Monotonicity up to radially symmetric cores of positive solutions to nonlinear elliptic equations: local moving planes and unique continuation in a non-Lipschitz case ⋆

Jean Dolbeault ∗

Ceremade (UMR CNRS no. 7534), Université Paris IX-Dauphine, Place de Lattre de Tassigny, 75775 Paris Cédex 16, France
Phone: (33) 1 44 05 46 78 – Fax: (33) 1 44 05 45 99

Patricio Felmer

Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático, UMR CNRS - UChile 2071, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile

---

Abstract

We prove local monotonicity and symmetry properties for nonnegative solutions of scalar field equations with nonlinearities which are not Lipschitz. Our main tools are a *local moving plane* method and a unique continuation argument.

Key words: Elliptic equations, scalar field equations, monotonicity, symmetry, positivity, non Lipschitz nonlinearities, comparison techniques, maximum principle, Hopf's lemma, unique continuation, dead cores, cores, local symmetry

AMS classification (2000). Primary: 35B50, 35B60. Secondary: 35B65, 35A07

⋆ Partially supported by ECOS-Conicyt under grant no. C02E08, Fondecyt Grant 1030929 and FONDAP de Matemáticas Aplicadas.

∗ Corresponding author.
Email addresses: `dolbeaul@ceremade.dauphine.fr` (Jean Dolbeault), `pfelmer@dim.uchile.cl` (Patricio Felmer).
URL: `http://www.ceremade.dauphine.fr/~dolbeaul/` (Jean Dolbeault).

6 February 2004

1 Introduction

We consider solutions of the semilinear elliptic partial differential equation

$$\Delta u + f(u) = 0 \ , \quad x \in D \subset \mathbb{R}^N \ ,$$

which are nonnegative and vanishing at the boundary. In many cases, these solutions are minimizers of an energy. This is why they are usually called *ground states*. The equation, which is often called the *nonlinear scalar field equation*, plays an important role in various domains of Physics, Chemistry and Population Dynamics, and it is fundamental from the mathematical point of view.

The purpose of this paper is to analyze the symmetry and the related monotonicity question for nonlinearities that are continuous but not Lipschitz continuous. The basic tools in the proof of our theorems are *a local* variant of the *moving plane method* and the *unique continuation principle*.

The moving plane method goes back to Alexandrov for the study of manifolds with constant mean curvature [1]. It was then applied to the study of the symmetry of positive solutions of elliptic PDEs by Serrin in [20], by Gidas, Ni and Nirenberg in [12,13], in case of a locally Lipschitz nonlinearity or at least a sum of Lipschitz and non decreasing functions, and then generalized: see [17] for more complete references. For instance in [15,4,5], symmetry results are obtained without Lipschitz property at zero, by assuming, in [5], that f is decreasing in a neighborhood of $u = 0$. Also see [9,10] for results in this direction, when the nonlinearity is not even assumed to be continuous. The monotonicity property of f has been used in a different context by Li and Ni [17] to overcome the lack of decay of the solutions when $f'(0) = 0$. We will use it here to obtain a result of *local radial symmetry* which will be defined below.

Such a notion of symmetry has already been introduced by Brock in [2,3] using a continuous Steiner symmetrization method. In the cases where Brock's result applies, our method is weaker, but on the other hand our description of how the global symmetry breaks is more detailed. It provides monotonicity results in unbounded domains as well, and can handle some nonlinear elliptic equations which are not in divergence form. For instance, the case of fully nonlinear elliptic operators with appropriate symmetries is covered. Moreover, we prove that monotonicity also holds in directions close to the direction of symmetry, which allows us to prove that when monotonicity holds up to "cores" these cores are radially symmetric, even for domains which are not balls. As far as we know, such results are certainly out of reach of symmetrization methods.

As a first step, we find what we call a *γ-core*, that is a subset of D on which the function u is symmetric with respect to a hyperplane orthogonal to the direction γ, and has some monotonicity properties. Then by choosing appropriate directions γ, we find a *radially symmetric core*, *i.e.* a ball on which the solution is radially symmetric and non increasing along any radius. All obstructions to monotonicity are shown to be due to radially symmetric cores. Finally we show under some further regularity assumptions on f that if D is a ball the function u is actually radially symmetric non increasing, or has monotonicity properties in the other cases. Our proof relies on some local unique continuation properties when the solution has a non zero gradient. The unique continuation principle in the context of symmetry results has already been used by Lopes in [18], for vector valued minimizers of an energy. Here we use it together with the generalization to PDEs of a trick which has been used in [19,11] for studying the uniqueness of radially symmetric solutions. We believe that this is a useful tool for symmetry methods, which has already proved its efficiency in the 2-dimensional case, see [6].

Let us state our main results. Consider the nonlinear elliptic problem

$$\begin{aligned} &\Delta u + f(u) = 0\ , \quad u > 0 \quad \text{in } D\ , \\ &u = 0 \qquad\qquad\qquad\qquad\ \ \text{on } \partial D\ , \end{aligned} \tag{1}$$

and assume that f satisfies the assumptions:

(f1) *$f : \mathbb{R}^+ \to \mathbb{R}$ is continuous,*

(f2) *For any $s \in [0, +\infty)$, there exists a positive constant η such that on $]s-\eta, s+\eta[\,\cap\,\mathbb{R}^+$, f is either (strictly) decreasing or it is the sum of a Lipschitz and of a non decreasing function (in the latter case, we shall say that f is Lipschitz + increasing in a neighborhood of s in $\mathbb{R}^+$).*

Also assume that D is a domain in $\mathbb{R}^N$ with one of the two following properties (the unit vector $e_1 \in S^{N-1}$ is given and we denote by x_1 the coordinate along this direction):

