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MONOTONICITY PROPERTIES OF THE POWER FUNCTIONS OF LIKELIHOOD RATIO TESTS FOR NORMAL MEAN HYPOTHESES CONSTRAINED BY A LINEAR SPACE AND A CONE[1]

BY XIAOMI HU AND F. T. WRIGHT

University of South Carolina and University of Missouri–Columbia

Anderson studied the monotonicity of the integral of a symmetric, unimodal density over translates of a symmetric convex set. Restricting attention to elliptically contoured, unimodal densities, Mukerjee, Robertson and Wright weakened the assumption of symmetry on the set and obtained monotonicity properties of power functions, including unbiasedness, for some likelihood ratio tests in order restricted inference for the variance-known case. For elliptically contoured, unimodal densities, we weaken the assumption of convexity to obtain similar results in the case of unknown variances. The results apply to situations in which the null hypothesis is a linear space and the alternative is a closed, convex cone.

1. Introduction. The study of the monotonicity of an integral, with a fixed integrand, over translates of a fixed set is motivated by one of the unresolved questions in order restricted statistical inference. The likelihood ratio test (LRT) for homogeneity of the components of a normal mean vector with the alternative restricted by quasiordering is one of the most extensively studied problems in order restricted inference. However, questions about the behavior of the power function of the LRT for the case in which the variances are unknown have not been answered in general. If the quasiordering is a simple ordering $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$, then the test has been proved to be unbiased, and if it is the simple tree ordering $\mu_1 \leq \mu_j$ for $j = 2, \ldots, k$, numerical calculations suggest that the test is unbiased. Robertson, Wright and Dykstra (1988) summarize these results and ask for "new techniques" to establish the unbiasedness of the LRT's in this case. We extend the monotonicity results of Mukerjee, Robertson and Wright (1986) to some nonconvex sets. Also, we apply these new results to show that the LRT is unbiased for the case of unknown variances and hypotheses more general than homogeneity with the alternative restricted by quasiordering.

Cohen, Kemperman and Sackrowitz (1993) study tests of $B\mu = 0$ with the alternative restricted by $B\mu \geq 0$, where μ is k-dimensional, B is a $(k - m)$-by-k matrix with rank $k - m$. They give sufficient conditions for a test to be unbiased, consider complete classes and show that the LRT is unbiased if $(BB')^{-1}$ has only nonnegative elements. Their results concerning the unbiasedness of the

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LRT include several interesting cases in order restricted inference but, as they point out, for independent random samples with unknown variance, they do not include the simple tree ordering or the umbrella ordering, that is, $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_h \geq \mu_{h+1} \geq \cdots \geq \mu_k$.

In this paper, independent random samples from k normal distributions with variances known or known up to a multiplicative constant as well as the analogous multivariate settings are considered. With L a linear space, C a closed, convex cone, $L \subset C$, H_0: $\mu \in L$ and H_1: $\mu \in C - L$, it is shown that the power functions of the LRT's of H_0 versus H_1 are nondecreasing along each line segment which starts at a point in L and continues in a direction in C. Hence, because the power functions are constant on L, the tests are unbiased. The setting considered here, which is quite general, does not require C to be polyhedral. While it includes testing homogeneity with the alternative restricted by an arbitrary quasiordering, it also includes testing $\mu = 0$ with the alternative restricted by a circular cone [see Pincus (1975)]. A set of means $\{\mu_{ij};\ 1 \leq i \leq r,\ 1 \leq j \leq c\}$ satisfies the matrix ordering if $\mu_{ij} \leq \mu_{kl}$ for $1 \leq i \leq k \leq r$ and $1 \leq j \leq l \leq c$. If one expresses this order restriction as $B\mu \geq 0$ in the natural way with $r = 2$ and $c = 3$, B has too many rows to apply the results in Cohen, Kemperman and Sackrowitz (1993), but the matrix order is a quasiorder.

