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Monotonicity results for fractional difference operators with discrete exponential kernels

Thabet Abdeljawad[1] and Dumitru Baleanu[2,3*]

*Correspondence:
dumitru@cankaya.edu.tr
[2]Department of Mathematics, Çankaya University, Ankara, 06530, Turkey
[3]Institute of Space Sciences, Magurele, Romania
Full list of author information is available at the end of the article

Abstract

We prove that if the Caputo-Fabrizio nabla fractional difference operator $({}^{\mathrm{CFR}}_{a-1}\nabla^{\alpha} y)(t)$ of order $0<\alpha\leq 1$ and starting at $a-1$ is positive for $t=a,a+1,\ldots$, then $y(t)$ is α-increasing. Conversely, if $y(t)$ is increasing and $y(a)\geq 0$, then $({}^{\mathrm{CFR}}_{a-1}\nabla^{\alpha} y)(t)\geq 0$. A monotonicity result for the Caputo-type fractional difference operator is proved as well. As an application, we prove a fractional difference version of the mean-value theorem and make a comparison to the classical discrete fractional case.

Keywords: discrete exponential kernel; Caputo fractional difference; Riemann fractional difference; discrete fractional mean value theorem

1 Introduction

The fractional calculus was successfully used during the last few years in many branches of engineering and science [1–5]. The core ideas of this type of nonlocal calculus were applied successfully to the so-called discrete fractional calculus (DFC) [6–18]. This new direction initiated about a decade ago is in continuous evolution, and it started recently to be considered as a powerful tool to extract new insides of the dynamics of complex discrete dynamical systems. The discrete diffusion equation within discrete Riesz derivative is one of the new results reported very recently [19, 20]. Therefore, the DFC is a natural generalization of the classical discrete ones. Very recently, Caputo and Fabrizio [21] introduced a new fractional derivative based on a nonsingular kernel. The discrete version of this operator was reported in [22]. In our opinion, the existence of various types of memory kernels increases the chances to formulate adequately different types of models where different types of memory appear. Very recently, some authors investigated the monotonicity properties of discrete functions via their discrete fractional operators. Some authors studied the monotonicity analysis of delta- or nabla-type fractional difference operators of order $0<\alpha<1$ (see [23]), whereas others studied fractional difference operators of order $\alpha>1$ [24–27]. These new results motivate us to discuss in this paper the monotonicity results for this nabla discrete fractional operator with discrete exponential kernel and compare them to the discrete classical ones. The fractional differences under consideration in this paper have kernels different from classical nabla fractional differences with kernels depending on the rising factorial powers, and we believe that they bring new kernels with new memories, which may be of different interest for applications.

2 Preliminaries

For two real numbers $a < b$ with $a \equiv b \pmod 1$, we denote $\mathbb{N}_a = \{a, a+1, \ldots\}$, ${}_b\mathbb{N} = \{b, b-1, \ldots\}$, and $\mathbb{N}_{a,b} = \mathbb{N}_a \cap {}_b\mathbb{N} = \{a, a+1, \ldots, b\}$. For details about concepts of discrete fractional calculus, we refer the reader to the nice text book [28].

Using the time scale notation, the nabla discrete exponential kernel can be expressed as $\widehat{e}_\lambda(t, \rho(s)) = (\frac{1}{1-\lambda})^{t-\rho(s)} = (1-\alpha)^{t-\rho(s)}$ [29], where $\lambda = \frac{-\alpha}{1-\alpha}$. The following discrete versions were proposed in [22]:

Definition 1 ([22]) For $\alpha \in (0,1)$ and f defined on $\mathbb{N}_a$, or ${}_b\mathbb{N}$ in right case, we have the following definitions:

- The left (nabla) new Caputo fractional difference is given by

$$\begin{aligned} \left({}_a^{\mathrm{CFC}}\nabla^\alpha f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \sum_{s=a+1}^{t} (\nabla_s f)(s)(1-\alpha)^{t-\rho(s)} \\ &= B(\alpha) \sum_{s=a+1}^{t} (\nabla_s f)(s)(1-\alpha)^{t-s}. \end{aligned} \tag{1}$$

