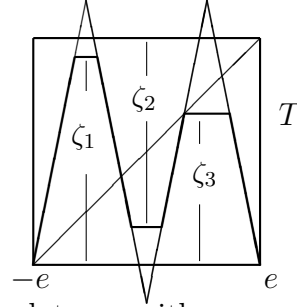
$$I_0 = [-e, c_1], I_1 = [c_1, c_2], \dots, I_b = [c_b, e];$$

- slope $\pm\lambda$ and extremal values $\pm\lambda$;
- and such that $S_0(\{-e, e\}) \subset \{-e, e\}$.



The space of \mathcal{S}^b of *stunted sawtooth* maps consists of continuous maps T with plateaus $Z_{i,T}$, $i = 1, \dots, b$, which are obtained from S_0 and satisfying

- $Z_{i,T}$ is a closed symmetric interval around c_i ;
- T and S_0 agree outside $\bigcup_i Z_{i,T}$;
- $T|_{Z_{i,T}}$ is constant and $T(Z_{i,T}) \in [-e, e]$;
- $Z_{i,T}$ have pairwise disjoint interiors.



Maps in \mathcal{S}^b are allowed to have touching plateaus (*i.e.*, plateaus with one point in common). We allow plateaus to touch because, as we will see later on, such maps T correspond to polynomial maps for which several critical points lie in one component of the basin of an attracting periodic point.

Note that if $T \in \mathcal{S}^b$ has touching plateaus, then the union of these touching plateaus is called a *block of plateaus*. If T has touching plateaus, then it is constant on at least one lap of S_0 . In this case T is b -modal only in a degenerate sense.

It is convenient to use the b ‘signed’ extremal values $\zeta \in [-e, e]^b$ to parametrize \mathcal{S}^b :

$$\zeta_i = \begin{cases} T(Z_{i,T}) & \text{if } S_0 \text{ assumes a maximum at } c_i, \\ -T(Z_{i,T}) & \text{if } S_0 \text{ assumes a minimum at } c_i. \end{cases}$$

Sometimes we will denote by T_ζ the map T with parameters $\zeta = (\zeta_1, \dots, \zeta_b)$. Note that decreasing ζ_i results in widening the corresponding plateau of T_ζ and that $\zeta_i + \zeta_{i+1}$ is equal to the length of the convex hull of $T(Z_i)$ and $T(Z_{i+1})$. Hence

$$\zeta_i \geq -\zeta_{i+1} \quad \text{for } i = 1, \dots, b - 1,$$

with equality when the plateaus Z_i and Z_{i+1} touch. Let us denote by $[Z_i, Z_{i+1}]$ the convex hull of the plateaus Z_i and Z_{i+1} . Thus we can identify \mathcal{S}^b with

$$\{\zeta = (\zeta_1, \dots, \zeta_b) \ ; \ \zeta_i \in [-e, e], \ \zeta_i \geq -\zeta_{i+1}\}.$$

We define $T < \tilde{T}$ if for the corresponding parameters $\zeta_i \leq \tilde{\zeta}_i$ for all $i = 1, \dots, b$ with at least one inequality.

Proposition 4.1. *The map $\zeta = (\zeta_1, \dots, \zeta_b) \rightarrow h_{top}(T_\zeta)$ is non-decreasing in each coordinate.*

Remark 4.2. *In [4] we prove that the analogous statement is false for P^b for $b \geq 2$. That is, if the cubic family is parametrized by its critical values a, b , then the map $(a, b) \mapsto h_{top}(f_{a,b})$ is not monotone in each of its parameters separately.*

A consequence of this proposition is that $\{T \in \mathcal{S}^b ; h_{top}(T) = s\}$ is contractible, see Theorem 6.1 in [31].

Proof of Proposition 4.1. Increasing a parameter ζ_i makes a plateau narrower, and affects none of the orbits that never enter Z_i . Therefore only new orbits are created, and none destroyed. Hence entropy is non-decreasing in each ζ_i . \square

4.2. The preplateau and the basin of a map T . We define the basin $B(T)$ of a map $T \in \mathcal{S}^b$ exactly as before, see Definition 3.1. Since maps T have plateaus, we also introduce a related notion: we define the *preplateau* $W(T)$ of a map T to be the set of points x which eventually map into the *interior* of the union of the plateaus of T , *i.e.*,

$$(5) \quad W(T) = \bigcup_{k \geq 0} T^{-k}(\text{int}(\cup_{i=1}^b Z_{i,T})).$$

Because we allow the possibility of plateaus touching each other, we take the interior of the union rather than the union of the interiors. We say that a component W of $W(T)$ is *periodic* of *period* s if $T^s(W) \subset \overline{W}$. A periodic point p of T is called *hyperbolic* if its orbit enters the interior of a plateau of T . The following elementary lemma explains how the sets $W(T)$ and $B(T)$ are related.

Lemma 4.3. *Let $T \in \mathcal{S}^b$. Then $W(T)$ is open and dense. Moreover, for W' and W'' are components of $W(T)$,*

- (1) *if $T^n(W')$ intersects a boundary point y of W'' , then $T^n(W') = \{y\}$;*
- (2) *if $T^n(W') \cap W'' \neq \emptyset$, then $T^n(W') \subset W''$;*
- (3) *if W' is periodic of period s , then $T^s(\partial W') \subset \partial W'$ and either $T^s(W') \subset W'$ or $T^s|_{W'}$ is constant;*
- (4) *W' is either eventually mapped into a periodic component W'' of $W(T)$ (with $T^s(W'') \subset W''$ for some s) or there exists n so that $T^n(W')$ is equal to a point;*
- (5) *if $T^s(W') \subset W'$, then W' contains precisely one periodic point p (so that some forward iterate is contained in the interior of a plateau) and $T^{ks}(x) \rightarrow p$ for every point in $x \in W'$;*

- (6) $B(T)$ is equal to the set of points which are eventually mapped into a periodic plateau. Moreover, for each component B of $B(T)$
- (a) there exists a sequence of touching components W_i of $W(T)$ so that $\bigcup_{i \in \mathcal{I}} W_i \subset B \subset \bigcup_{i \in \mathcal{I}} \overline{W}_i$, where \mathcal{I} is an at most countable index set;
 - (b) if each periodic orbit of T is hyperbolic (i.e., disjoint from $\partial(\cup_i Z_i)$), then B is equal to a component of $W(T)$;
- (7) if W' and W'' touch and one of them intersects a component B of the basin of a periodic attractor, then both of them are contained in B .

Proof. Openness of $W(T)$ follows from the definition. Since the complement of $W(T)$ is forward invariant and T is expanding on this complement, the set $W(T)$ is dense. To prove (1), take $x \in W'$ so that $y := T^n(x) \in \partial W''$. Note that we can assume that n is ‘minimal’, i.e., there exists no $0 < k < n$ such that $T^k(x)$ is in the boundary of a component of $W(T)$. Since $y \in \partial W''$ no iterate of y is mapped into the interior of a plateau. Since $x \in W$ and $y = T^n(x)$ there exists $0 \leq l < n$ so that $f^l(x)$ is mapped in the interior of a plateau. It follows that the interior H of the component of $T^{-n}(y)$ containing x is non-empty. Since $x \in W(T)$ and so x is eventually mapped into the $\text{int}(\cup_{i=1}^b Z_{i,T})$, it follows that $H \subset W(T)$ (here we use the minimality of n). Let x' be an endpoint of H . Then T^n is not locally constant near x' and so $x', T(x'), \dots, T^{n-1}(x') \notin \text{int}(\cup_{i=1}^b Z_{i,T})$. Since $T^n(x') = y \notin W(T)$ and therefore $y, T(y), \dots \notin \text{int}(\cup_{i=1}^b Z_{i,T})$, it follows that $T^k(x') \notin \text{int}(\cup_{i=1}^b Z_{i,T})$ for all $k \geq 0$ and therefore $x' \notin W(T)$. Hence $H = W'$ and so $T^n(W') = \{y\}$, proving Assertion (1). This implies (2), (3) and (4) because each component of $W(T)$ is open, because $W(T)$ is backward invariant and because T has only finitely many plateaus. Note that if $T^s(W') \subset W'$ then by Assertion (1) the one-sided slope of T^s at the endpoints of W is > 1 . Therefore T^s has a fixed point $p \in W'$ at which the map is locally constant. If $T^{2s}|W'$ has another fixed point, then $T^{2s}|W'$ also has a repelling fixed point which is impossible since $W' \subset W(T)$. Since T^s is locally constant at p , there exists an interval neighborhood U_0 of p so that $T^s(U_0) \subset U_0$ and so that $T^n(x) \rightarrow p$ for each $x \in U_0$. If we denote by U_n the component of $T^{-sn}(U_0)$ containing U_0 , we have $U_{n+1} \supset U_n \supset \dots \supset U_0$. Therefore T^s maps $U = \bigcup U_n$ into itself and $T^s(\partial U) \subset \partial U$. Since the only fixed point of $T^{2s}|J$ is p , it follows that $U = W$. This proves assertion (5). To prove (6) note that each attracting periodic orbit of T necessarily intersects $\cup_{i=1}^b Z_{i,T}$ because T and the unstunted sawtooth map S_0 agree outside this set. If this periodic orbit intersects $\text{int}(\cup_{i=1}^b Z_{i,T})$, then each component B of the basin of this periodic orbit coincides with a component W' of $W(T)$. On the other hand, if this periodic orbit does *not* intersect the interior of $\cup_{i=1}^b Z_{i,T}$, then the backward orbit of this periodic orbit is not contained in $W(T)$ and then components of $B(T)$ are contained in $\bigcup \overline{W}_i$ where W_i are adjacent components of $W(T)$. This situation is clarified in Example 4.8 below. The final statement holds because if W' and W'' have a boundary point in common, then $T^i(W \cup W')$ is a single point for some $i > 0$. \square

4.3. **The sets $\langle T \rangle$, $[T]$ and $[T]_b$.** As in Definition 3.2 we say that T and \tilde{T} are *partially conjugate* if there exists an orientation preserving homeomorphism which maps Z_i to \tilde{Z}_i , which maps $B(T)$ to $B(\tilde{T})$ and which conjugates T and \tilde{T} outside these sets. We define $\mathcal{PH}(T)$ to be the set of $\tilde{T} \in \mathcal{S}^b$ which are partially conjugate to T . Since such maps have plateaus, we also define

$$(6) \quad \langle T \rangle = \{\tilde{T} \in \mathcal{S}^b \ ; \ W(\tilde{T}) = W(T)\} \quad \text{and} \quad [T] = \text{closure}(\langle T \rangle).$$

Note that if $\tilde{T} \in \langle T \rangle$ then $\langle \tilde{T} \rangle = \langle T \rangle$.

Of course, $\langle T \rangle$ and $\mathcal{PH}(T)$ are closely related:

Lemma 4.4. *If each periodic orbit of T is hyperbolic, then $\langle T \rangle \subset \mathcal{PH}(T) \subset [T]$.*

Example 4.8 shows that the assumption that all periodic orbits of T are hyperbolic is required.

Proof. By Lemma 4.3(6)(b), if each periodic point of T is hyperbolic, then each component of $B(T)$ is a component of $W(T)$. It follows that if $W(\tilde{T}) = W(T)$ then the basins of \tilde{T} and T are the same, and therefore \tilde{T} and T are partially conjugate. Now assume that $\tilde{T} \in \mathcal{PH}(T)$. Since \tilde{T} and T agree outside plateaus, and periodic points are dense outside the basins, the partial conjugacy outside $B(\tilde{T})$ and $B(T)$ has to be the identity map. It follows that if \tilde{T} also has only hyperbolic periodic orbits, then $B(\tilde{T})$ is also open and $W(T) = W(\tilde{T})$. If \tilde{T} has one or more non-hyperbolic periodic orbits, then by widening the corresponding plateaus one obtains a sequence of maps $\hat{T}_n \in \langle T \rangle \cap \mathcal{PH}(T)$ with $\hat{T}_n \rightarrow \tilde{T}$. This implies the lemma. \square

Below we shall show that $\langle T \rangle$ is contained in a hyperplane V_T , and that each map $[T] \setminus \langle T \rangle$ either has touching plateaus or an orbit of one of the following special types:

Definition 4.5 (Homoclinic orbit). *We say that T has a homoclinic orbit (hc), if some iterate of a plateau is mapped to the boundary of a component of $W(T)$.*

Definition 4.6 (Hyperbolic/Saddle-node/Period Doubling/Pitchfork). *Assume that p is an attracting periodic point of T (of minimal period s) and therefore the orbit of p enters a plateau of T . We say that p is hyperbolic if its orbit enters the interior of a plateau of T . Otherwise, we say p is non-hyperbolic and a*

- (1) saddle-node (sn) if p is in the boundary of a plateau of T^s and S_0^s is orientation preserving near p ; if there exists a neighborhood U of p so that both components of $U \setminus \{p\}$ are in the immediate basin of a periodic attractor, then we say that p is a saddle-node merging two basins, see the left panel of Figure 7;
- (2) period-doubling (pd) if p is in a plateau of T^s and S_0^s is orientation reversing near p , see Figure 8;

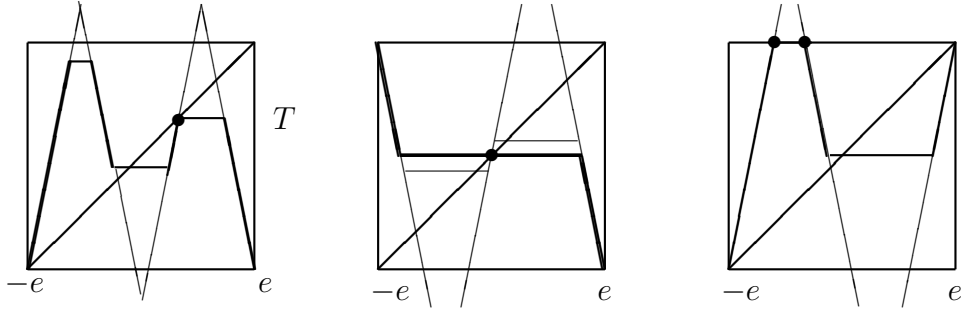


FIGURE 7. Left: a saddle-node merging two basins. Middle: a map with two adjacent plateaus with a pitch-fork fixed point; in this case, there are nearby maps with two attracting periodic points of the same period. Right: a map with a homoclinic orbit.

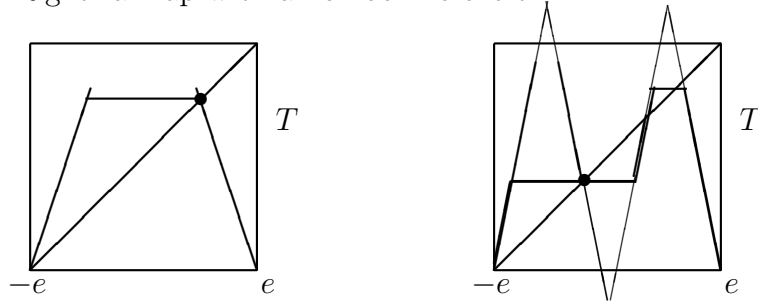


FIGURE 8. Left: a period doubling fixed point; nearby maps have a periodic point of period two. Right: another period doubling fixed point, this time between two plateaus.

- (3) pitch-fork (*pf*) if p is in the interior of a plateau of T^s and S_0^s is orientation preserving near p , see the middle panel of Figure 7.

In the latter case, p is either the common boundary of two touching plateaus of T , or the orbit of p hits at least twice a boundary point of a plateau.

Definition 4.7. ($\min[T]$ and $[T]_b$) Given $T \in \mathcal{S}^b$ we define the sets

$$\min[T] = \{T' \in [T] \ ; \ \text{there is no } T'' \in [T] \text{ with } T'' < T'\}.$$

Here $<$ is the partial ordering on the space \mathcal{S}^b defined above Proposition 4.1. To deal with the situation that plateaus of T touch, we also define

$$[T]_b = \left\{ T' \in \min[T] \ ; \ \left. \begin{array}{l} \text{no plateau } Z' \text{ of } T' \text{ is eventually mapped} \\ \text{into } \cup_{i=1}^b \text{int}(Z'_i) \text{ by some positive iterate.} \end{array} \right\}$$

Clearly

$$\emptyset \neq [T]_b \subset \min[T] \subset [T].$$

We illustrate these definitions in Examples 4.8-4.11, which helps in obtaining a general description of the set $\langle T \rangle$. We should emphasize that some parts of the boundary of $\langle T \rangle$ are contained in $\langle T \rangle$ whereas others are not. Figures 9, 10 and 11 give explicit descriptions of $\langle T \rangle$ in a number of situations.

Example 4.8. Figure 9 illustrates the definition of $\langle T \rangle$ in the *unimodal case*. Let $p \in (-e, e)$ be the orientation reversing fixed point of S_0 and $q \in (0, e)$ the periodic point of S_0 of period two. For $\zeta \in [-e, p)$, we have $W(T_\zeta) = (-e, e)$. For $\zeta \in [p, q)$, $W(T_\zeta)$ consists of a countable number of adjacent intervals: $W(T_\zeta) = (-e, e) \setminus \bigcup_{n \geq 0} T^{-n}(p)$. When $\zeta = q$, $W(T_\zeta)$ consists of a countable number of adjacent intervals: $W(T_\zeta) = (-e, e) \setminus \bigcup_{n \geq 0} (T^{-n}(p) \cup T^{-n}(q))$. In particular,

$$\langle T_{-e} \rangle = [-e, p), \mathcal{PH}(T_{-e}) = \{-e\}, \min[T_{-e}] = [T_{-e}]_b = \{-e\},$$

where $[-e, p)$ stands for the set of maps T_ζ with $\zeta \in [-e, p)$. Moreover,

$$\langle T_p \rangle = [p, q), \mathcal{PH}(T_p) = (-e, p], \min[T_p] = [T_p]_b = \{p\}.$$

This example shows that the assumption in Lemma 4.4 that T has only hyperbolic periodic orbits is essential.

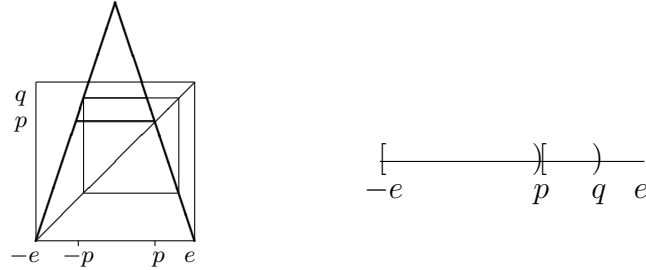


FIGURE 9. The set $\langle T \rangle$ in the unimodal case, see Example 4.8.

Example 4.9. In Figure 10 we illustrate the definition of $\langle T \rangle$ in the case when there is a component $W_i = (a, b)$ of the $W(T)$ containing two plateaus with the corresponding set $\langle T \rangle$ drawn in parameter space on the right. The map T depicted in the figure, is the first return map to (a, b) and has the property that $W(T) \cap (a, b) = (a, b)$. The set $\langle T \rangle$ for this map is shown in Figure 10 on the right, and is equal to the union of the two open triangles with the open interval Δ connecting the points marked 0 and 2 (corresponding to maps T_0 and T_2). To see this for T_0 , note that $W(T_0) \cap (a, b)$ is equal to $(a, b) \setminus Q$ where Q is a countable set made up of backward iterates of the left boundary point of the left plateau. Therefore $\langle T \rangle$ is neither open nor closed. Note that $\min[T] = \overline{\Delta}$ and $[T]_b = \{T_0, T_1, T_2\}$. Furthermore, $\langle T_2 \rangle = T_2$ since taking $\zeta_2 > 0$ (while ζ_1 is left unchanged) results in the right endpoint of Z_1 no longer belonging to $W(T)$. The open interval in Figure 10 connecting 2 to 3 is a single $\langle \tilde{T} \rangle$, whereas the line segment connecting 1 to 3 consists of countably many different cells $\langle \tilde{T} \rangle$ consisting of half-open line segments with endpoints corresponding to maps for which one plateau is mapped into the boundary of the other plateau.

Example 4.10. Figure 11 illustrates the definition of $\langle T \rangle$ in the bimodal case when there exists a periodic component W of $W(T)$ of period $s_1 + s_2$ so that W and the component W' of $W(T)$ containing $T^{s_1}(W)$ both contain a plateau. In this case $\min[T] = [T]_b = \{T_1\}$ where T_1 is the map corresponding to 1 in the figure. Note that only the left and bottom boundary is contained in $\langle T \rangle$ (not including the endpoints of these lines).