Suppose g is unimodal and elliptically contoured and let

$$f(\mu) = \int_{A - \mu} g(x)\,dx,$$

In Section 2, we prove that $f(\mu_0 + t\nu_0)$ is nonincreasing in $t \geq 0$ provided that A, μ_0 and ν_0 satisfy a suitable condition; see (2.5). This generalizes an earlier result of Mukerjee, Robertson and Wright (1986), which is a partial generalization of the inequality due to Anderson (1955). Anderson's work does not require an elliptically contoured g. In Section 3, we apply our result to sets A defined in terms of projections onto a linear space L and a convex cone $C \supset L$; see Theorem 2. Such sets A appear as acceptance regions of the LRT's of $\mu \in L$ versus $\mu \in C - L$, where μ is a normal mean vector and the underlying covariance matrix is known up to a scale factor. As a consequence, it is shown in Section 4 that such LRT's are unbiased.

2. The monotonicity property of integrals. Following Das Gupta, Eaton, Olkin, Perlman, Savage and Sobel (1972), the function $g(x)$: $R^k \to R$ is called unimodal and elliptically contoured if $g(x) = h(x'Wx)$ with $h(\cdot)$ a nonincreasing function on $[0, \infty)$ and W a positive definite matrix. For any x and y in R^k, $x'Wy$ defines an inner product. This inner product and the corresponding norm are denoted by $\langle \cdot, \cdot \rangle_W$ and $\|\cdot\|_W$. In this section we consider the behavior of the integral

$$f(\mu) = \int_{A - \mu} g(x)\,dx \tag{2.1}$$

as μ changes, assuming that $g(x)$ is unimodal and elliptically contoured and $f(\mu)$ is differentiable. The directional derivative of $f(\mu)$ in the direction of ν_0 at μ_0 exists when $f(\mu)$ is differentiable. Considering this function on the line $\mu = \mu_0 + t\nu_0$, we have a function of t, $f(\mu_0 + t\nu_0)$. By definition, the directional derivative at μ_0 can be expressed as

$$\frac{1}{\sqrt{\nu_0'\nu_0}}\frac{d}{dt}f(\mu_0 + t\nu_0)\bigg|_{t=0} \tag{2.2}$$

LEMMA 1. *Suppose that $\mu_0, \nu_0 \in R^k$ with $\nu_0 \neq 0, h(\cdot)$ is a nonincreasing function on $[0,\infty)$, W is a k-by-k positive definite matrix and A is a set such that*

$$f(\mu) = \int_{A-\mu} h\left(\|x\|_W^2\right)dx \tag{2.3}$$

is a differentiable function. If

$$\begin{gathered} x \in A \text{ and } \langle x - \mu_0, \nu_0\rangle_W \geq 0 \text{ imply that} \\ y = x - 2\langle x - \mu_0, \nu_0\rangle_W \langle \nu_0, \nu_0\rangle_W^{-1}\nu_0 \in A, \end{gathered} \tag{2.4}$$

then the directional derivative of $f(\mu)$ in the direction of ν_0 at μ_0 is nonpositive.

PROOF. Let

$$D = \{x \in A: \langle x - \mu_0, \nu_0\rangle_W \geq 0\}, \qquad E = \{y = x - 2\langle x - \mu_0, \nu_0\rangle_W\langle\nu_0,\nu_0\rangle_W^{-1}\nu_0 : x \in D\}$$

and $F = A \cap (D \cup E)^c$. Since $D \subset A, E \subset A$ and $D \cap E$ has Lebesque measure zero,

$$f(\mu) = \int_{A-\mu} h\left(\|x\|_W^2\right)dx = \int_{D-\mu} + \int_{E-\mu} + \int_{F-\mu} h\left(\|x\|_W^2\right)dx.$$

The directional derivative of $f(\mu)$ in the direction of ν_0 at μ_0 is given by (2.2) with

$$f(\mu_0 + t\nu_0) = \int_D + \int_E + \int_F h\left(\|x - \mu_0 - t\nu_0\|_W^2\right)dx.$$

We establish the following claims:

(i) $\int_D + \int_E h(\|x - \mu_0 - t\nu_0\|_W^2)\,dx$ is a symmetric function of t.

