Open Mathematics

Research Article

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# Monotonicity properties and inequalities related to generalized Grötzsch ring functions

https://doi.org/10.1515/math-2019-0064
Received August 21, 2018; accepted May 29, 2019

**Abstract:** In the paper, the authors present some monotonicity properties and some sharp inequalities for the generalized Grötzsch ring function and related elementary functions. Consequently, the authors obtain new bounds for solutions of the Ramanujan generalized modular equation.

**Keywords:** Gaussian hypergeometric function; generalized Hersch–Pfluger distortion function; sharp inequality; generalized Grötzsch ring function; generalized modular equation

**MSC:** Primary 33E05; Secondary 26A48, 26D15

## 1 Introduction and main results

For real numbers $a$, $b$, and $c$ with $c \neq 0, -1, -2, \dots$, the Gaussian hypergeometric function is defined [1, 4] by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad |x| < 1, \tag{1.1}$$

where

$$(x)_n = \begin{cases} x(x+1)\cdots(x+n-1), & n \geq 1 \\ 1, & n = 0 \end{cases} \tag{1.2}$$

is called [23] the rising factorial of $x \in \mathbb{C}$.

For $a, r \in (0,1)$ and $s = \sqrt{1-r^2}$, let $\mathcal{K}_a(r)$ and $\mathcal{E}_a(r)$ denote the generalized elliptic integrals of the first and second kinds which are defined [6] by

$$\begin{cases} \mathcal{K}_a = \mathcal{K}_a(r) = \dfrac{\pi}{2} F\left(a, 1-a; 1; r^2\right) \\ \mathfrak{K}_a = \mathfrak{K}_a(r) = \mathcal{K}_a(s) \\ \mathcal{K}_a(0) = \dfrac{\pi}{2} \\ \mathcal{K}_a(1) = \infty \end{cases} \tag{1.3}$$

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and

$$\begin{cases} \mathcal{E}_a = \mathcal{E}_a(r) = \dfrac{\pi}{2} F\left(a-1, 1-a; 1; r^2\right) \\ \mathfrak{E}_a = \mathfrak{E}_a(r) = \mathcal{E}_a(s) \\ \mathcal{E}_a(0) = \dfrac{\pi}{2} \\ \mathcal{E}_a(1) = \dfrac{\sin(\pi a)}{2(1-a)} \end{cases} \tag{1.4}$$

respectively. In the special case $a = \frac{1}{2}$, the functions $\mathcal{K}_a(r)$ and $\mathcal{E}_a(r)$ reduce to $\mathcal{K}(r)$ and $\mathcal{E}(r)$ which are the complete elliptic integrals of the first and second kinds [2, 5, 7, 8, 11, 15, 21, 35] respectively. The complete elliptic integrals have many important applications in physics, engineering, geometric function theory, quasi-conformal analysis, theory of mathematical means, number theory, and other fields [6, 8–10, 17, 31, 32].

In what follows, by the symmetry of (1.3), we assume that $a \in \left(0, \frac{1}{2}\right]$.

For real numbers $a \in \left(0, \frac{1}{2}\right]$ and $r \in (0, 1)$, the generalized Grötzsch ring function $\mu_a(r) : (0, 1) \to (0, \infty)$ is defined by

$$\mu_a(r) = \frac{\pi}{2\sin(\pi a)} \frac{\mathfrak{K}_a(r)}{\mathcal{K}_a(r)}. \tag{1.5}$$

In the special case $a = \frac{1}{2}$, we denote $\mu_{1/2}(r)$ by $\mu(r)$ which is the modulus of the plane Grötzsch ring $B^2 \setminus [0, r]$ for $r \in (0, 1)$ and $B^2$ is the unit disk in the plane [3, 6, 24, 28, 36].

It is known that the Ramanujan generalized modular equation with signature $\frac{1}{a}$ of degree $p$ can be expressed by

$$\frac{F(a, 1-a; 1; 1-s^2)}{F(a, 1-a; 1; s^2)} = p \frac{F(a, 1-a; 1; 1-r^2)}{F\left(a, 1-a; 1; r^2\right)}, \quad 0 < r < 1.$$

From (1.3) and (1.5), it follows that

$$\mu_a(s) = p\mu_a(r) \tag{1.6}$$

and the solution $s$ to the equation (1.6) can be written as

$$s = \varphi_K^a(r) = \mu_a^{-1}\left(\frac{\mu_a(r)}{K}\right), \quad K \in (0, \infty), \quad p = \frac{1}{K}.$$

In the special case $a = \frac{1}{2}$, the solution $\varphi_K^a(r)$ reduces to the Hersch–Pfluger distortion function $\varphi_K(r)$ which is important in the theory of the plane quasi-conformal mappings. As usual, we call $\varphi_K^a(r)$ the generalized Hersch–Pfluger distortion function [16, 26, 30, 34].