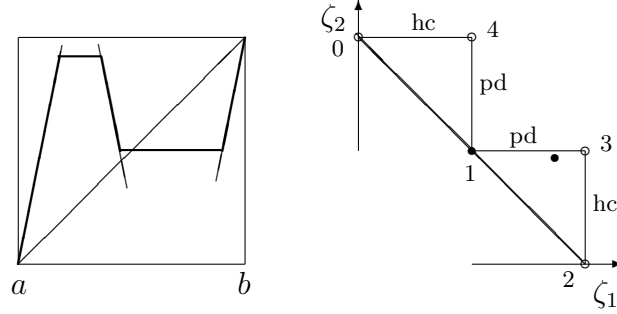


FIGURE 10. The set $\langle T \rangle$ in the case where two plateaus lie in one component of the basin, see Example 4.9. The \bullet on the right indicates the parameter of the map T on the left.

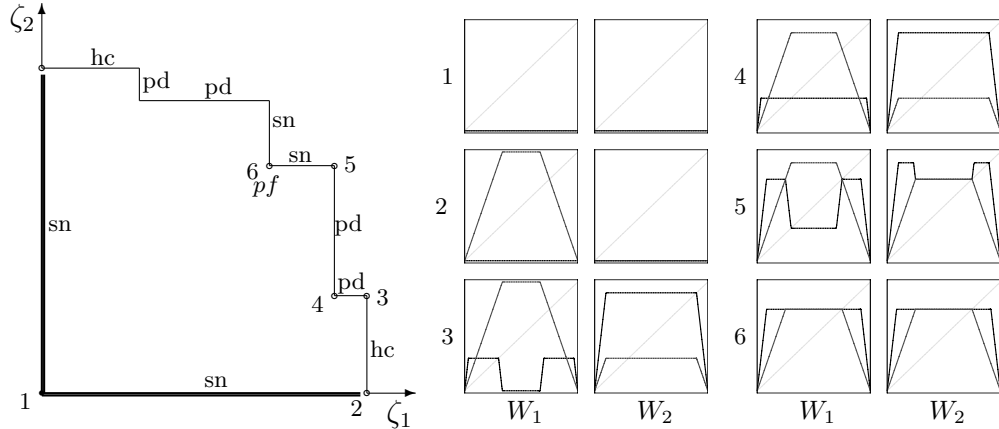


FIGURE 11. The set $\langle T \rangle$ in the bimodal case when an attractor contains two plateaus in its basin, see Example 4.10 is depicted on the left. For each of the 6 indicated parameters on the left, we draw on the right side, the corresponding graphs of T^{s_1} and return map $T^{s_2} \circ T^{s_1}$ to W_1 (respectively T^{s_2} and the return map $T^{s_1} \circ T^{s_2}$ to W_2). Here the graph with the larger slope corresponds to the first return map. Of course the diagonal has only meaning for the maps $T^{s_2} \circ T^{s_1}$ and $T^{s_1} \circ T^{s_2}$. The part of the boundary of the polygon which is contained in $\langle T \rangle$ is marked by (sn).

Example 4.11. In this example we show why we consider $\min[T]$; it is possible that (with the analogous definition) $\min \langle T \rangle = \emptyset$. Consider $T \in \mathcal{S}^b$ so that there exists an interval $[Z_i, Z_{i+1}]$ which is mapped into another plateau Z_k with $T(Z_i) \subset \partial Z_k$, with $T(Z_{i+1})$ contained in the interior of Z_k and so that no iterate of Z_k is contained in a plateau. Then ∂Z_k is not contained $W(T)$. This implies that the i -th projection of $W(T)$ is a point while the $i + 1$ -th projection of $W(T)$ is an open interval. Therefore $\min \langle T \rangle = \emptyset$. (However, $\min[T] \neq \emptyset$; it consists of maps for which Z_i and Z_{i+1} touch.

4.4. **Wandering pairs and the space \mathcal{S}_*^b .** The space \mathcal{S}^b contains maps that are of no use to us because they possess wandering intervals, a phenomenon that does not occur in P^b . For this reason, we define a subset \mathcal{S}_*^b to be used as a more faithful parameter space of P^b than \mathcal{S}^b .

An important property of \mathcal{S}_*^b , which will be used in the proof of the Main Theorem, is that isentropes $\{T \in \mathcal{S}_{\epsilon,*}^b ; h_{top}(T) = h\}$ are contractible in \mathcal{S}_*^b . The proof of this is deferred to Section 7. In this section, we will define \mathcal{S}_*^b and show some of its basic properties.

Definition 4.12. *A pair of plateaus (Z_i, Z_j) is called wandering if there exists $n \geq 1$ such that $T^n(\mathcal{J})$ is a point, where $\mathcal{J} := [Z_i, Z_j]$ is the convex hull of Z_i and Z_j . We say that T is non-degenerate if for every wandering pair (Z_i, Z_j) , the corresponding interval \mathcal{J} belongs to the closure of a component of the basin of a periodic plateau. Let \mathcal{S}_*^b denote the set of non-degenerate maps $T \in \mathcal{S}^b$.*

Remark 4.13. *If $T \in \mathcal{S}_*^b$ then $[T] \subset \mathcal{S}_*^b$.*

Remark 4.14. *Blocks of touching plateaus occur only at the boundary of the parameter space, and bimodal maps with touching plateaus have an attracting fixed point. For this reason, wandering pairs don't occur in Milnor & Tresser's paper [31], and $\mathcal{S}_*^b = \mathcal{S}^b$ for $b \leq 2$.*

Remark 4.15. *For $b \geq 3$, \mathcal{S}_*^b is clearly not closed, but it is not open either. Indeed, take T with a pair of adjacent plateaus Z_i, Z_{i+1} so that $[Z_i, Z_{i+1}]$ is mapped into the interior of a plateau Z_j and $T(U) \cap Z_j \subset \partial Z_j$ for some small open neighborhood U of Z_j . Since Z_j is in the basin of an attracting fixed point, such a map T exists in \mathcal{S}_*^b . Moreover, there exist maps \tilde{T} arbitrary close to T so that \tilde{Z}_j is no longer contained in the basin of a periodic attractor (just choose $\tilde{\zeta}_j < \zeta_j$ appropriately). It follows that there exists maps $\tilde{T} \notin \mathcal{S}_*^b$ arbitrarily close to T .*

Nevertheless, the space \mathcal{S}_*^b has the following useful property:

Lemma 4.16. (a) *Take $T \in \mathcal{S}^b$ and an interval U which is not eventually mapped into a plateau and is also not contained in the basin of a periodic attractor. Then there exists $n > m$ so that $T^n(U) \cap T^m(U) \neq \emptyset$ and $\liminf_j |T^j(U)| > 0$.*

(b) *Take $T \in \mathcal{S}_*^b$ and consider adjacent plateaus Z_i, Z_{i+1} so that the convex hull $[Z_i, Z_{i+1}]$ is not contained in the closure of a component of the basin of a periodic attractor. Then for each map $\tilde{T} \in \mathcal{S}^b$ sufficiently close to T , the adjacent plateaus $\tilde{Z}_i, \tilde{Z}_{i+1}$ do not form a wandering pair either.*

Proof. Take $T \in \mathcal{S}_*^b$ and interval U as in statement (a). Since T is expanding outside its plateaus, there exists a sequence $n_i \rightarrow \infty$ so that $T^{n_i}(U)$ intersects one of the plateaus of T (but is not contained in a plateau). It follows $T^{n_i}(U)$ contains a neighborhood of one of the endpoints of a plateau for infinitely many i . This means that there exist $n > m$ so that $T^n(U)$ and $T^m(U)$ intersect, and

therefore $T^{n+(j+1)k}(U) \cap T^{n+jk}(U) \neq \emptyset$ for $k := n - m$ and all $j \geq 0$. It follows that $V = \cup_{j \geq 0} T^{n+jk}(U)$ is an interval and $T^k(V) \subset V$. So either V contains a repelling fixed point of $T^k|_V$ or the interval V is the finite union of basins of basins of plateaus, separated by one-sided fixed points of $T^k|_V$. Since U is not eventually mapped into the basin of a periodic attractor, V contains in its interior a periodic point p of period k which is not attracting from both sides. Hence U contains in its interior a point x so that $T^l(x) = p$ for some l . The conclusion of statement (a) of the lemma follows.

Next consider $\mathcal{J} := [Z_i, Z_{i+1}]$ as in statement (b). By definition of \mathcal{S}_*^b , the interval U is never mapped into another plateau. Hence, by the proof of statement (a), some iterate of \mathcal{J} contains a repelling periodic point p of T in its interior, where p is either (i) repelling from both sides or (ii) p is attracting from one side and separates the immediate basin of two adjacent fixed points of T^k . Therefore, if (i) holds and \tilde{T} is sufficiently close to T , then p is still contained in the interior of some iterate of $[\tilde{Z}_i, \tilde{Z}_{i+1}]$ and p is still repelling for \tilde{T} . If (ii) holds then the two basins could merge, but will still contain the iterate of U . Hence in any case $[\tilde{Z}_i, \tilde{Z}_{i+1}]$ also does not form a wandering pair for \tilde{T} . \square

4.5. Further properties of the sets $\langle T \rangle$, $[T]$ and $[T]_b$. Given $T \in \mathcal{S}^b$, we define the following equivalence class \sim_T on $\{1, 2, \dots, b\}$: $i \sim_T j$ if and only if Z_i and Z_j are both in the immediate basin of the same periodic attractor. (We do not require that Z_i and Z_j are in the same component of the immediate basin.) Let $J_1, J_2, \dots, J_s \subset \{1, \dots, b\}$ be the corresponding equivalence classes. For $J = \{i_1, \dots, i_j\} \subset \{1, \dots, b\}$ define $\pi_J: \mathcal{S}^b \rightarrow \mathbb{R}^{\#J}$ be the projection of $\zeta = (\zeta_1, \dots, \zeta_b) \in \mathcal{S}^b$ to $(\zeta_{i_1}, \dots, \zeta_{i_j}) \in \mathbb{R}^{\#J}$.

Theorem 4.17. *There exists an affine space V_T so that, with respect to the coordinates $(\zeta_1, \dots, \zeta_b) \in \mathbb{R}^b$, $\langle T \rangle$ is a connected polygonal region in V_T . Moreover, the following properties hold:*

- (1) *If $[T_1] = [T_2]$ then $\langle T_1 \rangle = \langle T_2 \rangle$.*
- (2) *If $T_1 < T_2 < T_3$ and $T_1, T_3 \in \langle T \rangle$ then $T_2 \in \langle T \rangle$;*
- (3) *$\langle T \rangle$ has a product structure: if we take $J_1, \dots, J_s \subset \{1, \dots, b\}$ as above the statement of the theorem, then there exist connected (polygonal) sets $A_i \subset \mathbb{R}^{\#J_i}$, $i = 1, \dots, s$ so that*

$$\langle T \rangle = \{\zeta; \pi_{J_i}(\zeta) \in A_i \text{ for each } i = 1, \dots, s\}.$$

If $\#J_i = 1$, then the corresponding set A_i is an interval $[a_i, b_i)$ or a point $\{a_i\}$ whereas if $\#J_i > 1$, then A_i has a polygonal shape (see Figures 10 and 11 for representative examples).

- (4) *For each $T \in \mathcal{S}_*^b$, there exists $T' \in \mathcal{S}_*^b$ so that $T' \in [T']_b$ and $T \in [T']$.*
- (5) *If $[T_1] \neq [T_2]$ and $[T_1] \cap [T_2] \neq \emptyset$, then there exists $T' \in [T']_b$ with $T' \in \min([T_1] \cap [T_2])$.*

(6) If $\tilde{T} \in [T] \setminus \langle T \rangle$, then one of the following properties holds:

- \tilde{T} has a saddle-node merging two basins;
- \tilde{T} has a period doubling orbit;
- \tilde{T} has a pitch-fork orbit;
- \tilde{T} has a homoclinic orbit.

Proof. Let $T = T_\zeta$ be the map corresponding to $\zeta = (\zeta_1, \dots, \zeta_b)$, let $\tilde{\zeta} = (\tilde{\zeta}_1, \dots, \tilde{\zeta}_b)$, and assume that $\tilde{T} = T_{\tilde{\zeta}}$ is so that $\langle \tilde{T} \rangle = \langle T \rangle$, i.e., $W(\tilde{T}) = W(T)$. Consider a plateau Z_k of T and in order to be definite assume that S_0 has a maximum in Z_k . Let W_k be the component of $W(T)$ intersecting Z_k . We denote the k -th plateau of \tilde{T} by \tilde{Z}_k .

Step 1: First consider the case that $T(Z_k)$ is *not* contained in the closure of a component of $W(T)$. Let us show that in this case $\tilde{\zeta}_k = \zeta_k$. Indeed, W_k is equal to the interior of Z_k (or equal to the interior of the union of all plateaus which touch Z_k). If $\tilde{\zeta}_k < \zeta_k$ then \tilde{Z}_k strictly contains W_k and in particular the corresponding component of $W(\tilde{T})$ strictly contains W_k , contradicting $\langle \tilde{T} \rangle = \langle T \rangle$. Hence $\tilde{\zeta}_k \geq \zeta_k$. By assumption, there is a sequence of points converging from the right to $T(Z_k)$ which are not in $W(T)$. Hence if $\tilde{\zeta}_k > \zeta_k$ then the component of $W(T_{\tilde{\zeta}})$ intersecting \tilde{Z}_k is equal to \tilde{Z}_k and so is strictly inside Z_k , contradicting $W(\tilde{T}) = W(T)$. It follows that in this case $\tilde{\zeta}_k = \zeta_k$. Note that the freedom of choice of ζ_k does **not** depend on any of the other coordinates.

Step 2. Next consider the case W_k is not periodic and $T(Z_k)$ is contained in the closure of a component $W = (a, b)$ of $W(T)$.

Step 2a. Let us first assume that $T(Z_k) \subset [a, b)$. If $T(Z_k) = \{a\}$ then W_k is equal to the interior of Z_k and otherwise W_k is equal to the component of $T^{-1}(a, b)$ containing Z_k . In either case, W_k is equal to the component of $S_0^{-1}(a)$ containing Z_k . If $\tilde{T}(\tilde{Z}_k) \geq b$ then there exists $x \in Z_k \setminus \tilde{Z}_k$ so that $\tilde{T}(x) = S_0(x) = b \notin W$ and hence $x \notin W(T)$, contradicting that $W(\tilde{T}) = W(T)$. If $\tilde{T}(\tilde{Z}_k) < a$, then the interior of \tilde{Z}_k strictly contains W_k which is not possible either. On the other hand, since W_k is not periodic, changing $\tilde{\zeta}_k \in [a, b)$ does not change the component of $W(\tilde{T})$ containing $\tilde{T}(\tilde{Z}_k)$. It follows that in this case a necessary and sufficient condition on $\tilde{\zeta}_k$ for $W(\tilde{T}) = W(T)$ to hold is that $\tilde{\zeta}_k \in [a, b)$ and $\tilde{T} \in \mathcal{S}_*^b$.

Step 2b. If W_k is not periodic and $T(Z_k) = \{b\}$ where $W = (a, b)$ is a component of $W(T)$ then W_k is the interior of Z_k and as in Step 1 we get $\tilde{\zeta}_k = \zeta_k$. Again, the freedom of choice of ζ_k does **not** depend on any of the other coordinates.

Step 3. Now consider the case that W_k is periodic, i.e., $T^s(W_k) \subset \overline{W_k}$ for some $s > 0$. By Lemma 4.3 either $T^s(W_k) \subset \partial W_k$ or $T^s: W_k \rightarrow W_k$. In the latter case $T^s: W_k \rightarrow W_k$ has a unique fixed point $p \in W_k$ (and $T^j(p) \in \text{int}(\cup_{i=1}^b Z_{k,T})$ for some

$0 \leq j < s$) and $T^s(\partial W_k) \subset \partial W_k$. Because of Lemma 4.3(5)-(6), the component of $W(T_{\tilde{\zeta}})$ intersecting \tilde{Z}_k is equal to W_k if and only if

- (i) $\tilde{T}^s(W_k) \subset \overline{W_k}$;
- (ii) $\tilde{T}^s(W_k) \cap \partial W_k \neq \emptyset$ implies that $T^s|_{W_k}$ is constant, and
- (iii) $\tilde{T}^{2s}|_{W_k}$ has at most one fixed point (which is attracting).

Let (a, b) be the component of $W(T)$ which contains $T(W_k)$ in its closure. Properties (i), (ii) and (iii) persist while *decreasing* $\tilde{\zeta}_k \in [a, b)$, by Lemma 4.3(5)-(6). While *increasing* $\tilde{\zeta}_k$ in $[a, b)$ these properties are preserved until $\partial \tilde{Z}_k$ first hits a fixed point q of S_0^{2s} . If $S_0^j(q) \in \text{int}(\cup_{i=1}^b Z_{i,T})$ for at least one $0 \leq j < s$, then one can continue to increase $\tilde{\zeta}_k$ until one $\partial \tilde{Z}_k$ hits another fixed point q of S_0^{2s} . In this way, we can keep increasing $\tilde{\zeta}_k$ (*i.e.*, shrink the width of the plateau), until either ∂Z_i contains a fixed point q' of S_0^{2s} for which $S_0^j(q') \notin \text{int}(\cup_{i=1}^b Z_{i,T})$ for all $0 \leq j < s$ or until we no longer have $\tilde{T}^s(Z_k) \subset W_k$.

It follows that, fixing all ζ_j with $j \neq k$, the set $\langle T \rangle$ is equal to a line segment of the form $a_k \leq \zeta_k < b_k$. Here the left boundary a_k does **not** depend on the choice for the other ζ_j 's, because one can decrease $\tilde{\zeta}_k \in [a_k, b_k)$ and the latter interval does not depend on ζ_j , $j \neq k$. However, the right hand boundary b_k in general **will** depend on parameters ζ_j , $j \neq k$ for which Z_j is contained in (the closure of) one of the components of $W(T)$ containing $W_k, T(W_k), \dots, T^{s-1}(W_k)$. The location is determined by the position of the fixed points of S_0^{2s} .

Thus we have proved that $\langle T \rangle$ is contained in a polygonal region in some hyperplane V_T and that properties (1), (2) and (3) in the theorem are satisfied.

Step 4. We claim that we can choose $T' \in [T]_b$ (by only changing T inside the basin of periodic attractors) so that $T \in [T']$ and $T' \in [T']_b$. Indeed, if the number of plateaus within a component of the basin is even, then we choose T' analogously to the map T_1 in Example 4.9 and when the number of plateaus is odd then all these plateaus of T' necessarily touch and the attracting periodic point is at the boundary point of the union of the touching plateaus. This implies that, for the coordinates corresponding to attracting plateaus, $T \in [T']$ and $T' \in [T']_b$. If Z_i is contained in the basin, but not in the immediate basin of a periodic attractor, then for any map $\hat{T} \in [T]_b$ the corresponding plateau \hat{Z}_i is either mapped into the boundary of this component, or Z_i touches with one of its neighbouring plateaus. From this description it follows that $T \in [T']$ and $T' \in [T']_b$. Since T and T' only differ on the basin of periodic attractors, we will still have $T' \in \mathcal{S}_*^b$.

Step 5. To prove Property (5), assume that $[T_1] \neq [T_2]$ and $[T_1] \cap [T_2] \neq \emptyset$. Property (2) gives that $\min([T_1] \cap [T_2]) \subset [T_1] \cap [T_2]$. It is possible that $T \in \min([T_1] \cap [T_2])$ has two or more touching plateaus Z_i, Z_{i+1} which are mapped by some iterate T^s into the interior of another plateau Z_j . In this case choose a continuous deformation T_t

with $T_0 = T$ so that for each such pair of touching plateaus, $T_t^s([Z_i, Z_{i+1}]) \subset Z_j$ for all $t \in [0, 1]$ (leaving all other plateaus unchanged). Then $T' := T_1 \in \min([T_1] \cap [T_2])$ has the required properties.