(ii) With $r(t) = \int_F h(\|x - \mu_0 - t\nu_0\|_W^2)\,dx, r(t) \leq r(-t)$ for $t \geq 0$.

For the first claim, we make a linear transformation

$$\int_E h\left(\|y - \mu_0 - t\nu_0\|_W^2\right)dy = \int_D h\left(\|y(x) - \mu_0 - t\nu_0\|_W^2\right)\operatorname{abs}\left(|J|\right)dx,$$

where $y(x) = x - 2\langle x - \mu_0, \nu_0\rangle_W\langle\nu_0,\nu_0\rangle_W^{-1}\nu_0$ and $J = I - 2\nu_0(W\nu_0)'(\nu_0'W\nu_0)^{-1}$. However, $\operatorname{abs}(|J|) = 1$ since $J^2 = I$. It is straightforward to show that $\|y(x) - \mu_0 - t\nu_0\|_W^2 = \|x - \mu_0 + t\nu_0\|_W^2$. Thus,

$$\int_E h\left(\|x - \mu_0 - t\nu_0\|_W^2\right)dx = \int_D h\left(\|x - \mu_0 + t\nu_0\|_W^2\right)dx$$

and the first claim follows immediately. For the second claim, we note that if $x \in F$, then $\|x - \mu_0 - t\nu_0\|_W^2 = \|x - \mu_0\|_W^2 + t^2\|\nu_0\|_W^2 - 2t\langle x - \mu_0, \nu_0\rangle_W$ with $2t\langle x - \mu_0, \nu_0\rangle_W < 0$ for any $t > 0$. The claim follows because $h(\cdot)$ is nonincreasing. The lemma is proved by noting that $\|\nu_0\|_W$ times (2.2) is

$$\lim_{t \to 0+} \frac{f(t) - f(-t)}{2t} = \lim_{t \to 0+} \frac{r(t) - r(-t)}{2t} \leq 0. \qquad \square$$

Suppose μ_0 is a point in R^k that satisfies the conditions in Lemma 1. If all points on the line segment starting at μ_0 in the direction of ν_0 satisfy the same conditions, then we can establish the monotonicity property for $f(\mu)$ on this line. This thought leads to Lemma 2 that gives a sufficient condition for this desired result.

LEMMA 2. *Suppose that $\mu_0, \nu_0 \in R^k$ with $\nu_0 \neq 0$; $A \subset R^k$; and W is a positive definite matrix. If*

$$\begin{aligned} &x \in A \textit{ and } \langle x - \mu_0, \nu_0\rangle_W \geq 0 \textit{ imply that} \\ &y(s) = x - 2s\langle x - \mu_0, \nu_0\rangle_W \langle \nu_0, \nu_0\rangle_W^{-1}\nu_0 \in A \textit{ for any } s \textit{ in } (0, 1], \end{aligned} \tag{2.5}$$

then (2.4) *holds with μ_0 replaced by $\mu_1 = \mu_0 + t\nu_0$ for any $t \geq 0$.*

PROOF. Suppose that $\mu_1 = \mu_0 + t\nu_0, t \geq 0$; $x \in A$; and $\langle x - \mu_1, \nu_0\rangle_W \geq 0$. Since s can be 1, there is nothing to be proved for $t = 0$. If $\langle x - \mu_1, \nu_0\rangle_W = 0$, then the desired conclusion follows trivially. So we assume that $t > 0$ and $\langle x - \mu_1, \nu_0\rangle_W > 0$, which imply that $t\langle \nu_0, \nu_0\rangle_W \langle x - \mu_0, \nu_0\rangle_W^{-1} \in (0, 1)$. Then $s = 1 - t\langle \nu_0, \nu_0\rangle_W \langle x - \mu_0, \nu_0\rangle_W^{-1} \in (0, 1)$, but with this s,

