

Monotonicity Results for Coherent MIMO Rician Channels

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Abstract

The dependence of the Gaussian input information rate on the line-of-sight (LOS) matrix in multiple-input multiple-output coherent Rician fading channels is explored. It is proved that the outage probability and the mutual information induced by a multivariate circularly symmetric Gaussian input with any covariance matrix are monotonic in the LOS matrix D , or more precisely, monotonic in $D^\dagger D$ in the sense of the Loewner partial order. Conversely, it is also demonstrated that this ordering on the LOS matrices is a necessary condition for the uniform monotonicity over all input covariance matrices. This result is subsequently applied to prove the monotonicity of the isotropic Gaussian input information rate and channel capacity in the singular values of the LOS matrix. Extensions to multiple-access channels are also discussed.

1 Introduction and Main Result

It is well known that the capacity of a single-input single-output coherent Rician fading channel is monotonic in the magnitude of the line-of-sight (LOS) component. This can be easily deduced from the facts that the channel capacity is achieved by a zero-mean circularly-symmetric Gaussian input and that a non-central chi-square random variable is stochastically monotonic in the non-centrality parameter [1, Lemma 6.2 (b)], [2]. This result extends easily to the single-input multiple-output and, with a little more work, to multiple-input single-output scenarios, from the similar stochastic monotonicity for the non-central chi-square random variable of a higher degree.

The extension to the MIMO case, which may look straightforward at first, requires some extra care, however. The first difficulty one encounters is that in order to demonstrate the monotonicity, one has to introduce an ordering on the LOS matrices and it is *a priori* unclear what the natural ordering is for the problem at hand. The second difficulty is

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that there is no closed-form expression for the capacity-achieving input distribution. It is straightforward to demonstrate that the capacity is achieved by a circularly-symmetric multivariate Gaussian input, but no closed-form expressions for the eigenvalues of the optimal covariance matrix are known. Finally, as in the single-input case, under a fixed input distribution, one LOS matrix may give rise to a larger information rate for a given realization than another LOS matrix, but it may actually perform worse when averaged over all fading realizations.

In this paper we show that the natural ordering on the LOS matrices \mathbf{D} is given by the Loewner partial order on $\mathbf{D}^\dagger\mathbf{D}$, and through this ordering we extend the monotonicity results to the MIMO Rician channels. More specifically, we say that the $m \times n$ LOS matrix \mathbf{D} is “larger than or equal to” the $m \times n$ LOS matrix $\tilde{\mathbf{D}}$, if $\mathbf{D}^\dagger\mathbf{D}$ is greater than or equal to $\tilde{\mathbf{D}}^\dagger\tilde{\mathbf{D}}$ in the Loewner sense, i.e., if $\mathbf{D}^\dagger\mathbf{D} - \tilde{\mathbf{D}}^\dagger\tilde{\mathbf{D}}$ is a positive semidefinite $n \times n$ matrix.¹ (Here \mathbf{D}^\dagger is the Hermitian conjugate of \mathbf{D} .) Under this ordering on the LOS matrices, we shall show the monotonicity of channel capacity, the monotonicity of the isotropic Gaussian information rate, and the monotonicity of outage probability.

We shall also extend the discussion to the multiple-access channel (MAC). The MAC poses an additional challenge in that the capacity region depends not only on the LOS matrices of different users individually, but also on how these matrices relate to each other. This requires a joint preorder on LOS matrices, as will be made clear in the next section.

It should be emphasized that our monotonicity results are proved when the distribution of the granular component is held fixed. Consequently, as we vary the LOS matrix the output power is not held fixed. See [3, 4, 5, 6] for studies where the output power is held fixed.

We state our main result, from which the monotonicity results will follow.

Theorem 1.1. *Let \mathbb{H} be a random $m \times n$ matrix whose components are independent, each with a zero-mean unit-variance circularly symmetric complex Gaussian distribution. If two deterministic complex $m \times n$ matrices $\mathbf{D}, \tilde{\mathbf{D}}$ are such that*

$$\mathbf{D}^\dagger\mathbf{D} \succeq \tilde{\mathbf{D}}^\dagger\tilde{\mathbf{D}}$$

then we have

$$\Pr \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \mathbf{D})\mathbf{K}(\mathbb{H} + \mathbf{D})^\dagger \right) \leq t \right] \leq \Pr \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \tilde{\mathbf{D}})\mathbf{K}(\mathbb{H} + \tilde{\mathbf{D}})^\dagger \right) \leq t \right]$$

for any $t \geq 0$ and any positive semidefinite $n \times n$ matrix \mathbf{K} .

In this theorem and throughout, the notation $\mathbf{A} \succeq \mathbf{B}$ indicates that $\mathbf{A} - \mathbf{B}$ is positive semidefinite. The notation \mathbf{I}_m denotes the m -dimensional identity matrix. We use $\mathcal{H}^+(n)$

¹We point out that the Loewner partial order on $\mathbf{D}^\dagger\mathbf{D}$ induces a *preorder* on the LOS matrices \mathbf{D} , for $\mathbf{D}^\dagger\mathbf{D} \succeq \tilde{\mathbf{D}}^\dagger\tilde{\mathbf{D}}$ and $\tilde{\mathbf{D}}^\dagger\tilde{\mathbf{D}} \succeq \mathbf{D}^\dagger\mathbf{D}$ implies $\mathbf{D}^\dagger\mathbf{D} = \tilde{\mathbf{D}}^\dagger\tilde{\mathbf{D}}$, but *not* $\tilde{\mathbf{D}} = \mathbf{D}$. It only implies $\tilde{\mathbf{D}} = \mathbf{U}\mathbf{D}$ for some unitary matrix \mathbf{U} .

to denote the set of all $n \times n$ positive semidefinite Hermitian matrices and use $\mathcal{U}(n)$ for the set of all unitary $n \times n$ matrices. For a complex matrix \mathbf{A} , \mathbf{A}^\top denotes its transpose while \mathbf{A}^\dagger denotes its Hermitian conjugate (i.e., elementwise complex conjugate of \mathbf{A}^\top). We extend the usual notion of diagonality to non-square matrices by saying that any matrix \mathbf{A} is *diagonal* if $A_{ij} = 0$ for all $i \neq j$. All vectors are column vectors unless specified otherwise. All logarithms are natural, i.e., to the base e .