Step 6. Property (6) holds because $T' \in [T] \setminus \langle T \rangle$ implies that at least one periodic orbit is in the boundary of a plateau. By Property (2) this periodic orbit cannot be a saddle-node, unless it corresponds to a situation where two basins are merged. \square

5. THE MAP $\Psi: P^b \rightarrow \mathcal{S}^b$.

Let us review some basic kneading theory, see [30], and also [26, 31]. Given an interval I and piecewise monotone b -modal map $f: I \rightarrow I$ with turning points $c_1 < \dots < c_b$ in the interior of I , one can associate to each point $x \in I$ an *itinerary* $\underline{i}_f(x)$ consisting of a sequence (i_0, i_1, \dots) of symbols from the alphabet $\{I_0, c_1, I_1, c_2, \dots, c_b, I_b\}$. Here I_0, \dots, I_b are the components of $I \setminus \{c_1, \dots, c_b\}$ ordered from left to right. It is well-known that $x \mapsto \underline{i}_f(x)$ is monotone w.r.t. the signed lexicographic ordering and that therefore the i -th *kneading sequence*

$$\nu_i := \lim_{x \downarrow c_i} \underline{i}_f(x)$$

is well-defined. Note that the sequence ν_i does not contain any of the symbols c_1, \dots, c_b . Let σ be the left shift on the space of symbol sequences. The kneading invariant $\nu(f)$ of f is defined as

$$\nu(f) := (\nu_1, \dots, \nu_b).$$

Any kneading invariant which is realized by some piecewise monotone b -modal map is called *admissible*.

To each map $f \in P^b$ one can associate *uniquely* a stunted sawtooth map as follows. Let $\nu(f) = (\nu_1, \dots, \nu_b)$ be the kneading invariant of f , and let s_i be the unique point in the $(i+1)$ -th lap I_i of S_0 such that

$$(7) \quad \lim_{y \downarrow s_i} \underline{i}_{S_0}(y) = \nu_i := \lim_{x \downarrow c_i} \underline{i}_f(x).$$

Let Z_i be the symmetric interval around the i -th turning point of S with right endpoint s_i . Let us define

$$(8) \quad \Psi: P^b \rightarrow \mathcal{S}^b, \quad f \mapsto \Psi(f),$$

by associating to f the unique stunted sawtooth map $\Psi(f)$ which agrees with S_0 outside $\cup_{i=1}^b Z_i$ and which is constant on Z_i with value $S_0(s_i)$.

5.1. Some good properties of Ψ . The main reason for working with the map Ψ is that it allows us to work with the Euclidean space \mathcal{S}^b rather than with the space of kneading invariants.

Lemma 5.1. *The map $\Psi: P^b \rightarrow \mathcal{S}^b$*

- (1) *is well-defined;*
- (2) *the kneading invariant of f and $T := \Psi(f)$ are the same in the sense that $\lim_{y \downarrow Z_i} i_T(y) = \nu_i$.*
- (3) *f and $\Psi(f)$ have the same topological entropy;*
- (4) *$\Psi(P_\epsilon^b) \subset \mathcal{S}_{\epsilon,*}^b$.*

Proof. Since S_0 allows every sequence in $\{I_0, \dots, I_b\}^{\mathbb{N}}$ as itinerary, we can always find a stunted version T with the required kneading sequences. In fact, because S_0 is expanding and so distinct points have different itineraries, the stunted version is unique, so Ψ is well-defined. It also follows that the orbits of the boundary points of Z_i under T and the sawtooth map S_0 agree and therefore statement (2) holds. Entropy is fully determined by kneading sequences, so Ψ preserves entropy. For the last statement, assume by contradiction that $\Psi(f) \in \mathcal{S}^b \setminus \mathcal{S}_*^b$. Since f has no wandering intervals, $\Psi(f) \in \mathcal{S}^b \setminus \mathcal{S}_*^b$ implies that there exists an interval connecting two adjacent critical points, so that the n -th iterate of this interval is another critical point. Clearly this is impossible. \square

5.2. Some bad properties of Ψ . The next example shows that $f \mapsto \Psi(f)$ is neither continuous, nor injective nor surjective, but later on we shall see that this map is ‘almost’ continuous, injective and surjective. It also shows that $\Psi(f) = \Psi(\tilde{f})$ does *not* imply that f, \tilde{f} are partially conjugate.

Example 5.2. Consider the family $f_\lambda(x) = \lambda x(1-x)$ and let T_ζ be as in Example 4.8. Then there are parameters $0 < \lambda'_1 = 2 < \lambda_1 = 3 < \lambda'_2 < \lambda_2 < \lambda'_3$ so that λ_1, λ_2 are the first two period doubling parameters, and $\lambda'_1, \lambda'_2, \lambda'_3$ are the first three parameters at which the critical point of f_λ is periodic. Then

$$\Psi(f_\lambda) = \begin{cases} T_{-e} & \text{for } \lambda \in [0, \lambda'_1], \\ T_p & \text{for } \lambda \in (\lambda'_1, \lambda'_2], \\ T_q & \text{for } \lambda \in (\lambda'_2, \lambda'_3], \end{cases}$$

see Figure 12. As in Example 4.8, T_p and T_q are the maps for which the right endpoint of the plateau is a fixed point and has period two respectively. The discontinuities of $f \mapsto \Psi(f)$ occur when the critical point becomes periodic, rather than when a period doubling bifurcation occurs. In particular, for $\delta > 0$ small, $\Psi(f_{\lambda_1+\delta}) = \Psi(f_{\lambda_1-\delta}) = T_p$ but $f_{\lambda_1+\delta}$ and $f_{\lambda_1-\delta}$ are not partially conjugate. On the other hand, for each $\lambda \in [0, \lambda_1]$, $f_\lambda \in \mathcal{PH}(f_0)$ and $\Psi(f_\lambda) \in \{T_{-e}, T_p\} \subset [T_{-e}] = [\Psi(f_{\lambda'_1})]$; note that T_{-e} and T_p are partially conjugate.

5.3. The definition of \mathcal{A}_b .

Definition 5.3. *The set \mathcal{A}_b is the collection of polynomials $f \in P^b$ such that for each component B of its basin $B(f)$ containing a critical point the following holds:*

- (1) *if $f(\partial B)$ consists of a single point, then each points in $f(B)$ has the same itinerary.*

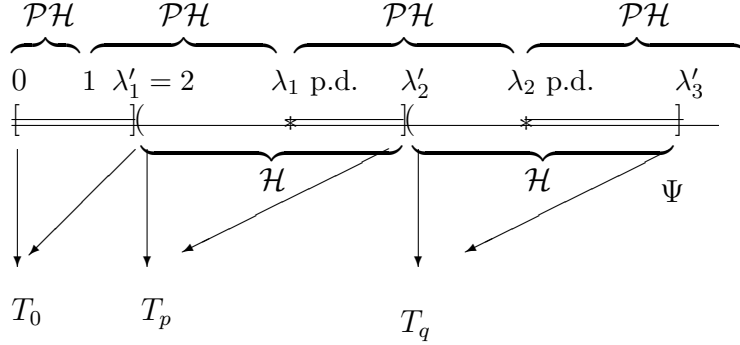


FIGURE 12. The map $\lambda \mapsto \Psi(f_\lambda)$ is discontinuous at parameters λ'_i where the critical point of f_λ is periodic, see Example 5.2. These parameters are alternated with period doubling parameters λ_i , and at $\lambda = 1$, the stability of 0 changes, so \mathcal{PH} changes too. The equivalence classes \mathcal{PH} (of partially conjugate maps) and \mathcal{H} (of maps with the same kneading invariant) are also shown. The segments with an additional line correspond to maps in the set \mathcal{A}_b , defined in Section 5.3.

- (2) if $f(\partial B)$ consists of two points, then (i) each point in $f(B')$ has the same itinerary where B' be the convex hull of the critical points in B and (ii) if B contains an attracting periodic point p , then p is in the interior of B' .

Note that $f(\partial B)$ consists of a single point if and only if the interior of B contains an odd number of critical points. The reason for introducing \mathcal{A}_b is:

Proposition 5.4. *If $f \in \mathcal{A}_b$, then $\Psi(\mathcal{PH}(f)) \subset [\Psi(f)]$.*

Before proving this proposition we will motivate the definition of \mathcal{A}_b by considering three examples.

Example 5.5. If $f_\lambda(x) = \lambda x(1 - x)$ is a quadratic map with an attracting fixed point, then $\Psi(f_\lambda)$ is a stunted sawtooth map which is either equal to the constant map T_0 corresponding to the parameter $\zeta = -e$ or to the map T_p which has plateau $[-p, p]$, see Example 4.8 and Figure 9. Now $T_p \in [T_{-e}]$ while $T_{-e} \notin [T_p]$. Moreover $\mathcal{PH}(f_{3/2}) = \{f_\lambda; \lambda \in (1, 3)\}$ and $\Psi(\mathcal{PH}(f_{3/2})) \subset \{T_{-e}, T_p\} \subset [T_{-e}]$. Note that $f_\lambda \in \mathcal{A}_b$ when $\lambda \in [0, 2]$ and then $\Psi(f) = T_{-e}$ and so $\Psi(\mathcal{PH}(f)) \subset [\Psi(f)]$. If $f_\lambda \notin \mathcal{A}_b$ then this inclusion does **not** hold, so the assumption that $f \in \mathcal{A}_b$ is essential for Proposition 5.4 to hold. Note that it also **not** true that $\Psi(\mathcal{PH}(f)) \subset \mathcal{PH}(\Psi(f))$; Example 4.8 (see also Figure 12) is the simplest counter-example.

The next example, shows why in the definition of \mathcal{A}_b we add condition 2(ii) if $f(B)$ consists of two points,

Example 5.6. Assume that f is a cubic map with an attracting fixed point which attracts both critical points (say with the left critical point a maximum). Then $\Psi(f)$

is equal to one of the following five maps T_0, T_1, T_2, T_3, T_4 determined by (ζ_1, ζ_2) equal to $(-e, e), (0, 0), (e, -e), (e, 0)$ or $(0, e)$ as indicated in Example 4.9 and Figure 10 (when taking $a = -e$ and $b = e$). Note that $T_0, T_1, \dots, T_4 \in [T_i]$ when $i = 1$ but not when $i = 0, 2, 3, 4$. Also note that $\Psi(\mathcal{PH}(f)) \subset \{T_0, T_1, T_2\} \subset [T_1]$. If $f \in \mathcal{A}_b$ then $\Psi(f) = T_1$ and so $\Psi(\mathcal{PH}(f)) \subset [\Psi(f)]$.

Example 5.7. Finally consider the example of a bimodal map such that there exist $s_1, s_2 \in \mathbb{N}$ so that $f^{s_1}(B_1) \subset B_2$ and $f^{s_2}(B_2) \subset B_1$ where B_i are distinct components of $B(f)$ and so that B_1 and B_2 both contain exactly one critical point. Then the situation is as in Example 4.9 and $\Psi(f)$ is contained in the polygon drawn in Figure 11. Maps in $\{\Psi(\tilde{f}) ; \tilde{f} \in \mathcal{PH}(f)\}$ correspond to the maps indicated by 1, 2, 4, 6 in Figure 11 and their symmetric counterpart under reflexion in the diagonal of the (ζ_1, ζ_2) -space. There are six such maps, all belonging to $\partial[\Psi(f)]$. The fact that $f \in \mathcal{A}_b$ ensures that $\Psi(f)$ corresponds to the lower corner of the region, *i.e.*, the map T denoted by 1 in Figure 11. This map has the property that $[T]$ is equal to this polygon (this is false for maps denoted by **hc** and **pd**).

Proof of Proposition 5.4. Let $T = \Psi(f)$ for an arbitrary $f \in P^b$. The definition of Ψ ensures that the orbit of $T(Z_i)$ under T is the same as an orbit of S_0 , and therefore no plateau can be mapped into the interior of a plateau by T . On the other hand, if $T^k(Z_i) \in \partial Z_j$ for some minimal k , then a small changes of ζ_i can move $T^k(Z_i)$ into the interior of Z_j . Therefore $T \in \partial\langle T \rangle$, and in particular, $\Psi(f) \in [\Psi(f)]_b$ for $f \in \mathcal{A}_b$.

Take $\tilde{f} \in \mathcal{PH}(f)$. For each i so that c_i is not in the basin of a periodic attractor, c_i and \tilde{c}_i have the same kneading invariant and so the i -th component of $\Psi(f)$ and $\Psi(\tilde{f})$ agree.

Now let us consider critical points in the basin of a periodic attractor. Although kneading sequences within $\mathcal{PH}(f)$ are not constant, all itineraries in the basin of a periodic attractor (in the limit sense of (7)) are (pre)periodic to the same periodic sequence in $\{I_0, \dots, I_b\}^{\mathbb{N}}$. Since every periodic attractor of \tilde{f} has a critical point in its immediate basin, there is $\tilde{T} \in \mathcal{S}^b$ that realizes the corresponding periodic itineraries by orbits that intersect the interior of a plateau. Additionally, \tilde{T} can be chosen such that all kneading sequences of $c_i \in B(f)$ are indeed achieved by points in the interior of plateaus. For this \tilde{T} and the semiconjugacy h between \tilde{T} and \tilde{f} , we have $h^{-1}(B(\tilde{f})) \subset W(\tilde{T})$. Also $W(\tilde{T}) = W(T)$, so $\tilde{T} \in \langle T \rangle$. Naturally, the periodic attractors of $\Psi(\tilde{f})$ lie on the boundary of plateaus, but that still means that $\Psi(\tilde{f}) \in [\tilde{T}] = [T]$. Since $\tilde{f} \in \mathcal{PH}(f)$ is arbitrary, $\Psi(\mathcal{PH}(f)) \subset [\Psi(f)]$. \square

5.4. Almost surjectivity of Ψ . In Proposition 5.9 below, we shall prove that Ψ is almost surjective. In order to prove this we need a result from [26] concerning full families. Let us say that a periodic attractor of a piecewise monotone interval map

$g: I \rightarrow I$ is *essential* if it contains a turning point in its immediate basin. We say that g has *no wandering intervals*, if each interval J for which $J, g(J), g^2(J), \dots$ are all pairwise disjoint necessarily intersects the basin of some periodic attractor. If g has no wandering interval, then each interval J for which $g^n|_J$ is a homeomorphism for all n is necessarily contained in the basin of periodic attractor, see [26]. It is well-known, see Theorem IV.A in [26], that maps in P^b do not have wandering intervals and that all their attractors are essential.

Theorem 5.8 (Fullness of Families). *Each piecewise monotone map g with b turning points is topologically conjugate to a polynomial in P^b , provided the following two properties are met:*

- (1) g has no wandering intervals and no inessential attractors;
- (2) each periodic turning point is an attractor (this is automatically satisfied if g is C^1).

Moreover, assume that g has an attracting periodic point, then one can find $f, \tilde{f} \in P^b$ which are both topologically conjugate to g so that the corresponding attracting periodic point is hyperbolic for f and parabolic for \tilde{f} .

Proof. The first part of this theorem is Theorem II.4.1 in [26]. The second part requires a slightly modifying the proof in [26] on page 124-125. \square

The next proposition gives the required surjectivity:

Proposition 5.9 (Ψ is almost surjective). *For each $T \in \mathcal{S}_*^b$ there exists a polynomial $f \in P^b \cap \mathcal{A}_b$ such that $T \in [\Psi(f)]$.*

Proof of Proposition 5.9. By Theorem 4.17(4), for each $T \in \mathcal{S}_*^b$ there exists $T' \in \mathcal{S}_*^b$ so that $T' \in [T']_b$ and $T \in [T']$. Therefore, if we can prove that there exists a polynomial $f \in P^b \cap \mathcal{A}_b$ such that $T' = \Psi(f)$, then the proposition follows as well.

Since T is not piecewise monotone (because of its plateaus), we cannot apply Fullness Theorem 5.8 directly. In order to obtain a piecewise monotone map, we first replace T on each component B of its basin which contains plateaus, by an affinely scaled copy of a map L_q as in Figure 13 of the appropriate type. Here q is the number of plateaus in B . Let us call the resulting map T'' . We can choose L_q so that T' and T'' have the same kneading invariants (and hence $[T']_b = [T'']_b$), and so that if $T \in [T']$ has an attracting periodic point in the common boundary point of two plateaus, then T'' has an attracting periodic point between the two corresponding turning points.

Since T'' may still have plateaus, we next define $x \sim y$ for $x, y \in [-e, e]$, if there exists $n \geq 0$ so that T''^n maps the convex hull $[x, y]$ into one of the (remaining) plateaus of T'' . Collapse each such interval $[x, y]$ to a point and let T''' be the corresponding map. From the definition it follows that T''' is continuous and since $T \in \mathcal{S}_*^b$, it follows that T''' is b -modal and has no wandering intervals. It also has

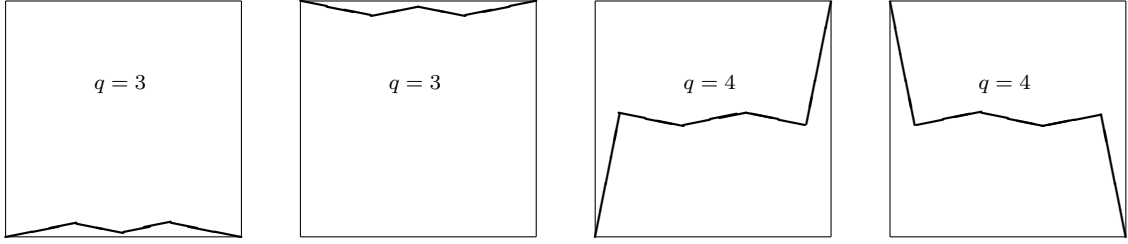


FIGURE 13. The map L_q where q is the number of touching plateaus in the component B of the basin of $T \in [T''']_b$. We choose L_q so that it is continuous, piecewise affine and so that the slope between its turning points is at most $1/4$.

no inessential attractors (since T does not have these either). Hence we can apply Theorem 5.8 and there exists $f \in P^b$ that is topologically conjugate to T''' , and therefore have the same kneading invariants. The choice of the L_q 's corresponds exactly with the definition of \mathcal{A}_b , so indeed $f \in \mathcal{A}_b$. \square

5.5. Almost injectivity of Ψ .

Proposition 5.10 (Ψ is almost injective). *The map $\Psi: P^b \rightarrow \mathcal{S}_*^b$ is ‘almost injective’ in the sense that if $f, \tilde{f} \in \mathcal{A}_b$ and $[\Psi(f)] \cap [\Psi(\tilde{f})] \neq \emptyset$, then at least one of $\overline{\mathcal{PH}(f)} \cap \overline{\mathcal{PH}(\tilde{f})}$ and $\overline{\mathcal{PH}(f)} \cap \mathcal{PH}(\tilde{f})$ is non-empty.*

Remark 5.11. *Here the assumption that $f, \tilde{f} \in \mathcal{A}_b$ is needed. Indeed, using the notation from Example 5.2, take $f = f_{\lambda_1 + \epsilon}$ and $\tilde{f} = f_{\lambda_3}$. Then $\Psi(f) = T_p$ and $\Psi(\tilde{f}) = T_q$, and therefore $[\Psi(f)] \cap [\Psi(\tilde{f})] \neq \emptyset$ while $\overline{\mathcal{PH}(f)} \cap \overline{\mathcal{PH}(\tilde{f})} = \emptyset$.*

Proof. If $\Psi(f) = \Psi(\tilde{f})$ then $f, \tilde{f} \in P^b$ have the same kneading invariants. Hence there exists an order preserving bijection $h: \cup_c \cup_{n \in \mathbb{Z}} f^n(c) \rightarrow \cup_{\tilde{c}} \cup_{n \in \mathbb{Z}} \tilde{f}^n(\tilde{c})$ (where the outer union runs over the critical points c of f and \tilde{c} of \tilde{f}), so that $h \circ f = \tilde{f} \circ h$. Here h maps each critical points of f to the corresponding critical point of \tilde{f} . It follows that if f has no periodic attractors, then \tilde{f} also has no periodic attractors and so f and \tilde{f} are topologically conjugate. Rigidity Theorem 3.7 then gives that $f = \tilde{f}$.

If f has a periodic attractor p , then define $H \ni p$ to be the largest interval such that $f^s(H) \subset H$ and $f^s|_H$ preserves orientation. That is, s is either the period of p if p is orientation preserving and twice the period otherwise. Since $f \in P^b$, by taking an iterate of p is necessary, we can assume that H contains a critical point c , and $(f^{ks}(c))_{k \geq 1}$ is a monotone sequence in H converging to p . In this case \tilde{f} also has a periodic attractor in a corresponding interval \tilde{H} . However, it is possible

that $f^s|_H$ has a unique attracting fixed point, while $\tilde{f}^s|\tilde{H}$ has two attracting fixed points, or vice versa. It follows that either $\mathcal{PH}(f) = \mathcal{PH}(\tilde{f})$ or there exists a map $g \in P^b$ with at least one parabolic periodic point such that $g \in \mathcal{PH}(f) \cap \overline{\mathcal{PH}(\tilde{f})}$ or $g \in \overline{\mathcal{PH}(f)} \cap \mathcal{PH}(\tilde{f})$, see the last part of Theorem 5.8. Since $\mathcal{PH}(f)$ and $\mathcal{PH}(\tilde{f})$ are connected, in particular it follows that the space $\mathcal{H}(f)$ from Theorem 1 is connected.