$$x - 2s\langle x - \mu_0, \nu_0\rangle_W \langle \nu_0, \nu_0\rangle_W^{-1}\nu_0 = x - 2\langle x - \mu_1, \nu_0\rangle_W \langle \nu_0, \nu_0\rangle_W^{-1}\nu_0,$$

and hence Lemma 2 is proved. $\square$

REMARK. Let $c = c(\mu_0, \nu_0)$ be the point on the line $x + t\nu_0, t \in R$, which is closest to μ_0. The point y in (2.4) is the reflection of x relative to c, and $y(s)$ in (2.5) with $0 < s < 1$ are the points between x and y.

The conditions imposed in Lemma 2 are quite general. It is clear that these conditions hold if the condition of Lemma 1 is true and A is convex, or only convex in the direction of ν_0, that is, $\alpha x + (1 - \alpha)y \in A$ for any $\alpha \in [0, 1]$ and $x, y \in A$ with $y - x = b\nu_0$ and $b \in R$. The next theorem establishes the monotonicity property of the integral of a unimodal, elliptically contoured function over translates of set A. The proof, which is a direct application of Lemmas 1 and 2, is omitted.

THEOREM 1. *Suppose that $\mu_0, \nu_0 \in R^k$ with $\nu_0 \neq 0$; $h(\cdot)$ is a nonincreasing function on $[0, \infty)$; W is a k-by-k positive definite matrix; and A is a set such that $f(\mu)$ given by* (2.3) *is a differentiable function. If* (2.5) *holds, then $f(\mu)$ is nonincreasing on the line segment $\mu = \mu_0 + t\nu_0$ for $t \geq 0$ as t increases.*

3. The monotonicity of probabilities. The projection of a vector x onto a set D is defined as a vector which minimizes $\|x-y\|_W$ over $y \in D$ and is denoted by $E_w(x \mid D)$. For a closed, convex cone $C, E_W(x \mid C)$ exists and is unique. Furthermore, $E_W(x \mid C) = x^*$ if and only if $x^* \in C$, $\langle x - x^*, x^* \rangle_W = 0$ and $\langle x - x^*, y^* \rangle_W \leq 0$ for each $y \in C$. For a linear space $L, L^\perp$ denotes its orthogonal complement. The polar cone associated with C is the closed, convex cone $C^p = \{x \in R^k : \langle x, y \rangle_W \leq 0$, for each $y \in C\}$. For any $x \in R^k, x = E_W(x \mid C) + E_W(x \mid C^p)$ and $\langle E_W(x \mid C^p), E_W(x \mid C) \rangle_W = 0$.

In this section we consider applications of Theorem 1 to probabilities.

LEMMA 3. *Suppose L is a linear space, C is a closed, convex cone, $L \subset C$ and*

$$A = \left\{x \in R^k : \left\|E_W(x \mid C) - E_W(x \mid L)\right\|_W^2 \leq a + b\left\|x - E_W(x \mid C)\right\|_W^2\right\},$$

where $a > 0$ and $b > 0$. If $\mu_0 \in R^k, \nu_0 \in C \cap L^\perp, \nu_0 \neq 0$ and $\langle \mu_0, \nu_0 \rangle_W \geq 0$, then (2.5) *holds.*

PROOF. Suppose that $\langle \mu_0, \nu_0 \rangle_W \geq 0, \nu_0 \in C \cap L^\perp, \nu_0 \neq 0$, $x \in A$ and $\langle x - \mu_0, \nu_0 \rangle_W \geq 0$. Fix s in $(0, 1]$ and let y denote the vector $y(s) = x - 2s\langle x - \mu_0, \nu_0 \rangle_W \langle \nu_0, \nu_0 \rangle_W^{-1} \nu_0$. We need to show that $y \in A$. First, we establish the following claims: (i) $\|y\|_W^2 \leq \|x\|_W^2$; (ii) $\|E_W(y \mid (C \cap L^\perp)^p)\|_W^2 \geq \|E_W(x \mid (C \cap L^\perp)^p)\|_W^2$; and (iii) $\|E_W(y \mid C^p)\|_W^2 \geq \|E_W(x \mid C^p)\|_W^2$. Note that