For real number $x > 0$, the Euler gamma function $\Gamma$ and its logarithmic derivative $\psi$, the so-called digamma or psi function, are defined [1, 19, 20, 22, 29] by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t \quad \text{and} \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

for $\Re(x) > 0$ respectively. For $a \in \left(0, \frac{1}{2}\right]$, the so-called Ramanujan constant $R(a)$ is defined [27] by

$$R(a) = -2\gamma - \psi(a) - \psi(1-a), \tag{1.7}$$

where $\gamma$ is the Euler–Mascheroni constant which can be defined [12–14, 18] by

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \ln n\right) = 0.5772156649\ldots.$$

By [1, 6.3.4], we have $R\left(\frac{1}{2}\right) = \ln 16$.

In 2000, Anderson, Qiu, Vamanamurthy, and Vuorinen discovered relations between bounds of $s = \varphi_K^a(r)$, $\mu_a(r)$, and $m_a(r)$ by establishing in [6, Theorem 6.6] the double inequality

$$1 < \exp\{(K-1)[m_a(r) + \ln r]\} < \frac{r^K}{\varphi_{1/K}^a(r)} < \exp\{(K-1)[\mu_a(r) + \ln r]\} \tag{1.8}$$

for all $r \in (0,1)$ and $K \in (1,\infty)$, where

$$m_a(r) = \frac{2}{\pi \sin(a\pi)} s^2 \mathcal{K}_a(r)\mathfrak{K}_a(r)$$

and $m_a(r) + \ln r$ is the so-called Hüber function.

During the past decades, the function $\mu_a(r)$ plays an important role in several fields of mathematics. For instance, it is indispensable in the theory of mathematical means, the theory of geometric functions, quasi-conformal theory, and the theory of the Ramanujan modular equations. See [3, 6, 24, 28, 33]. In recent years, convexity and Hölder mean property of the function $\mu_a(r)$ were obtained. Especially, many remarkable properties and sharp inequalities can be found in the literature [3, 33, 36, 37].

The main purpose of this paper is to present some monotonicity properties and some sharp inequalities for the generalized Grötzsch ring function $\mu_a(r)$ and related elementary functions. By applying these results, we establish new bounds for solutions to the Ramanujan generalized modular equation.

Our main results in this paper can be stated as follows.

**Theorem 1.** *For $r \in (0,1)$, $a \in \left(0, \frac{1}{2}\right]$, $b = 1-a$, and $C = \frac{R(a)}{2}$, define*

$$F(r) = \frac{C - [\mu_a(r) + \ln r]}{1 - (s^2 \operatorname{artanh} r)/r}, \quad r \in (0,1),$$

*where* artanh *denotes the inverse of the hyperbolic tangent function. Then the function $F(r)$ is strictly increasing from $(0,1)$ onto $\left(\frac{3(a^2+b^2)}{4}, C\right)$. In particular, the double inequality*

$$C\left(1 - A_1 \sum_{n=1}^{\infty} a_n r^{2n}\right) < \mu_a(r) + \ln r < C\left(1 - A_2 \sum_{n=1}^{\infty} a_n r^{2n}\right) \tag{1.9}$$

*holds for $A_1 = 1$, $A_2 = \frac{3(a^2+b^2)}{4C}$, $a_n = \frac{2}{4n^2-1}$, and $r \in (0,1)$.*

**Theorem 2.** *For $B_1 = \frac{R(a)-\ln 16}{2}$, $B_2 = \frac{B_1}{\ln 4}$, $B_3 = \frac{3(1-2a)^2}{8}$, $B_4 = e^{B_1}$, $B_5 = e^{B_2}$, $B_6 = \frac{B_3}{B_1}$, $a \in \left(0, \frac{1}{2}\right)$, and $r \in (0,1)$, the following conclusions hold true:*

1. *The function*

$$G_1(r) = \frac{\mu_a(r) - \mu(r)}{s^2 \ln(4/s)} \tag{1.10}$$

*is strictly increasing from $(0,1)$ onto $(B_2, \infty)$. In particular, for $r \in (0,1)$,*

$$B_2 s^2 \ln\frac{4}{s} < \mu_a(r) - \mu(r) < B_1. \tag{1.11}$$

2. *The function*

$$G_2(r) = \frac{B_1 - [\mu_a(r) - \mu(r)]}{1 - (s^2 \operatorname{artanh} r)/r}$$

*is strictly increasing from $(0,1)$ onto $(B_3, B_1)$. In particular, for $r \in (0,1)$,*

$$B_1 \frac{s^2 \operatorname{artanh} r}{r} < \mu_a(r) - \mu(r) < B_1\left[1 - B_6\left(1 - \frac{s^2 \operatorname{artanh} r}{r}\right)\right]. \tag{1.12}$$

3. *Let $r_0 = s$, $r_n = \frac{2\sqrt{r_{n-1}}}{1+r_{n-1}} = \varphi_{2^n}(s)$ for $n \in \mathbb{N}$, $A(r) = \frac{s^2 \operatorname{artanh} r}{r}$, $B(r) = s^2 \ln\frac{4}{s}$, and $P(r) = \prod_{n=0}^{\infty}(1+r_n)^{2^{-n}}$. For $a \in \left(0, \frac{1}{2}\right]$ and $r \in (0,1)$, we have*

$$P(r)\max\left\{{B_4}^{A(r)}, {B_5}^{B(r)}\right\} \le \exp[\mu_a(r) + \ln r] \le P(r){B_4}^{1-B_6[1-A(r)]}. \tag{1.13}$$