Assume now that $[\Psi(f)] \cap [\Psi(\tilde{f})] \neq \emptyset$ and $[\Psi(f)] \neq [\Psi(\tilde{f})]$. By Theorem 4.17(6), there exists a map $T_* \in [T_*]_b$ so that $T_* \in \min([\Psi(f)] \cap [\Psi(\tilde{f})])$. According to Theorem 4.17(6) there are four possibilities:

- (1) T_* has a saddle-node merging two basins (or splitting one into two);
- (2) T_* has a period doubling orbit;
- (3) T_* has a pitch-fork orbit;
- (4) T_* has a homoclinic orbit.

Proposition 5.9 produces a map $f_* \in P^b$ such that $\Psi(f_*) = T_*$ and with the corresponding bifurcations.

Claim: In all these cases, $f_* \in \mathcal{PH}(f) \cap \overline{\mathcal{PH}(\tilde{f})}$ or $f_* \in \overline{\mathcal{PH}(f)} \cap \mathcal{PH}(\tilde{f})$.

Proof of Claim. This follows from Theorem 3.6. Indeed, let $i = 1, \dots, b$ and assume that the coordinate $\zeta_i(\Psi(f)) < \zeta_i(\Psi(\tilde{f}))$. Since $f \in \mathcal{A}_b$, by Proposition 5.4, $\Psi(\mathcal{PH}(f)) \subset [\Psi(f)]$.

Claim: One can find a continuous path f_t in $P^b(f)$, $t \in [-1, 0]$ so that $f_t \in \mathcal{PH}^o(f)$ for $t \in (-1, 0)$, $f_{-1} = f$, $f_0 = f_*$.

Proof of Claim: This follows from Theorem 3.6. Indeed, consider $T = \Psi(f)$ and for each of its non-hyperbolic basins, consider what type of bifurcations $T_* \in [T]$ undergoes: merging or splitting components of touching basins, see (1)-(4) above. Then choose the sign σ_i for each neutral orbit of f_t so that Theorem 3.6 ensures that f_t undergoes the corresponding bifurcation as $t \uparrow 0$: it splits into two components by a saddle-node, a period doubling bifurcation, a pitch-fork bifurcation or a critical point moves from the boundary into the interior of one of the components of the basin.

Similarly, since $\tilde{f} \in \mathcal{A}_b$ we have $\zeta_i(\Psi(\tilde{f})) = \zeta_i(T_*) = \zeta_i(\Psi(f_*))$ (in fact $\zeta_{i'}(\Psi(\tilde{f})) = \zeta_{i'}(T_*) = \zeta_{i'}(\Psi(f_*))$ holds for each $i' \in \{1, \dots, b\}$ when $c_{i'}$ is a critical point attracted to the same periodic orbit as c_i .) Applying Theorem 3.6 again, one can find a continuous path f_t in $P^b(\tilde{f})$, $t \in [0, 1]$ so that $f_t \in \mathcal{PH}^o(\tilde{f})$ for $t \in (0, 1)$, $f_0 = f_*$ and $f_1 = \tilde{f}$. Since $\zeta_i(\Psi(\tilde{f})) = \zeta_i(T_*) = \pi_i(\Psi(f_*))$, we can even make sure that $\zeta_i(\Psi(f_t)) = \zeta_i(T_*) = \zeta_i(\Psi(f_*))$ for each $t \in [0, 1]$ (so the kneading invariant of the i -th critical point does not change as t varies in $[0, 1]$). If $\zeta_i(\Psi(f)) > \zeta_i(\Psi(\tilde{f}))$, these bifurcations occur in reverse. If $\zeta_i(\Psi(f)) = \zeta_i(\Psi(\tilde{f}))$, then the i -th kneading invariant of f and \tilde{f} are the same, and the argument from the beginning of the proof

applies for this critical point. Since one can apply this argument for each periodic attractor simultaneously, the proposition follows. \square

The previous proof in particular showed:

Theorem 5.12. *Given a map $f \in P^b$, the set $\mathcal{H}(f)$ of maps $\tilde{f} \in P^b$ with the same kneading invariant as f forms a connected set. Moreover, if $\tilde{f} \in \mathcal{H}(f)$ then $\overline{\mathcal{PH}(f)} \cap \overline{\mathcal{PH}(\tilde{f})} \neq \emptyset$.*

5.6. Almost continuity of Ψ .

Proposition 5.13 (Ψ is almost continuous). $\Psi: P^b \rightarrow \mathcal{S}_*^b$ is ‘almost continuous’ in the following sense. Assume that $f_n \rightarrow f$ where $f_n, f \in P^b$ and $f_n \in \mathcal{A}_b$. Then there exists $T_* \in \mathcal{S}_*^b$ so that any limit of $T_n \in [\Psi(f_n)]$ is contained in $[T_*]$ and so that $\Psi(f) \in [T_*]$.

Remark 5.14. *It is not necessarily true that the limit of $T_n \in [\Psi(f_n)]$ is contained in $[\Psi(f)]$. Indeed, let f be a cubic map so that its left critical point is a fixed point (and is a maximum). Then $\Psi(f)$ is equal to the map T_0 from Figure 10. Since $[T_0] = \{T_0\}$ whereas for maps $f_n \rightarrow f$ with $f_n \in \mathcal{A}_b$ one has that $[\Psi(f_n)]$ is equal to the union of the two triangles in the figure. Note that $f \notin \mathcal{A}_b$.*

Proof of Proposition 5.13. Take $T_n = \Psi(f_n)$ and $T = \Psi(f)$. By taking a subsequence, we can assume that T_n converges to some map \tilde{T} . Let $Z_i, Z_{i,n}$ be the plateaus associated to T and T_n respectively. Note that $\langle T \rangle$ has a product structure, see Theorem 4.17(3) and let π_i and π_j be the projections as defined in that theorem. If c_i is not eventually mapped onto another critical point, then $\nu_i(f_n) \rightarrow \nu_i(f)$ as $n \rightarrow \infty$ (in the usual topology on sequence spaces) and so $Z_{i,n} \rightarrow Z_i$ as $n \rightarrow \infty$. That is, $\pi_i(\Psi(f_n)) \rightarrow \pi_i(\Psi(f))$. In this case define $\zeta_i := \zeta_i(\Psi(f))$. If c_i is mapped onto another critical point, say $f^k(c_i) = c_j$ but is not in the immediate basin of a periodic attractor, then $T^k(Z_i) \subset \partial Z_j$ and for a sufficiently small neighborhood U_i of Z_i one has $T^k(U_i) \cap Z_j \subset \partial Z_j$. That is, if T has a maximum (minimum) at Z_i then $T(Z_i)$ is the left (respectively right) endpoint of a component of $W(T)$. Since $f_n \rightarrow f$ and c_i is not in the basin of a periodic attractor, one has that $\tilde{T}^k(\tilde{Z}_i) \subset \partial \tilde{Z}_j$ where $\tilde{T}^k(\tilde{Z}_i)$ is possibly the ‘other’ endpoint of \tilde{Z}_j . It follows that $\pi_i([\Psi(f)])$ is equal to an interval of the form $[a_i, b_i]$ and that $\pi_i(\Psi(f_n))$ converges to an endpoint of this interval. In this case define $\zeta_i := a_i$. If c_i is periodic, then f_n also has an attracting periodic point p_n near c_i . Let B_n be the component of the immediate basin containing p_n and let B be the component of the immediate basin containing c_i and we define $\zeta_i := \zeta_i(\Psi(f))$. If $f_n(\partial B_n)$ consists of one point, then $f_n \in \mathcal{A}_b$ implies that the itinerary w.r.t. f_n of each critical point in B_n agrees with the itinerary w.r.t. f of the corresponding critical point in B . In this case $\pi_i(\Psi(f_n)) = \pi_i(\Psi(f))$. If $f_n(\partial B_n)$ consists of two points, then this no longer needs to be the case, see Example 5.6 and Remark 5.14. In this case $\pi_i([\Psi(f_n)])$ corresponds to a set as in Figure 10 and Figure 11 and $\pi_i(\Psi(f))$ is in the closure of this set. Since $f_n \in \mathcal{A}_b$ it follows that

$\Psi(f_n)$ is equal to the point marked 1 in these figures. Since $\Psi(f_n) \in \langle \Psi(f_n) \rangle_{\mathfrak{b}}$, we get that $[\Psi(f_n)]$ is equal to the closure of this set and we set $\zeta_i = \Psi(f_n)$. If c_i is mapped to another critical point and in the immediate basin of a periodic attractor, then the same argument goes through. The map T_* for which $\zeta_t(T_*) = \zeta_i$ for $i = 1, \dots, d$ and where ζ_i is chosen as above, has the required properties. \square

6. PROOF OF THE MAIN THEOREM

In this section we shall prove the Main Theorem, assuming Theorem 2.4 (which will be proved as Theorem 7.1 in the next section). First let us prove the following

Theorem 6.1 (‘Fibers’ of Ψ are connected). *Assume that K is a closed and connected subset of \mathcal{S}_*^b with the property that if $T \in K$ then $[T] \subset K$. Then $\Psi^{-1}(K)$ is connected.*

Remark 6.2. *Note that \mathcal{S}_*^b is not a closed subset of \mathcal{S}^b . Therefore we merely assume that K is a closed subset in the relative topology of \mathcal{S}_*^b meaning that if $T_n \in K$ converges to $T \in \mathcal{S}_*^b$ then $T \in K$.*

Proof. Take a closed connected set $K \subset \mathcal{S}_*^b$, and assume by contradiction that $\Psi^{-1}(K)$ is not connected. This means that there are disjoint open sets $U_1, U_2 \subset P^b$ so that $U_1 \cup U_2 \supset \Psi^{-1}(K)$ and $C_i := U_i \cap \Psi^{-1}(K) \neq \emptyset$, $i = 1, 2$. Write $[\Psi(C_i)] := \cup_{f \in C_i} [\Psi(f)]$.

Claim 1: $[\Psi(C_1)] \cup [\Psi(C_2)] \supset K$. Indeed, it follows from surjectivity (Proposition 5.9) that for every $T \in K$ there exists $f \in P^b$ such that $T \in [\Psi(f)]$. Since $[\Psi(f)] \cap K \neq \emptyset$ we have by assumption that $[\Psi(f)] \subset K$ and therefore $f \in \Psi^{-1}(K)$. Therefore $f \in C_1 \cup C_2$ and since $T \in [\Psi(f)]$, the claim follows.

Claim 2: $[\Psi(C_i)] \cap K$ is closed (again in the relative topology of \mathcal{S}_*^b). To see this, take a sequence $T_n \in [\Psi(f_n)] \cap K$ with $f_n \in C_i$. By Theorem 4.17(4) and Proposition 5.9 we can assume that $f_n \in \mathcal{A}_{\mathfrak{b}}$. By considering subsequences we may assume that $T_n \rightarrow T$ for some $T \in K$ and $f_n \rightarrow f$ for some $f \in U_i$. Since $[\Psi(f_n)] \cap K \neq \emptyset$ we have $[\Psi(f_n)] \subset K$. Because of this and because $f_n \in \mathcal{A}_{\mathfrak{b}}$, continuity (Proposition 5.13) implies that $[\Psi(f)] \subset K$. Hence $f \in \Psi^{-1}(K) \subset U_1 \cup U_2$ and, since $f_n \in C_i \subset U_i$ converges to f , also $f \in U_i \cap \Psi^{-1}(K) = C_i$. This completes the proof of Claim 2.

Claim 3: $[\Psi(C_1)] \cap [\Psi(C_2)] \neq \emptyset$. This follows from the connectedness of K and Claims 1 and 2.

By Proposition 5.9, there exist therefore $f_i \in C_i \cap \mathcal{A}_{\mathfrak{b}}$ such that $[\Psi(f_1)] \cap [\Psi(f_2)] \neq \emptyset$. By injectivity (Proposition 5.10), this implies that

$$\mathcal{PH}(f_1) \cap \overline{\mathcal{PH}(f_2)} \neq \emptyset \quad \text{or} \quad \overline{\mathcal{PH}(f_1)} \cap \mathcal{PH}(f_2) \neq \emptyset.$$

Moreover, by Proposition 5.4, $\Psi(\mathcal{PH}(f_i)) \subset [\Psi(f_i)] \subset K$. Hence $\mathcal{PH}(f_i) \subset \Psi^{-1}([\Psi(f_i)]) \subset C_i$. Since $\mathcal{PH}(f_1)$ and $\mathcal{PH}(f_2)$ are both connected, this contradicts that $C_i \subset U_i$ with U_1, U_2 disjoint. \square

Proof of Main Theorem. By Theorem 7.1 (and the remark below) it follows that level sets of $h_{top}: \mathcal{S}_*^b \rightarrow \mathbb{R}$ are connected. Moreover, $h_{top}: P^b \rightarrow \mathbb{R}$ agrees with $h_{top} \circ \Psi$. Because the topological entropy of each map in $[T]$ is the same, Theorem 6.1 shows that the isentropes lift to connected sets in P^b . Similarly, the set $I(h_0)$ is connected. \square

7. ISENTROPES IN \mathcal{S}_*^b ARE CONTRACTIBLE.

Recall from Definition 4.12 that \mathcal{S}_*^b is the collection of *non-degenerate* stunted sawtooth maps $T \in \mathcal{S}^b$. That is, by definition, if $\mathcal{J} := [Z_i, Z_j]$ is the convex hull of Z_i and Z_j , and there is $n \geq 0$ such that $T^n(\mathcal{J})$ is a point (so (Z_i, Z_{i+1}) forms a wandering pair), then $T \in \mathcal{S}_*^b$ means that $T^n(\mathcal{J})$ is eventually mapped into the closure of the immediate basin a periodic plateau.

Of course, if (Z_i, Z_j) is a wandering pair, then all plateaus between Z_i and Z_j form wandering pairs as well. The subset $\mathcal{S}_*^b \subset \mathcal{S}^b$ is chosen because $\Psi: P^b \rightarrow \mathcal{S}^b$ fails to be surjective in a serious way (whereas $\Psi: P^b \rightarrow \mathcal{S}_*^b$ is almost surjective in the sense of Proposition 5.9). Indeed, if $T \in \mathcal{S}^b \setminus \mathcal{S}_*^b$ has a non-preperiodic wandering pair (Z_i, Z_j) and $\Psi(f) \in [T]$, then f has a wandering interval $[c_i, c_j]$. However, it is well-known (see *e.g.* [26]) that polynomials, and in fact C^2 interval maps with non-flat critical points, have no wandering intervals.

This section is devoted to proving Theorem 2.4, which we restate as

Theorem 7.1. *Let $L(h) = \{T \in \mathcal{S}_\epsilon^b \ ; \ h_{top}(T) = h\}$ and $L_*(h) = L(h) \cap \mathcal{S}_{\epsilon,*}^b$. Then*

- *for every $h_0 \in [0, \log(b+1)]$, the level set $L_*(h_0)$ is a contractible subset of $L(h_0)$;*
- *$L_*(h_0^+) := L_*(h_0) \cap \text{closure}(\{T \in \mathcal{S}_*^b \ ; \ h_{top}(T) > h_0\})$ is contractible.*

Remark 7.2. *The sets $L_*(h_0)$ has the property that if T is contained in one of these sets, then $[T]$ is contained also in this set. This property holds for $L_*(h_0)$ since each map in $[T]$ has the same topological entropy. Since $L_*(h_0^+)$ is connected and each set of the form $[T]$ is connected, the set*

$$[L_*(h_0^+)] := \cup_{T \in L_*(h_0^+)} [T]$$

is also connected.

Throughout this section we fix $h_0 \in [0, \log(b+1)]$, although we separate the easier cases $h_0 = 0$ and $h_0 = \log(b+1)$, see Sections 7.4 and 7.5.

That $L(h_0)$ is contractible was already proved in Theorem 6.1 in [31]. Contractibility of $L_*(h_0)$ is much more difficult, and we have to adjust the proof of [31] in a delicate way. The proof involves the construction of a retract R composed of entropy decreasing deformations (to contract $L(h)$ to a single point) and entropy increasing deformations (to keep $L(h)$ within itself). The problem is to keep $L_*(h)$ within itself under continuous action of the retract. To this end we are forced to compose R of altogether five deformations, with some auxiliary deformations. We use the letters $\Gamma, \gamma, \hat{\Gamma}$ to indicate entropy increasing deformations, and $\delta, \hat{\delta}, \Delta$ for entropy decreasing deformations. The deformation β will not change entropy. The letters R and r stand for retract.

Before we are able to give the proof of this theorem we will develop the necessary ingredients.

7.1. The piecewise affine case. An interval K is a *renormalization interval* an interval map f if $f^n(K) \subset K$ for some $n \geq 1$ and $f^n(\partial K) \subset \partial K$. If $n = 1$, and $K = I$, then this is a renormalization interval only in a trivial sense, but we still want to consider it as such. The set $\text{orb}(K) = K \cup f(K) \cup \dots \cup f^{n-1}(K)$ is called a *renormalization cycle*.

It is well-known [30], that every interval map of entropy $h > 0$ is semi-conjugate to a piecewise affine interval map with slope $\pm e^h$. The semi-conjugacy is a monotone map, and collapses every interval that doesn't contribute to the exponential growth rate of the lapnumber; these are wandering intervals, basins of periodic attractors and possibly renormalization intervals, as well as intervals that map into those.

The following lemma will be used at several places in the rest of the proof.

Lemma 7.3. *Assume that $F: [0, 1] \rightarrow [0, 1]$ is a continuous, piecewise affine map with at most finitely many plateaus Z_i , and slope $|F'| > 1$ outside these plateaus. Suppose that Z is a turning plateau (or point) in a minimal¹ renormalization cycle $\text{orb}(K)$, such that no neighborhood of Z is ever mapped into a plateau. Let J be a neighborhood of Z so that $F(\partial J)$ is a single point, then the function \tilde{F} defined by*

$$\tilde{F}(x) = \begin{cases} F(\partial J) & \text{if } x \in \bar{J}, \\ F(x) & \text{if } x \notin \bar{J}, \end{cases}$$

satisfies $h_{\text{top}}(\tilde{F}|_{\text{orb}(K)}) < h_{\text{top}}(F|_{\text{orb}(K)})$.

Remark 7.4. *The origins of the following proof are somewhat nebulous to us. Jozef Bobok drew our attention to the argument, ascribing it to Sasha Blokh, but we haven't been able to locate a precise source. Related results were proved by Boyland [5] and Block & Ledis [2].*

¹*i.e.*, K contains no strictly smaller renormalization interval.

Proof. The inequality $\tilde{h} := h_{top}(\tilde{F}) \leq h := h_{top}(F)$ follows directly from the definition of \tilde{F} ; in fact $|J| \mapsto h_{top}(\tilde{F})$ is a decreasing function. However, we need to prove strict inequality. Let $Y = \text{orb}(K) \setminus \cup_n F^{-n}(\text{int}(\cup_i Z_i))$. Since $\text{orb}(K)$ is a minimal cycle and the derivatives $|F'| > 1$ on Y , the restriction $F : Y \rightarrow Y$ is transitive and supports a unique measure of maximal entropy, see [17]. The assumption on Z implies that $\partial Z \subset Y$ and in fact $\mu(J \setminus Z) > 0$.

Now $\tilde{F}|_{\text{orb}(K)}$ is entropy-preservingly semi-conjugate (say via ψ) to a map with slope $\pm e^{\tilde{h}}$. Let $\tilde{\nu}$ be the measure of maximal entropy of this map, and $\nu = \tilde{\nu} \circ \psi$. Then $0 = \nu(\tilde{F}(J)) \geq \nu(J)$, because ν is non-atomic. It follows that $\text{supp}(\nu) \cap \text{int}(J) = \emptyset$, and definitely $\nu \neq \mu$, whilst at the same time ν is not only \tilde{F} -invariant, but F -invariant as well. Since μ is the unique measure of maximal entropy of $F|_{\text{orb}(K)}$, it follows that $\tilde{h} < h$. \square

7.2. Increasing the entropy of maps in \mathcal{S} : γ_t and Γ_t . The stunted seesaw map $T : [-e, e] \rightarrow [-e, e]$ is uniquely determined by the parameters $(\zeta_1, \dots, \zeta_b)$, and so we can define the norm

$$(9) \quad \text{dist}(T, \tilde{T}) = \max_{i \in \{1, \dots, b\}} |\zeta_i - \tilde{\zeta}_i|.$$

Write $T \leq \tilde{T}$ if the parameters satisfy $\zeta_i \leq \tilde{\zeta}_i$ for all i . Similarly $T < \tilde{T}$ if $T \leq \tilde{T}$ and $\zeta_i < \tilde{\zeta}_i$ for at least one i . Notice that

$$(10) \quad T \leq \tilde{T} \text{ implies } h_{top}(T) \leq h_{top}(\tilde{T}).$$

Construction of γ_t : Let γ_t linearly increase all parameters: $\gamma_t : \zeta_i \mapsto \zeta_i + 2et$, as long as they do not map to $\pm e$.