In the following section we shall describe the single-user and the multiple-access Rician fading channels and present the main corollaries of Theorem 1.1. The proof of Theorem 1.1 is given in Section 3.

2 Applications

We introduce two functions that will simplify the notation in our subsequent discussion. In the notation of Theorem 1.1, we define for any $t \geq 0$ and $\mathbf{K} \in \mathcal{H}^+(n)$

$$F(t; \mathbf{K}, \mathbf{D}) \triangleq \Pr \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \mathbf{D})\mathbf{K}(\mathbb{H} + \mathbf{D})^\dagger \right) \leq t \right]$$

and

$$\mathcal{I}(\mathbf{K}, \mathbf{D}) \triangleq \mathbb{E} \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \mathbf{D})\mathbf{K}(\mathbb{H} + \mathbf{D})^\dagger \right) \right].$$

Noting that

$$\mathcal{I}(\mathbf{K}, \mathbf{D}) = \int_0^\infty (1 - F(t; \mathbf{K}, \mathbf{D})) dt \tag{1}$$

we obtain the following corollary of Theorem 1.1.

Corollary 2.1. *If $\mathbf{D}^\dagger \mathbf{D} \succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}}$, then*

$$\mathcal{I}(\mathbf{K}, \mathbf{D}) \geq \mathcal{I}(\mathbf{K}, \tilde{\mathbf{D}}), \quad \forall \mathbf{K} \in \mathcal{H}^+(n).$$

The following converse to Corollary 2.1 also holds, which shows that the preorder on the LOS matrices is natural:

Proposition 2.2. *If $\mathcal{I}(\mathbf{K}, \mathbf{D}) \geq \mathcal{I}(\mathbf{K}, \tilde{\mathbf{D}})$ for all $\mathbf{K} \in \mathcal{H}^+(n)$, then $\mathbf{D}^\dagger \mathbf{D} \succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}}$.*

Proof. See Appendix A. □

We further note the rotational symmetry in $F(t; \mathbf{K}, \mathbf{D})$ and $\mathcal{I}(\mathbf{K}, \mathbf{D})$. First observe that the law of \mathbb{H} is invariant under left and right rotations, i.e., for any $\mathbf{U} \in \mathcal{U}(m)$ and $\mathbf{V} \in \mathcal{U}(n)$,

$$\mathbf{U}\mathbb{H}\mathbf{V}^\dagger \stackrel{\mathcal{L}}{\cong} \mathbb{H}.$$

Consequently, we have for any $\mathbf{U} \in \mathcal{U}(m)$ and $\mathbf{V} \in \mathcal{U}(n)$

$$\begin{aligned}
F(t; \mathbf{K}, \mathbf{U}\mathbf{D}\mathbf{V}^\dagger) &= \Pr \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \mathbf{U}\mathbf{D}\mathbf{V}^\dagger)\mathbf{K}(\mathbb{H} + \mathbf{U}\mathbf{D}\mathbf{V}^\dagger)^\dagger \right) \leq t \right] \\
&= \Pr \left[\log \det \left(\mathbf{I}_m + (\mathbf{U}\mathbb{H}\mathbf{V}^\dagger + \mathbf{U}\mathbf{D}\mathbf{V}^\dagger)\mathbf{K}(\mathbf{U}\mathbb{H}\mathbf{V}^\dagger + \mathbf{U}\mathbf{D}\mathbf{V}^\dagger)^\dagger \right) \leq t \right] \\
&= \Pr \left[\log \det \left(\mathbf{U} \left(\mathbf{I}_m + (\mathbb{H} + \mathbf{D})\mathbf{V}^\dagger\mathbf{K}\mathbf{V}(\mathbb{H} + \mathbf{D})^\dagger \right) \mathbf{U}^\dagger \right) \leq t \right] \\
&= \Pr \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \mathbf{D})\mathbf{V}^\dagger\mathbf{K}\mathbf{V}(\mathbb{H} + \mathbf{D})^\dagger \right) \leq t \right] \\
&= F(t; \mathbf{V}^\dagger\mathbf{K}\mathbf{V}, \mathbf{D}).
\end{aligned} \tag{2}$$

From this and (1), we thus have

$$\mathcal{I}(\mathbf{K}, \mathbf{U}\mathbf{D}\mathbf{V}^\dagger) = \mathcal{I}(\mathbf{V}^\dagger\mathbf{K}\mathbf{V}, \mathbf{D}). \tag{3}$$

2.1 The Single-User Rician Fading Channel

The output (\mathbb{H}, \mathbf{Y}) of the coherent single-user Rician (or Ricean in certain dialects) fading channel consists of a random $m \times n$ matrix \mathbb{H} whose components are independent and identically distributed (IID) according to the zero-mean unit-variance circularly symmetric complex Gaussian distribution $\mathcal{N}_{\mathbb{C}}(0, 1)$, and of a random m -vector $\mathbf{Y} \in \mathbb{C}^m$ given by

$$\mathbf{Y} = (\mathbb{H} + \mathbf{D})\mathbf{x} + \mathbf{Z} \tag{4}$$

where $\mathbf{x} \in \mathbb{C}^n$ is the channel input; \mathbf{D} is a deterministic $m \times n$ complex LOS matrix; and $\mathbf{Z} \in \mathbb{C}^m$ is drawn according to the zero-mean circularly symmetric complex multivariate Gaussian distribution $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \sigma^2 \mathbf{I}_m)$ for some $\sigma^2 > 0$. It is assumed that \mathbb{H} and \mathbf{Z} are independent of each other, and that their joint law does not depend on the channel input \mathbf{x} .

Since the law of \mathbb{H} does not depend on \mathbf{x} , we can express the mutual information between the channel input and output as

$$I(\mathbf{X}; \mathbb{H}, \mathbf{Y}) = I(\mathbf{X}; \mathbf{Y} | \mathbb{H}). \tag{5}$$

Of all input distributions of a given covariance matrix, the zero-mean circularly symmetric multivariate complex Gaussian maximizes the conditional mutual information $I(\mathbf{X}; \mathbf{Y} | \mathbb{H} = \mathbf{H})$, irrespective of the realization $\mathbb{H} = \mathbf{H}$. Consequently, it also maximizes the average mutual information $I(\mathbf{X}; \mathbf{Y} | \mathbb{H})$. We shall therefore consider in this paper zero-mean circularly symmetric Gaussian input distributions $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{K})$ only. focus on the dependence of mutual information on the LOS matrix \mathbf{D} when the input covariance matrix \mathbf{K} is held fixed. Also, since we can absorb the dependence on σ^2 into \mathbf{K} , we assume $\sigma^2 = 1$ without loss of generality.