**Theorem 3.** *For $C_1 = \frac{R(a)-3\ln 2}{2\ln 4}$, the function*

$$H(r) = \frac{\mu_a(r) - \operatorname{artanh}\sqrt{s}}{s^2 \ln(4/s)}$$

*is strictly increasing from $(0,1)$ onto $(C_1, \infty)$. In particular, for $r \in (0,1)$,*

$$C_1 s^2 \ln\frac{4}{s} < \mu_a(r) - \operatorname{artanh}\sqrt{s} < C_1 \ln 4. \tag{1.14}$$

## 2 Lemmas

For proving our main results, we need the following formulas and lemmas.

The following derivative formulas in [6, Theorem 4.1] and [29] hold true:

$$\frac{\mathrm{d}\mathcal{K}_a}{\mathrm{d}r} = \frac{2(1-a)}{rs^2}(\mathcal{E}_a - s^2\mathcal{K}_a), \quad \frac{\mathrm{d}\mathcal{E}_a}{\mathrm{d}r} = \frac{2(a-1)}{r}(\mathcal{K}_a - \mathcal{E}_a), \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(\mathcal{K}_a - \mathcal{E}_a) = \frac{2(1-a)r\mathcal{E}_a}{s^2}, \quad \frac{\mathrm{d}}{\mathrm{d}r}(\mathcal{E}_a - s^2\mathcal{K}_a) = 2ar\mathcal{K}_a, \tag{2.2}$$

$$\frac{\mathrm{d}\mu_a(r)}{\mathrm{d}r} = -\frac{\pi^2}{4rs^2\mathcal{K}_a^2}, \quad \mathcal{K}_a\mathcal{E}_a' + \mathcal{K}_a'\mathcal{E}_a - \mathcal{K}_a'\mathcal{K}_a = \frac{\pi\sin(a\pi)}{4(1-a)}. \tag{2.3}$$

**Lemma 1** ([7, Theorem 1.25]). *For $-\infty < a < b < \infty$, let $g, h : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$ and let $h'(x) \neq 0$ on $(a,b)$. If $\frac{g'(x)}{h'(x)}$ is increasing (or decreasing respectively) on $(a,b)$, then so are*

$$\frac{g(x)-g(a)}{h(x)-h(a)} \quad \text{and} \quad \frac{g(x)-g(b)}{h(x)-h(b)}.$$

**Lemma 2** ([2, Lemma 2]). *Let $r_n$ and $s_n$ for $n \in \mathbb{N}$ be real numbers and the power series*

$$R(x) = \sum_{n=0}^{\infty} r(n)x^n \quad \text{and} \quad S(x) = \sum_{n=0}^{\infty} s(n)x^n$$

*be convergent for $|x| < 1$. If $s_n > 0$ for $n \in \mathbb{N}$ and if $\frac{r_n}{s_n}$ is strictly increasing (or decreasing respectively) for $n \in \mathbb{N}$, then the function $\frac{R(x)}{S(x)}$ is strictly increasing (or decreasing respectively) on $(0,1)$.*

**Lemma 3** ([6, Lemmas 5.2 and 5.4] and [25, Theorem 2.2]). *For $r \in (0,1)$ and $b = 1-a$, the following conclusions hold true:*

1. *the function $\frac{\mathcal{E}_a - s^2\mathcal{K}_a}{r^2}$ is strictly increasing from $(0,1)$ onto $\left(\frac{a\pi}{2}, \frac{\sin(a\pi)}{2b}\right)$;*
2. *the function $\frac{\mathcal{K}_a - \mathcal{E}_a}{\ln(1/s)}$ is strictly decreasing from $(0,1)$ onto $(\sin(a\pi), (1-a)\pi)$;*
3. *the function $\frac{\pi^2/4 - s^2\mathcal{K}_a^2}{r^2}$ is strictly increasing from $(0,1)$ onto $\left(\frac{(a^2+b^2)\pi^2}{4}, \frac{\pi^2}{4}\right)$;*
4. *the function $s^c\mathcal{K}_a$ is decreasing from $(0,1)$ onto $\left(0, \frac{\pi}{2}\right)$ if and only if $c \geq 2a(1-a)$; the function $\sqrt{s}\,\mathcal{K}_a(r)$ is decreasing for each $a \in \left(0, \frac{1}{2}\right]$;*
5. *the function $\frac{\mathcal{E}-1}{s^2\ln(4/s)}$ is strictly increasing from $(0,1)$ onto $\left(\frac{\pi-2}{2\ln 4}, \frac{1}{2}\right)$.*

**Lemma 4** ([6, Theorem 5.5] and [24, Theorems 1 and 2]). *Let $R(a)$ be the Ramanujan constant defined in (1.7). Then*

1. *the function $\mu_a(r) + \ln r$ is strictly decreasing from $(0,1)$ onto $\left(0, \frac{R(a)}{2}\right)$;*
2. *the function $\mu_a(r) - \mu(r)$ is strictly decreasing from $(0,1)$ onto $\left(0, \frac{R(a)-\ln 16}{2}\right)$;*
3. *the function $\mu_a(r) - \operatorname{artanh}\sqrt{s}$ is strictly decreasing from $(0,1)$ onto the interval $\left(0, \frac{R(a)-3\ln 2}{2}\right)$.*