Let $W^o(T)$ be the components of the domain of the first entry map to the interior of plateaus of T :

$$W^o(T) := \{x; \exists j \in \{1, \dots, b\} \text{ and } k \geq 0 \text{ such that } T^k(x) \in \text{int}(Z_j) \text{ and } x, T(x), \dots, T^{k-1}(x) \notin Z_j\}.$$

We say that T satisfies the **β -property** if

$$(11) \quad \text{no interval of the form } \mathcal{J}_i = [Z_i, Z_{i+1}] \text{ is contained in } W^o(T).$$

Lemma 7.5. *Assume that T satisfies the β -property (11). Then $\gamma_t(T) \in \mathcal{S}_*^b$ for all $t > 0$.*

Proof. For every fixed integer $u \geq 0$, as t increases, the image of the interval $\mathcal{J}_{i,t} = [Z_{i,t}, Z_{i+1,t}]$ under the u -th iterate of the map $\gamma_t(T)$ becomes larger while the sizes of plateaus shrink. If the β -property (11) holds then it follows that $\mathcal{J}_{i,t}$ cannot be mapped into a non-periodic plateau for $t > 0$ and therefore no new wandering pairs can be created by the deformation γ_t as t increases. Also $\gamma_t(T)$ has no non-trivial blocks of touching plateaus for $t > 0$. This implies the lemma. \square

Construction of Γ_t : The deformation Γ_t uses γ_t and the following observation. If $\text{dist}(T, \tilde{T}) < \varepsilon/(2e)$, then $\text{dist}(\gamma_t(T), \gamma_t(\tilde{T})) < \varepsilon$ for every $t > 0$. It follows that

$$\gamma_{t-\varepsilon}(T) < \gamma_t(\tilde{T}) < \gamma_{t+\varepsilon}(T)$$

and so by (10),

$$h_{\text{top}}(\gamma_{t-\varepsilon}(T)) \leq h_{\text{top}}(\gamma_t(\tilde{T})) \leq h_{\text{top}}(\gamma_{t+\varepsilon}(T)).$$

Hence the function

$$(12) \quad t_{\text{max}}(T) := \max\{t \geq 0 \ ; \ h_{\text{top}}(\gamma_t(T)) = h_0\}$$

is continuous in T provided $h_{\text{top}}(T) \leq h_0$. In particular, $\Gamma_t(T) := \gamma_{t \cdot t_{\text{max}}(T)}(T)$ is continuous in t and T as well. By construction,

$$(13) \quad \{T \in \mathcal{S}^b \ ; \ h_{\text{top}}(T) < h_0, T \text{ satisfies the } \beta\text{-property (11)}\} \implies$$

$$\Gamma_t(T) \in \cup_{s \leq h_0} L_*(s), \forall t \in (0, 1].$$

Moreover, if T satisfies the β -property (11) then $\Gamma_t(T)$ also satisfies the β -property (11) for $t \in (0, 1]$.

7.3. Decreasing entropy of maps in \mathcal{S}^b : δ_t , $\hat{\delta}_t$ and r_t . In this section we define the basic operations to decrease entropy, although later we will need refined versions of them.

Construction of δ_t : Define the ‘sign’

$$(14) \quad \text{sgn}(Z_i) = \begin{cases} 0 & \text{if } Z_i \text{ touches another plateau,} \\ 1 & \text{otherwise.} \end{cases}$$

Now deform T according to the flow defined by the system of differential equations on the parameters, where we will now indicate the t -dependence by $\zeta_{i,t}(T)$.

$$\begin{aligned} \frac{d\zeta_{i,t}(T)}{dt} &= \begin{cases} -2 \text{sgn}(Z_i)e & \text{if } \zeta_{i,t}(T) \in (-e, e), \\ 0 & \text{otherwise,} \end{cases} \\ \zeta_{i,t}(T)|_{t=0} &= \zeta_i(T). \end{aligned}$$

Let us denote the resulting deformation by δ_t ; it continuously decreases/increases the height of a plateau if it is local maximum/minimum of T until this plateau touches a neighboring plateau, or reaches the boundary of I . Furthermore, if T has no touching plateaus then $h_{\text{top}}(\gamma_s \circ \delta_t(T)) \leq h_{\text{top}}(T)$ for $s \in (0, t)$ and $t > 0$ sufficiently small. Note, however, that $\delta_t(T) = \delta_s(T)$ for $s, t \geq 1$ but this might mean that each plateau of $\delta_t(T)$ touches another plateau, and therefore this is not a guarantee that $h_{\text{top}}(\delta_t(T)) = 0$. Therefore this deformation, although necessary (see Section 7.9), is not sufficient for our purposes.

Construction of $\hat{\delta}_t$: A natural variant is the deformation $\hat{\delta}_t$ which will widen plateaus in order to decrease entropy to 0. The difference in the deformations δ_t and $\hat{\delta}_t$ is that δ_t decreases the parameter ζ_i until the corresponding plateau Z_i touches another plateau (or reaches e); blocks (consisting of more than one plateau) do not

move under δ_t . By contrast, the deformation $\hat{\delta}_t$ will also move blocks of plateaus, provided they form a local extremum, which happens whenever the block consists of an odd number of plateaus. Blocks of an even number of plateaus are not moved by $\hat{\delta}_t$ (unless an extra plateau joins the block). As it turns out, this may introduce new wandering pairs.

Define the ‘sign’ for plateaus that are part of a block of plateaus:

$$(15) \quad \widehat{\text{sgn}}(Z_i) = \begin{cases} 0 & \text{if } Z_i \text{ is part of a block of an even number of plateaus,} \\ 1 & \text{if } Z_i \text{ touches no other plateau or is an odd-numbered} \\ & \text{plateau in a block of an odd number of plateaus,} \\ -1 & \text{if } Z_i \text{ is an even-numbered plateau in a block of an odd} \\ & \text{number of plateaus.} \end{cases}$$

Note that $\widehat{\text{sgn}}(Z_i)$ depends not only on i but also on T , and $\widehat{\text{sgn}}(Z_i) = \pm 1$ means that T has a local extremum at the block of plateaus that Z_i is part of. We deform T according to the flow defined by the system of differential equations on the parameters.

$$\begin{aligned} \frac{d\zeta_{i,t}(T)}{dt} &= \begin{cases} -2 \cdot \widehat{\text{sgn}}(Z_i) \cdot e & \text{if } \zeta_{i,t}(T) \in (-e, e), \\ 0 & \text{otherwise,} \end{cases} \\ \zeta_{i,t}(T)|_{t=0} &= \zeta_i(T). \end{aligned}$$

The differential equation defines a continuous deformation $\hat{\delta}_t$ with the property that $t \mapsto \hat{\delta}_t(T)$ (not necessarily strictly) decreases the topological entropy. During the deformation blocks can collide, and then the combined larger blocks are deformed according to the same rule. (As a result $\widehat{\text{sgn}}(Z_i)$ can change during the deformation.)

Construction of the retract r_t : If b is odd and $t \geq 1$, then all plateaus of $\hat{\delta}_t(T)$ will touch and the map $\hat{\delta}_t(T)$ is constant $\pm e$. If b is even and $t \geq 1$, then each map $\hat{\delta}_t(T)$ will be monotone (with some blocks of touching plateaus). More precisely, if $t \geq 1$ then $\hat{\delta}_t(T) \in \Sigma_\epsilon^b$ where

$$\Sigma_\epsilon^b = \begin{cases} \{T_0(x) \equiv \pm e\} & \text{if } b \text{ is odd and } \epsilon = \mp 1; \\ \{\text{monotone maps in } \mathcal{S}_\epsilon^b\} & \text{if } b \text{ is even.} \end{cases}$$

Since Σ_ϵ^b is a singleton in the first case and a simplex in the second case, there exists a continuous retract $r_t: \Sigma_\epsilon^b \rightarrow \Sigma_\epsilon^b$ with $r_0 = id$ and $r_1 \equiv T_0$ for T_0 some map in Σ_ϵ^b .

Construction of a retract R_t of an isentrope of \mathcal{S}^b : If we only had to construct a retract of an isentrope of \mathcal{S}^b then we could finish the construction of a deformation R_t as follows. Define

$$R_t = \begin{cases} \Gamma_{3t} & \text{for } t \in [0, 1/3], \\ \Gamma_1 \circ \hat{\delta}_{3t-1} & \text{for } t \in [1/3, 2/3], \\ \Gamma_1 \circ r_{3t-2} \circ \hat{\delta}_1 & \text{for } t \in [2/3, 1]. \end{cases}$$

Obviously this defines the required retract of an isentrope with the space \mathcal{S}^b to a point. However, as we will see, this is still insufficient for our purposes because it is not a retract of an isentrope within the smaller space \mathcal{S}_*^b . In the remainder of this section we will show how to modify this construction to obtain a retract of an isentrope in \mathcal{S}_*^b .

7.4. The case $h_0 = \log(b + 1)$. There is only one stunted sawtooth map with entropy $h_{top}(T) = \log(b + 1)$ (just as there is only one polynomial of give ϵ with entropy $h_{top}(f) = \log(b + 1)$, namely the Chebyshev polynomial. Hence this level is trivial.

7.5. The retract R_t for the case $h_0 = 0$. Let us give the proof in the case that $h_0 = 0$, as the construction will be much easier in this case. Consider a map $T \in \mathcal{S}_*^b$ with zero topological entropy. Let us first review some results on the renormalization structure of such maps.

We say that $T \in \mathcal{S}^b$ has a *2-renormalizable*, if there exists an interval K so that K and $T(K)$ have disjoint interiors and $T^2(K) \subset K$.

Lemma 7.6. *Let $T \in \mathcal{S}^b$ and assume that $h_{top}(T) = 0$. If T is not 2-renormalizable, then the ω -limit set of each point in I is a fixed point of T .*

Proof. Since T has zero topological entropy, it follows that there exists no interval J so that $T^2(\partial J) \subset \partial J$ and $T^2(J) \supset J$. From this, and since T is not 2-renormalizable, this well-known lemma easily follows. \square

Remark: If T is 2-renormalizable then one can apply the lemma again to $T^2: K \rightarrow K$. If one can repeat this infinitely often, then the map is *infinitely renormalizable*. In this case, for each $k \geq 0$, the map T has one or more periodic points of period 2^k and no other periodic points. If T is not infinitely often renormalizable, then each point is eventually mapped into a periodic point of T or is in the basin of a periodic orbit of period 2^k .

Lemma 7.7. *Let $T \in \mathcal{S}_*^b$ with $h_0 = h_{top}(T) = 0$ and assume that $t > 0$. Then there exists $k_0 < \infty$ such that all periodic attractors of $\hat{\delta}_t(T)$ have period 2^k , $k \leq k_0$. Moreover, each point is either (pre-)periodic or in the basin of one of the periodic attractors of $\hat{\delta}_t(T)$.*

Proof. From the previous lemma it follows that each interval map of zero entropy and finite modality has only periodic points of period 2^k for $k \in \mathbb{N}$. Take $T \in \mathcal{S}_*^b$ such that $h_{top}(T) = 0$. If Z_i is a plateau with an infinite orbit, then T must be infinitely renormalizable, *i.e.*, there exists a sequence of periodic intervals K_u , $u \in \mathbb{N}$, with period 2^u such that $\cap_u K_u \supset Z_j$ for some j , and $\omega(Z_i) = \omega(Z_j)$. In fact, $\text{orb}(K_u)$ can contain more plateaus, but since the period of K_u tends to infinity as $u \rightarrow \infty$,

and there are only b plateaus, we can assume (by an appropriate choice of K_u) that there exists $n_u \rightarrow \infty$ as $u \rightarrow \infty$ such that $T^n(K_u)$ does not intersect any plateau for $0 < n < n_u$. Therefore $|T(K_u)| \rightarrow 0$ as $u \rightarrow \infty$.

Since $T \in \mathcal{S}_*^b$ there exists $t_1 \in (0, t)$ and $\eta > 0$ such that all plateaus of T move at least η when t' moves from 0 to t_1 (for t' small $\hat{\delta}_{t'}$ agrees with $\delta_{t'}$). For u so large that $|T(K_u)| < \eta$, this means that K_u is no longer invariant under $\hat{\delta}_{t_1}(T)$ and so this map is not infinitely renormalizable anymore. Instead, there is k_0 such that every plateau of $\hat{\delta}_{t_1}(T)$ is (eventually) periodic with period 2^k for some $k \leq k_0$. If we increase t' further from t_1 to t , each periodic attractor remains but can undergo period halving bifurcations. So all periodic orbits of $\hat{\delta}_t(T)$ have period 2^k , $k \leq k_0$. \square

Now define a retract R_t of the zero-entropy level set of \mathcal{S}_*^b as follows:

$$R_t = \begin{cases} \hat{\delta}_{2t} & \text{for } t \in [0, 1/2], \\ r_{2t-1} \circ \hat{\delta}_1 & \text{for } t \in [1/2, 1]. \end{cases}$$

Lemmas 7.7 and 7.6 (and the remark above) imply that under $\hat{\delta}_{2t}(T)$ (resp. $r_{2t-1} \circ \hat{\delta}_1(T)$), each plateau is contained in the closure of a component of the basin of one of the periodic attractors of $\hat{\delta}_{2t}(T)$ (resp. $r_{2t-1} \circ \hat{\delta}_1(T)$). Hence $\hat{\delta}_t(T), r_t \circ \hat{\delta}_1(T) \in \mathcal{S}_*^b$. Thus we obtain a retract of \mathcal{S}_*^b and proved Theorem 7.1 in the zero entropy case.

The remainder of this section will deal with the case $h_0 > 0$, which is plagued with additional difficulties.

7.6. The retract R_t when $h_0 > 0$ and the trouble with $\hat{\delta}_t$. As mentioned at the end of Subsection 7.3 the retract is insufficient for our purposes: we need to construct a retract of an isentrope of \mathcal{S}_*^b (so the deformation is not allowed to leave the space \mathcal{S}_*^b). The hurdle we have to overcome is that if $T \in \mathcal{S}_*^b$, then $\hat{\delta}_t(T)$ need no longer be in \mathcal{S}_*^b for $t > 0$, because the deformation $t \mapsto \hat{\delta}_t(T)$ can create wandering pairs (Z_i, Z_j) . To resolve this issue, the aim is to ensure that the deformation Γ_t (or a similar deformation) will be able to ‘undo’ these wandering pairs. In view of property (13) on page 47 we will construct a deformation β_t with the property that if $T \in \mathcal{S}^b$ then $\beta_1(T) \in \overline{\mathcal{S}_*^b}$. It does this by deforming T in such a way that $\beta_1(T)$ never eventually maps an interval of the form $[Z_i, Z_{i+1}]$ into the interior of another plateau. However, (13) only applies to maps $T \in \overline{\mathcal{S}_*^b}$ with topological entropy h_0 . For this reason we need to define a more subtle way of ‘decreasing’ and ‘increasing’ the map T while the topological entropy remains equal to h_0 . These analogues of $\hat{\delta}$ and Γ_1 may act on some of the plateaus while leaving some others alone. The challenge will be to define this as a continuous deformation. To achieve this, we will introduce three additional deformations.

- $\hat{\Gamma}_t$: To increase the topological entropy more carefully by increasing some (but possibly not all) ζ_i 's, so we essentially increase each ζ_i 'as far as possible'. This is the purpose of $\hat{\Gamma}_t$ defined in Section 7.8.
- $\hat{\Delta}_t$: To decrease the topological entropy more carefully in such a way that if $\hat{\Delta}_t(T')$ or $\Gamma_t(T')$ does not move certain plateaus (because otherwise the entropy would become too large), then we 'can assume' that $T' \in \mathcal{S}_*^b$. This is the purpose of the deformations Δ_t and $\hat{\Delta}_t$ defined in Section 7.9.
- β_t : Finally, we want to ensure that we only need to apply $\hat{\Gamma}_s$ to maps $T' \in \mathcal{S}^b$ with the property that a convex hull $[Z_i, Z_{i+1}]$ is never eventually mapped into the interior of another plateau (*i.e.*, only to maps with $T' \in \overline{\mathcal{S}_*^b}$). This means that $\hat{\Gamma}_s(T') \in \mathcal{S}_*^b$ for *every* $s > 0$. Unfortunately, this may not be enough because it may happen that $h_{top}(\hat{\Gamma}_s(T')) > h_0$ for any $s > 0$. The deformation β_t , defined in Section 7.7, prevents this, but will not change the entropy.

7.7. The construction of β_t . Consider a map $T' = \hat{\delta}_t(T)$ so that the convex hull $\mathcal{J}_i := [Z_i, Z_{i+1}]$ of two neighboring plateaus is eventually mapped into the interior of another plateau. If this happens then for $s > 0$ small, $\gamma_s(T')$ will still have this property and so in general $\gamma_s(T') \notin \mathcal{S}_*^b$. To overcome this problem we define another deformation β_t . This deformation does not change topological entropy, and only moves (certain) plateaus which are mapped into other plateaus.

As before, let

$$W^o(T) = \{x; \exists j \in \{1, \dots, b\} \text{ and } k \geq 0 \text{ such that } T^k(x) \in \text{int}(Z_j) \text{ and } x, T(x), \dots, T^{k-1} \notin Z_j\}.$$

Let $\mathbf{I}(T)$ be the collection of integers $i \in \{1, \dots, b\}$ such that $T(Z_i) \in W^o(T)$ and Z_i is not periodic. Next we say that T has a *local maximum* (*resp. minimum*) at Z_i if the sawtooth map S_0 has a local maximum (*resp. minimum*) at the midpoint of Z_i . Moreover, given an interval J , we define $\partial_l J$ and $\partial_r J$ to be the left and right endpoint of J respectively. For $i \in \mathbf{I}(T)$ define $v_i = T(Z_i)$, let $W_i = W_i(T)$ be the component of $W^o(T)$ containing v_i and

$$(16) \quad \tau_i(T) = \begin{cases} \frac{\text{dist}(v_i, \partial_l W_i)}{|W_i|} & \text{if } T \text{ has a maximum at } Z_i, \\ \frac{\text{dist}(v_i, \partial_r W_i)}{|W_i|} & \text{if } T \text{ has a minimum at } Z_i. \end{cases}$$

We define the deformation $T_t = \beta_t(T)$ with parameters $(\zeta_{1,t}, \dots, \zeta_{b,t})$ of T as the flow of the following differential equation:

$$\frac{d\zeta_{i,t}}{dt} = \begin{cases} 4 \cdot \tau_i(T_t) \cdot e & \text{if } i \in \mathbf{I}(T_t), \\ 0 & \text{if } i \notin \mathbf{I}(T_t), \end{cases} \quad i = 1, \dots, b.$$

Note that if Z_i contains a periodic attractor or Z_i is never mapped into another plateau, then $Z_{i,t}$ does not vary with t . As $W^o(T)$ consists of preimages of the

interiors of such plateaus, it follows that $W^o(T_t)$ is independent of t except if some plateau $Z_{i,t}$ is mapped by T_t into the boundary of a component W of $W^o(T_t)$ as in the right panel of Figure 14. Also note that $W^o(T)$ depends continuously on T except if some plateau Z_i maps into the boundary of a component of $W^o(T)$, *i.e.*, eventually maps to the boundary of another plateau. In fact only if it is only discontinuous if Z_i ‘arches over’ this component as in the right panel of Figure 14.

Even though W_i does not necessarily depend continuously on T , the deformation $(T, t) \mapsto \beta_t(T)$ turns out to be well-defined and continuous, see the lemma below. Since T_t and T agree outside the union of the closures of components of $W^o(T)$, we have $h_{top}(\beta_t(T)) = h_{top}(T)$.

Lemma 7.8. *The deformation $(T, t) \mapsto \beta_t(T)$ is well-defined. Furthermore:*

- (a) $(T, t) \mapsto \beta_t(T)$ is continuous.
- (b) The map $\beta_1(T)$ satisfies the β -property (11).