- The right (nabla) new Caputo fractional difference is given by

$$\begin{aligned} \left({}^{\mathrm{CFC}}\nabla_b^\alpha f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \sum_{s=t}^{b-1} (-\Delta_s f)(s)(1-\alpha)^{s-\rho(t)} \\ &= B(\alpha) \sum_{s=t}^{b-1} (-\Delta_s f)(s)(1-\alpha)^{s-t}. \end{aligned} \tag{2}$$

- The left (nabla) new Riemann fractional difference is given by

$$\begin{aligned} \left({}_a^{\mathrm{CFR}}\nabla^\alpha f\right)(t) &= \frac{B(\alpha)}{1-\alpha} \nabla_t \sum_{s=a+1}^{t} f(s)(1-\alpha)^{t-\rho(s)} \\ &= B(\alpha) \nabla_t \sum_{s=a+1}^{t} f(s)(1-\alpha)^{t-s}. \end{aligned} \tag{3}$$

- The right (nabla) new Riemann fractional difference is given by

$$\begin{aligned} \left({}^{\mathrm{CFR}}\nabla_b^\alpha f\right)(t) &= \frac{B(\alpha)}{1-\alpha} (-\Delta_t) \sum_{s=t}^{b-1} f(s)(1-\alpha)^{s-\rho(t)} \\ &= B(\alpha)(-\Delta_t) \sum_{s=t}^{b-1} f(s)(1-\alpha)^{s-t}, \end{aligned} \tag{4}$$

where $B(\alpha)$ is a normalizing positive constant depending on α and satisfying $B(0) = B(1) = 1$.

Remark 1 ([22]) In the limiting cases $\alpha \to 0$ and $\alpha \to 1$, we remark the following:

- $$\left({}_a^{\mathrm{CFC}}\nabla^\alpha f\right)(t) \to f(t) - f(a) \quad \text{as } \alpha \to 0,$$

and

$$\left({}_{a}^{\mathrm{CFC}}\nabla^{\alpha}f\right)(t)\to\nabla f(t)\quad\text{as }\alpha\to 1.$$

•

$$\left({}^{\mathrm{CFC}}\nabla_{b}^{\alpha}f\right)(t)\to f(t)-f(b)\quad\text{as }\alpha\to 0,$$

and

$$\left({}^{\mathrm{CFC}}\nabla_{b}^{\alpha}f\right)(t)\to -\Delta f(t)\quad\text{as }\alpha\to 1.$$

•

$$\left({}_{a}^{\mathrm{CFR}}\nabla^{\alpha}f\right)(t)\to f(t)\quad\text{as }\alpha\to 0,$$

and

$$\left({}_{a}^{\mathrm{CFR}}\nabla^{\alpha}f\right)(t)\to\nabla f(t)\quad\text{as }\alpha\to 1.$$

•

$$\left({}^{\mathrm{CFR}}\nabla_{b}^{\alpha}f\right)(t)\to f(t)\quad\text{as }\alpha\to 0,$$

and

$$\left({}^{\mathrm{CFR}}\nabla_{b}^{\alpha}f\right)(t)\to -\Delta f(t)\quad\text{as }\alpha\to 1.$$

Remark 2 ([22] (the action of the discrete Q-operator)) The Q-operator acts regularly between left and right new fractional differences as follows:

- $(Q{}_{a}^{\mathrm{CFR}}\nabla^{\alpha}f)(t)=({}^{\mathrm{CFR}}\nabla_{b}^{\alpha}Qf)(t)$,
- $(Q{}_{a}^{\mathrm{CFC}}\nabla^{\alpha}f)(t)=({}^{\mathrm{CFC}}\nabla_{b}^{\alpha}Qf)(t)$,

where $(Qf)(t)=f(a+b-t)$.