**Lemma 5.** *For $r \in (0,1)$ and $b = 1-a$, the following conclusions hold true:*

1. *the function*

$$I_1(r) = \frac{\ln(1/s)}{(1+r^2)(\operatorname{artanh} r)/r - 1}$$

*is strictly increasing from $(0,1)$ onto $\left(\frac{3}{8}, \frac{1}{2}\right)$;*

2. *the function*

$$I_2(r) = \frac{\mathcal{K}_a - \mathcal{E}_a - (1-2a)(\mathcal{E}_a - s^2\mathcal{K}_a)}{r^2}$$

*is strictly increasing from $(0,1)$ onto $\left(\frac{(a^2+b^2)\pi}{2}, \infty\right)$;*

3. *the function*

$$I_3(r) = \frac{\pi^2/(4s^2\mathcal{K}_a^2) - 1}{\ln(1/s)}$$

*is strictly increasing from* $(0,1)$ *onto* $(2(a^2+b^2),\infty)$.

*Proof.* Utilizing (1.1) and [29, 2.2.5] and using power series expansion lead to

$$\ln\frac{1}{s}=\sum_{n=0}^{\infty}\frac{r^{2n+2}}{2(n+1)} \tag{2.4}$$

and

$$\operatorname{artanh} r=rF\left(\frac{1}{2},1;\frac{3}{2};r^2\right)=\sum_{n=0}^{\infty}\frac{r^{2n+1}}{2n+1}. \tag{2.5}$$

Applying (2.5) yields

$$\begin{aligned}\frac{(1+r^2)\operatorname{artanh} r}{r}-1&=\sum_{n=0}^{\infty}\frac{1}{2n+1}r^{2n}+\sum_{n=0}^{\infty}\frac{1}{2n+1}r^{2n+2}-1\\&=\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+3}\right)r^{2n+2}=\sum_{n=0}^{\infty}\frac{4(n+1)}{(2n+1)(2n+3)}r^{2n+2},\end{aligned}$$

from which and (2.4), it follows that

$$I_1(r)=\frac{\ln(1/s)}{(1+r^2)(\operatorname{artanh} r)/r-1}=\frac{\sum_{n=0}^{\infty}a_1(n)r^{2n}}{\sum_{n=0}^{\infty}b_1(n)r^{2n}},$$

where

$$a_1(n)=\frac{1}{2(n+1)}\quad\text{and}\quad b_1(n)=\frac{4(n+1)}{(2n+1)(2n+3)}.$$

Let $c_1(n)=\frac{a_1(n)}{b_1(n)}$. Then

$$\frac{c_1(n)}{c_1(n+1)}=\frac{(2n+1)(n+2)^2}{(2n+5)(n+1)^2}<1. \tag{2.6}$$

Hence, the inequality (2.6) implies that $c_1(n)$ is strictly increasing in $n$. From Lemma 2, it follows that $I_1(r)$ is strictly increasing in $(0,1)$.

By virtue of L'Hôpital's rule and Lemma 5, we easily obtain the limits $I_1(0^+)=\frac{3}{8}$ and $I_1(1^-)=\frac{1}{2}$.

From (1.1) to (1.4), it is easy to verify that

$$\mathcal{K}_a-\mathcal{E}_a=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(a)_n(1-a)_{n+1}}{(n+1)(n!)^2}r^{2n+2}$$

and

$$\mathcal{E}_a-s^2\mathcal{K}_a=\frac{a\pi}{2}\sum_{n=0}^{\infty}\frac{(a)_n(1-a)_n}{(n+1)(n!)^2}r^{2n+2}$$

for $r\in(0,1)$. Hence, after simplifying and utilizing (1.2), the function $I_2(r)$ can be rewritten as

$$\begin{aligned}I_2(r)&=\frac{\pi}{2}\left[\sum_{n=0}^{\infty}\frac{(a)_n(1-a)_{n+1}}{(n+1)(n!)^2}r^{2n}-a(1-2a)\sum_{n=0}^{\infty}\frac{(a)_n(1-a)_n}{(n+1)(n!)^2}r^{2n}\right]\\&=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{J_n}{(n+1)(n!)^2}r^{2n},\end{aligned}$$

where $J_n=[a^2+b^2+n](a)_n(1-a)_n$ and $b=1-a$. A conclusion in [6, Lemma 7.1] states that the function $(x)_{n+1}(1-x)_{n+1}$ for $n\ge 0$ is positive, increasing on $[0,\frac{1}{2}]$, and decreasing on $[\frac{1}{2},1]$. This implies that $J_n>0$. Thus, the monotonicity of $I_2(r)$ follows immediately.

The limits $\lim_{x\to 0^+}I_2(x)=\frac{(a^2+b^2)\pi}{2}$ and $\lim_{x\to 1^-}I_2(x)=\infty$ are straightforward.