Proof. Part (a): Well-defined and continuity. If $i \in \mathbf{I}(T)$ and plateaus are mapped into the interior of components of $W_i(T)$, then $T \mapsto \tau_i(T)$ is locally Lipschitz and there is a unique solution. Otherwise, if $i \notin \mathbf{I}(T)$ the following considerations show that $t \mapsto \beta_t(T)$ is well-defined. Note that it suffices to consider $t \approx 0$ and maps \tilde{T} near some $T \in \mathcal{S}^b$. Let $\tilde{\zeta}_{i,t}$ denote the i -th parameter of $\beta_t(\tilde{T})$. If $T(Z_i)$ is contained in a component of $W^o(T)$, then continuity of $(t, \tilde{T}) \mapsto \tilde{\zeta}_{i,t}$ obviously holds. Next assume that $v_i := T(Z_i)$ is contained in the boundary of a component W of $W^o(T)$. By definition this means that $t \mapsto \zeta_{i,t}$ is constant for $t \geq 0$. Moreover, for a nearby map \tilde{T} , there are two possibilities: (i) $\tilde{T}(\tilde{Z}_i)$ is not contained in $W^o(\tilde{T})$, which means that $t \mapsto \tilde{\zeta}_{i,t}$ is also constant, or (ii) $\tilde{T}(\tilde{Z}_i)$ is contained in a component \tilde{W} of $W^o(\tilde{T})$. In this case, since \tilde{T} is close to T , either (iia) $\tilde{T}(\tilde{Z}_i)$ is near the boundary of a component \tilde{W} of $W^o(\tilde{T})$ or (iib) there exists k so that $T^k(v_i) \in \partial Z_j$ and $T^{k+1}(v_i) \in \partial W'$ where W' is a component of $W^o(T)$. In the latter case $\tilde{T}(\tilde{Z}_j)$ is near the boundary of a component \tilde{W}' of $W^o(\tilde{T})$ and $\tau_j(\tilde{T}) \approx 1$. These two situations are illustrated in Figures 14 and 15.

If (iia) holds, then either $\tau_i(\tilde{T}) \approx 0$ which means that $\tilde{\zeta}_{i,t}$ remains close to $\tilde{\zeta}_{0,i}$ for all $t \in [0, 1]$ (see the left panel in Figure 14) or $\tau_i(\tilde{T}) \approx 1$ which means that $\tilde{\zeta}_{i,t}$ is moving with speed $\approx 4e$ towards the nearest boundary point of \tilde{W} (see the right panel in Figure 14). Therefore there exists $\tilde{t} > 0$ close to zero, so that $\tilde{T}_{\tilde{t}}(\tilde{Z}_i) \in \partial \tilde{W}$ and therefore $\tilde{\zeta}_{i,t}$ remains constant for $t \geq \tilde{t}$.

If (iib) holds, see Figure 15, then there exists $\tilde{t} > 0$ small so that $\tilde{T}_{\tilde{t}}(\tilde{Z}_{j,t}) \in \partial \tilde{W}'$ and \tilde{W}^* no longer is a component of $W^o(\tilde{T}_{\tilde{t}})$ (the interval \tilde{W}^* is split into three components of $W^o(\tilde{T}_{\tilde{t}})$). This means that $\tilde{T}_{\tilde{t}}(Z_i)$ is near a boundary of a component \tilde{W}_- of $W^o(\tilde{T}_{\tilde{t}})$ and we can argue as before.

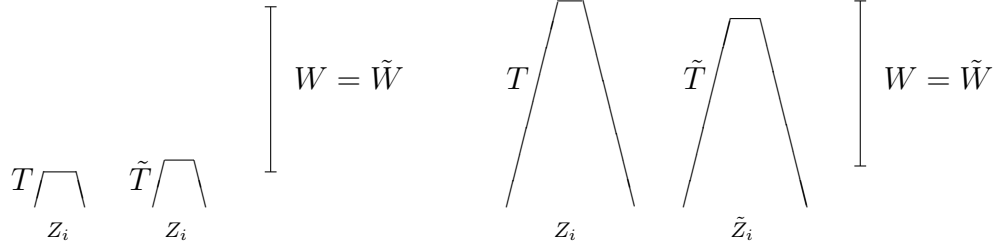


FIGURE 14. Case (iia) in the proof of continuity. On the left the situation where $\tilde{T}(\tilde{Z}_i)$ is contained in a component W of $W^o(T)$ with $\tau_i(\tilde{T}) \approx 0$. In this case, $\tilde{\zeta}_{i,t}$ will increase only very slowly with t . On the right the situation where $\tilde{T}(\tilde{Z}_i)$ is close to the boundary of an endpoint of $W^o(T)$ when $\tau_i(\tilde{T}) \approx 1$. In this case $\tilde{\zeta}_{i,t}$ increases with speed $\approx 4e$ which means that there exists $t > 0$ close to 0 so that $\tilde{T}_t(\tilde{Z}_i)$ is contained in the boundary of W and then stops. In this figure we set $W = \tilde{W}$, but also when these are different intervals, the same argument holds. Note that in the situation on the right, the component of $W^o(T)$ containing Z_i does not depend continuously on T .

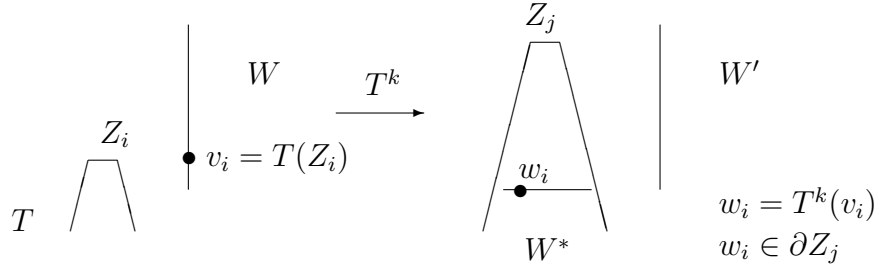


FIGURE 15. Case (iib) in the proof of continuity. In this case W is a component of $T^{-k}(W^*)$ where W^* is a component of $W^o(T)$ containing a plateau Z_j and $T(Z_j)$ is close to the boundary of the component W' containing $T(Z_j)$. There exists $t > 0$ close to zero, so that $T_t(Z_{j,t})$ hits the boundary of W' and then W_t^* splits into three components: to the left and right of $Z_{j,t}$ and the interior of $Z_{j,t}$. This means that W_t also splits into three components, and at this moment $Z_{i,t}$ moves until $T_t(Z_{i,t})$ belongs to the boundary of one of these plateaus (which in this situation means close to v_i).

Finally, assume that $T(Z_i)$ is not contained in the closure of a component of $W^o(T)$. If \tilde{T} is near T while $\tilde{T}(\tilde{Z}_i)$ is contained in a component \tilde{W} of $W^o(\tilde{T})$, then this

component \tilde{W} is small and so for $t > 0$ small $\beta_t(\tilde{T})(\tilde{Z}_i)$ is contained in the boundary of \tilde{W} (and for $t' > t$ this plateau no longer moves).

It follows that in all cases, $(T, t) \mapsto \beta_t(T)$ is continuous.

Part (b). If Z_i is non-periodic and \mathcal{J}_i is mapped into a component W of $W^o(T)$, then either $\tau_i(T) \geq 1/2$ or $\tau_{i+1}(T) \geq 1/2$, see Figure 16. By construction, $\tau_i(\beta_t(T)) \geq 1/2$ or $\tau_{i+1}(\beta_t(T)) \geq 1/2$ for all $t \geq 0$. Hence $\frac{d\zeta_i}{dt}(\beta_t(T)) \geq 2e$ or $\frac{d\zeta_{i+1}}{dt}(\beta_t(T)) \geq 2e$, for all $t \in [0, t_0]$ where t_0 is chosen so that $\beta_t(T)(\mathcal{J}_{i,t})$ is still contained in W for all $t \in [0, t_0]$. Since for each map $T \in \mathcal{S}^b$, the corresponding ζ_i can be at most $2e$, there exists $t \leq 1$ so that $\beta_t(T)(\mathcal{J}_{i,t}) \in \partial W$. Note that $x \in \partial W$ implies that no iterate of x can be mapped into the interior of a plateau. Hence the claim follows. \square

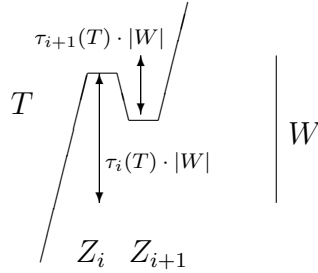


FIGURE 16. In this example, $T(Z_i)$ and $T(Z_{i+1})$ are both contained in the same component W of $W^o(T)$. We show a situation where $\tau_i(T) \approx 1$ whereas $\tau_{i+1}(T)$ is much smaller. As a result $\zeta_{i,t}$ changes much faster than $\zeta_{i+1,t}$ under the deformation, but as soon as $\beta_t(T)(Z_{i,t})$ hits the boundary of W then $\zeta_{i,t}$ stops moving. The component of $W^o(T_t)$ containing Z_i splits in three as soon as $T_t(Z_{i,t})$ hits the boundary of W : one to the left of Z_i , one to the right of Z_i and the interior of Z_i .

7.8. Increasing entropy of maps more carefully: $\hat{\Gamma}_t$. We use the entropy increasing deformation $(T, t) \mapsto \Gamma_t(T) = \gamma_{t \cdot t_{max}(T)}(T)$ (with t_{max} as in (12) on page 47) until $h_{top}(\Gamma_t(T)) = h_0$. But it is possible that only part of the phase space is responsible for reaching this entropy bound, while in other parts (namely in renormalization cycles), plateaus have not been lifted ‘sufficiently’ yet. It is essential to perform some version of γ_s for at least some time $s > 0$, so as to resolve (destroy) wandering pairs that may have been created by δ_t . Thus in the presence of renormalization intervals, we may need to lift some plateaus faster than others. This subsection explains how this is done.

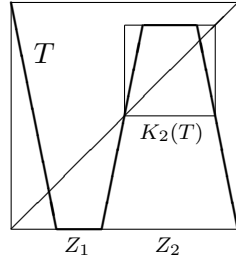
Remark 7.9. Recall that $T \in \mathcal{S}^b$ has renormalization interval K if there is $n \geq 1$ such that $T^n(K) \subset K$ and $T^n(\partial K) \subset \partial K$. The orbit $\text{orb}(K) = K \cup T(K) \cup \dots \cup T^{n-1}(K)$ is called a renormalization cycle. Note that ∂K consists of (pre)periodic points which do not depend on T , unless they disappear in a saddle node bifurcation.

The only other way by which K can disappear is when $T^n(Z) \subset \partial K$ for some plateau Z compactly contained in K .

Fix $h_0 \in (0, \log(b+1))$. Let $K_i(T)$ be the smallest renormalization interval for T which contains the omega-limit set $\omega(Z_i)$. If there exists no smallest renormalization interval then we take $K_i(T) = \emptyset$. Let

$$\begin{cases} L_i(h_0^-) = \text{closure}(\{T \in \mathcal{S}^b ; h_{top}(T|\text{orb}(K_i(T))) \leq h_0\}), \\ L_i(h_0^+) = \text{closure}(\{T \in \mathcal{S}^b ; h_{top}(T|\text{orb}(K_i(T))) > h_0\}). \end{cases}$$

Note that in this definition $\text{orb}(K_i(T))$ does indeed depend on T . Consequently, the common boundary of these sets contains maps T for which $h_{top}(T|\text{orb}(K_i(T))) \leq h_0$ but which can be increased by an arbitrarily small change in parameter ζ_j for some $Z_j \subset \text{orb}(K_i(T))$. It does not follow, however, that $h_{top}(T|\text{orb}(K_i(T))) = h_0$, see Figure 17. In fact, $T \mapsto h_{top}(T|\text{orb}(K_i(T)))$ is discontinuous because if Z_i is not in the basin of an attractor, a small change in T can make Z_i periodic itself, and then $h_{top}(T|\text{orb}(K_i(T))) = 0$.



$T \in L_2(h_0^-) \cap L_2(h_0^+)$ for
 $h_0 = \log(1 + \sqrt{2})$, but
 $h_{top}(T|\text{orb}(K_2(T))) = \log 2 \neq h_0$.

FIGURE 17. For this T with $h_{top}(T) = h_0 = \log(1 + \sqrt{2})$ we have $h_{top}(T|\text{orb}(K_2(T))) = \log 2 < h_0$, but lifting Z_2 by any amount pushes $h_{top}(T) = h_{top}(T|\text{orb}(K_2(T)))$ above h_0 .

If the period of $K_i(T)$ is m , then since the intervals $K_i(T), \dots, T^{m-1}(K_i(T))$ have disjoint interiors, the first return map of $\text{orb}(K_i(T))$ has at most 2^b branches. It follows that if $h_{top}(T) = h_0$ then

$$(17) \quad 0 < h_0 = h_{top}(T|\text{orb}(K)) \leq (b \log 2)/m$$

which gives the upper bound $m \leq (b \log 2)/h_0$.

Define

$$\Phi_j(T) = \text{dist}(T, L_j(h_0^-) \cap L_j(h_0^+))$$

where dist is as in (9) on page 46. Define the following deformation of maps $T \in \mathcal{S}^b$:

$$(18) \quad \hat{\Gamma}_t(\zeta_1, \dots, \zeta_b) = (\min(\zeta_1 + \Phi_1(T)t, e), \dots, \min(\zeta_b + \Phi_b(T)t, e)).$$

Let us say that $T \in \mathcal{S}_{*,j}^b$ if whenever $\mathcal{J} = [Z_{j-1}, Z_j]$ or $\mathcal{J} = [Z_j, Z_{j+1}]$ (assuming $1 \leq j-1$ and $j+1 \leq b$ respectively) is eventually mapped into a plateau, then \mathcal{J} is contained in the closure of a component of the basin of T .

Lemma 7.10. *Let $\hat{\Gamma}_t$ be as above and let $T \in \mathcal{S}^b$ be such that $h_{top}(T) \leq h_0$. Then the following hold:*

- (1) *The deformation $(T, t) \mapsto \hat{\Gamma}_t(T)$ is continuous in T and t , and $t \mapsto h_{top}(\hat{\Gamma}_t(T))$ is non-decreasing.*
 - (2) *$h_{top}(\hat{\Gamma}_t(T)) \leq h_0$ for all $0 \leq t \leq 1$.*
 - (3) *Assume that T satisfies*
 - (a) *the β -property (11) on page 46 and*
 - (b) *for any $j = 1, \dots, b$,*
- (19) $T \in L_j(h_0^-) \cap L_j(h_0^+)$ *implies* $T \in \mathcal{S}_{*,j}^b$.

Then $\hat{\Gamma}_t(T) \in \mathcal{S}_^b$ for each $t > 0$.*

Proof. The continuity and monotonicity of statement (1) are obvious.

For statement (2), take $t \in [0, 1]$ and $T \in \mathcal{S}^b$ with $h_{top}(T) \leq h_0$. Let j_1 be such that $m_1 := \Phi_{j_1}(T)$ is maximal among $\{\Phi_1(T), \dots, \Phi_b(T)\}$, and let M_1 be the open b -dimensional m_1 -cube centered at T , parallel to the coordinate hyperplanes and of side length $2m_1$. Then M_1 is disjoint from $L_{j_1}(h_0^-) \cap L_{j_1}(h_0^+)$, and in particular disjoint from $L_{j_1}(h_0^+)$. Therefore $h_{top}(T' | \text{orb}(K_{j_1}(T))) \leq h_0$ for any $T' \in M_1$ and in particular for $\hat{\Gamma}_t(T)$.

Now let j_2 be such that $m_2 := \Phi_{j_2}(T)$ is second largest among $\{\Phi_1(T), \dots, \Phi_b(T)\}$. The corresponding m_2 -cuboid M_2 is the set of T' with parameters $\{\zeta_1, \dots, \zeta_b\}$ such that $|\zeta_j - \zeta_j(T)| < m_2$ for all $j \neq j_1$ and $|\zeta_{j_1} - \zeta_{j_1}(T)| < m_1$. (This is the Cartesian product of a $b - 1$ -dimensional cube and an arc of length $2m_1$ in the ζ_{j_1} -direction.)

Claim 1: M_2 is disjoint from $L_{j_2}(h_0^+)$. To prove this claim, consider $T' \in M_2$. Let M_2'' be the m_2 -cube centered at T , and choose $T'' \in M_2''$ be so that T' and T'' agree except at Z_{j_1} . Let $T_t, t \in [0, 1]$ be the one-parameter family of maps connecting T'' to T' corresponding to maps for which the parameter ζ_{t,j_1} varies linearly and so that $\zeta_{t,i} = \zeta_{0,i}$ for all $t \in [0, 1]$ and $i \neq j_1$. Denote $K_{j_2,t}^* := \text{orb}(K_{j_2}(T_t))$ and $K_{j_1,t}^* := \text{orb}(K_{j_1}(T_t))$. By definition of m_1 and m_2 , we have $h_{top}(T_t | K_{j_1,t}^*) \leq h_0$ for each $t \in [0, 1]$ and $h_{top}(T_0 | K_{j_2,0}^*) \leq h_0$.

Next let

$$X_1 = \{t \in [0, 1]; K_{j_2,t}^* \subseteq K_{j_1,t}^*\}, \quad X_2 = \{t \in [0, 1]; K_{j_2,t}^* \not\subseteq K_{j_1,t}^*\}$$

and

$$X_3 = \{t \in [0, 1]; K_{j_2,t}^* \cap K_{j_1,t}^* = \emptyset\}.$$

From the above properties we obtain $h_{top}(T_t | K_{j_2,t}^*) \leq h_{top}(T_t | K_{j_1,t}^*) \leq h_0$ for each $t \in X_1$. Consider a component C of X_3 or of X_2 . If there exists $t \in C$ so that $h_{top}(T_t | K_{j_2,t}^*) \leq h_0$ then $h_{top}(T_s | K_{j_2,s}^*) \leq h_0$ for each $s \in C$; here we use that $h_{top}(T_t | K_{j_1,t}^*) \leq h_0$ for each $t \in [0, 1]$ if C is a component of X_2 .

Claim 1 follows if there exists a component of X_1 , X_2 or X_3 containing both 0 and 1. So from now on we assume that this is not the case, and we can assume that t' in the next claim is not 0 or 1.

Subclaim 1: for each boundary point t' of C there exists $t_n \in X_1$ so that $t_n \rightarrow t'$. In fact, $K_{j_1, t_n} \supset K_{j_1, t_n} = K_{j_1, t'}$.

To prove this, let $\tilde{K}_{j_1, t'}$ be the *maximal* renormalization interval containing $K_{j_1, t'}$ which is either contained in $K_{j_2, t'}$ (if $t' \in X_2$) or which is disjoint from $K_{j_2, t'}$ (if $t' \in X_3$). Observe that the maximality of $\tilde{K}_{j_1, t'}$ implies that there are backwards iterates of $K_{j_2, t'}$ accumulating to the boundary points of $\tilde{K}_{j_1, t'}$. That t' is a boundary point of C means that some iterate of $Z_{j_1, t'}$ under $T_{t'}$ is mapped into the boundary of a component of $\tilde{K}_{j_1, t'}$. Because of the above observation this implies that there exists a sequence of points x_n converging to either boundary point $\tilde{K}_{j_1, t'}$ such that the omega-limit set $\omega(x_n) \subset K_{j_2, t'}^*$. One can choose x_n even so that $K_{j_2, t'}$ is the smallest renormalization interval whose orbit contains $\omega(x_n)$. It follows that there exists a sequence $t_n \rightarrow t'$ with $x_n \in \text{orb}(Z_{j_1, t_n})$ so that $K_{j_1, t_n} \supseteq K_{j_2, t_n} = K_{j_2, t'}$. This completes the proof of Subclaim 1.

Proof of Claim 1 continued. Let us say that a renormalization K of T_t is *created* at parameter t , if K is not a renormalization interval for T_s for all $s \in (t - \delta, t)$ with $\delta > 0$ small. Since ζ_{j_1} is the only parameter moving, this implies that

$$(20) \quad Z_{j_1, t} = K \text{ has a one-sided attracting periodic boundary point.}$$

Again, since ζ_{j_1} is the only parameter moving, $Z_{j_1, t}$ cannot be part of block of plateaus if K is created.