Definition 2 ([22]) For $0<\alpha<1$ and $u:\mathbb{N}_a\to\mathbb{R}$, $a<b$, $a\equiv b\ (\mathtt{mod}\ 1)$, we define:

- the corresponding left fractional sum by

$$\left({}_{a}^{\mathrm{CF}}\nabla^{-\alpha}u\right)(t)=\frac{1-\alpha}{B(\alpha)}u(t)+\frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}u(s)\,ds;\tag{5}$$

- the right fractional sum by

$$\left({}^{\mathrm{CF}}\nabla_{b}^{-\alpha}u\right)(t)=\frac{1-\alpha}{B(\alpha)}u(t)+\frac{\alpha}{B(\alpha)}\sum_{s=t}^{b-1}u(s)\,ds.\tag{6}$$

In [22], it was shown that $({}_{a}^{\mathrm{CF}}\nabla^{-\alpha}{}_{a}^{\mathrm{CF}}\nabla^{\alpha}f)(t)=f(t)$ and $({}^{\mathrm{CF}}\nabla_{b}^{-\alpha}{}^{\mathrm{CF}}\nabla_{b}^{\alpha}f)(t)=f(t)$. Also, it was shown that $({}_{a}^{\mathrm{CF}}\nabla^{\alpha}{}_{a}^{\mathrm{CF}}\nabla^{-\alpha}f)(t)=f(t)$ and $({}^{\mathrm{CF}}\nabla_{b}^{\alpha}{}^{\mathrm{CF}}\nabla_{b}^{-\alpha}f)(t)=f(t)$.

Proposition 1 ([22] (the relation between Riemann- and Caputo-type fractional differences with exponential kernels))

- $({}^{\mathrm{CFC}}_{a}\nabla^{\alpha}f)(t) = ({}^{\mathrm{CFR}}_{a}\nabla^{\alpha}f)(t) - \frac{B(\alpha)}{1-\alpha}f(a)(1-\alpha)^{t-a}$;
- $({}^{\mathrm{CFC}}\nabla^{\alpha}_{b}f)(t) = ({}^{\mathrm{CFR}}\nabla^{\alpha}_{b}f)(t) - \frac{B(\alpha)}{1-\alpha}f(b)(1-\alpha)^{b-t}$.

Some parts of the following lemma are essential to proceed.

Lemma 1 *For* $0 < \alpha < 1$ *and* g *defined on* $\mathbb{N}_a$, *we have*:

(i)

$$\bigl({}^{\mathrm{CF}}_{a}\nabla^{-\alpha}(1-\alpha)^{t}\bigr)(t) = \frac{(1-\alpha)^{a+1}}{B(\alpha)}; \tag{7}$$

(ii) $\nabla_s(1-\alpha)^{t-s} = \alpha(1-\alpha)^{t-s}$;

(iii) $({}^{\mathrm{CF}}_{a}\nabla^{-\alpha}\nabla g)(t) = (\nabla{}^{\mathrm{CF}}_{a}\nabla^{-\alpha}g)(t) - \frac{\alpha}{B(\alpha)}g(a)$;

(iv) $\nabla(1-\alpha)^{t} = -\alpha(1-\alpha)^{t-1}$;

(v) $({}^{\mathrm{CFR}}_{a}\nabla^{\alpha}(1-\alpha)^{t})(t) = B(\alpha)(1-\alpha)^{t-1}[1-\alpha(t-a)]$;

(vi) $({}^{\mathrm{CFR}}_{a}\nabla^{\alpha}1)(t) = B(\alpha)(1-\alpha)^{t-a-1}$.

Proof We just give the proof of (i), (iii), (v), and (vi). The other parts are direct and easy.