(B) *<u>Bounded case:</u> D is x_1-convex, bounded, symmetric with respect to the hyperplane*

$$T^{e_1} := \{x \in \mathbb{R}^N \ : \ x \cdot e_1 = 0\}$$

and has the property

$$\forall\, \epsilon > 0 \quad \exists\, \eta > 0 \quad \textit{such that} \qquad \forall\, \lambda > \epsilon$$
$$\nu \in S^{N-1},\ |\nu - e_1| < \eta \Longrightarrow \{x + 2(\lambda - x\cdot\nu)\,\nu \ : \ x \in D,\ x\cdot\nu > \lambda\} \subset D\ ,$$

(C) Case with <u>Cone property:</u> *there exists* $\eta > 0$ *such that for any* $\lambda \in \mathbb{R}$ *and* $\nu \in S^{N-1}$ *such that* $|\nu - e_1| < \eta$, *the set* $\{x \in D \, : \, x \cdot \nu > \lambda\}$ *is bounded and*

$$D = \bigcup_{\nu \in S^{N-1}, \, |\nu - e_1| < \eta} \{ y - t\nu \, : \, y \in \partial D \, , \, t > 0 \} \, .$$

Assumption (B) means that D is symmetric with respect to T^{e_1} and that for directions ν which are close to e_1, the image of the reflection by the hyperplane $\{x \in \mathbb{R}^N \, : \, x \cdot \nu = \lambda\}$ of the domain $\{x \in D \, : \, x \cdot \nu > \lambda\}$ is contained in D provided $\lambda > \epsilon$. Ellipsoids are an example of such a set. Assumption (C) essentially means that ∂D is the graph of a uniformly Lipschitz function of $x' = (x_2, x_3, ... \, x_N)$ which goes to $-\infty$ as $|x'| \to +\infty$. In order to describe our results we introduce the following notion of local monotonicity.

Definition. A nonnegative function u is said to be *monotone up to cores* on $\tilde{D}$ in the direction e_1, where $\tilde{D} \subset D$ is a bounded subdomain, if there are nonnegative functions $\tilde{u}, u_1, ..., u_k$ defined on $\tilde{D}$ such that:

(i) $u|_{\tilde{D}} = \tilde{u} + \sum_{j=1}^{k} u_j$,
(ii) the functions u_j have support in balls B_j intersecting $\tilde{D}$ and they are radially symmetric non increasing, with respect to the center of B_j, $j = 1, ...k$,
(iii) if $B_i \cap B_j \neq \emptyset$, $i \neq j$, then either $B_i \subset B_j$ and u_j is constant on B_i or $B_j \subset B_i$ and u_i is constant on B_j,
(iv) $\tilde{u}$ is monotone non increasing on $\tilde{D}$ in the direction e_1, and it is constant on any B_j, $j = 1, ...k$.

Theorem 1 *Assume that f satisfies (f1), (f2) and D satisfies (B) (resp. (C)). Let $u \in C^2(D) \cap C^0(\bar{D})$ be a solution of (1). Then u is monotone up to cores on $\tilde{D} = \{x \in D \, : \, x \cdot e_1 \geq 0\}$ in the direction e_1 (resp. on $\tilde{D} = D$). Moreover in case (B), with the above notations, $\tilde{u}$ is symmetric with respect to T^{e_1}.*

Actually, the result is a little bit stronger, and the monotonicity up to cores is true in any of the entering directions in case (C), or in any of the directions ν such that $|\nu - e_1| < \eta$, on the domain $\{x \in D \, : \, x \cdot \nu \geq \epsilon\}$, in case (B). Under the following additional assumption

(f3) *For any* $u > 0$ *such that* $f(u) = 0$, $\liminf_{v \to u, \, v > u} \frac{f(v)}{v-u} > -\infty$,

we obtain global monotonicity and symmetry results:

Theorem 2 *Assume that f satisfies (f1), (f2), (f3) and D satisfies either (B) or (C). Let $u \in C^2(D) \cap C^0(\bar{D})$ be a solution of (1). Then u is decreasing in any direction ν given by Conditions (B) or (C), on $\tilde{D}$ defined as in Theorem 1. In case (B), u is symmetric with respect to T^{e_1}.*

The paper is organized as follows. In Section 2, we give some definitions, develop a framework for a *local moving plane* method and prove a crucial lemma that allows us to obtain the cores. In Section 3 we present the main proofs and give monotonicity and symmetry results using the *unique continuation principle*. Section 4 is devoted to some extensions (weaker conditions on f, whole space results, fully nonlinear case). Some of the results of this paper were announced in [7].

2 A technical lemma for local moving planes

In this section we set up the basic notation and give some definitions. Then we state and prove a crucial technical lemma.

Consider a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of

$$\begin{aligned} &\Delta u + f(u) = 0 \ , \quad u \geq u_0 && \text{in } \Omega \ , \\ &u = u_0 && \text{on } \partial\Omega \ , \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$.

Definition. Given $\gamma \in S^{N-1}$, Ω is said to be a $\gamma-$ *core* of u if and only if

(i) There exists a real number λ_Ω such that Ω *and* u *are symmetric* with respect to the hyperplane $T_{\lambda_\Omega} := \{x \in \mathbb{R}^N \ : \ x \cdot \gamma = \lambda_\Omega\}$. In other words, for any $x \in \Omega$, we have: $x_{\lambda_\Omega} := x - 2(x \cdot \gamma - \lambda_\Omega)\,\gamma \in \Omega$ and $u(x_{\lambda_\Omega}) := u(x)$,
(ii) Ω is *convex in the* $\gamma-$ *direction* (or $\gamma-$ *convex*), which means that for any $x \in \Omega$, the set $\{t \in \mathbb{R} \,:\, (x + t\gamma) \in \Omega\}$ is an interval,
(iii) $\nabla u(x) \cdot \gamma \leq 0$ for any $x \in \Omega$ such that $x \cdot \gamma > \lambda_\Omega$.

The domain Ω is said to be a *radially symmetric core* of u, or simply a *core* of u, if it is a γ-core of u for any direction $\gamma \in S^{N-1}$. We may observe that such a core of u is a ball on which u is radially symmetric with respect to the center of the ball and non increasing along any radius.