$$\|y\|_W^2 = \|x\|_W^2 - 4s\langle x - \mu_0, \nu_0 \rangle_W \langle \nu_0, \nu_0 \rangle_W^{-1} \left[(1-s)\langle x - \mu_0, \nu_0 \rangle_W + \langle \mu_0, \nu_0 \rangle_W\right]$$

and that $\langle x - \mu_0, \nu_0 \rangle_W \geq 0, \langle \mu_0, \nu_0 \rangle_W \geq 0$, $s > 0$ and $1 - s \geq 0$. Thus, the first claim is proved. Claims (ii) and (iii) follow from the fact that $2s\langle x - \mu_0, \nu_0 \rangle_W \langle \nu_0, \nu_0 \rangle_W^{-1} \nu_0 \in C \cap L^\perp$ and Lemma 2.2 in Mukerjee, Robertson and Wright (1986). From the proof of Lemma 3.3 and the first corollary to Theorem 3.6 of Raubertas, Lee and Nordheim (1986) and from our first two claims, we have that

$$\begin{aligned}
\left\|E_W(y \mid C) - E_W(y \mid L)\right\|_W^2 &= \left\|E_W\left(y \,\middle|\, C \cap L^\perp\right)\right\|_W^2 \\
&= \|y\|_W^2 - \left\|E_W\left(y \,\middle|\, (C \cap L^\perp)^p\right)\right\|_W^2 \\
&\leq \|x\|_W^2 - \left\|E_W\left(y \,\middle|\, (C \cap L^\perp)^p\right)\right\|_W^2 \\
&\leq \|x\|_W^2 - \left\|E_W\left(x \,\middle|\, (C \cap L^\perp)^p\right)\right\|_W^2 \\
&= \left\|E_W(x \mid C) - E_W(x \mid L)\right\|_W^2.
\end{aligned}$$

By the definition of A and the third claim, we further have that

$$\begin{aligned}
\left\|E_W(y \mid C) - E_W(y \mid L)\right\|_W^2 &\leq a + b\left\|x - E_W(x \mid C)\right\|_W^2 = a + b\left\|E_W(x \mid C^p)\right\|_W^2 \\
&\leq a + b\left\|E_W(y \mid C^p)\right\|_W^2 = a + b\left\|y - E_W(y \mid C)\right\|_W^2.
\end{aligned}$$

Thus, $y \in A$ and the proof is complete. □

In the next lemma we consider the probability $P(X + \mu \in A)$, where X has a unimodal, elliptically contoured density and A is defined as in Lemma 3. Of course, if X has multivariate normal distribution, then it has such a density.

LEMMA 4. *Suppose X has a unimodal, elliptically contoured density and A is the set defined in Lemma 3. Then the following conclusions hold.*

(i) *The probability $P(X + \mu \in A)$, as a function of μ, is nonincreasing on the line segment $\mu = \mu_0 + t\nu_0$ for $t \geq 0$ as t increases, provided $\mu_0 \in L$ and $\nu_0 \in C \cap L^{\perp}$.*

(ii) *For any $\nu \in L, P(X + \mu \in A) = P(X + \mu \pm \nu \in A)$.*

PROOF. The first conclusion follows immediately from Lemma 3 and Theorem 1.