- The proof of (i):

$$\begin{aligned}
\bigl({}^{\mathrm{CF}}_{a}\nabla^{-\alpha}(1-\alpha)^{t}\bigr)(t) &= \frac{1-\alpha}{B(\alpha)}(1-\alpha)^{t} + \frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}(1-\alpha)^{s} \\
&= \frac{1-\alpha}{B(\alpha)}(1-\alpha)^{t} + \frac{\alpha}{B(\alpha)}(1-\alpha)^{a+1}\frac{1-(1-\alpha)^{t-a}}{1-(1-\alpha)} \\
&= \frac{1}{B(\alpha)}\bigl[(1-\alpha)^{t+1} + (1-\alpha)^{a+1} - (1-\alpha)^{t+1}\bigr] \\
&= \frac{(1-\alpha)^{a+1}}{B(\alpha)}.
\end{aligned} \tag{8}$$

- The proof of (iii):

$$\begin{aligned}
\bigl({}^{\mathrm{CF}}_{a}\nabla^{-\alpha}\nabla g\bigr)(t) &= \frac{1-\alpha}{B(\alpha)}\nabla g(t) + \frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}\nabla g(s) \\
&= \frac{1-\alpha}{B(\alpha)}\nabla g(t) + \frac{\alpha}{B(\alpha)}\bigl[g(t)-g(a)\bigr] \\
&= \nabla\left[\frac{1-\alpha}{B(\alpha)}g(t) + \frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}g(s)\right] - \frac{\alpha}{B(\alpha)}g(a) \\
&= \bigl(\nabla{}^{\mathrm{CF}}_{a}\nabla^{-\alpha}g\bigr)(t) - \frac{\alpha}{B(\alpha)}g(a).
\end{aligned}$$

- The proof of (v): By (iv) we have

$$\begin{aligned}
\bigl({}^{\mathrm{CFR}}_{a}\nabla^{\alpha}(1-\alpha)^{t}\bigr)(t) &= B(\alpha)\nabla\sum_{s=a+1}^{t}(1-\alpha)^{t-s}(1-\alpha)^{s} \\
&= B(\alpha)\nabla\bigl[(t-a)(1-\alpha)^{t}\bigr]
\end{aligned}$$

$$= B(\alpha)\big[(1-\alpha)^{t-1} - \alpha(t-a)(1-\alpha)^{t-1}\big]$$
$$= B(\alpha)(1-\alpha)^{t-1}\big[1-\alpha(t-a)\big]. \tag{9}$$

- The proof of (vi):

$$\begin{aligned}
\big({}_a^{\mathrm{CFR}}\nabla^{\alpha}1\big)(t) &= B(\alpha)\nabla_t \sum_{s+a+1}^{t}(1-\alpha)^{t-s} \\
&= B(\alpha)\left[1 + \sum_{s=a+1}^{t-1}\nabla_t(1-\alpha)^{t-s}\right] \\
&= B(\alpha)\left[1 - \alpha\sum_{s=a+1}^{t-1}(1-\alpha)^{t-1-s}\right] = B(\alpha)\left[1-\alpha\sum_{i=0}^{t-a-2}(1-\alpha)^{i}\right] \\
&= B(\alpha)\left[1-\alpha\frac{1-(1-\alpha)^{t-a-1}}{1-(1-\alpha)}\right] \\
&= B(\alpha)(1-\alpha)^{t-a-1}.
\end{aligned} \tag{10}$$

□

Definition 3 (See also [23]) Let $y:\mathbb{N}_a \to \mathbb{R}$ be a function satisfying $y(a) \geq 0$. Then y is called an α-increasing function on $\mathbb{N}_a$ if

$$y(t+1) \geq \alpha y(t) \quad \text{for all } t \in \mathbb{N}_a.$$

Note that if y is increasing on $\mathbb{N}_a$, then y is an α-increasing function on $\mathbb{N}_a$, and if $\alpha = 1$, then the increasing and α-increasing concepts coincide.

Definition 4 (See also [23]) Let $y:\mathbb{N}_a \to \mathbb{R}$ be a function satisfying $y(a) \leq 0$. Then y is called an α-decreasing function on $\mathbb{N}_a$, if

$$y(t+1) \leq \alpha y(t) \quad \text{for all } t \in \mathbb{N}_a.$$

Note that if y is decreasing on $\mathbb{N}_a$, then y is an α-decreasing function on $\mathbb{N}_a$, and if $\alpha = 1$, then the decreasing and α-decreasing concepts coincide.