For a given realization $\mathbb{H} = \mathbf{H}$, we can express the conditional mutual information $I(\mathbf{X}; \mathbf{Y} | \mathbb{H} = \mathbf{H})$ for a $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{K})$ input as

$$I(\mathbf{X}; \mathbf{Y} | \mathbb{H} = \mathbf{H}) = \log \det \left(\mathbf{I}_m + (\mathbf{H} + \mathbf{D})\mathbf{K}(\mathbf{H} + \mathbf{D})^\dagger \right). \tag{6}$$

By taking the expectation with respect to \mathbb{H} , we can express the average conditional mutual information as an explicit function of \mathbf{K} and \mathbf{D} as

$$\begin{aligned} I(\mathbf{X}; \mathbf{Y} | \mathbb{H}) &= \mathbb{E} \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \mathbf{D})\mathbf{K}(\mathbb{H} + \mathbf{D})^\dagger \right) \right] \\ &= \mathcal{I}(\mathbf{K}, \mathbf{D}). \end{aligned}$$

Thus Corollary 2.1 can be interpreted as the monotonicity of the average conditional mutual information of the Rician fading channel (4) with fixed input covariance matrix. We can also give a more direct interpretation of Theorem 1.1 through the notion of *outage probability*. Consider the probability

$$\Pr \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \mathbf{D})\mathbf{K}(\mathbb{H} + \mathbf{D})^\dagger \right) \leq R \right] = F(R; \mathbf{K}, \mathbf{D}).$$

We can interpret this quantity as the probability that the realization \mathbf{H} of \mathbb{H} will be such that the information rate on the Gaussian channel $\mathbf{Y} = (\mathbf{D} + \mathbf{H})\mathbf{x} + \mathbf{Z}$ for the input distribution $\mathcal{N}_{\mathbb{C}}(0, \mathbf{K})$ does not exceed R . Under this interpretation, Theorem 1.1 can be viewed as the monotonicity of the outage probability in the channel LOS matrix.

These monotonicity results can be used to study the power- \mathcal{E} isotropic Gaussian input information rate

$$I^{\text{IG}}(\mathcal{E}, \mathbf{D}) \triangleq \mathcal{I} \left(\frac{\mathcal{E}}{n} \mathbf{I}_n, \mathbf{D} \right)$$

and the capacity $C(\mathcal{E}, \mathbf{D})$ of the Rician channel under the average input power constraint $\mathbb{E}[\mathbf{X}^\dagger \mathbf{X}] \leq \mathcal{E}$:

$$C(\mathcal{E}, \mathbf{D}) \triangleq \max_{\mathbf{K}} \mathcal{I}(\mathbf{K}, \mathbf{D}) \quad (7)$$

where the maximum is taken over the set of all input covariance matrices \mathbf{K} satisfying the trace constraint

$$\text{tr}(\mathbf{K}) \leq \mathcal{E}. \quad (8)$$

It follows immediately from Corollary 2.1 that, if $\mathbf{D}^\dagger \mathbf{D} \succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}}$, then $I^{\text{IG}}(\mathcal{E}, \mathbf{D}) \geq I^{\text{IG}}(\mathcal{E}, \tilde{\mathbf{D}})$ and $C(\mathcal{E}, \mathbf{D}) \geq C(\mathcal{E}, \tilde{\mathbf{D}})$.

Theorem 1.1 can also be used to study the rate- R outage probability corresponding to the isotropic Gaussian input of power- \mathcal{E}

$$P_{\text{out}}^{\text{IG}}(R, \mathcal{E}, \mathbf{D}) \triangleq F \left(R, \frac{\mathcal{E}}{n} \mathbf{I}_n, \mathbf{D} \right)$$

and the optimal power- \mathcal{E} rate- R outage probability $P_{\text{out}}^*(R, \mathcal{E}, \mathbf{D})$, which is the smallest outage probability that can be achieved for the rate R and the average power \mathcal{E} :

$$P_{\text{out}}^*(R, \mathcal{E}, \mathbf{D}) \triangleq \min_{\mathbf{K}} F(R, \mathbf{K}, \mathbf{D}) \quad (9)$$

where the minimum is over all positive semidefinite matrices \mathbf{K} satisfying (8). From Theorem 1.1 we now obtain that $\mathbf{D}^\dagger \mathbf{D} \succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}}$ implies that $P_{\text{out}}^{\text{IG}}(R, \mathcal{E}, \mathbf{D}) \leq P_{\text{out}}^{\text{IG}}(R, \mathcal{E}, \tilde{\mathbf{D}})$ and $P_{\text{out}}^*(R, \mathcal{E}, \mathbf{D}) \leq P_{\text{out}}^*(R, \mathcal{E}, \tilde{\mathbf{D}})$.²

Using the rotational invariance (3), we can strengthen these results by stating them in terms of the singular values of the LOS matrices. Indeed, for any unitary matrix \mathbf{V} , we have $\text{tr}(\mathbf{V}^\dagger \mathbf{K} \mathbf{V}) = \text{tr}(\mathbf{K})$, and hence it follows from (3) that for any $\mathbf{U} \in \mathcal{U}(m)$ and $\mathbf{V} \in \mathcal{U}(n)$

$$I^{\text{IG}}(\mathcal{E}, \mathbf{U} \mathbf{D} \mathbf{V}^\dagger) = I^{\text{IG}}(\mathcal{E}, \mathbf{D})$$

and

$$C(\mathcal{E}, \mathbf{U} \mathbf{D} \mathbf{V}^\dagger) = C(\mathcal{E}, \mathbf{D})$$

i.e., that the isotropic Gaussian input information rate and channel capacity depend on the LOS matrix only via its singular values. By a similar argument, it can be verified that, by (2), both the outage probability corresponding to the isotropic Gaussian input $P_{\text{out}}^{\text{IG}}(R, \mathcal{E}, \mathbf{D})$ and the optimal outage probability $P_{\text{out}}^*(R, \mathcal{E}, \mathbf{D})$ depend on the LOS matrix \mathbf{D} only via its singular values. Consequently, all these quantities are monotonic in the singular values of the LOS matrix:

Corollary 2.3. *Let $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\min\{m,n\}}$ and $\tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \dots \geq \tilde{\sigma}_{\min\{m,n\}}$ be the singular values of the LOS matrices \mathbf{D} and $\tilde{\mathbf{D}}$, respectively. Suppose that $\sigma_i \geq \tilde{\sigma}_i$ for all i . Then*

$$\begin{aligned} I^{\text{IG}}(\mathcal{E}, \mathbf{D}) &\geq I^{\text{IG}}(\mathcal{E}, \tilde{\mathbf{D}}) \\ C(\mathcal{E}, \mathbf{D}) &\geq C(\mathcal{E}, \tilde{\mathbf{D}}) \\ P_{\text{out}}^{\text{IG}}(R, \mathcal{E}, \mathbf{D}) &\leq P_{\text{out}}^{\text{IG}}(R, \mathcal{E}, \tilde{\mathbf{D}}) \end{aligned}$$

and

$$P_{\text{out}}^*(R, \mathcal{E}, \mathbf{D}) \leq P_{\text{out}}^*(R, \mathcal{E}, \tilde{\mathbf{D}}).$$

We can obtain an alternative proof (cf. [7]) of this corollary based on the observation that, if the LOS matrix \mathbf{D} is diagonal, the capacity-achieving covariance matrix \mathbf{K} is also diagonal. (See also [8].) Since this structural theorem on the capacity-achieving input distribution is of independent interest, we restate it here.

Theorem 2.4. *Suppose that $\mathbf{D}^\dagger \mathbf{D}$ has the eigenvalue decomposition $\mathbf{D}^\dagger \mathbf{D} = \mathbf{V} \mathbf{L} \mathbf{V}^\dagger$ for some unitary matrix \mathbf{V} and diagonal matrix \mathbf{L} . Then the capacity-achieving covariance matrix \mathbf{K}_* is given by*

$$\mathbf{K}_* = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^\dagger$$

for some diagonal matrix $\mathbf{\Lambda}$.

²Note that from the definition of power- \mathcal{E} ϵ -outage capacity $C_{\text{out}}^*(\epsilon, \mathcal{E}, \mathbf{D}) \triangleq \sup\{R : P_{\text{out}}^*(R, \mathcal{E}, \mathbf{D}) < \epsilon\}$ we immediately get the monotonicity $C_{\text{out}}^*(\epsilon, \mathcal{E}, \mathbf{D}) \geq C_{\text{out}}^*(\epsilon, \mathcal{E}, \tilde{\mathbf{D}})$ if $\mathbf{D}^\dagger \mathbf{D} \succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}}$. A similar monotonicity holds for $C_{\text{out}}^{\text{IG}}(\epsilon, \mathcal{E}, \mathbf{D}) \triangleq \sup\{R : P_{\text{out}}^{\text{IG}}(R, \mathcal{E}, \mathbf{D}) < \epsilon\}$.

Proof. We show that if \mathbf{D} is diagonal, the capacity-achieving input covariance matrix \mathbf{K}_* is diagonal. The general case follows from (3) and (7).

Fix some $1 \leq j \leq n$. Let $\mathbf{V} \in \mathcal{U}(n)$ be a diagonal matrix with all diagonal entries equal to 1 except the j -th entry, which is -1 . Similarly, let $\mathbf{U} \in \mathcal{U}(m)$ be diagonal with all diagonal entries equal to 1 except for the j -th entry being -1 . (In case $j > m$, $\mathbf{U} = \mathbf{I}_m$.) Since \mathbf{D} is diagonal, we have

$$\mathbf{U}\mathbf{D}\mathbf{V}^\dagger = \mathbf{D}. \quad (10)$$

Let $\tilde{\mathbf{K}} = \mathbf{V}^\dagger \mathbf{K} \mathbf{V}$. From (10) and the rotational invariance (3), we have

$$\begin{aligned} \mathcal{I}(\tilde{\mathbf{K}}, \mathbf{D}) &= \mathcal{I}(\mathbf{V}^\dagger \mathbf{K} \mathbf{V}, \mathbf{D}) \\ &= \mathcal{I}(\mathbf{K}, \mathbf{U}\mathbf{D}\mathbf{V}^\dagger) \\ &= \mathcal{I}(\mathbf{K}, \mathbf{D}). \end{aligned} \quad (11)$$

Now consider the matrix $\hat{\mathbf{K}} = \frac{1}{2}(\mathbf{K} + \tilde{\mathbf{K}})$. We note that the entries of $\hat{\mathbf{K}}$ are identical to those of \mathbf{K} except that its off-diagonal elements in the j -th row and in the j -th column are zero. In particular, $\text{tr}(\mathbf{K}) = \text{tr}(\hat{\mathbf{K}})$. On the other hand, it follows from (11), the strict concavity of $\mathcal{I}(\mathbf{K}, \mathbf{D})$ in \mathbf{K} , and Jensen's inequality that

$$\begin{aligned} \mathcal{I}(\hat{\mathbf{K}}, \mathbf{D}) &\geq \frac{1}{2}(\mathcal{I}(\mathbf{K}, \mathbf{D}) + \mathcal{I}(\tilde{\mathbf{K}}, \mathbf{D})) \\ &= \mathcal{I}(\mathbf{K}, \mathbf{D}) \end{aligned}$$

with equality if, and only if, $\mathbf{K} = \hat{\mathbf{K}}$. Repeating this procedure for each $j = 1, \dots, n-1$ shows that an optimal covariance matrix must be diagonal. \square

2.2 The Rician Multiple-Access Fading Channel

The coherent MIMO Rician multiple-access channel (MAC) with k senders is modeled as follows. The channel output consists of k independent random matrices $\mathbb{H}_1, \dots, \mathbb{H}_k$, where \mathbb{H}_i is a random $m \times n_i$ matrix whose components are IID $\mathcal{N}_{\mathbb{C}}(0, 1)$, and of a random vector $\mathbf{Y} \in \mathbb{C}^m$ of the form

$$\mathbf{Y} = \sum_{i=1}^k (\mathbb{H}_i + \mathbf{D}_i) \mathbf{x}_i + \mathbf{Z} \quad (12)$$

where $\mathbf{x}_i \in \mathbb{C}^{n_i}$ is the i -th transmitter's input vector, \mathbf{D}_i is a deterministic $m \times n_i$ complex matrix corresponding to the LOS matrix of the i -th user, and $\mathbf{Z} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \sigma^2 \mathbf{I}_m)$ corresponds to the additive noise vector. It is assumed that all fading matrices $\{\mathbb{H}_i\}_{i=1}^k$ are independent of \mathbf{Z} and that the joint distribution of $(\mathbb{H}_1, \dots, \mathbb{H}_k, \mathbf{Z})$ does not depend on the inputs $\{\mathbf{x}_i\}_{i=1}^k$. Without loss of generality, we will assume $\sigma^2 = 1$.