Let $I_3(r)=\frac{I_4(r)}{I_5(r)}$, where $I_4(r)=\frac{\pi^2}{4s^2\mathcal{K}_a^2}-1$ and $I_5(r)=\ln\frac{1}{s}$. By (1.3), it follows that $\lim_{x\to 0^+}I_4(x)=\lim_{x\to 0^+}I_5(x)=0$.

From the formula (2.1) and elementary computation, it follows that

$$\frac{I_4'(r)}{I_5'(r)} = \frac{\pi^2}{2}\frac{1}{s^4\mathcal{K}_a^3}\frac{r^2\mathcal{K}_a - 2(1-a)(\mathcal{E}_a - s^2\mathcal{K}_a)}{r}\frac{s^2}{r} = \frac{\pi^2}{2}\frac{1}{s^2\mathcal{K}_a^3}\frac{\mathcal{K}_a - \mathcal{E}_a - (1-2a)(\mathcal{E}_a - s^2\mathcal{K}_a)}{r^2}.$$

From Lemma 1, the fourth item in Lemma 3, and the second item in Lemma 5, the monotonicity of $I_3(r)$ follows immediately.

By L'Hôpital's rule and Lemmas 3 and 5, we arrive at

$$\lim_{r\to 0^+} I_3(r) = \lim_{r\to 0^+}\frac{I_4'(r)}{I_5'(r)} = \frac{\pi^2}{2}\frac{1}{(\pi/2)^3}\frac{(a^2+b^2)\pi}{2} = 2(a^2+b^2),$$

while $\lim_{r\to 1^-} I_3(r) = \infty$ directly. The proof of Lemma 5 is complete. □

**Lemma 6.** *For $r \in (0,1)$ and $b = 1-a$, we have the following conclusions:*

1. *the function $L_1(r) = \frac{\mathcal{K}-\mathcal{K}_a}{\ln(1/s)}$ is strictly increasing from $(0,1)$ onto the interval $\left(\frac{\pi(1-2a)^2}{4}, 1-\sin(a\pi)\right)$;*
2. *the function $L_2(r) = \frac{\mathcal{K}-\mathcal{K}_a}{\mathcal{K}_a-\mathcal{E}_a}$ is strictly increasing from $(0,1)$ onto the interval $\left(\frac{(1-2a)^2}{4b}, \frac{1}{\sin(a\pi)} - 1\right)$; the function $\frac{\mathcal{K}-\mathcal{K}_a}{\mathcal{K}-\mathcal{E}}$ is strictly increasing from $(0,1)$ onto $\left(\frac{(1-2a)^2}{2}, 1-\sin(a\pi)\right)$;*
3. *the function $L_3(r) = \frac{\mathcal{K}-\mathcal{K}_a}{(1+r^2)(\operatorname{artanh} r)/r - 1}$ is strictly increasing from $(0,1)$ onto $\left(\frac{3\pi(1-2a)^2}{32}, \frac{1-\sin(a\pi)}{2}\right)$.*

*Proof.* Let $\ell_1(r) = \mathcal{K} - \mathcal{K}_a$ and $\ell_2(r) = \ln\frac{1}{s}$. It is obvious that $L_1(r) = \frac{\ell_1(r)}{\ell_2(r)}$ and $\lim_{r\to 0^+}\ell_1(r) = \lim_{r\to 0^+}\ell_2(r) = 0$.

It follows from (1.3) and (2.1) that

$$\frac{\ell_1'(r)}{\ell_2'(r)} = \frac{\mathcal{E} - s^2\mathcal{K} - 2(1-a)(\mathcal{E}_a - s^2\mathcal{K}_a)}{r^2} = \frac{\ell_3(r)}{\ell_4(r)},$$

where

$$\ell_3(r) = \mathcal{E} - s^2\mathcal{K} - 2(1-a)(\mathcal{E}_a - s^2\mathcal{K}_a) \quad \text{and} \quad \ell_4(r) = r^2.$$

It is clear that $\lim_{r\to 0^+}\ell_3(r) = \lim_{r\to 0^+}\ell_4(r) = 0$. Applying (2.2) and (1.3) and differentiating give

$$\frac{\ell_3'(r)}{\ell_4'(r)} = \frac{\mathcal{K} - 4a(1-a)\mathcal{K}_a}{2} = \frac{\pi}{4}\sum_{n=0}^{\infty}\frac{T_n}{(n!)^2}r^{2n},$$

where $T_n = \left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n - 4a(1-a)(a)_n(1-a)_n$. From [6, Lemma 7.1], it is easy to see that $T_n \geq 0$. Therefore, the monotonicity of $L_1(r)$ follows from Lemma 2.