Because L is a linear space, it is well known that, for $\nu \in L$, $E_W(x \pm \nu \mid C) = E_W(x \mid C) \pm \nu$ and $E_W(x \pm \nu \mid L) = E_W(x \mid L) \pm \nu$. Thus,

$$\begin{aligned}
P(X + \mu \in A) &= P\Big(\big\|E_W(X + \mu \mid C) - E_W(X + \mu \mid L)\big\|_W^2 \\
&\qquad \leq a + b\big\|X + \mu - E_W(X + \mu \mid C)\big\|_W^2\Big) \\
&= P\Big(\big\|E_W(X + \mu \pm \nu \mid C) - E_W(X + \mu \pm \nu \mid L)\big\|_W^2 \\
&\qquad \leq a + b\big\|X + \mu \pm \nu - E_W(X + \mu \pm \nu \mid C)\big\|_W^2\Big) \\
&= P(X + \mu \pm \nu \in A).
\end{aligned}$$

The proof is complete. □

We now present the main result about the monotonicity of probabilities.

THEOREM 2. *Suppose that X has unimodal, elliptically contoured density $g(x) = h(x'Wx)$ with $h(\cdot)$ nonincreasing, L is a linear space, C is a closed, convex cone, $L \subset C$ and*

$$A = \Big\{x \in R^k \colon \big\|E_W(x \mid C) - E_W(x \mid L)\big\|_W^2 \leq a + b\big\|x - E_W(x \mid C)\big\|_W^2\Big\},$$

where $a > 0$ and $b > 0$. If $\mu_0 \in L$ and $\nu_0 \in C$, then the probability $P(X + \mu \in A)$, as a function of μ, is nonincreasing in t on the line segment $\mu = \mu_0 + t\nu_0$ with $t \geq 0$.

PROOF. Because $\mu_0 + tE_W(\nu_0 \mid L) \in L$, we may apply Lemma 4(ii) to obtain

$$\begin{aligned}
P(X + \mu \in A) &= P(X + \mu_0 + t\nu_0 \in A) \\
&= P\Big(X + \mu_0 + tE_W(\nu_0 \mid L) + tE_W(\nu_0 \mid L^{\perp}) \in A\Big) \\
&= P\Big(X + tE_W(\nu_0 \mid L^{\perp}) \in A\Big).
\end{aligned}$$

Since $E_W(\nu_0 \mid L^{\perp}) = \nu_0 - E_W(\nu_0 \mid L)$, $\nu_0 \in C$, $-E_W(\nu_0 \mid L) \in L \subset C$ and C is closed under addition, $E_W(\nu_0 \mid L^{\perp}) \in C \cap L^{\perp}$ and by Lemma 4(i) this is a nonincreasing function of t on $t \geq 0$, that is, the probability $P(X + \mu \in A)$, as a function of μ, is nonincreasing on the line segment $\mu = \mu_0 + t\nu_0$ with $t \geq 0$. The theorem is proved. □

REMARK. Define A as $\{x \in R^k: \|E_W(x \mid C) - E_W(x \mid L)\|_W^2 < a\}$. Then Lemma 3, Lemma 4 and Theorem 2 also are true with this A. Actually, the condition of Lemma 1 is satisfied and A is convex. This is the case studied in Mukerjee, Robertson and Wright (1986).

4. The monotonicity of power functions. Suppose μ is a vector of normal means, L is a linear space, C is a closed, convex cone and $L \subset C$. Let H_0: $\mu \in L$ and H_1: $\mu \in C - L$, and consider the LRT of H_0 versus H_1. We consider the following two cases.

CASE 1 (Independent random samples). Suppose Y_{ij}, for $j = 1, 2, \ldots, n_i$ and $i = 1, 2, \ldots, k$, are independent and $Y_{ij} \sim N(\mu_i, \sigma_i^2)$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)'$ and $\overline{Y} = (\overline{Y}_1, \overline{Y}_2, \ldots, \overline{Y}_k)'$ with $\overline{Y}_i$ the mean of the ith random sample and $\Sigma = \operatorname{diag}(\sigma_1^2/n_1, \sigma_2^2/n_2, \ldots, \sigma_k^2/n_k)$.