As in the single-user scenario, it can be shown [9, 10] that Gaussian inputs achieve the capacity region of the MIMO Rician MAC. The rate region $\mathcal{R}(\mathbf{K}_1, \dots, \mathbf{K}_k; \mathbf{D}_1, \dots, \mathbf{D}_k)$

achieved by independent Gaussian inputs $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{K}_i)$ over the MIMO Rician MAC with LOS matrices $\{\mathbf{D}_i\}_{i=1}^k$ is given as the set of all rate vectors (R_1, \dots, R_k) satisfying

$$\sum_{i \in \mathcal{S}} R_i \leq \mathbb{E} \left[\log \det \left(\mathbf{I}_m + \sum_{i \in \mathcal{S}} (\mathbb{H}_i + \mathbf{D}_i) \mathbf{K}_i (\mathbb{H}_i + \mathbf{D}_i)^\dagger \right) \right] \quad (13)$$

for all $\mathcal{S} \subseteq \{1, \dots, k\}$. The capacity region of the MIMO Rician MAC, denoted as an explicit function of the input power constraints on the different users and of their corresponding LOS matrices, can be written as

$$\mathcal{C}(\mathcal{E}_1, \dots, \mathcal{E}_k; \mathbf{D}_1, \dots, \mathbf{D}_k) = \bigcup_{\{\mathbf{K}_i\}_{i=1}^k} \mathcal{R}(\mathbf{K}_1, \dots, \mathbf{K}_k; \mathbf{D}_1, \dots, \mathbf{D}_k) \quad (14)$$

where the union is over all input covariance matrices $\{\mathbf{K}_i\}_{i=1}^k$ that satisfy the trace constraints $\text{tr}(\mathbf{K}_i) \leq \mathcal{E}_i$, $i = 1, \dots, k$.

For each set $\mathcal{S} \subseteq \{1, \dots, k\}$ of elements $1 \leq i_1 < i_2 < \dots < i_s \leq k$, define the block matrices

$$\begin{aligned} \mathbf{D}_{\mathcal{S}} &\triangleq [\mathbf{D}_{i_1}, \dots, \mathbf{D}_{i_s}] \\ \mathbb{H}_{\mathcal{S}} &\triangleq [\mathbb{H}_{i_1}, \dots, \mathbb{H}_{i_s}] \end{aligned}$$

and

$$\mathbf{K}_{\mathcal{S}} \triangleq \text{diag}(\mathbf{K}_{i_1}, \dots, \mathbf{K}_{i_s}).$$

Further define $\mathbf{D} = [\mathbf{D}_1, \dots, \mathbf{D}_k]$. Under this simplified notation, the rate region (13) can be expressed as

$$\sum_{i \in \mathcal{S}} R_i \leq \mathbb{E} \left[\log \det \left(\mathbf{I}_m + (\mathbb{H}_{\mathcal{S}} + \mathbf{D}_{\mathcal{S}}) \mathbf{K}_{\mathcal{S}} (\mathbb{H}_{\mathcal{S}} + \mathbf{D}_{\mathcal{S}})^\dagger \right) \right] = \mathcal{I}(\mathbf{K}_{\mathcal{S}}, \mathbf{D}_{\mathcal{S}}).$$

Since the condition $\mathbf{D}^\dagger \mathbf{D} \succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}}$ implies that $\mathbf{D}_{\mathcal{S}}^\dagger \mathbf{D}_{\mathcal{S}} \succeq \tilde{\mathbf{D}}_{\mathcal{S}}^\dagger \tilde{\mathbf{D}}_{\mathcal{S}}$ for all $\mathcal{S} \subseteq \{1, \dots, k\}$, it follows from Corollary 2.1 that

$$\mathcal{R}(\mathbf{K}_1, \dots, \mathbf{K}_k; \mathbf{D}_1, \dots, \mathbf{D}_k) \supseteq \mathcal{R}(\mathbf{K}_1, \dots, \mathbf{K}_k; \tilde{\mathbf{D}}_1, \dots, \tilde{\mathbf{D}}_k)$$

and consequently, by (14),

$$\mathcal{C}(\mathcal{E}_1, \dots, \mathcal{E}_k; \mathbf{D}_1, \dots, \mathbf{D}_k) \supseteq \mathcal{C}(\mathcal{E}_1, \dots, \mathcal{E}_k; \tilde{\mathbf{D}}_1, \dots, \tilde{\mathbf{D}}_k).$$

We can strengthen this result using the symmetry of the problem as in the single-user case. The utility of the rotational invariance (3) is, however, rather limited since the LOS matrices cannot be assumed to be jointly diagonalizable. Thus, the monotonicity cannot be simply stated in terms of the singular values of LOS matrices. Instead, we have the following.

Corollary 2.5. Let $\mathbf{D} = [\mathbf{D}_1, \dots, \mathbf{D}_k]$ and $\tilde{\mathbf{D}} = [\tilde{\mathbf{D}}_1, \dots, \tilde{\mathbf{D}}_k]$ be LOS matrices such that

$$[\mathbf{D}_1 \mathbf{U}_1, \dots, \mathbf{D}_k \mathbf{U}_k]^\dagger [\mathbf{D}_1 \mathbf{U}_1, \dots, \mathbf{D}_k \mathbf{U}_k] \succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}}$$

for some $\mathbf{U}_i \in \mathcal{U}(n_i), i = 1, \dots, k$. Then

$$\mathcal{C}(\mathcal{E}_1, \dots, \mathcal{E}_k; \mathbf{D}_1, \dots, \mathbf{D}_k) \supseteq \mathcal{C}(\mathcal{E}_1, \dots, \mathcal{E}_k; \tilde{\mathbf{D}}_1, \dots, \tilde{\mathbf{D}}_k).$$