3 The monotonicity results

Theorem 1 *Let* $y:\mathbb{N}_{a-1} \to \mathbb{R}$. *Suppose that, for* $0 < \alpha \leq 1$,

$$\big({}_{a-1}^{\mathrm{CFR}}\nabla^{\alpha}y\big)(t) \geq 0, \qquad t \in \mathbb{N}_{a-1}.$$

Then $y(t)$ *is* α*-increasing.*

Proof Rewrite $({}_{a-1}^{\mathrm{CFR}}\nabla^{\alpha}y)(t) = B(\alpha)\nabla S(t)$, where $S(t) = \sum_{s=a}^{t} y(s)(1-\alpha)^{t-s}$. By the assumption we have

$$S(t) - S(t-1) = y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s} \geq 0. \tag{11}$$

Substituting $t = a$ into (11), we see that $y(a) \geq 0$. Substituting $t = a + 1$ into (11), we get

$$y(a+1) - \frac{\alpha}{1-\alpha} y(a)(1-\alpha) = y(a+1) - \alpha y(a) \geq 0,$$

and hence $y(a+1) \geq \alpha y(a) \geq 0$. We shall proceed by induction on $t \in \mathbb{N}_a$. Assume that $y(i+1) \geq \alpha y(i) \geq 0$ for all $i < t$. Let us show that $y(t+1) \geq \alpha y(t)$. Replacing t with $t+1$ in (11), we have

$$y(t+1) \geq \frac{\alpha}{1-\alpha}\left[(1-\alpha)^{t+1-a} y(a) + (1-\alpha)^{t-a} y(a+1) + \cdots + (1-\alpha) y(t)\right],$$

or

$$y(t+1) \geq \left[\alpha(1-\alpha)^{t-a} y(a) + \alpha(1-\alpha)^{t-a-1} y(a+1) + \cdots + \alpha y(t)\right] \geq \alpha y(t),$$

which completes the proof. □

Using Proposition 1 and Theorem 1, we can state the following Caputo fractional difference monotonicity result.

Theorem 2 *Let a function* $y : \mathbb{N}_{a-1} \to \mathbb{R}$ *satisfy* $y(a) \geq 0$. *Suppose that, for* $0 < \alpha \leq 1$,

$$\left({}^{\mathrm{CFC}}_{a-1}\nabla^{\alpha} y\right)(t) \geq \frac{-B(\alpha)}{1-\alpha} f(a-1)(1-\alpha)^{t-a+1}, \quad t \in \mathbb{N}_{a-1}.$$

Then $y(t)$ *is* α*-increasing.*

Theorem 3 *Let a function* $y : \mathbb{N}_{a-1} \to \mathbb{R}$ *satisfy* $y(a) \geq 0$ *and be increasing on* $\mathbb{N}_a$. *Then, for* $0 < \alpha \leq 1$,

$$\left({}^{\mathrm{CFR}}_{a-1}\nabla^{\alpha} y\right)(t) \geq 0, \quad t \in \mathbb{N}_{a-1}.$$

Proof Again, rewriting $({}^{\mathrm{CFR}}_{a-1}\nabla^{\alpha} y)(t) = B(\alpha)\nabla S(t)$, where $S(t) = \sum_{s=a}^{t} y(s)(1-\alpha)^{t-s}$, it suffices to show that $S(t)$ is increasing on $\mathbb{N}_a$. Substituting $t = a$ into (11) implies that $S(a) - S(a-1) = y(a) \geq 0$ by assumption. Assume that $S(i) - S(i-1) \geq 0$ for all $i < t$. We shall show that $S(t) - S(t-1) \geq 0$. By the assumption that y is increasing we conclude that $y(t) \geq y(t-1) \geq y(a) \geq 0$ for all $t = a + k \in \mathbb{N}_a$. Now, we have

$$\begin{aligned}
S(t) - S(t-1) &= y(t) - \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s} \\
&= y(t) - \alpha y(t-1) - \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-2} y(s)(1-\alpha)^{t-s} \\
&= y(t) - \alpha y(t-1) \\
&\quad - \frac{\alpha}{1-\alpha}\left[\sum_{s=a}^{t-2}\bigl(y(s) - y(t-1)\bigr)(1-\alpha)^{t-s} + \sum_{s=a}^{t-2} y(t-1)(1-\alpha)^{t-s}\right] \\
&\geq y(t) - \alpha y(t-1) - \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-2} y(t-1)(1-\alpha)^{t-s}
\end{aligned}$$