First consider the case that $\zeta_{j_1, 1} > \zeta_{j_1, 0}$. If $t \in (X_2 \cup X_3)$, then $Z_{j_2, t}$ is never mapped into $Z_{j_1, t}$. It follows that for each $s \in [t, 1]$ no iterate under T_s of $Z_{j_2, s} = Z_{j_2, t}$ is mapped into $Z_{j_1, s}$. Hence $\omega(Z_{j_2, s})$ remains the same for each $s \in [t, 1]$ where we can take $t^* := \inf(X_2 \cup X_3)$. If K is a renormalization interval for $T_{s'}$ which is created at parameter s' and K intersects $\omega(Z_{j_2, s'})$, then by remark (20) it follows that $\omega(Z_{j_2, s'})$ would have to intersect $Z_{j_1, s'}$, which is impossible when $s \geq t'$. Hence, if the renormalization interval $K_{j_2, s}$ changes at some $s \in [t^*, 1]$, then it is replaced by some larger renormalization interval $K_{j_2, s'}$ for $T_{s'}$ with $s' > s$ close to s . Thus the set $K_{j_2, s}$ only gets larger as s increases from t^* to 1. Let C be a component as before and let t' be an endpoint of C (with $t' \neq 0, 1$). By Subclaim 1 there exists $t_n \rightarrow t'$ with $t_n \in X_1$ and so $K_{j_1, t_n} \supset K_{j_1, t_n} = K_{j_1, t'}$. This, together with the fact that $h_{\text{top}}(T_{t_n}|K_{j_1, t_n}^*) \leq h_0$, implies that $h_{\text{top}}(T_{t'}|K_{j_2, t'}^*) \leq h_0$. Using the first part of the proof (above Subclaim 1), Claim 1 follows.

Now consider the case that $\zeta_{j_1, 1} < \zeta_{j_1, 0}$. In this case, $K_{j_2, s}$ can shrink as s increases, namely when $Z_{j_1, s}$ is contained in $K_{j_2, s}^*$ and the image of $Z_{j_1, s}$ is mapped to a boundary point of $K_{j_2, s}^*$. Since in this case $K_{j_2, s}^*$ can only shrink as s increases, and $Z_{j_2, s}$ only gets wider as s increases, it follows that $h_{\text{top}}(T_{t'}|K_{j_2, s}^*)$ is a decreasing

function of s in this case. The set $K_{j_2, s}^*$ can also change if at some parameter s' it is *destroyed* (*i.e.*, created in reverse direction), but remark (20) implies that some iterate of $Z_{j_2, s'}$ is contained in $Z_{j_1, s'}$, so $\omega(Z_{j_2, s'}) = \omega(Z_{j_1, s'})$ and we are in the parameter set X_1 and in particular $h_{top}(T_{s'}|K_{j_2, s'}^*) = h_{top}(T_{s'}|K_{j_1, s'}^*) \leq h_0$ for all $s'' \geq s'$. Thus Claim 1 holds also if $\zeta_{j_1, 1} < \zeta_{j_1, 0}$.

Continuing inductively, we see that if $T' \in \cap_{k=1}^b M_k$, then $h_{top}(T'|\text{orb}(K_j(T')))) \leq h_0$ for each j , and this holds in particular for $\hat{\Gamma}_t(T)$ for each $t \in [0, 1]$. If $K_j(\hat{\Gamma}_t(T)) = [-e, e]$ for some j (*i.e.*, if $\hat{\Gamma}_t(T)$ has no renormalization interval), then this proves statement (2).

If, however, every plateau belongs to some renormalization cycle and the entropy of $\hat{\Gamma}_t(T)$ is carried by the Cantor set of points that never enter these renormalization cycles, then we argue as follows. Write $T_t = \hat{\Gamma}_t(T)$ and assume by contradiction that $h_{top}(T_{t_1}) > h_0$ for some $t_1 \in [0, 1]$. Take $t_0 \in [0, t_1)$ maximal such that $h_{top}(T_{t_0}) \leq h_0$. Since $T_t \in \cap_{k=1}^b M_k$ for each $t \in [0, 1]$, the first part of the proof gives

$$(21) \quad h_{top}(T_t|\text{orb}(K_j(T_t))) \leq h_0 \text{ for each } t \in [0, 1] \text{ and each } j = 1, \dots, b.$$

Now we need the following

Claim 2: There exists $j \in \{1, \dots, b\}$ and $t' \in (t_0, t_1)$ so that $h_{top}(T_{t'}|\text{orb}(K_j(T_{t'}))) > h_0$ for the minimal interval of renormalization $K_j(T_{t'})$ containing $\omega(Z_{j, T_{t'}})$.

Proof of Claim 2: For each $t \in (t_0, t_1)$ consider the semi-conjugacy of T_t with the map F_t with slope $\pm \exp(h_{top}(T_t))$ as above Lemma 7.3. Since $h_{top}(F_t) = h_{top}(T_t)$ depends continuously on t , and is not constant on (t_0, t_1) , $h_{top}(F_t)$ assumes uncountably many values as t moves through (t_0, t_1) . But there are only countably many slopes for which all turning points of F_t are periodic. Therefore there exists $t' \in (t_0, t_1)$ so that at least one of the turning points of $F_{t'}$, say c_j , is non-periodic. Let $X_{j, t'}$ be the smallest renormalization interval of $F_{t'}$ containing this turning point. Since $F_{t'}$ has constant slope, $h_{top}(F_{t'}|\text{orb}(X_{j, t'})) = h_{top}(F_{t'}) > h_0$. Since the j -th critical point of $F_{t'}$ is not periodic, the smallest renormalization $K_j(T_{t'})$ of $T_{t'}$ containing $\omega(Z_{j, t'})$ is mapped by the semi-conjugacy onto $X_{j, t'}$. It follows that $h_{top}(T_{t'}|\text{orb}(K_j(T_{t'}))) > h_0$, completing the proof of the claim.

Obviously the claim contradicts (21), so we can conclude that $h_{top}(T_1) \leq h_0$. Statement (2) follows.

Finally we turn statement to statement (3). For this we need to show that $\hat{\Gamma}_t(T) \in \mathcal{S}_{*, j}^b$ for each j and each $t > 0$. If $T \in L_j(h_0^-) \cap L_j(h_0^+)$ then this holds by assumption (19). So assume that $T \notin L_j(h_0^-) \cap L_j(h_0^+)$. Then the β -property (11) implies that whenever $T^n([Z_j, Z_{j+1}])$ is contained in a plateau Z_k then $T^n([Z_j, Z_{j+1}]) \cap \partial Z_k \neq \emptyset$. Since the plateau $Z_{j, t}$ shrinks at t increases (as $T \notin L_j(h_0^-) \cap L_j(h_0^+)$), this means that for $t > 0$, $(\hat{\Gamma}_t(T))^n$ maps $[Z_j, Z_{j+1}]$ at least partially outside Z_k . So again $\hat{\Gamma}_t(T) \in \mathcal{S}_{*, j}^b$. This proves statement (3). \square

7.9. Decreasing the entropy more carefully: Δ_t . Take $T \in \mathcal{S}_*^b$ with $h_{top}(T) = h_0 > 0$. Even though $t \mapsto h_{top}(\hat{\delta}_t(T))$ is non-increasing, it is possible that for fixed $t > 0$, $h_{top}(\gamma_s \circ \hat{\delta}_t(T)) > h_0$ for all $s > 0$. The reason is that (in the notation of Section 7.3) $\widehat{\text{sgn}}(Z_i)$ can change from 1 to -1 (or vice versa) during the deformation. To explain what can happen, let us discuss two examples.

Example 7.11. Consider the map from Figure 18 on page 60. Although $T \in \mathcal{S}_*^b$, the map $T' = \hat{\delta}_t(T)$ has a wandering pair that does not map into a periodic basin (so T' is no longer in \mathcal{S}_*^b). There is a periodic interval \hat{K} (here of period 1) and T' maps the convex hull $\mathcal{J} = [Z'_1, Z'_3]$ into $\partial\hat{K}$. (Note that, in this example, $\hat{\delta}_t(T)$ first decreases ζ_2 and then, after the plateaus Z_1 and Z_3 touch Z_2 , increases it again. Even though γ_t initially is ‘the inverse’ of the deformation $\hat{\delta}_t$, the map $T' = \hat{\delta}_t(T)$ will have some touching plateaus.) Because the entropy within the renormalization interval \hat{K} is $\leq h_0$, the movements of plateaus Z_1, Z_2, Z_3 under $\hat{\delta}_t$ have no effect on the global entropy. Therefore $h_{top}(T') = h_{top}(T)$, whereas $h_{top}(\gamma_s(T')) > h_{top}(T)$ for any $s > 0$, because γ_s decouples the plateaus Z_1, Z_2, Z_3 again and $\gamma_s(T')([Z_1, Z_3])$ is a closed neighbourhood of the left endpoint of (and therefore not entirely contained in) \hat{K} . The effect is that within $\hat{\Gamma}_s$, the deformation γ_s will not be applied at all, and hence it will not be able to remove the wandering pair created by $\hat{\delta}_t$.

Example 7.12. In fact, a similar problem can occur even when we consider the family $\beta_1 \circ \delta_t(T)$. It is possible that for some $t \in (0, 1)$ a pair of plateaus $[Z_{1,t}, Z_{2,t}]$ of $\delta_t(T)$ is mapped by $\delta_t(T)$ into a plateau $Z_{3,t}$, which in turn is mapped into a renormalization interval \hat{K}_t . As t increases, the parameters ζ_t associated to $\beta_1 \circ \delta_t(T)$ no longer decrease with t and a similar situation as in Figure 18 can arise (but with $Z_{3,t}$ the image of $[Z_{1,t}, Z_{2,t}]$ and $Z_{3,t}$ mapped into the boundary of a periodic plateau).

To overcome the issues caused by such examples, we introduce the deformation Δ_t .

Construction of Δ_t : Let us define a modification Δ_t of the deformation $\hat{\delta}_t$, which allows some of the plateaus (namely those within renormalization intervals of ‘low entropy’) to move before others.

Fix $h_0 \in (0, \log(b+1))$.

Definition 7.13. *Given a periodic interval K of period m , we can find intervals $K_i \supset T^i(K)$ such that $T(\partial K_i) \subset \partial K_{i+1 \bmod m}$ for all $0 \leq i < m$. For the minimal choice of such intervals K_i , we write $\text{cycle}(K) = \cup_{i=0}^{m-1} K_i$. Given a renormalization interval K of T , we define*

$$\Omega(K) = \left\{ z \in \text{cycle}(K); \quad z, T(z), T^2(z), \dots \notin \hat{K} \text{ for any} \right. \\ \left. \text{renormalization interval } \hat{K} \subsetneq K \right\}.$$

We say that K has entropy h_0 (for T) if $h_{top}(T|\Omega(K)) = h_0$.

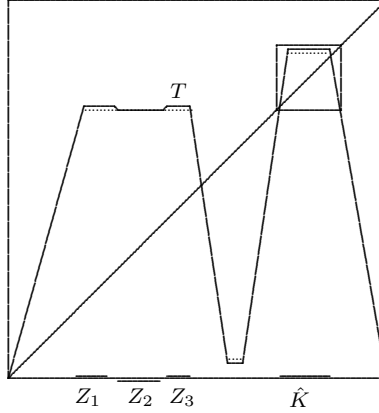


FIGURE 18. The maps T and $T' := \hat{\delta}_t(T)$ (in dotted lines). For $t > 0$ small, $t \mapsto \hat{\delta}_t(T)$ increases the height of the plateau Z_2 and decreases those of Z_1, Z_3 . Once they merge, this deformation decreases the height of all of them together. In this example, the plateaus Z_1, Z_2 and Z_3 are mapped into $\partial\hat{K}$, i.e., $T'(Z_1) = T'(Z_2) = T'(Z_3) \in \partial\hat{K}$. The map is constructed so that $T'|_{\hat{K}}$ is unimodal with entropy $h_{top}(T'|_{\hat{K}}) < h_{top}(T)$. In this case, $h_{top}(\gamma_s(T')) > h_{top}(T)$ for any $s > 0$, because points near Z_2 will then be mapped outside (i.e., to the left of) \hat{K} . Therefore $\hat{\delta}_t(T) \in L_2(h_0^-) \cap L_2(h_0^+)$ where $h_0 = h_{top}(T)$ and $\hat{\delta}_t(T) \notin \mathcal{S}_*^b$ unless the left boundary point of \hat{K} is in the boundary of a component of the basin of a periodic attractor.

It is possible that $h_{top}(T|\Omega(K)) < h_0$ but that there exists a renormalization interval $\hat{K} \subsetneq K$ so that $h_{top}(T|\Omega(\hat{K})) = h_{top}(T|\text{cycle}(\hat{K})) = h_0$.

Definition 7.14. Given an interval L , we say that a plateau Z_j arches over ∂L if there exists m so that $T^m(Z_j) \subset \partial L$ and if $k < m$ is maximal such that $T^k(Z_j)$ is contained in a plateau Z , then $T^{-(m-k)}(L)$ strictly contains the maximal block of touching plateaus containing Z , see Figure 19.

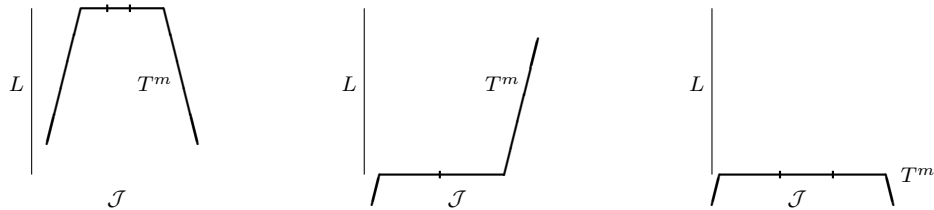


FIGURE 19. Three basic possibilities how a block of plateaus can map onto ∂L . In the situations described in the left two panels, the interval \mathcal{J} arches over ∂L .

Definition 7.15. We say that $T \in \xi_i$ if there exists renormalization interval K of entropy h_0 and a convex hull $\mathcal{J} \subset K$ of plateaus which is **non-trivial**, i.e., $T(\mathcal{J})$ is not a singleton and so that

- (1) there exists a maximal renormalization interval $\hat{K} \subsetneq K$ so that $T^m(\mathcal{J}) \subset \partial\hat{K}$ for some $m \geq 2$ (in particular $T^m(\mathcal{J})$ is a singleton);
- (2) the first return map to \hat{K} is non-monotone;
- (3) $T^j(\mathcal{J})$ intersects Z_i for some $0 \leq j < m - 1$;
- (4) $Z_i \cap \text{cycle}(\hat{K}) = \emptyset$.

It will be useful to stratify the space \mathcal{S}^b , dividing the boundary of this space into subspaces according to which plateaus touch. To do this, let Ξ be a partition $\{1, \dots, b\}$. Then we define the *stratum* $\mathcal{S}^b(\Xi) \subset \mathcal{S}^b$ as follows: $T \in \mathcal{S}^b(\Xi)$ if and only if for each $i \in \{1, \dots, b-1\}$ the plateaus Z_i and Z_{i+1} touch whenever $i, i+1$ are in the same partition element of Ξ . For example $\mathcal{S}^b(\{1\}, \{2\}, \dots, \{b\})$ consists of all maps for which none of the plateaus touch, and $\mathcal{S}^b(\{1, 2\}, \{3\}, \dots, \{b\})$ consists of the space of maps for which the first two plateaus touch (and no other two plateaus do). Note that when Ξ_1, Ξ_2 are two distinct partitions, then $\mathcal{S}^b(\Xi_1)$ and $\mathcal{S}^b(\Xi_2)$ are disjoint. Clearly \mathcal{S}^b is the disjoint union of $\mathcal{S}^b(\Xi)$ where the union runs over all partitions Ξ of $\{1, \dots, b\}$, and each $T \in \mathcal{S}^b$ is associated to a partition $\Xi(T)$ of $\{1, \dots, b\}$.

Note that in Definition 7.15, a non-trivial convex hull \mathcal{J} always contains two plateaus Z_i, Z_j where i, j are in distinct subsets from the partition $\Xi(T)$. Also note that property (2) Definition 7.15 in particular implies that the first return map to \hat{K} is non-constant.

For each partition Ξ of $\{1, \dots, b\}$, choose a C^∞ function $\rho_i^\Xi: \mathcal{S}^b(\Xi) \rightarrow [0, 1]$ which is zero on $\overline{\xi_i \cap \mathcal{S}^b(\Xi)}$ and positive elsewhere. Next define $\rho_i: \mathcal{S}^b \rightarrow [0, 1]$ by $\rho_i(T) = \rho_i^\Xi(T)$ whenever $T \in \mathcal{S}^b(\Xi)$ for some partition Ξ of $\{1, \dots, b\}$.

Define the modification Δ_t of $\hat{\delta}_t$ as the flow of the differential equation

$$(22) \quad \begin{aligned} \frac{d\zeta_{i,t}}{dt} &= \begin{cases} -\rho_i(T_t) \cdot \widehat{\text{sgn}}(Z_{i,t}) & \text{when } \zeta_{i,t} \in (-e, e), \\ 0 & \text{otherwise,} \end{cases} \\ \zeta_{i,t}|_{t=0} &= \zeta_i(T). \end{aligned}$$

Here $\widehat{\text{sgn}}$ is defined as in (15) on page 48 and T_t is the map corresponding to $\zeta_{i,t}$ and so $T_t = \Delta_t(T)$.

Proposition 7.16. For each $T \in \mathcal{S}^b$, $\Delta_t(T)$ exists for all $t > 0$ and

$$\mathcal{S}_*^b \times \mathbb{R}^+ \ni (T, s) \mapsto \Delta_s(T)$$

is continuous. Moreover:

- (a) For each $T \in \mathcal{S}_*^b$ there exists $t_\Delta(T) \in (0, \infty)$ so that the map $\Delta_{t_\Delta(T)}(T)$ is trivial (i.e., monotone).

- (b) $\mathcal{S}_*^b \ni T \mapsto t_\Delta(T)$ is continuous.
- (c) For each $t \in [0, t(T)]$, $\Delta_t(T) \in L_j(h_0^-) \cap L_j(h_0^+)$ implies that if Z_j is part of a wandering pair then Z_j is contained in the closure of a component of the basin of a periodic attractor of $\Delta_t(T)$. In particular, $\Delta_t(T) \in L_j(h_0^-) \cap L_j(h_0^+)$ implies that $\Delta_t(T) \in \mathcal{S}_{*,j}^b$.

Proof. Note that the right hand side of differential equation (22) is smooth on each stratum $\mathcal{S}^b(\Xi)$. The discontinuities of the right hand side occur when two plateaus start to touch, and the nature of the equation is that once they touch they remain touching. From this and the existence and uniqueness theorem of differential equations (applied to each stratum separately), it follows that the flow of the differential equation is well-defined. However, continuity of $\mathcal{S}_*^b \times \mathbb{R}^+ \ni (T, t) \rightarrow \Delta_t(T)$ still needs to be proved.

For simplicity write $T_s = \Delta_s(T)$ and let Ξ_s be the partition associated to T_s . As s increases, each plateau only widens under the flow T_s , until it touches another plateau in which case these plateaus widen jointly unless the corresponding block has an even number of touching plateaus (or touches $\pm e$). In this sense each coordinate of T_s depends monotonically on $s > 0$, and therefore the limit $\tilde{T} := \lim_{s \rightarrow \infty} T_s \in \mathcal{S}^b$ exist. Also $s \mapsto h_{top}(T_s)$ is non-increasing in s .

Now take $T \in \mathcal{S}_*^b$ with $h_{top}(T) = h_0$, and assume by contradiction that property (a) does not hold.

Step 1. Assume that K is a renormalization interval of entropy h_0 for T_t so that there exists no renormalization interval $K' \not\supseteq K$ of the same period. We claim that K is a renormalization interval for T_s for each $s \in [0, t]$. Indeed, otherwise there would exist $0 \leq s_0 < t$ so that one of the plateaus $Z_{j,s} \subset K$ of T_{s_0} arches over ∂K under the first return map of T_{s_0} to K . If $Z_{j,s}$ is independent of $s \in [0, s_0]$, then K is still a renormalization interval for T_s and there is nothing to show. If $Z_{j,s}$ does depend on s , then there exists $s' \in [0, s_0)$ so that the first return map to K under $T_{s'}$ has another branch, see the left panel in Figure 20. Here we use that if for T_{s_0} there exists a plateau outside K , but which touches an endpoint of K (as is shown in the right panel of Figure 20), then there would have existed a renormalization interval $K' \not\supseteq K$ with the same return time, contradicting the choice of K . Since the first return map to K under $T_{s'}$ has another branch, it follows by Lemma 7.3 that $h_0 \geq h_{top}(T_{s'}|\Omega(K)) > h_{top}(T_t|\Omega(K)) = h_0$, contradicting the assumption made in this step.

Step 2. Take $t > 0$. We claim that there exists $\epsilon > 0$ so that for each $s \in [t - \epsilon, t + \epsilon]$ there exists $i \in \{1, \dots, b\}$ with $T_s \notin \overline{\xi_i} \cap \mathcal{S}^b(\Xi_s)$ and $Z_i \subset \text{cycle}(K)$.