3 Proof of Theorem 1.1

Recall that given any $\mathbf{K} \in \mathcal{H}^+(n)$ and $\mathbf{D}, \tilde{\mathbf{D}} \in \mathbb{C}^{m \times n}$ satisfying

$$\mathbf{D}^\dagger \mathbf{D} \succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}} \tag{15}$$

we wish to show that for all $t \geq 0$,

$$F(t; \mathbf{K}, \mathbf{D}) \leq F(t; \mathbf{K}, \tilde{\mathbf{D}}) \tag{16}$$

where

$$F(t; \mathbf{K}, \mathbf{D}) = \Pr \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \mathbf{D}) \mathbf{K} (\mathbb{H} + \mathbf{D})^\dagger \right) \leq t \right].$$

Without loss of generality, we can assume that the matrices \mathbf{D} and $\tilde{\mathbf{D}}$ satisfy

$$\tilde{\mathbf{D}} = \Phi \mathbf{D}, \quad \Phi = \text{diag}(\alpha, 1, \dots, 1) \tag{17}$$

for some $0 \leq \alpha \leq 1$. We justify this reduction as follows. Suppose that the desired inequality (16) holds under the condition (17). Then from the rotational invariance (2), for any permutation matrix \mathbf{P} ,

$$\begin{aligned} F(t; \mathbf{K}, \mathbf{P} \Phi \mathbf{P}^\dagger \mathbf{D}) &= F(t; \mathbf{K}, \Phi \mathbf{P}^\dagger \mathbf{D}) \\ &\geq F(t; \mathbf{K}, \mathbf{P}^\dagger \mathbf{D}) \\ &= F(t; \mathbf{K}, \mathbf{D}) \end{aligned} \tag{18}$$

and consequently the result must also hold when $\Phi = \text{diag}(1, \dots, 1, \alpha, 1, \dots, 1)$. Expressing $\text{diag}(\alpha_1, \dots, \alpha_m)$ as a product

$$\text{diag}(\alpha_1, \dots, \alpha_m) = \text{diag}(\alpha_1, 1, \dots, 1) \cdot \text{diag}(1, \alpha_2, 1, \dots, 1) \cdot \dots \cdot \text{diag}(1, \dots, 1, \alpha_m)$$

and applying the inequality (18) $m - 1$ times yields that the result (16) must also hold for any \mathbf{D} and $\tilde{\mathbf{D}}$ such that $\tilde{\mathbf{D}} = \Phi \mathbf{D}$ with arbitrary diagonal contraction matrix Φ with $0 \leq \Phi_{ii} \leq 1, i = 1, \dots, m$. Now applying the rotational invariance (2) once again to

arbitrary unitary matrices $U \in \mathcal{U}(m)$, $V \in \mathcal{U}(m)$ and nonnegative diagonal contraction matrix Φ , we obtain

$$\begin{aligned} F(t; K, U\Phi V^\dagger D) &= F(t; K, \Phi V^\dagger D) \\ &\geq F(t; K, V^\dagger D) \\ &= F(t; K, D). \end{aligned}$$

Thus the desired inequality (16) holds for any D, \tilde{D} , and Φ such that

$$\tilde{D} = \Phi D, \quad \Phi^\dagger \Phi \preceq I_m. \quad (19)$$

But (19) is equivalent to the original condition (15) (see, for example, [11]). Therefore, in order to prove the theorem, it suffices to establish the inequality (16) under the simplified condition (17).

For the rest of our discussion, we need the following result by T. W. Anderson [12] [13, Theorem 8.10.5].

Lemma 3.1. (*Anderson's Theorem*) *Let \mathcal{H} be a convex set in \mathbb{C}^n , symmetric about the origin (i.e., $\boldsymbol{\xi} \in \mathcal{H}$ implies $-\boldsymbol{\xi} \in \mathcal{H}$). Let $f(\boldsymbol{\xi}) \geq 0$ be a function on \mathbb{C}^n such that (i) $f(-\boldsymbol{\xi}) = f(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$, (ii) the set $\{\boldsymbol{\xi} \in \mathbb{C}^n : f(\boldsymbol{\xi}) \geq u\}$ is convex for every $u > 0$; and (iii) $\int_{\mathcal{H}} f(\boldsymbol{\xi}) d\boldsymbol{\xi} < \infty$. Then*

$$\int_{\mathcal{H}} f(\boldsymbol{\xi} + \alpha \boldsymbol{\eta}) d\boldsymbol{\xi} \geq \int_{\mathcal{H}} f(\boldsymbol{\xi} + \boldsymbol{\eta}) d\boldsymbol{\xi} \quad (20)$$

for every vector $\boldsymbol{\eta} \in \mathbb{C}^n$ and $0 \leq \alpha \leq 1$.

The proof of this celebrated result is based on the Brunn-Minkowski inequality [14]. An interested reader can refer to a nice review by Perlman [15] for further generalizations and applications in multivariate statistics.

Returning to our problem, for any $t \geq 0$, we define a set of matrices

$$\mathcal{G}_t = \left\{ \mathbf{G} \in \mathbb{C}^{m \times n} : \log \det \left(\mathbf{I}_m + \mathbf{G} \mathbf{G}^\dagger \right) \leq t \right\}. \quad (21)$$

For any fixed vectors $\mathbf{g}_2, \dots, \mathbf{g}_m \in \mathbb{C}^n$, let

$$\mathcal{H}_t(\mathbf{g}_2, \dots, \mathbf{g}_m) = \{ \boldsymbol{\xi} \in \mathbb{C}^n : [\boldsymbol{\xi}, \mathbf{g}_2, \dots, \mathbf{g}_m]^\top \in \mathcal{G}_t \}. \quad (22)$$

In other words, $\mathcal{H}_t(\mathbf{g}_2, \dots, \mathbf{g}_m)$ is the set of the first rows $\boldsymbol{\xi}^\top$ that belong to \mathcal{G}_t with given values of other rows $\mathbf{g}_2^\top, \dots, \mathbf{g}_m^\top$. As will be checked later at the end of this section, for any $\mathbf{g}_2, \dots, \mathbf{g}_m$, the set $\mathcal{H}_t(\mathbf{g}_2, \dots, \mathbf{g}_m)$ is convex and symmetric about the origin.