By L'Hôpital's rule and Lemmas 1 and 3, we have

$$\lim_{r\to 0^+} L_1(r) = \lim_{r\to 0^+}\frac{\ell_1'(r)}{\ell_2'(r)} = \lim_{r\to 0^+}\frac{\ell_3'(r)}{\ell_4'(r)} = \frac{\pi(1-2a)^2}{4}.$$

It is known [4, (1.6)] that $F(a,b;a+b;x)$ satisfies the Ramanujan asymptotic relation

$$B(a,b)F(a,b;a+b;x) + \ln(1-x) = R(a,b) + O((1-x)\ln(1-x)), \quad x \to 1$$

for $a, b \in (0,\infty)$, where $R(a,b) = -2\gamma - \psi(a) - \psi(b)$ and

$$\lim_{x\to 1^-}\frac{F(a,1-a;1;x)}{\ln[1/(1-x)]} = \frac{1}{B(a,1-a)}, \tag{2.7}$$

where

$$B(a,1-a) = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(a\pi)}. \tag{2.8}$$

Hence, the limit $\lim_{r\to 1^-} L_1(r) = 1 - \sin(a\pi)$ follows from (1.3), (2.7), and (2.8).

The function $L_2(r)$ can be rewritten as

$$L_2(r) = \frac{\mathcal{K} - \mathcal{K}_a}{\ln(1/s)} \frac{\ln(1/s)}{\mathcal{K}_a - \mathcal{E}_a}.$$

Hence, the monotonicity property of the function $L_2(r)$ follows from the second item in Lemma 3 and the first item in 6. Furthermore, the limits

$$\lim_{r\to 0^+} L_2(r) = \frac{(1-2a)^2}{4(1-a)} \quad \text{and} \quad \lim_{r\to 1^-} L_2(r) = \frac{1}{\sin a\pi} - 1$$

are easily obtained. Similarly, we can prove that the function $\frac{\mathcal{K}-\mathcal{K}_a}{\mathcal{K}-\mathcal{E}}$ is strictly increasing from $(0,1)$ onto $\left(\frac{(1-2a)^2}{2}, 1-\sin(a\pi)\right)$.

The function $L_3(r)$ can be rewritten as

$$L_3(r) = \frac{\mathcal{K} - \mathcal{K}_a}{\ln(1/s)} \frac{\ln(1/s)}{(1+r^2)(\operatorname{artanh} r)/r - 1}.$$

From the first items in Lemmas 5 and 6, the monotonicity of $L_3(r)$ follows immediately. Moreover, the limits

$$\lim_{r\to 0^+} L_3(r) = \frac{3\pi(1-2a)^2}{32} \quad \text{and} \quad \lim_{r\to 1^-} L_3(r) = \frac{1-\sin(a\pi)}{2}$$

can be obtained from the first items in Lemmas 5 and 6. The proof of Lemma 6 is complete. □

## 3 Proofs of main results

Now we are in a position to prove our main results.

*Proof of Theorem 1.* Let

$$f_1(r) = C - [\mu_a(r) + \ln r] \quad \text{and} \quad f_2(r) = 1 - s^2 \frac{\operatorname{artanh} r}{r}.$$

Then $F(r) = \frac{f_1(r)}{f_2(r)}$ and, by Lemma 4, $f_1(0^+) = f_2(0^+) = 0$.

Differentiating and making use of (2.3) give

$$\frac{f_1'(r)}{f_2'(r)} = \frac{\pi^2/(4s^2\mathcal{K}_a^2) - 1}{(1+r^2)(\operatorname{artanh} r)/r - 1} = \frac{\pi^2/(4s^2\mathcal{K}_a^2) - 1}{\ln(1/s)} \frac{\ln(1/s)}{(1+r^2)(\operatorname{artanh} r)/r - 1}.$$

From Lemmas 1 and the first and third conclusions in Lemma 5, we see that the function $F(r)$ is strictly increasing on $(0,1)$.

By L'Hôpital's rule and the first and third items in Lemma 5, we obtain

$$\lim_{r\to 0} F(r) = \lim_{r\to 0} \frac{f_1'(r)}{f_2'(r)} = \frac{3(a^2+b^2)}{4}.$$

Clearly, the limit $F(1^-) = \frac{R(a)}{2}$ follows from the first item in Lemma 4.

Finally, by (2.5), the double inequality in (1.9) follows from the monotonicity property of $F(r)$. The proof of Theorem 1 is complete. □

**Corollary 1.** *For $r \in (0,1)$ and $K \in (1,\infty)$, the inequality*

$$\varphi_{1/K}^a(r) > r^K \exp\left\{C(1-K)\left[1 - \sum_{n=1}^{\infty} a_n r^{2n}\right]\right\} \tag{3.1}$$

*holds true, where $C = \frac{R(a)}{2}$ and $a_n = \frac{2}{4n^2-1}$.*

*Proof.* This follows from combining (1.8) with the double inequality (1.9). □

*Remark* 1. The upper and lower bounds in (1.9) are better than corresponding bounds in

$$C\left[1-\frac{ab\pi}{\sin(a\pi)}\sum_{n=0}^{\infty}a_n r^{2n+2}\right]<\mu_a(r)+\ln r<C\left[1-\frac{a^2+b^2}{2c}\sum_{n=0}^{\infty}a_n r^{2n+2}\right]$$

obtained in [34, Theorem 2].

The inequality (3.1) gives an elementary and infinite series estimates for $\varphi^a_{1/K}(r)$ and, consequently, the bound of solutions to the Ramanujan generalized modular equations is refined.