CASE 2 (A multivariate random sample). Suppose $Y_j = (Y_{1j}, Y_{2j}, \ldots, Y_{kj})'$, for $j = 1, 2, \ldots, n$, are independent and identically distributed k-dimensional normal random vectors with mean $\mu = (\mu_1, \mu_2, \ldots, \mu_k)'$ and covariance matrix V. Then $\overline{Y} = (\overline{Y}_1, \overline{Y}_2, \ldots, \overline{Y}_k)'$, with $\overline{Y}_i$ the mean of Y_{ij} for $j = 1, 2, \ldots, n$, has covariance $\Sigma = V/n$.

In both cases, set $W = \Sigma^{-1}$. When W is known, by a standard argument one sees that the LRT rejects the null hypothesis for large values of $\|E_W(\overline{Y} \mid C) - E_W(\overline{Y} \mid L)\|_W^2$. If A is the acceptance region for the LRT, then from the remark following Theorem 2 we have that the probability of A is nonincreasing on each line segment starting at a point in L and continuing in the direction of a vector in C. Thus, the power is nondecreasing on this line segment. Because the power is a constant on L, the test is unbiased.

Suppose that W is unknown. For independent random samples, that is, Case 1, one commonly assumes that $\sigma_i^2 = a_i\sigma^2$ with a_i known and σ^2 unknown. Thus with $u_i = n_i/a_i$, for $i = 1, 2, \ldots, k$, and $U = \operatorname{diag}(u_1, u_2, \ldots, u_k)$, $\Sigma = \sigma^2 U^{-1}$ and $W = U/\sigma^2$. In Case 2, it is commonly assumed that $V = \sigma^2\Sigma_0$, where Σ_0 is known and σ^2 is unknown. Then $W = U/\sigma^2$, where $U = n\Sigma_0^{-1}$. In both cases, U is known and σ^2 is unknown. For Cases 1 and 2, respectively, define

$$R = \sum_{i=1}^{k} \frac{u_i}{n_i} \sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_i)^2 \quad \text{and} \quad R = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \overline{Y})' U (Y_i - \overline{Y}),$$

and note that the degrees of freedom associated with R are $\nu = n_1 + n_2 + \cdots + n_k - k$ and $\nu = nk - k$, respectively. It can be shown that the LRT rejects the null hypothesis for large values of

$$T = \frac{\|E_U(\overline{Y}\,|\,C) - E_U(\overline{Y}\,|\,L)\|_U^2}{R + \|\overline{Y} - E_U(\overline{Y}\,|\,C)\|_U^2}.$$

Thus, the LRT accepts H_0 with probability

$$P\Big(\|E_W(\overline{Y}\,|\,C) - E_W(\overline{Y}\,|\,L)\|_W^2 \le d\frac{R}{\sigma^2} + d\|\overline{Y} - E_W(\overline{Y}\,|\,C)\|_W^2\Big),$$

where d is a positive constant, $\overline{Y}$ and R/σ^2 are independent, $\overline{Y} \sim N(\mu, W^{-1})$ and $R/\sigma^2 \sim \chi_\nu^2$. By Theorem 2, conditioned on R, this probability is nonincreasing on each line segment starting at a point in L and continuing in the direction of a vector in C. Consequently, the probability of acceptance is nonincreasing on this line and the power is nondecreasing on this line. By Lemma 4(ii), the power is constant on L and thus the LRT is unbiased.

The following theorem has been proved.

THEOREM 3. *Suppose that $\overline{Y} \sim N(\mu, \sigma^2 U^{-1})$ with U known; $R/\sigma^2 \sim \chi_\nu^2$; and $\overline{Y}$ and R are independent. Suppose that L is a linear space, C is a closed, convex cone, $L \subset C$, H_0: $\mu \in L$ and H_1: $\mu \in C - L$. Whether σ^2 is known or not, the power function of the LRT of H_0 versus H_1 is nondecreasing on each line segment starting at a point in L and continuing in the direction of a vector of C. Furthermore, these tests are unbiased.*

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DEPARTMENT OF STATISTICS
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, SOUTH CAROLINA 29208

DEPARTMENT OF STATISTICS
UNIVERSITY OF MISSOURI–COLUMBIA
COLUMBIA, MISSOURI 65211