The rest of the proof proceeds along the lines similar to those of Das Gupta, Anderson, and Mudholkar [16]. We represent \mathbb{H} as $[\mathbf{H}_1, \dots, \mathbf{H}_m]^\top$, where \mathbf{H}_j^\top is the j -th row of \mathbb{H} . Similarly, let \mathbf{d}_j^\top denote the j -th row of \mathbf{D} . Let $f(\boldsymbol{\xi} | \mathbf{h}_2, \dots, \mathbf{h}_m)$ be the conditional density

of \mathbf{H}_1 conditioned on $\mathbf{H}_j = \mathbf{h}_j, j = 2, \dots, m$. Since the rows of \mathbb{H} are mutually independent, $f(\boldsymbol{\xi}|\mathbf{h}_2, \dots, \mathbf{h}_m) = f(\boldsymbol{\xi})$ is multivariate Gaussian $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_n)$, which satisfies the conditions (i) to (iii) of Anderson's Theorem. Combining the conditions on f and \mathcal{H}_t with the standing assumption (17), we can invoke Anderson's Theorem for the first row of \mathbb{H} after conditioning on the other rows $\mathbf{H}_2^T, \dots, \mathbf{H}_m^T$ as follows:

$$\begin{aligned}
& \Pr \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \mathbf{D}) \mathbf{K} (\mathbb{H} + \mathbf{D})^\dagger \right) \leq t \mid \mathbf{H}_i = \mathbf{h}_i, i = 2, \dots, m \right] \\
&= \int_{\mathcal{H}_t(\mathbf{h}_2 + \mathbf{d}_2, \dots, \mathbf{h}_m + \mathbf{d}_m)} f(\boldsymbol{\xi} - \mathbf{d}_1) d\boldsymbol{\xi} \\
&\leq \int_{\mathcal{H}_t(\mathbf{h}_2 + \mathbf{d}_2, \dots, \mathbf{h}_m + \mathbf{d}_m)} f(\boldsymbol{\xi} - \alpha \mathbf{d}_1) d\boldsymbol{\xi} \\
&= \Pr \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \tilde{\mathbf{D}}) \mathbf{K} (\mathbb{H} + \tilde{\mathbf{D}})^\dagger \right) \leq t \mid \mathbf{H}_i = \mathbf{h}_i, i = 2, \dots, m \right]. \tag{23}
\end{aligned}$$

By taking the expectation on both sides of (23) with respect to the joint density of $\mathbf{H}_2, \dots, \mathbf{H}_m$, we establish the desired inequality (16).

It remains to check the convexity and symmetry of the set $\mathcal{H}_t = \mathcal{H}_t(\mathbf{g}_2, \dots, \mathbf{g}_m)$. Let $\mathbf{G} = [\boldsymbol{\xi}, \mathbf{g}_2, \dots, \mathbf{g}_m]^T$. We show that $\det(\mathbf{I}_m + \mathbf{G}\mathbf{K}\mathbf{G}^\dagger)$ is convex and symmetric in $\boldsymbol{\xi}$, which clearly implies the convexity and symmetry of \mathcal{H}_t . For the symmetry, observe that

$$\det(\mathbf{I}_m + \mathbf{G}\mathbf{K}\mathbf{G}^\dagger) = \det(\mathbf{I}_m + \mathbf{U}\mathbf{G}\mathbf{K}\mathbf{G}^\dagger\mathbf{U}^\dagger)$$

for any unitary matrix \mathbf{U} ; in particular, $\mathbf{U} = \text{diag}(-1, 1, \dots, 1)$.

For the convexity, let $\mathbf{F} = \mathbf{G}\mathbf{K}^{\frac{1}{2}}$ where $\mathbf{K}^{\frac{1}{2}}$ is any matrix satisfying $\mathbf{K}^{\frac{1}{2}}(\mathbf{K}^{\frac{1}{2}})^\dagger = \mathbf{K}$. Recall the identity

$$\det(\mathbf{I}_k + \mathbf{A}\mathbf{B}) = \det(\mathbf{I}_j + \mathbf{B}\mathbf{A}) \tag{24}$$

for any $\mathbf{A} \in \mathbb{C}^{k \times j}, \mathbf{B} \in \mathbb{C}^{j \times k}$. Then we have

$$\begin{aligned}
\det(\mathbf{I}_m + \mathbf{G}\mathbf{K}\mathbf{G}^\dagger) &= \det(\mathbf{I}_m + \mathbf{F}\mathbf{F}^\dagger) \\
&= \det(\mathbf{I}_n + \mathbf{F}^\dagger\mathbf{F}) \\
&= \det \left(\mathbf{I}_n + \sum_{j=2}^m (\mathbf{f}_j^\dagger)^\dagger \mathbf{f}_j^\dagger + (\mathbf{f}_1^\dagger)^\dagger \mathbf{f}_1^\dagger \right) \\
&= \det(\mathbf{M} + (\mathbf{f}_1^\dagger)^\dagger \mathbf{f}_1^\dagger) \\
&= \det(\mathbf{M}) \det(\mathbf{I}_n + \mathbf{M}^{-1}(\mathbf{f}_1^\dagger)^\dagger \mathbf{f}_1^\dagger) \\
&= \det(\mathbf{M}) \left(1 + \mathbf{f}_1^\dagger \mathbf{M}^{-1}(\mathbf{f}_1^\dagger)^\dagger \right) \\
&= \det(\mathbf{M}) \left(1 + \boldsymbol{\xi}^\dagger \mathbf{K}^{\frac{1}{2}} \mathbf{M}^{-1} \mathbf{K}^{\frac{1}{2}} (\boldsymbol{\xi}^\dagger)^\dagger \right) \tag{25}
\end{aligned}$$

where \mathbf{f}_j^T denotes the j -th row of \mathbf{F} and the positive definite matrix \mathbf{M} is defined as $\mathbf{M} = \mathbf{I}_n + \sum_{j=2}^m (\mathbf{f}_j^T)^\dagger \mathbf{f}_j^T$. The last line of (25) is a positive semidefinite quadratic form in $\boldsymbol{\xi}$, and hence it is convex.