Remark 1 *In order to prove that a given set Ω is a ball, it is sufficient to prove that it is symmetric with respect to N independent hyperplanes corresponding to N independent orthogonal directions $\gamma_i \in S^{N-1}$, $i = 1, 2, \ldots N$ such that the angle (γ_i, γ_j) is 2π-irrational for any (i, j) with $i \neq j$.*

In dimension $N = 2$, if two lines make a 2π-irrational angle θ_0, then the composition of two orthogonal reflections with respect to each of these two lines *gives a rotation of angle $\pm 2\theta_0$ which is 2π-irrational too. The set $\{n\theta_0\}_{n\in\mathbb{Z}}$ is dense in S^1 so that Ω is a disk.*

In dimension $N \geq 3$, let $x^0 \in \Omega$: $x^0 = \sum_{i=1}^{N} x_i^0 \gamma_i$, since the $\{\gamma_i\}_{i=1,2,\dots N}$ are linearly independent. Here we take the origin to be the unique point in the intersection of the hyperplanes T_{λ_i} associated to the directions γ_i. Next we consider the affine plane $\Pi_{1,N}(x^0) = \{x \in \mathbb{R}^N : x = x^0 + y,\ y \in span(\gamma_1, \gamma_N)\}$. By using the 2-dimensional argument given above we see that we can rotate x^0 in $\Pi_{1,N}(x^0)$ to obtain $x^1 = \sum_{i=1}^{N-1} x_i^1 \gamma_i$. Of course $|x^0| = |x^1|$. We can repeat this argument $N-1$ times until getting $x^{N-1} = x_1^{N-1}\gamma_1$, where $x_1^{N-1} = |x^0|$. Since x_0 is arbitrary, Ω is a ball.

If Ω is a γ_i-core of u for $\{\gamma_i\}_{i=1,2,\dots N}$ as above, then $u(x^0) = u(|x^0|\gamma_1)$. Thus u is radially symmetric and we shall say that Ω is a radially symmetric core *of u.*

Now we set up some notational conventions. Whenever possible, given $\gamma \in S^{N-1}$ we will choose a system of coordinates so that $\gamma = e_1$. In that case we write x_1-*core* for a γ-core. Following the usual notations we consider

$$T_\lambda := \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \ : \ x_1 = \lambda\} \ ,$$
$$\Sigma_\lambda := \{x = (x_1, x') \in (\mathbb{R} \times \mathbb{R}^{N-1}) \ : \ x_1 > \lambda\} \ .$$

If $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}$, then we write

$$x_\lambda := (2\lambda - x_1, x') \quad \text{and} \quad u_\lambda(x) := u(x_\lambda)$$

for any $x \in \mathbb{R}^N$ such that $x_\lambda \in \Omega$.

Let Ω be a non empty bounded domain in $\mathbb{R}^N$. We say that Ω satisfies *property $\mathcal{P}$* if and only if the following conditions are satisfied:

(i) Ω is symmetric with respect to the hyperplane $T_{\lambda_\Omega} = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \ : \ x_1 = \lambda_\Omega\}$ for some $\lambda_\Omega \in \mathbb{R}$, that is, $x_{\lambda_\Omega} \in \Omega$ for any $x \in \mathbb{R} \times \mathbb{R}^{N-1} \cap \Omega$,
(ii) Ω is x_1-convex,
(iii) There is a constant u_0 such that $u_{|\partial\Omega} \equiv u_0$ and $u > u_0$ on Ω,
(iv) There exists an $\epsilon > 0$ such that f is decreasing on $[u_0, u_0 + \epsilon)$.

Remark 2 *If Ω satisfies property $\mathcal{P}$, then it is an x_1-core of u if and only if $u_{\lambda_\Omega}(x) = u(x)$ for any $x \in \Omega$, and $\frac{\partial u}{\partial x_1}(x) \leq 0$, for any $x \in \overline{\Sigma}_{\lambda_\Omega} \cap \Omega$.*

We are now in a position to state a technical lemma which is the key tool of our approach.

Lemma 3 *Under assumptions (f1) and (f2), if Ω is a non-empty subdomain of D satisfying property $\mathcal{P}$, then there exists $\overline{\lambda} \geq \lambda_\Omega$ such that*

$$\frac{\partial u}{\partial x_1} \leq 0 \quad on \quad \Omega \cap \Sigma_{\overline{\lambda}} \,,$$

and either $\overline{\lambda} = \lambda_\Omega$, or $\overline{\lambda} > \lambda_\Omega$ and there exists $\overline{x} \in \overline{\Sigma_{\overline{\lambda}}} \cap \Omega$ such that

$$u_{\overline{\lambda}}(\overline{x}) = u(\overline{x}) \quad if \quad \overline{x} \in \Sigma_{\overline{\lambda}} \,, \qquad \frac{\partial u}{\partial x_1}(\overline{x}) = 0 \quad if \quad \overline{x} \in T_{\overline{\lambda}} \,.$$

If $\overline{\lambda} = \lambda_\Omega$, the same properties hold if we replace the direction x_1 by the direction $-x_1$, so that, up to this change of coordinates, there are two possibilities:

(Case a) *either $\overline{\lambda} = \lambda_\Omega$ and $u_{\overline{\lambda}}(x) = u(x)$ for any $x \in \Omega$,*
(Case b) *or there exist $u_1 > u_0$ and $u_2 > u_1$, with $u(\overline{x}) \in (u_1, u_2)$, such that f is locally Lipschitz + increasing on (u_1, u_2).*

Assume that Case b holds and let (u_1, u_2) be the maximal interval containing $u(\overline{x})$ in $(u_0, +\infty)$, on which f is locally Lipschitz + increasing. Then the two following properties hold.