$$\begin{aligned}
&= y(t) - y(t-1) + y(t-1) - \frac{\alpha}{1-\alpha} y(t-1) \sum_{s=a}^{t-1} (1-\alpha)^{t-s} \\
&\geq y(t-1) \left[1 - \frac{\alpha}{1=\alpha} \sum_{s=a}^{t-1} (1-\alpha)^{t-s} \right] \\
&= y(t-1) \left[1 - \alpha(1-\alpha)^k \sum_{s=1}^{k} (1-\alpha)^{-s} \right] \\
&= y(t-1)(1-\alpha)^k \geq 0, \qquad (12)
\end{aligned}$$

which completes the proof. □

Similarly, can prove the following result.

Theorem 4 *Let a function* $y : \mathbb{N}_{a-1} \to \mathbb{R}$ *satisfy* $y(a) > 0$ *and be strictly increasing on* $\mathbb{N}_a$. *Then, for* $0 < \alpha \leq 1$,

$$\left({}_{a-1}^{\mathrm{CFR}}\nabla^{\alpha} y\right)(t) > 0, \quad t \in \mathbb{N}_{a-1}.$$

The following results can also be proved in a similar way.

Theorem 5 *Let a function* $y : \mathbb{N}_{a-1} \to \mathbb{R}$ *satisfy* $y(a) \leq 0$. *Suppose that, for* $0 < \alpha \leq 1$,

$$\left({}_{a-1}^{\mathrm{CFR}}\nabla^{\alpha} y\right)(t) \leq 0, \quad t \in \mathbb{N}_{a-1}.$$

Then $y(t)$ *is* α*-decresing.*

Theorem 6 *Let a function* $y : \mathbb{N}_{a-1} \to \mathbb{R}$ *satisfy* $y(a) \leq 0$ *and be decreasing on* $\mathbb{N}_a$. *Then, for* $0 < \alpha \leq 1$,

$$\left({}_{a-1}^{\mathrm{CFR}}\nabla^{\alpha} y\right)(t) \leq 0, \quad t \in \mathbb{N}_{a-1}.$$

4 Application: mean value theorem

We know that $({}_{a}^{\mathrm{CF}}\nabla^{-\alpha}\,{}_{a}^{\mathrm{CFR}}\nabla^{\alpha} y)(t) = y(t)$. However, the next result, which provides an initial condition $y(a)$, will be a tool to prove our fractional difference mean value theorem.

Theorem 7 *For* $0 < \alpha \leq 1$, *we have*

$$\left({}_{a}^{\mathrm{CF}}\nabla^{-\alpha}\,{}_{a-1}^{\mathrm{CFR}}\nabla^{\alpha} y\right)(t) = y(t) - \alpha y(a). \qquad (13)$$

Proof By definition and Lemma 1 we have

$$\begin{aligned}
\left({}_{a}^{\mathrm{CF}}\nabla^{-\alpha}\,{}_{a-1}^{\mathrm{CFR}}\nabla^{\alpha} y\right)(t) &= {}_{a}^{\mathrm{CF}}\nabla^{-\alpha} \left[B(\alpha)\nabla_t \sum_{s=a}^{t} y(s)(1-\alpha)^{t-s} \right] \\
&= B(\alpha)\,{}_{a}^{\mathrm{CF}}\nabla^{-\alpha}\nabla_t \left[y(a)(1-\alpha)^{t-a} + \sum_{s=a+1}^{t} f(s)(1-\alpha)^{t-s} \right]
\end{aligned}$$

$$
\begin{aligned}
&= B(\alpha)y(a)(1-\alpha)^{-a}\,{}^{CF}_{a}\nabla^{-\alpha}\nabla(1-\alpha)^{t} + {}^{CF}_{a}\nabla^{-\alpha}\,{}^{CF}_{a}\nabla^{\alpha}y(t) \\
&= -\alpha B(\alpha)y(a)(1-\alpha)^{-a}\,{}^{CF}_{a}\nabla^{-\alpha}(1-\alpha)^{t-1} + y(t) \\
&= y(t) - \alpha y(a).
\end{aligned}
$$