4 Concluding Remarks

In this paper we have found a natural ordering of MIMO Rician channels via their LOS matrices. We have shown that for two LOS matrices $\mathbf{D}, \tilde{\mathbf{D}} \in \mathbb{C}^{m \times n}$

$$\mathbf{D}^\dagger \mathbf{D} \succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}} \iff \left(\mathcal{I}(\mathbf{K}, \mathbf{D}) \geq \mathcal{I}(\mathbf{K}, \tilde{\mathbf{D}}) \quad \forall \mathbf{K} \in \mathcal{H}^+(n) \right)$$

where $\mathcal{I}(\mathbf{K}, \mathbf{D}) = I(\mathbf{X}; \mathbf{Y} | \mathbb{H})$ is the mutual information induced by a $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{K})$ input over a coherent MIMO Rician channel with LOS matrix \mathbf{D} . From this result we obtained monotonicity results for isotropic Gaussian input information rate and for channel capacity, not only for the single-user channel but also for the multiple-access channel.

In some sense the results of this paper may not be surprising because the relation $\mathbf{D}^\dagger \mathbf{D} \succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}}$ implies $\text{tr}(\mathbf{D}^\dagger \mathbf{D}) \geq \text{tr}(\tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}})$ and hence a larger output power. Note, however, that some care must be exercised because in MIMO communications a larger output power need not imply a larger capacity. For instance, if

$$\mathbf{D}_1 = \begin{pmatrix} 10 & 10 \\ 10 & 0 \end{pmatrix}, \quad \mathbf{D}_2 = \begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix}$$

then although the power in the LOS component increases while changing from \mathbf{D}_1 to \mathbf{D}_2 , one can numerically show that the isotropic Gaussian input information rate and channel capacity are larger on the channel with LOS matrix \mathbf{D}_1 than on the channel with LOS matrix \mathbf{D}_2 . The intuition is that \mathbf{D}_1 has full rank with singular values 16.18 and 6.18, whereas \mathbf{D}_2 is rank deficient with singular values 20 and 0, thus providing only one LOS eigenmode.

A Proof of Proposition 2.2

Instead of proving Proposition 2.2 directly, we will prove the equivalent statement

$$\mathbf{D}^\dagger \mathbf{D} \not\succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}} \quad \Rightarrow \quad \mathcal{I}(\mathbf{K}, \mathbf{D}) < \mathcal{I}(\mathbf{K}, \tilde{\mathbf{D}}) \text{ for some } \mathbf{K} \in \mathcal{H}^+(n).$$

We first note that $\mathbf{D}^\dagger \mathbf{D} \not\succeq \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}}$ means that there exists a vector $\mathbf{a} \in \mathbb{C}^n$ such that

$$\mathbf{a}^\dagger \mathbf{D}^\dagger \mathbf{D} \mathbf{a} < \mathbf{a}^\dagger \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}} \mathbf{a}. \tag{26}$$

For such a vector \mathbf{a} , let $\mathbf{K}_0 = \mathbf{a} \mathbf{a}^\dagger \in \mathcal{H}^+(n)$. We will show that for \mathbf{K}_0 the strict inequality $\mathcal{I}(\mathbf{K}_0, \mathbf{D}) < \mathcal{I}(\mathbf{K}_0, \tilde{\mathbf{D}})$ holds.

By (1) it suffices to show that $F(t; \mathbf{K}_0, \mathbf{D}) > F(t; \mathbf{K}_0, \tilde{\mathbf{D}})$ for all $t > 0$. Define $\mathbf{G} = \mathbb{H}\mathbf{a}$, $\mathbf{b} = \mathbf{D}\mathbf{a}$, and $\tilde{\mathbf{b}} = \tilde{\mathbf{D}}\mathbf{a}$. Then we have for any $t > 0$

$$\begin{aligned} F(t; \mathbf{K}_0, \mathbf{D}) &= \Pr \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \mathbf{D})\mathbf{a}\mathbf{a}^\dagger(\mathbb{H} + \mathbf{D})^\dagger \right) \leq t \right] \\ &= \Pr \left[\log \left(1 + \mathbf{a}^\dagger(\mathbb{H} + \mathbf{D})^\dagger(\mathbb{H} + \mathbf{D})\mathbf{a} \right) \leq t \right] \end{aligned} \quad (27)$$

$$\begin{aligned} &= \Pr \left[\log \left(1 + (\mathbf{G} + \mathbf{b})^\dagger(\mathbf{G} + \mathbf{b}) \right) \leq t \right] \\ &> \Pr \left[\log \left(1 + (\mathbf{G} + \tilde{\mathbf{b}})^\dagger(\mathbf{G} + \tilde{\mathbf{b}}) \right) \leq t \right] \quad (28) \\ &= \Pr \left[\log \left(1 + \mathbf{a}^\dagger(\mathbb{H} + \tilde{\mathbf{D}})^\dagger(\mathbb{H} + \tilde{\mathbf{D}})\mathbf{a} \right) \leq t \right] \\ &= \Pr \left[\log \det \left(\mathbf{I}_m + (\mathbb{H} + \tilde{\mathbf{D}})\mathbf{a}\mathbf{a}^\dagger(\mathbb{H} + \tilde{\mathbf{D}})^\dagger \right) \leq t \right] \\ &= F(t; \mathbf{K}_0, \tilde{\mathbf{D}}) \end{aligned}$$

where (27) follows from (24) and (28) follows from the strict monotonicity result for the single-antenna case [1, Lemma 6.2 (b)]. Indeed, \mathbf{G} is distributed according to $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{a}^\dagger\mathbf{a}\mathbf{I}_m)$ and $(\mathbf{G} + \mathbf{b})^\dagger(\mathbf{G} + \mathbf{b})$ has a scaled non-central chi-square distribution with (scaled) non-centrality parameter $\mathbf{b}^\dagger\mathbf{b}$. Now $(\mathbf{G} + \tilde{\mathbf{b}})^\dagger(\mathbf{G} + \tilde{\mathbf{b}})$ in (28) is also a scaled non-central chi-square random variable, which, from (26), has a strictly larger non-centrality parameter $\tilde{\mathbf{b}}^\dagger\tilde{\mathbf{b}} > \mathbf{b}^\dagger\mathbf{b}$. Hence, $(\mathbf{G} + \tilde{\mathbf{b}})^\dagger(\mathbf{G} + \tilde{\mathbf{b}})$ is stochastically strictly larger than $(\mathbf{G} + \mathbf{b})^\dagger(\mathbf{G} + \mathbf{b})$, so that the strict inequality in (28) is justified for any $t > 0$.

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