(i) *Let $\mathcal{C}$ be the connected component of $\{x \in \Omega \,:\, u_1 < u(x) < u_2\}$ containing $\overline{x}$. Then we have $u_{\overline{\lambda}}(x) = u(x)$ for any $x \in \mathcal{C}$,*
(ii) *Let $\tilde{\mathcal{C}}$ be the x_1-*convexified *of $\mathcal{C}$, i.e. the set*

$$\tilde{\mathcal{C}} := \{x \in \Omega \,:\, \exists\, (y,z) \in \mathcal{C} \times \mathcal{C} \text{ such that } z - y \text{ is parallel to } x_1 \\ \text{and } \exists\, t \in]0,1[\text{ such that } x = ty + (1-t)z\} \,,$$

and $\tilde{\Omega} := \{x \in \tilde{\mathcal{C}} \,:\, u(x) > u_2\}$. Then either $\tilde{\Omega} = \emptyset$ or $\tilde{\Omega}$ satisfies property $\mathcal{P}$.

Remark 3 *In Case a, the set Ω is an x_1-core of u. In Case b, if $\tilde{\Omega} = \emptyset$, then $\mathcal{C} = \tilde{\mathcal{C}}$ is an x_1-core of u. And if $\tilde{\Omega} \neq \emptyset$, let $\hat{\Omega} := \{x \in \Omega \,:\, u(x) > \tilde{u}_0\} \supset \tilde{\Omega}$, where $\tilde{u}_0 := \inf\{u \in [u_1, u_2] \;:\; f \text{ is strictly decreasing on } [u, u_2]\}$. If $\overline{u} := \max_{x \in \hat{\Omega}} u(x)$ is such that f is decreasing on $[\tilde{u}_0, \overline{u}]$, then $\hat{\Omega}$ is an x_1-core of u. The proof is direct: on $\partial\hat{\Omega} \subset \mathcal{C}$, $u_{\overline{\lambda}} \equiv u$, $u_{\overline{\lambda}} \geq u$ in $\hat{\Omega}$ according to Lemma 3 and since f is decreasing, $-\Delta(u_{\overline{\lambda}} - u) \leq 0$, which means $u_{\overline{\lambda}} \leq u$.*

Proof of Lemma 3. The proof relies on the moving plane technique. We say that Ω satisfies *property* Π_λ if

$$w_\lambda(x) := u_\lambda(x) - u(x) \geq 0 \quad \forall\, x \in \Omega \cap \Sigma_\lambda \,.$$

Let $\quad \lambda^* := \sup\{\lambda \in \mathbb{R} \,:\, \exists\, x' \in \mathbb{R}^{N-1} \text{ such that } (\lambda, x') \in \Omega\}$,

$$\overline{\lambda} := \inf\{\lambda \in [\lambda_\Omega, \lambda^*] \,:\, \forall\, \mu \in (\lambda, \lambda^*) \; \Pi_\mu \text{ is true }\} \,.$$

We will first see that $\overline{\lambda} < \lambda^*$. Assume by contradiction that $\overline{\lambda} = \lambda^*$. Then there exists an increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ converging to λ^* such that

$$\forall\, k \in \mathbb{N} \quad \exists\, x_k \in \Sigma_{\lambda_k} \cap \Omega \quad w_{\lambda_k}(x_k) < 0 \; .$$

On $\partial(\Sigma_{\lambda_k} \cap \Omega)$, $w_{\lambda_k} \geq 0$, so that w_{λ_k} reaches its minimum value at some point of $\Sigma_{\lambda_k} \cap \Omega$ and we may assume that x_k realizes this minimum. Then we have

$$0 \geq -\Delta w_{\lambda_k}(x_k) = f(u_{\lambda_k}(x_k)) - f(u(x_k)) = f(u(x_k) + w_{\lambda_k}(x_k)) - f(u(x_k)) > 0 \, ,$$

for k large enough, since $u_0 < u(x_k) < u_0 + \epsilon$, a contradiction. Thus $\overline{\lambda} < \lambda^*$.

Assume now that $\overline{\lambda} > \lambda_\Omega$, where λ_Ω is defined in part (i) of property $\mathcal{P}$. We recall that Ω is symmetric with respect to T_{λ_Ω}. Again we may find an increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ converging to $\overline{\lambda}$ (with $\lambda_\Omega < \lambda_k < \overline{\lambda}$), and a sequence of points $(x_k)_{k\in\mathbb{N}}$ such that

$$x_k \in \Sigma_{\lambda_k} \cap \Omega \; , \quad w_{\lambda_k}(x_k) = \min_{x\in\Sigma_{\lambda_k}\cap\Omega} w_{\lambda_k}(x) < 0 \; .$$

As above, $x_k \notin u^{-1}(\mathcal{U})$ where $\mathcal{U}$ is a neighborhood of $u(x)$ in $[u_0, +\infty)$ on which f is decreasing, because $-\Delta w_{\lambda_k}(x_k) \leq 0$. Up to the extraction of a subsequence, we may assume

$$\lim_{k\to+\infty} x_k = \overline{x} \in \overline{\Omega \cap \Omega_0^c} \cap \overline{\Sigma_{\overline{\lambda}}} \; .$$

On the one hand we have $0 \geq u_{\overline{\lambda}}(\overline{x}) - u(\overline{x}) = \lim_{k\to+\infty} w_{\lambda_k}(x_k)$, and on the other hand $u_{\overline{\lambda}}(\overline{x}) \geq u(\overline{x})$ because of $\Pi_{\overline{\lambda}}$. Note indeed that $\Pi_{\overline{\lambda}}$ is true since Π_λ is true for any $\lambda - \overline{\lambda} > 0$, small enough. Also note that either $\overline{x} \in \Sigma_{\overline{\lambda}}$, or $\overline{x} \in T_{\overline{\lambda}}$. In the latter case, $\frac{\partial u}{\partial x_1}(\overline{x}) = -\frac{1}{2} \lim_{k\to+\infty} \frac{\partial w_{\lambda_k}}{\partial x_1}(x_k) = 0$.