*Proof of Theorem 2.* Write $G_1(r)$ as

$$G_1(r)=\frac{\mu_a(r)-\mu(r)}{\mathcal{E}-1}\frac{\mathcal{E}-1}{s^2\ln(4/s)}=g_1(r)g_2(r),$$

where

$$g_1(r)=\frac{\mu_a(r)-\mu(r)}{\mathcal{E}-1}\quad\text{and}\quad g_2(r)=\frac{\mathcal{E}-1}{s^2\ln(4/s)}.$$

Let $g_3(r)=\mu_a(r)-\mu(r)$ and $g_4(r)=\mathcal{E}-1$. By (1.4) and the second item in Lemma 4, we obtain

$$g_1(r)=\frac{g_3(r)}{g_4(r)}\quad\text{and}\quad g_3(1)=g_4(1)=0.$$

Direct computation and utilization of (2.1) and (2.3) result in

$$\frac{g_3'(r)}{g_4'(r)}=\frac{\pi^2}{4}\frac{\mathcal{K}+\mathcal{K}_a}{s\mathcal{K}^2 s\mathcal{K}_a^2}\frac{\mathcal{K}-\mathcal{K}_a}{\mathcal{K}-\mathcal{E}}. \tag{3.2}$$

Hence, from the fourth item in Lemma 3 and the second item in Lemma 6, it follows that the function $g_1(r)$ is strictly increasing on $(0,1)$. Using L'Hôpital's rule together with the fifth item in Lemma 3 and the second item in Lemma 6, the limits $g_1(0)=\frac{R(a)-\ln 16}{\pi-2}$ and $g_1(1^-)=\infty$ follows readily.

By (3.2), the function $G_1(r)$ is a product of two positive and strictly increasing functions, so the monotonicity of $G_1(r)$ follows from the fifth item in Lemma 3. From the fifth item in Lemma 3 and the limit of $g_1(r)$, we gain $G_1(0^+)=\frac{R(a)-\ln 16}{2\ln 4}$ and $G_1(1^-)=\infty$. Moreover, the double inequality (1.11) is obvious.

Let $g_5(r)=B_1-[\mu_a(r)-\mu(r)]$ and $g_6(r)=1-\frac{s^2\operatorname{artanh} r}{r}$. Then $G_2(r)=\frac{g_5(r)}{g_6(r)}$ and $g_5(0)=g_6(0)=0$. By (2.3), simple computation leads to

$$\begin{aligned}\frac{g_5'(r)}{g_6'(r)}&=\frac{\pi^2}{4}\frac{\mathcal{K}^2-\mathcal{K}_a^2}{s^2\mathcal{K}^2\mathcal{K}_a^2}\frac{1}{(1+r^2)(\operatorname{artanh} r)/r-1}\\&=\frac{\pi^2}{4}\frac{\mathcal{K}+\mathcal{K}_a}{(s\mathcal{K}^2)(s\mathcal{K}_a^2)}\frac{\mathcal{K}-\mathcal{K}_a}{(1+r^2)(\operatorname{artanh} r)/r-1}.\end{aligned}$$

Hence, by Lemma 1, the monotonicity of $G_2(r)$ follows from the fourth item in Lemma 3 and the third item in Lemma 6.

Clearly, the limit $G_2(1^-)=\frac{R(a)-\ln 16}{2}$ is valid. By L'Hôpital's rule and the third item in Lemma 6, we readily obtain

$$\lim_{r\to 0}G_2(r)=\lim_{r\to 0}\frac{g_5'(r)}{g_6'(r)}=\frac{3(1-2a)^2}{8}.$$

By the monotonicity of $G_2(r)$, the double inequality (1.12) follows immediately.

By the formula (1.11) in [28, Theorem 1], we have

$$\exp\left(\mu(r)+\ln r\right)=\prod_{n=0}^{\infty}(1+r_n)^{2^{-n}}=P(r). \tag{3.3}$$

Consequently, the third item in Theorem 2 follows from (1.11) and (1.12). The proof of Theorem 2 is complete. □

**Corollary 2.** *For $r \in (0, 1)$ and $K \in (1, \infty)$, the inequality*

$$\varphi^a_{1/K}(r) > \left[\max\{B_4{}^{A(r)}, B_5{}^{B(r)}\}\prod_{n=0}^{\infty}(1 + r_n)^{1/2^n}\right]\frac{r^K}{e^K} \tag{3.4}$$

*holds true, where* $A(r) = \frac{s^2 \operatorname{artanh} r}{r}$ *and* $B(r) = s^2 \ln \frac{4}{s}$.

*Proof.* This follows from combining the double inequality (1.8), the equality (3.3), and the inequality (1.13). □

*Remark* 2. The lower bound in (1.11) is better than the corresponding bound in the equation (11) in [24, Theorem 1] which is referenced in item (2) of Lemma 4.

The upper and lower bounds in (1.12) are better than corresponding bounds in the equation (11) in [24, Theorem 1] which is referenced in item (2) of Lemma 4.

The inequality (3.4) provides an elementary and an infinite product estimates for $\varphi^a_{1/K}(r)$ and a new bound of solutions to the Ramanujan generalized modular equations is given.