We may assume that K is a renormalization interval of entropy h_0 for T_t because otherwise this claim holds trivially. To prove this claim, we first note that by Step



FIGURE 20. The situation considered in Step 1 of the proof of Proposition 7.16. The first return map to K for $\Delta_s(T)$ and (schematically) for $\Delta_{s'}(T)$ for $s' < t$ is drawn in the left panel. The situation as in the right cannot occur, see the proof in Step 1.

1, K is a renormalization interval for T_s for each $s \in [0, t]$. Let us consider all the possible situations.

Case A. T_t does not have a renormalization interval $\hat{K} \subsetneq K$. We claim that in this case $h_{top}(T_t|\Omega(K)) \leq h_{top}(T_s|\Omega(K)) < h_{top}(T_0|\Omega(K)) \leq h_0$. By the definition of ξ_i this implies that $T_s \notin \xi_i \cap \mathcal{S}^b(\Xi_s)$ for each $0 \leq s \leq t$ and each i so that $Z_{i,t} \subset K$; therefore, in this case Step 2 follows from the claim. To prove this, first note that T_s has no renormalization interval $\hat{K} \subsetneq K$ for any $s \in [0, t]$. Indeed, if there exists $s \in [0, t]$ so that T_s does have a renormalization interval $\hat{K} \subsetneq K$, then by Lemma 7.3 we obtain $h_{top}(T_t|\text{cycle}(K)) < h_{top}(T_s|\text{cycle}(K)) \leq h_0$ for any $s < t$, which gives a contradiction. Next take T_s with $s > 0$ small, and let $T_s(n) \in \mathcal{S}^b(\Xi_s)$ be a sequence of maps so that $T_s(n) \rightarrow T_s$. Since T_t has no renormalization intervals $\hat{K} \subsetneq K$, the map T_t and therefore $T_0 = T$ has no periodic attractors in K either (and since $T \in \mathcal{S}_*^b$, therefore no two plateaus of T touch). Since $T \in \mathcal{S}_*^b$, the convex hull \mathcal{J}_n of two adjacent plateaus $Z_{i,n}, Z_{i+1,n}$ for $T_s(n)$ for n large and $s > 0$ small, does not form a wandering pair, see Lemma 4.16(b). It follows that $T_s \notin \overline{\xi_i \cap \mathcal{S}^b}$ for each i and each $s \geq 0$ small, and therefore for each i so that $Z_i \subset K$ we have that $\rho_i(T_s) \cdot \text{sgn}(Z_{i,s}) \neq 0$ for each $s \geq 0$ small. But this implies by Lemma 7.3 that $h_{top}(T_t|\Omega(K)) \leq h_{top}(T_s|\Omega(K)) < h_{top}(T_0|\Omega(K))$ as claimed.

Case B. T_t has a maximal renormalization interval $\hat{K} \subsetneq K$ on which the map T_t is non-monotone. We claim that $T_s \notin \overline{\xi_i \cap \mathcal{S}^b(\Xi_s)}$ for each $s \in [0, t]$ and for each i so that $Z_i \subset \hat{K}$. To see this, first note that we may assume that each maximal renormalization interval $\hat{K} \subsetneq K$ is a renormalization interval for T_s for all $s \in [0, t]$. Indeed, otherwise by Lemma 7.3, $h_{top}(T_t|\Omega(K)) < h_{top}(T_0|\Omega(K)) \leq h_0$, and then the claim follows immediately. To prove the claim, we consider the following cases:

(i) Each plateau $Z_{i,t}$ in \hat{K} is mapped by T_t into the interior of $\text{cycle}(\hat{K})$. Then a nearby map \tilde{T} also has $\tilde{Z} \subset \text{cycle}(\hat{K})$ and so part (4) of Definition 7.15 fails, and $\tilde{T} \notin \xi_i$. It follows $T_s \notin \overline{\xi_i \cap \mathcal{S}^b(\Xi_s)}$ for each i associated to a plateau $Z_{i,s} \subset \text{cycle}(\hat{K})$ and for each s near t .

(ii) One of the plateaus $Z_{i,t}$ in \hat{K} arches by T_t over the boundary of a component of $\text{cycle}(\hat{K})$. Then consider the following two subcases:

(iia) There exists $\epsilon > 0$ so that the cardinality of the block of plateaus touching $Z_{i,s}$ does not change for $s \in [t - \epsilon, t]$. Next take a convex hull \mathcal{J}_s connecting $Z_{i,s}$ and another plateau (outside the block of plateaus touching $Z_{i,s}$). (By Definition 7.15 we need to take \mathcal{J}_s non-trivial, so \mathcal{J}_s contains two plateaus $Z_{i,s}, Z_{j,s}$ from distinct subsets of the partition Ξ_s .) Because $Z_{i,s}$ arches, either \mathcal{J}_s is contained in \hat{K} or the block of plateaus touching $Z_{i,s}$ (and that of $Z_{i,t}$) contains an even number of plateaus. In the former case for $s \in [t - \epsilon, t)$ the width of this block is smaller, and in the latter case, $\widehat{\text{sgn}}(Z_{i,s}) = 0$ for each $s \in [t - \epsilon, t]$. It follows that in both cases $T_s(\mathcal{J}_s)$ contains a repelling (pre-)periodic boundary point of \hat{K} for each $s \in [t - \epsilon, t]$. The same holds for each $\tilde{T} \in \Xi(T_s)$ near T_s and therefore $\tilde{T} \notin \xi_i \cap \mathcal{S}^b(\Xi_s)$ for each $s \in [t - \epsilon, t]$. It follows that $T_s \notin \xi_i \cap \mathcal{S}^b(\Xi_s)$ for each $s \in [t - \epsilon, t]$. Provided $\epsilon > 0$ is small, for each $s \in [t, t + \epsilon]$ the cardinality of the block of plateaus touching $Z_{i,s}$ does not change either, and for the same reason $T_s \notin \xi_i \cap \mathcal{S}^b(\Xi_s)$ for each $s \in [t, t + \epsilon]$.

(iib) There exists no such $\epsilon > 0$. Then there exists a plateau $Z_{j,s} \subset \hat{K}$ (so that $Z_{j,t}$ is in the block of plateaus touching $Z_{i,t}$) which creates an extra branch for the return map of T_s to K (compared to the return map of T_t). Hence, by Lemma 7.3, $h_{\text{top}}(T_t, \Omega(K)) < h_{\text{top}}(T_s, \Omega(K)) \leq h_0$, which gives a contradiction.

(iii) T_t is constant on one of the components of $\text{cycle}(\hat{K})$ and for simplicity assume that \hat{K} is this component. Let $Z_{i,t}$ be contained in \hat{K} .

(iiia) There exists $\epsilon > 0$ so that the cardinality of the block of plateaus touching $Z_{i,s}$ (in \hat{K}) does not change for $s \in [t - \epsilon, t]$. In this case the number of plateaus in this block is odd and so this block of plateaus is mapped into the interior of \hat{K} for $s \in [t - \epsilon, t)$. By case (i) it follows that for $s \in [t - \epsilon, t)$ one has that $T_s \notin \xi_i \cap \mathcal{S}^b(\Xi_s)$. To prove this conclusion also for $s = t$, consider a sequence of maps $T_t(n) \rightarrow T_t$ where $T_t(n) \in \mathcal{S}^b(\Xi_t)$. By maximality of $\hat{K} \subsetneq K$ it follows that each boundary point of \hat{K} is a repelling periodic or pre-periodic point which is not the common boundary point of two basins. Hence there exists a sequence of repelling periodic points accumulating on $\partial\hat{K}$ (from outside \hat{K}). It follows that any non-trivial convex hull $\mathcal{J}(n)$ connecting $Z_{i,t}$ with another plateau (outside this component of \hat{K}) will contain a repelling periodic point of $T_t(n)$ and therefore iterates of $\mathcal{J}(n)$ are not singletons. It follows that $T_t(n) \notin \xi_i$ and therefore $T_t \notin \xi_i \cap \mathcal{S}^b(\Xi_t)$. Moreover, for $s \in (t, t + \epsilon]$ one has $h_{\text{top}}(T_s|_{\text{orb}(K)}) < h_{\text{top}}(T_t|_{\text{orb}(K)}) = h_0$, where we choose $\epsilon > 0$ so that K remains periodic for T_s for all $s \in [t, t + \epsilon]$. It again follows that $T_s \notin \xi_i \cap \mathcal{S}^b(\Xi_s)$ for $s \in (t, t + \epsilon]$ for each i with $Z_i \subset K$.

(iiib) There exists no such $\epsilon > 0$. Then there exists a plateau $Z_{j,s} \subset \hat{K}$ (so that $Z_{j,t}$ is in the block of plateaus touching $Z_{i,t}$) which creates an extra branch for the

return map of T_s to K (compared to the return map of T_t), and so Lemma 7.3 yields $h_{top}(T_t, \Omega(K)) < h_{top}(T_s, \Omega(K)) \leq h_0$, which again shows that $T_s \notin \overline{\xi_i \cap \mathcal{S}^b(\Xi_s)}$ for $s \in [t, t + \epsilon]$.

Case C. T_t is monotone on each component of $\text{cycle}(\hat{K})$ for each maximal renormalization interval $\hat{K} \subsetneq K$. We claim that in this case there exists $\epsilon > 0$ so that $T_s \notin \overline{\xi_i \cap \mathcal{S}^b(\Xi_s)}$ for $s \in [t, t + \epsilon]$ and for each i so that $Z_i \subset K$. Indeed, in this setting there exists $\epsilon > 0$ so that the cardinality of the block of plateaus touching $Z_{i,s}$ in $\text{cycle}(\hat{K})$ is constant for $s \in [t, t + \epsilon]$. Moreover, for each sequence $T_s(n) \rightarrow T_s$ with $T_s(n) \in \mathcal{S}^b(\Xi_s)$, the map $T_s(n)$ is also monotone on each component of $\text{cycle}(\hat{K})$. It follows that for each plateau $Z_{i,s}$ which is not contained in a renormalization interval $\hat{K} \subsetneq K$ one has $T_s(n) \notin \xi_i$ (because of Definition 7.15(2)). Hence $T_s \notin \overline{\xi_i \cap \mathcal{S}^b(\Xi_s)}$ for each such i and $s \in [t, t + \epsilon]$. Note that for $s \in [t - \epsilon, t]$ either T_s is also monotone on each component of $\text{cycle}(\hat{K})$ (and therefore $T_s(n) \notin \overline{\xi_i \cap \mathcal{S}^b(\Xi_s)}$ for each i as above) or one of the plateaus $Z_{i,s} \subset \hat{K}$ corresponds to a local extremum and then $T_s(n) \notin \overline{\xi_i \cap \mathcal{S}^b(\Xi_s)}$.

Step 3. We claim that if $h_{top}(T_t|\Omega(K)) = h_0$ for some $t > 0$, then for each $Z_i \subset K$ and $s \in [0, t]$ some iterate of $Z_{i,s}$ is contained in a renormalization interval $\hat{K} \subsetneq K$ (and this renormalization interval \hat{K} does not depend on s). Indeed, if $Z_{i,t}$ is not eventually mapped into some renormalization interval $\hat{K} \subset K$ but this is the case for some $s \in [0, t]$ then Lemma 7.3 implies that $h_0 \geq h_{top}(T_s|\Omega(K)) > h_{top}(T_t|\Omega(K))$, and so we are done. Therefore it suffices to consider the case that no iterate of $Z_{i,s}$, $s \in [0, t]$, is contained in a renormalization interval \hat{K} . Let us show that this implies that $Z_{i,s}$ moves when s small. Indeed, consider a sequence of maps $T_s(n) \rightarrow T_s$ as $n \rightarrow \infty$ and a non-trivial convex hull $\mathcal{J}_s(n)$ containing the plateau corresponding to Z_i . Since $T \in \mathcal{S}_*^b$, Lemma 4.16(b) implies that if $\mathcal{J}_s(n)$ forms a wandering pair for T_s for $s > 0$ small, then the corresponding convex hull \mathcal{J} is contained in the basin of a periodic attractor for T . Since we assumed that Z_i is not eventually mapped into a renormalization interval \hat{K} , this is impossible. It follows that $T_s(n) \notin \xi_i \cap \mathcal{S}^b(\Xi_s)$ and therefore $T_s \notin \overline{\xi_i \cap \mathcal{S}^b(\Xi_s)}$ when s is small. Hence $Z_{i,s}$ moves when s small, and so Lemma 7.3 implies again that $h_0 \geq h_{top}(T_s|\Omega(K)) > h_{top}(T_t|\Omega(K))$.

Step 4. We claim that for each $T \in \mathcal{S}_*^b$ there exists $t > 0$ so that T_t is trivial. Indeed, consider a renormalization interval K for T . By Step 3, either $h_{top}(T_t|\Omega(K)) < h_0$ or each plateau $Z_{i,s}$ in K is eventually mapped into a renormalization interval $\hat{K} \subsetneq K$ for each $s \in [0, t]$. If $h_{top}(T_t|\Omega(K)) < h_0$ for each $t > 0$, then each plateau in K will move with positive speed and since $h_{top}(T_t|\Omega(K))$ is decreasing, this speed will not tend to zero as t increases (unless this plateau becomes part of a block consisting of an even number of plateaus). If each plateau $Z_{i,s}$ in K is eventually mapped into a renormalization interval $\hat{K} \subsetneq K$ for each $s \in [0, t]$, then the period of these intervals \hat{K} will only depend on T (and not on t). It follows by Step 2, Case B that the speed

of at least one plateau in \hat{K} is bounded away from zero (until all plateaus are in blocks consisting of an even number plateaus), and so in finite time the first return map to \hat{K} will be either monotone or constant. In the former case other plateaus in K start moving, see Step 2, Case C. In the latter case the renormalization interval \hat{K} disappears, see Step 2 Case C(iii), and $h_{top}(T_t|\Omega(K))$ becomes $< h_0$.

Note that $h_{top}(T_t|\Omega(K)) = h_0$ implies that the period of K is bounded from below, and so there are only a bounded number of intervals K to be considered in the previous paragraph. Step 4 and therefore part (a) of the proposition follow.

Step 5. From Step 4 it follows that if $T \in \mathcal{S}_*^b$ then T_t never enters the set where the speed of a plateau is actually zero, unless plateaus touch. It follows that if T and T' are nearby maps in \mathcal{S}_*^b so that for at time $s = t$, two plateaus of T_s start to touch, then the same two plateaus will start touching for the map T'_s , for some $s' \approx t$. It follows that the map $(T, t) \rightarrow \Delta_t(T)$ is continuous and that the map $\mathcal{S}_*^b \ni T \mapsto t_\Delta(T)$ is continuous. This finished part (b) of the proposition.

Step 6. Let us now prove part (c) of the proposition and assume that $t > 0$ and $\Delta_t(T) \in L_j(h_0^-) \cap L_j(h_0^+)$. Amongst other things, this implies that there exists a renormalization interval K for $\Delta_t(T)$ on which the entropy is h_0 . By Lemma 7.3, this implies that each plateau is either contained in a maximal renormalization interval $\hat{K} \subsetneq K$ or is mapped into such an interval. Since $\Delta_t(T) \in L_j(h_0^-) \cap L_j(h_0^+)$, the plateau $Z_{j,t}$ is eventually mapped into the boundary of such a renormalization interval \hat{K} . If $Z_{j,t}$ maps in an arching way over $\partial\hat{K}$, then this implies that either there exists $s \in [0, t)$ so that $Z_{j,s}$ is not mapped into \hat{K} , contradicting that the entropy of T_t on K is h_0 (using Lemma 7.3), or $Z_{j,s}$ maps to $\partial\hat{K}$ for each $s \in [0, t]$. Since $T \in \mathcal{S}_*^b$, this implies that $Z_{j,t}$ maps to the boundary of the basin of a periodic attractor of T_t , and so we are done. Next assume that $Z_{j,t}$ maps in a non-arching way over $\partial\hat{K}$ and that $Z_{j,t}$ is part of a wandering pair. In this case, either the first return map to \hat{K} is monotone or $T_t \in \xi_j$, in which case the speed of $Z_{j,s}$ at $s = t$ is zero. If the former holds, each point in \hat{K} is in the boundary of a component of the basin of a periodic attractor and we are done. If the latter holds, the speed of $Z_{j,s}$ is zero for each $s \in [0, t]$ and $T_s \in \xi_j$ for *each* $s \in [0, t]$. But since $T \in \mathcal{S}_*^b$, this implies that a boundary point of \hat{K} for T (and therefore for T_s for each $s \in [0, t]$) is in the boundary of a component of the basin of a periodic attractor, and again we are done. \square

Let us define

$$\hat{\Delta}_t(T) = \Delta_{t_\Delta(T) \cdot t}(T)$$

so that $\hat{\Delta}_1(T)$ is a trivial map for each $T \in \mathcal{S}_*^b$.

7.10. The proof of Theorem 7.1. Now that we have developed the ingredients of the proof, we can define the retract for a fixed $h_0 \in (0, \log(b+1))$. (The cases

$h_0 = \log(b + 1)$ and $h_0 = 0$ were dealt with in Sections 7.4 and 7.5.)

$$R_t = \begin{cases} \beta_{5t} & \text{for } t \in [0, \frac{1}{5}], \\ \Gamma_{5t-1} \circ \beta_1 & \text{for } t \in [\frac{1}{5}, \frac{2}{5}], \\ \Gamma_1 \circ \hat{\Gamma}_{5t-2} \circ \beta_1 & \text{for } t \in [\frac{2}{5}, \frac{3}{4}], \\ \Gamma_1 \circ \hat{\Gamma}_1 \circ \beta_1 \circ \hat{\Delta}_{(5t-3)} & \text{for } t \in [\frac{3}{5}, \frac{4}{5}], \\ \Gamma_1 \circ \hat{\Gamma}_1 \circ \beta_1 \circ r_{5t-4} \circ \hat{\Delta}_1 & \text{for } t \in [\frac{4}{5}, 1]. \end{cases}$$

Obviously, $R_0(T) = T$, and since for $t = 1$, the retract r_{6t-5} has been carried out completely, $R_1(T)$ is the same map for each $T \in L_*(h_0)$ of the same shape ϵ . All components of R_t are continuous in t and T , so the same holds for R_t .

Let us show that R_t keeps maps within \mathcal{S}_*^b . First note that the only deformation which takes a map outside the space \mathcal{S}_*^b is $\hat{\Delta}_t$. Take T' of the form $T' = \hat{\Delta}_t(T)$ or $T' = r_t \circ \hat{\Delta}_1(T)$. The deformation $\beta_t(T')$ moves plateaus Z_i, Z_{i+1} whose convex hull is mapped into other plateaus. It does so in such a way that $\beta_1(T')$ never eventually maps $[Z_i, Z_{i+1}]$ into the interior of another plateau and so the β -property (11) on page 46 will hold.

If $h_{top}(T') < h_0$ then $h_{top}(\beta_1(T')) < h_0$ and of course $\beta_1(T)$ satisfies the β -property (11). Because of (13) this gives that $\hat{\Gamma}_1 \circ \beta_1(T') \in \mathcal{S}_*^b$. In particular, we are done if $T' = r_t \circ \hat{\Delta}_1(T)$ satisfies $h_{top}(T') < h_0$.

Let us now consider the case that $h_{top}(T') = h_0$. By the third part of Lemma 7.10, $\Gamma_1 \circ \hat{\Gamma}_1 \circ \beta_1(T') \in \mathcal{S}_*^b$ provided that $T' = \beta_1(T') \in L_j(h_0^-) \cap L_j(h_0^+)$ implies $T' \in \mathcal{S}_{*,j}^b$. But in Proposition 7.16 it is shown that any map T' of the form $T' = \beta_1 \circ \hat{\Delta}_t(T)$ with $t > 0$ and $T \in \mathcal{S}_*^b$ has indeed this property, and so again the resulting map belongs to \mathcal{S}_*^b . This concludes the proof of the first part of Theorem 7.1. The 2nd part follows from the construction of Γ_t and β_t .

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HENK BRUIN, FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR MORGENSTERN PLATZ 1, A-1090 VIENNA, AUSTRIA.

SEBASTIAN VAN STRIEN, DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, 180 QUEEN’S GATE, LONDON SW7 2AZ, UK.

E-mail address: `henk.bruin@univie.ac.at` and `s.van-strien@imperial.ac.uk`

URL: <http://www.mat.univie.ac.at/~bruin/> and <http://www2.imperial.ac.uk/~svanstri/>