Since f is decreasing in $[u_0, u_0 + \epsilon)$ for some $\epsilon > 0$ (and since $u_{|\partial\Omega} = u_0$), again $\overline{x}$ cannot belong to $\partial\Omega$. Then $\overline{x} \in \Omega\backslash\Omega_0$ and f is therefore Lipschitz + increasing in a neighborhood of $u(\overline{x})$, which proves the properties of Case b.

If $\overline{\lambda} = \lambda_\Omega$, we may exchange the direction x_1 and $-x_1$. We observe that property $\mathcal{P}$ is invariant under the transformation $(x_1, x') \mapsto (2\lambda_\Omega - x_1, x')$. Then either we find a $\overline{\lambda} \neq \lambda_\Omega$ and we are back to the previous case, or we get $\overline{\lambda} = \lambda_\Omega$, which proves that $u_{\lambda_\Omega}(x) = u(x)$ for any $x \in \Omega$.

Since $\Pi_{\overline{\lambda}}$ is true, the monotonicity of u with respect to x_1 on $\Omega \cap \Sigma_{\overline{\lambda}}$ follows. In fact, for any $\lambda \in [\overline{\lambda}, \lambda^*)$, for any $x = (\lambda, x') \in (T_\lambda \cap \Omega) \subset (\overline{\Sigma_{\overline{\lambda}}} \cap \Omega)$,

$$0 \leq \frac{1}{\varepsilon} \cdot w_\lambda(\lambda + \varepsilon, x') \to -2\frac{\partial u}{\partial x_1}(\lambda, x') \quad \text{as } \varepsilon \searrow 0 \; .$$

Assertion (i) is obtained, in case $\overline{x} \in \Sigma_{\overline{\lambda}}$ as a consequence of the maximum principle applied to $w_{\overline{\lambda}}$. Note that $w_{\overline{\lambda}} \geq 0$ and $w_{\overline{\lambda}}(\overline{x}) = 0$ with $\overline{x} \in \mathcal{C}$. When $\overline{x} \in T_{\overline{\lambda}}$, assertion (i) is a consequence of Hopf's Lemma, since in this case, $\frac{\partial w_{\overline{\lambda}}}{\partial x_1}(\overline{x}) = -2\frac{\partial u}{\partial x_1}(\overline{x}) = 0$.

To finish with the proof of Lemma 3, one has to check that in Case b, $\tilde{\Omega}$ satisfies property $\mathcal{P}$ if it is not empty. The symmetry and the x_1-convexity follow from the definition of $\tilde{\Omega}$, and f is decreasing on a neighborhood of u_2 in $[u_2, +\infty)$ again because of assumption (f2). □

3 Unique continuation and proofs of the main results

Before proving Theorem 1 and a slightly more general result in Theorem 5, let us state an important property of the radially symmetric cores, which is based on a *unique continuation argument*. Consider

$$\begin{aligned} &\Delta u + f(u) = 0\ , \quad u \geq 0 \quad &&\text{in} \quad D\ , \\ &u = 0 &&\text{on} \quad \partial D\ . \end{aligned} \tag{2}$$

Lemma 4 *Under Assumtion (f1), let u be a nonnegative solution of (2) which is monotone on $\Sigma_{\overline{\lambda}}$ for some $\overline{\lambda} \in \mathbb{R}$. Assume that either condition (B) or condition (C) is satisfied. If $\Omega \subset D$ is a radially symmetric core of u such that $\Omega \cap \Sigma_{\overline{\lambda}} \neq \emptyset$, then either $\Omega = D$ or u is constant on $\partial\Omega$, $f(u_{|\partial\Omega}) = 0$ and $\nabla u_{|\partial\Omega} = 0$.*

As we shall see below, the only property needed to prove that the solutions corresponding to a continuous nonlinearity f are locally symmetric is

$$\{u(x)\ :\ x \in D\,,\ \nabla u(x) = 0\} \subset f^{-1}(0)\ .$$

This has been exploited in [6] in the case of the dimension $N = 2$ but still needs to be proved in higher dimensions.

Proof. For simplicity, we assume that the center of the core is $x = 0$. This is easily achieved by means of a translation. Let us define

$$\rho := \max\{r > 0\ :\ B(0,r) \subset D \text{ and } u \text{ is radially symmetric, non increasing in } B(0,r)\}\ .$$

By non increasing, we mean, with an evident abuse of notations,

$$\frac{du}{dr} = \frac{x}{|x|} \cdot \nabla u \leq 0 \ .$$

We will prove that either $B(0,\rho) = D$ or $\frac{du}{dr}(\rho) = 0$. In this last case, for any $\overline{x} \in \partial B(0,\rho) \cap \Sigma_{\overline{\lambda}}$,

$$\nabla u(\overline{x}) = \frac{du}{dr}(\rho)\,\frac{\overline{x}}{\rho} = 0 \ .$$

By definition of the cores, we know that $\frac{du}{dr} \leq 0$ for any $r \in (0,\rho)$, so that $\frac{d^2u}{dr^2}(\rho) \geq 0$. If $\frac{d^2u}{dr^2}(\rho) > 0$, we immediately get a contradiction with the monotonicity property of u in $\Sigma_{\overline{\lambda}}$. Thus

$$\Delta u(\overline{x}) = \Big(\frac{d^2u}{dr^2} + \frac{N-1}{r}\frac{du}{dr}\Big)_{|r=\rho} = 0 \ ,$$

and then $f(u(\overline{x})) = 0$. From now on, assume that $\frac{du}{dr}(\rho) \neq 0$, which actually means $\frac{du}{dr}(\rho) < 0$.

If $D \cap \partial B(0,\rho) \neq \emptyset$, then $B(0,\rho) = D$. Otherwise, $\frac{du}{dr}(\rho) < 0$ would contradict the condition $u \geq 0$ in D. In fact, since u is radially symmetric in $\overline{B(0,\rho)}$, $u_{|\partial B(0,\rho)} = 0$.