The proof is completed. □

Theorem 8 (The fractional difference MVT) *Let f and g be functions defined on $\mathbb{N}_{a,b}$, where $a \equiv b \pmod{1}$. Assume that g is strictly increasing and $\alpha \in (0,1)$. Then, there exist $s_1, s_2 \in \mathbb{N}_{a,b}$ such that*

$$
\frac{({}^{CFR}_{a-1}\nabla^{\alpha}f)(s_1)}{({}^{CFR}_{a-1}\nabla^{\alpha}g)(s_1)} \le \frac{f(b)-\alpha f(a)}{g(b)-\alpha g(a)} \le \frac{({}^{CFR}_{a-1}\nabla^{\alpha}f)(s_2)}{({}^{CFR}_{a-1}\nabla^{\alpha}g)(s_2)}. \tag{14}
$$

Proof We follow by contradiction. Suppose (14) is not true. Then, either

$$
\frac{f(b)-\alpha f(a)}{g(b)-\alpha g(a)} > \frac{({}^{CFR}_{a-1}\nabla^{\alpha}f)(t)}{({}^{CFR}_{a-1}\nabla^{\alpha}g)(t)} \quad \text{for all } t \in \mathbb{N}_{a,b}, \tag{15}
$$

or

$$
\frac{f(b)-\alpha f(a)}{g(b)-\alpha g(a)} < \frac{({}^{CFR}_{a-1}\nabla^{\alpha}f)(t)}{({}^{CFR}_{a-1}\nabla^{\alpha}g)(t)} \quad \text{for all } t \in \mathbb{N}_{a,b}. \tag{16}
$$

Since g is strictly increasing, by Theorem 4 we conclude that $({}^{CFR}_{a}\nabla^{\alpha}g)(t) > 0$. Hence, (15) becomes

$$
\frac{f(b)-\alpha f(a)}{g(b)-\alpha g(a)}\big({}^{CFR}_{a-1}\nabla^{\alpha}g\big)(t) > \big({}^{CFR}_{a-1}\nabla^{\alpha}f\big)(t).
$$

Applying the fractional sum operator evaluated at $t = b$ to both sides of the last inequality and using (13) in Theorem 7 lead to

$$
f(b) - \alpha f(a) > f(b) - \alpha f(a),
$$

which is a contradiction. In a similar way, we can show that (16) leads to a contradiction. This completes the proof. □

Remark 3

- Since $\alpha < 1$ and g is strictly increasing, clearly, the quantity $g(b) - \alpha g(a)$ in Theorem 8 is not equal to zero.
- The corresponding coefficient of $g(b) - \alpha g(a)$ in the classical discrete fractional calculus in case of delta analysis is of the form $g(b) - \frac{\Gamma(b-a+\alpha)}{\Gamma(\alpha)\Gamma(b-a+1)}g(a)$ [23], where both $\frac{\Gamma(b-a+\alpha)}{\Gamma(\alpha)\Gamma(b-a+1)}$ and α tend to 1 as $\alpha \to 1$. The coefficient in this paper for discrete fractional differences with discrete exponential kernels is simpler, free of $\Gamma(\alpha)$, and does not depend on the end points a and b. This reflects the absence of the memory in the corresponding fractional sum.
- The results in this paper can be carried over the right fractional case by using the action of the Q-operator.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors have equal contributions. Both authors read and approved the final form of the manuscript.

Author details
[1]Department of Mathematics and General Sciences, Prince Sultan University, P. O. Box 66833, Riyadh, 11586, Saudi Arabia. [2]Department of Mathematics, Çankaya University, Ankara, 06530, Turkey. [3]Institute of Space Sciences, Magurele, Romania.

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