*Proof of Theorem 3.* It is easy to see that the function $H(r)$ can be written as

$$H(r) = \frac{\mu_a(r) - \mu(r)}{s^2 \ln(4/s)} + \frac{\mu(r) - \operatorname{artanh}\sqrt{s}}{s^2 \ln(4/s)} = G_1(r) + H_1(r), \tag{3.5}$$

where $G_1(r)$ is defined by (1.10) and

$$H_1(r) = \frac{\mu(r) - \operatorname{artanh}\sqrt{s}}{s^2 \ln(4/s)} \tag{3.6}$$

which can be equivalently written as the product of two functions

$$H_1(r) = \frac{\mu(r) - \operatorname{artanh}\sqrt{s}}{\mathcal{E} - 1}\frac{\mathcal{E}(r) - 1}{s^2 \ln(4/s)}. \tag{3.7}$$

Denote

$$h_1(r) = \frac{\mu(r) - \operatorname{artanh}\sqrt{s}}{\mathcal{E} - 1} = \frac{h_2(r)}{h_3(r)},$$

where $h_2(r) = \mu(r) - \operatorname{artanh}\sqrt{s}$ and $h_3(r) = \mathcal{E} - 1$. By the third item in Lemma 4 and (1.4), we obtain $h_2(1^-) = h_3(1^-) = 0$. Applying (2.1) and (2.3) and simply computing yield

$$\frac{h_2'(r)}{h_3'(r)} = \frac{1}{2}\frac{\frac{\pi^2}{2} - \sqrt{s}(1 + s)\mathcal{K}^2}{s^2\mathcal{K}^2(\mathcal{K} - \mathcal{E})} = \frac{1}{2}\frac{\frac{\pi^2}{2} - \sqrt{s}(1 + s)\mathcal{K}^2}{r^2}\frac{1}{s\mathcal{K}^2(r)}\frac{r^2}{s(\mathcal{K} - \mathcal{E})}.$$

Let

$$h_4(r) = \frac{\frac{\pi^2}{2} - \sqrt{s}(1 + s)\mathcal{K}^2(r)}{r^2} \tag{3.8}$$

and $s = \sqrt{1 - r^2}$. Using the substitution

$$r = \frac{2\sqrt{u}}{1 + u} \quad \text{and} \quad u = \frac{2\sqrt{t}}{1 + t}. \tag{3.9}$$

Then $u = \frac{1 - \sqrt{t}}{1 + \sqrt{t}}$. By Landen's transformation formula

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1 + r}\right) = (1 + r)\mathcal{K}(r)$$

in [30] and (3.9), we have

$$\mathcal{K}(r) = (1 + u)\mathcal{K}(u) = (1 + u)(1 + t)\mathcal{K}(t). \tag{3.10}$$

By (3.10), the identity (3.8) is equivalent to

$$h_4(r) = \frac{\left(t + \sqrt{t}\right)^4}{4(1 + t)\sqrt{t}}\frac{(\pi/2)^2 - [t'\mathcal{K}(t)]^2}{t^2}. \tag{3.11}$$

It is easy to show that the first factor in the right hand side of (3.11) is strictly increasing in $t$ on $(0, 1)$. Hence, by virtue of the third item in Lemma 3 and the relation between $r$ and $t$, the function $h_4(r)$ is strictly increasing on $(0, 1)$.

It was given in [2, Theorem 15] that the function $r \to \frac{s(\mathcal{K}-\mathcal{E})}{r^2}$ is strictly decreasing from $(0, 1)$ onto $\left(0, \frac{\pi}{4}\right)$. Therefore, by (3.7) and Lemma 1, the function $h_1(r)$ is positive and strictly increasing.

From the fifth item in Lemma 3 and (3.6), we conclude that the function $H_1(r)$ is strictly increasing on $(0, 1)$. Hence, the monotonicity of $H(r)$ follows from the first item in Theorem 2 and (3.5).

It is clear that the limits $H_1(0^+) = \frac{1}{4}$ and $H_1(1^-) = \infty$ follow from item (5) in Lemma 3, item (1) in Lemma 4, and item (1) in Theorem 2. Additionally we note that $H(0^+) = G_1(o^+) + H_1(o^+) = \frac{R(a)-\ln 16}{2\ln 4}$. The double inequality (1.14) follows immediately. The proof of Theorem 3 is complete. □

*Remark* 3. The lower bound in (1.14) is better than corresponding bounds in the equation (14) in [24, Theorem 2] which is presented in item (3) of Lemma 4.

**Acknowledgements** This research was partially supported by the Natural Science Foundation of China (Grant No. 11171307 and 11401041), the Natural Science Foundation of the Educational Department of Zhejiang (Grant No. Y201635387 and Y201840023), the Science and Technology Foundation of Shandong (Grant No. J16li52), and Zhejiang Higher Education Visiting Scholar Foundation (Grant No. FX2018093).

The authors appreciate anonymous referees for their careful corrections and valuable comments on the original version of this paper.

**Authors' Contributions** All authors contributed equally to the manuscript and read and approved the final manuscript.

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