If $\partial D \cap \partial B(0,\rho) = \emptyset$, then there exists a sequence of points $(x_k)_{k\in\mathbb{N}}$ of D such that $(|x_k|)_{k\in\mathbb{N}}$ is decreasing and converges to ρ and such that $u(x_k) \neq u(R_k x_k)$, where R_k is the reflection with respect to some hyperplane containing the origin and defined by a direction ν_k close to e_1. Without loss of generality, we may assume that $x_k \to \overline{x}$ for some $\overline{x} \in \partial B(0,\rho)$, and $\nu_k \to \nu$ for some $\nu \in S^{N-1}$. Thus $R_k \to \overline{R}$, where $\overline{R}$ is the reflection with respect to the hyperplane defined by ν.

For notational convenience, we can perform a rotation such that $\overline{x} = \rho \cdot e_1$. The monotonicity with respect to e_1 is true at least locally because $\nabla u(\overline{x}) \neq 0$: since the rest of the argument is local, we do not have to take care of the geometrical restrictions corresponding to assumptions (B) or (C). For some $\sigma > 0$ small enough, we have then $\frac{\partial u}{\partial x_1}(x) < 0$ for any $x \in B(\overline{x},\sigma)$. If we denote $\overline{u}(x) = u(\overline{R}x)$, we have that $\overline{u}$ provides another solution of

$$\Delta u + f(u) = 0 \ , \quad \forall\, x \in B(0,\rho) \cup B(\overline{x},\sigma)$$

such that $u \not\equiv \overline{u}$ in $B(\overline{x},\sigma)$. We observe that, taking σ smaller if necessary, $\frac{\partial \overline{u}}{\partial x_1}(x) < 0$ for any $x \in B(\overline{x},\sigma)$.

Here we shall use a local argument which involves a local change of coordinates. This transformation is the extension to $N \geq 2$ of the one used in [6] in the case $N = 2$. By the Implicit Function Theorem, there exists a neighborhood $\mathcal{V}$ of $(u(\overline{x}), 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and two functions v and $\overline{v}$ of class C^2 such that

$$t = u(v(t,x'),x') \quad \text{and} \quad t = \overline{u}(\overline{v}(t,x'),x') \qquad \forall\,(t,x') \in \mathcal{V}\ ,$$

with $\frac{\partial v}{\partial t} \neq 0$ and $\frac{\partial \overline{v}}{\partial t} \neq 0$ in $\mathcal{V}$. After some computations, we find that the function v satisfies in $\mathcal{V}$ the quasilinear equation

$$\left[1 + \sum_{i=2}^{N}\left(\frac{\partial v}{\partial x_i}\right)^2\right]\frac{\partial^2 v}{\partial t^2} - 2\frac{\partial v}{\partial t}\sum_{i=2}^{N}\frac{\partial v}{\partial x_i}\frac{\partial^2 v}{\partial x_i \partial t} + \left(\frac{\partial v}{\partial t}\right)^2\sum_{i=2}^{N}\frac{\partial^2 v}{\partial x_i^2} = \left(\frac{\partial v}{\partial t}\right)^3 f(t)\ .$$

A similar equation is satisfied by the function $\overline{v}$. It is easy to see that these equations are elliptic in $\mathcal{V}$.

We may now consider the function $z(t,x') = v(t,x') - \overline{v}(t,x')$ that satisfies in $\mathcal{V}$ the equation

$$a\frac{\partial^2 z}{\partial t^2} - 2\frac{\partial v}{\partial t}\sum_{i=2}^{N}\frac{\partial v}{\partial x_i}\frac{\partial^2 z}{\partial x_i \partial t} + \left(\frac{\partial v}{\partial t}\right)^2\sum_{i=2}^{N}\frac{\partial^2 z}{\partial x_i^2}b_1\frac{\partial z}{\partial t} + \sum_{i=2}^{N} b_i\frac{\partial z}{\partial x_i} = 0\ ,$$

where the coefficients a and b_i are given by

$$\begin{aligned}
a(t,x') &= 1 + \sum_{i=2}^{N}\left(\frac{\partial v}{\partial x_i}\right)^2\ ,\\
b_1(t,x') &= -2\sum_{i=2}^{N}\Big(\frac{\partial v}{\partial x_i}\frac{\partial^2 \overline{v}}{\partial x_i \partial t}\Big) + \Big(\frac{\partial v}{\partial t} + \frac{\partial \overline{v}}{\partial t}\Big)\Big(\sum_{i=2}^{N}\frac{\partial^2 \overline{v}}{\partial x_i^2}\Big)\\
&\qquad -f(t)\Big\{\left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\cdot\frac{\partial \overline{v}}{\partial t}\right) + \left(\frac{\partial \overline{v}}{\partial t}\right)^2\Big\}\ ,\\
b_i(t,x') &= \frac{\partial^2 \overline{v}}{\partial t^2}\left(\frac{\partial v}{\partial x_i} + \frac{\partial \overline{v}}{\partial x_i}\right) - 2\frac{\partial^2 \overline{v}}{\partial x_i \partial t}\cdot\frac{\partial \overline{v}}{\partial t}, \quad i = 2, 3, \ldots N\ .
\end{aligned}$$

We observe that the coefficients of the second order term are all of class C^1, while the b_i's are of class C^0. Thus the equation satisfied by z has the unique continuation property, see [14] for instance.

We conclude that since u and $\overline{u}$ coincide on the open set $B(0,\rho) \cap B(\overline{x},\sigma)$, the functions v and $\overline{v}$ coincide in the corresponding open set. Therefore $u \equiv \overline{u}$ in $B(\overline{x},\sigma)$, a contradiction. $\square$

Now we can state a more refined version of Theorem 1, under a weaker assumption: we do not assume the strict positivity of u anymore.

Theorem 5 *Assume that f satisfies (f1) and (f2). Let $u \in C^2(D) \cap C^0(\bar{D})$ be a solution of (2). If condition (B) is satisfied, there exists a finite number $\mathcal{N}$ of balls B_i, $i \in \mathcal{I} = \{1, 2, \ldots \mathcal{N}\}$, contained in D such that there exists at least one $i_0 \in \mathcal{I}$ satisfying*

(i) *for any $j \in \mathcal{I} \setminus \{i_0\}$, if $B_j \cap B_{i_0} \neq \emptyset$, then $B_{i_0} \subset B_j$,*
(ii) *$u|_{B_{i_0}}$ is radially symmetric and decreasing along any radius of B_{i_0},*
(iii) *if $\mathcal{N} > 1$, the C^2 function defined on D by*

$$\tilde{u} = u \qquad \text{in } D \backslash B_{i_0}$$

$$\tilde{u} = u_{|\partial B_{i_0}} = Const \qquad \text{on } \partial B_{i_0}$$

is still a solution of (2).

In case (iii), we can then iterate and apply again the above result to $\tilde{u}$ with the set of $\mathcal{N} - 1$ balls B_i, $i \in \mathcal{I} \setminus \{i_0\}$.

In case of assumption (C), the same result is true except that $\mathcal{N}$ might be infinite. However, for any $\lambda \in \mathbb{R}$, $\mathcal{I}(\lambda) := \{j \in \mathbb{N} \,:\, B_j \cap \Sigma_\lambda \neq \emptyset\}$ is finite and the same result as above applies to $u_{|D(\lambda)}$ where $D(\lambda) = (D \cap \Sigma_\lambda) \cup (\cup_{j \in \mathcal{I}(\lambda)} B_j)$.

On $D \setminus (\cup_{j \in \mathcal{I}} B_j \cap \Sigma_0)$ in case (B), on $D(\lambda) \setminus (\cup_{j \in \mathcal{I}(\lambda)} B_j \cap \Sigma_\lambda)$ for any $\lambda \in \mathbb{R}$ in case (C), u is monotone non increasing.

Proof of Theorem 5. We first obtain an x_1-core, and then a radially symmetric core, which can be removed by the procedure described in the statement of Theorem 5. By iteration and since the possible number of cores is finite in case (B), locally finite in case (C), we prove the theorem using Lemma 4.

First step : Obtaining an x_1-core

Let u be a solution of equation (2) and define $\eta(u) := \sup\{\eta \geq 0 \,:\, \text{on } (u-\eta, u+\eta),\ u \text{ is either decreasing or Lipschitz + increasing}\}$, $\overline{\eta} := \inf\{\eta(v) \,:\, v \in [0, \max_{x \in D} u(x)]\}$. If f satisfies (f2), then $\overline{\eta} > 0$.

With the notations of Section 2, define

$$\overline{\lambda} := \inf\{\lambda > 0 \,:\, w_\lambda(x) \geq 0 \quad \forall\, x \in D \cap \Sigma_\lambda\}\,.$$

For the same reasons as in Lemma 3 if f is decreasing on $[0, \epsilon)$ for some $\epsilon > 0$, or because of the standard moving plane method (see [12]) if f is Lipschitz +

increasing on $[0,\epsilon)$ for some $\epsilon > 0$, $D \cap \Sigma_{\overline{\lambda}}$ is non empty. If f is decreasing on $[0,\epsilon)$, we take $\Omega = D \cap \Sigma_{\overline{\lambda}}$. If not, let us consider $\Omega = \{x \in D \cap \Sigma_{\overline{\lambda}} \,:\, u(x) > u_*\}$, where $u_* = \inf\{u > 0 \,:\, f$ is not Lipschitz + increasing on $(u, u+\epsilon)$ for any $\epsilon > 0$ $\}$. To prove that in both cases Ω satisfies property $\mathcal{P}$ of Lemma 3, we have more or less to repeat the arguments of the proof of Lemma 3.

Of course, if $u_* > \max_D u(x)$, the usual methods apply and the conclusions of Theorem 5 hold. The symmetry $u_\lambda \equiv u$ at $\lambda = 0$ is proved if $0 = \overline{\lambda} := \inf\{\lambda > 0 \,:\, u_\lambda \geq u$ in $D \cap \Sigma_\lambda\}$ in case of assumption (B) and if the property also holds after changing the direction x_1 to $-x_1$. The monotonicity property is also proved if $\overline{\lambda} = -\infty$ in case (C). Assume that $\overline{\lambda} > 0$ in case of assumption (B), up to a change in the coordinate direction x_1, and $\overline{\lambda} > -\infty$ in case (C): there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ with $\lambda_k < \overline{\lambda}$, $\lim_{k\to+\infty} \lambda_k = \overline{\lambda}$, and a sequence $(x_k)_{k\in\mathbb{N}}$ such that

$$u_{\lambda_k}(x_k) - u(x_k) = \min_{x \in \Sigma_{\lambda_k} \cap D} \Big(u_{\lambda_k}(x) - u(x) \Big) < 0 \ .$$

After the extraction of a subsequence, we can define $\overline{x} := \lim_{k\to+\infty} x_k$. Then f has to be Lipschitz + increasing on $(u(\overline{x}) - \overline{\eta}, u(\overline{x}) + \overline{\eta})$ for the same reason as in Lemma 3.

Exactly as in Lemma 3, $u_{\overline{\lambda}} \equiv u$ on the connected component of $\{x \in D \,:\, u_1 < u(x) < u_2\}$ where (u_1, u_2) is the maximal interval on which u is Lipschitz + increasing and such that $u(\overline{x}) \in (u_1, u_2)$. Moreover, by construction of $\overline{\lambda}$, $\frac{\partial u}{\partial x_1} \leq 0$ on $\Omega \cap \Sigma_{\overline{\lambda}}$, we have that Ω satisfies property $\mathcal{P}$.

We may now apply Lemma 3 to $\Omega = \Omega_1$ and iterate n times to find an x_1-core, n being at most the integer part of $(\overline{\eta})^{-1} \cdot \max_{x \in D \cap \Sigma_{\overline{\lambda}}} u(x)$. Here $\Sigma_{\overline{\lambda}}$ is the domain corresponding to the $\overline{\lambda}$ obtained at the first iteration. In the following, with the notations of Lemma 3, we note $\Omega_{k+1} = \tilde{\Omega}_k$ for $1 \leq k \leq n$.

Second step : *Obtaining a radially symmetric core*

If Ω_n is the last non-empty x_1-core given by the iteration procedure of Step 1, we may notice that u is constant on $\partial\Omega_n$ and strictly bigger than $u_{|\partial\Omega_n}$ in Ω_n: u reaches its maximum in Ω_n at some interior point $\overline{x}$. According to assumption (f2), two cases are possible: either there exists some $u \in (u_{|\partial\Omega_n}, u(\overline{x})]$ such that f is Lipschitz + increasing on $]u - \overline{\eta}, u + \overline{\eta})$, or not.

In the first case, by construction of Ω_n, $u_{|\partial\Omega_n} < u - \overline{\eta} \leq u(\overline{x}) - \overline{\eta}$. In the second case, we may use the set $\hat{\Omega}_n$ defined as in Remark 3: $u(\overline{x}) > u_{|\partial\hat{\Omega}_n} + \overline{\eta}$. In both cases, the method shows the existence of an x_1-core ω such that u reaches its maximum at some interior point $\overline{x}$ and $u(\overline{x}) = \max_{x\in\omega} u(x) > u_{|\partial\omega} + \overline{\eta}$. Let $M = \|\nabla u\|_{L^\infty(D)}$. Then $B(\overline{x}, \overline{r}) \subset \omega$ with $\overline{r} = \overline{\eta}/M$. The number $\mathcal{N}$ of the connected components which are x_1-cores is therefore finite and bounded by $\mathcal{N} \leq C\,(M/\overline{\eta})^N$ for some constant C which depends on the volume of $D \cap \Sigma_{\overline{\lambda}}$ (where $\overline{\lambda}$ was defined in the first step of the proof).

Let us take $(N-1)\mathcal{N}+1$ directions $\gamma_i \in S^{N-1}$, $i = 1, 2, \ldots\ (N-1)\mathcal{N}+1$, satisfying the conditions defined by assumptions (B) or (C), such that the angle (γ_i, γ_j) is 2π-irrational for any (i,j) with $i \neq j$, and such that any subfamily of N such unit vectors generates $\mathbb{R}^N$. Then, applying the method of the first two steps successively to each of these directions (for each i, choose the direction x_1 as the one of γ_i), we find at least one core ω which is symmetric with respect to at least N directions. According to Remark 1, ω is a radially symmetric core of u: u is radially symmetric with respect to the center $\overline{x}$ of ω and $(x-\overline{x}) \cdot \nabla u(x) \leq 0$ for any $x \in \omega$.

Note that in case (B), the maximum of the core obtained by iterating Lemma 3 can be arbitrarily close to the hyperplane T^{e_1}. We have to choose a direction ν close enough to e_1 so that it can be reached by moving hyperplanes along the direction ν.

Third step : *Removing a radially symmetric core.*

According to Lemma 4, on the boundary of a radially symmetric core satisfying the conditions of Lemma 4, ∇u has to be equal to 0 and the value of u is that of a zero of f. Thus it is possible to apply the procedure described in the statement of the theorem. The function $\tilde{u}$ is also a solution of $\Delta u + f(u) = 0$. Since there are only finitely many cores, Theorem 5 is proved by repeating the procedure as many times as the number of cores. □

Remark 4 *The assumption $u > 0$ in Theorem 1 has been replaced by the weaker assumption $u \geq 0$ in Theorem 5. In this case any connected component of the support which is strictly included in D is a ball on which u is radially symmetric and decreasing up to cores. Note that this is possible only if f is not Lipschitz + increasing on a neighborhood of $u = 0+$. Otherwise, we would get a contradiction with Hopf's lemma.*

With this remark, the proof of Theorem 1 is straightforward.

Proof of Theorem 2. We have to prove that the solution is radially symmetric under assumption (f3): there exists a constant $C > 0$ such that for $v - u(\overline{x}) > 0$ small enough, $\frac{f(v)}{v-u(\overline{x})} > -C$, and Hopf's applied to $-\Delta(u - u(\overline{x})) + C(u - u(\overline{x})) > 0$ in $B((\rho-\epsilon)\frac{\overline{x}}{|\overline{x}|}, \epsilon)$ at $\overline{x}$ for $\epsilon > 0$, small enough, is in contradiction with: $\nabla u(\overline{x}) = \frac{\overline{x}}{|\overline{x}|}\frac{du}{dr}(\rho) = 0$.

If $B(0,\rho)$ is a radially symmetric core of u, the only possibility is therefore that $u_{|\partial B(0,\rho)} = 0$. Thus $B(0,\rho) = D$, which ends the proof of Theorem 2 if D is a ball. Otherwise, we obtain a contradiction with the assumption $u > 0$ by considering $\partial B(0,\rho) \cap D$. $\square$

Remark 5 *When D is a ball, the solution in Theorem 5 is locally radially symmetric, with a finite number of cores: there exists a finite partition in balls and annuli on which the solution is radially symmetric and decreasing, and complementary domains on which the solution is constant. Because of Condition (f2), the number of possible cores is finite (Step 2 of the proof of Theorem 5).*

If $u > 0$ is a solution of (2), which is not radially symmetric, there exists therefore a $\rho \in (0,1)$ such that u is radially symmetric in the annulus $\{x \in \mathbb{R}^N : \rho < |x| < 1\}$ and $0 = \frac{d^2u}{dr^2}(\rho) = \frac{du}{dr}(\rho) = f(u(\rho))$. We can now notice that if assumption (f3) is replaced by

(f3') *For any $u > 0$ such that $f(u) = 0$, $\liminf_{v\to u,\, v<u} \frac{f(v)}{v-u} > -\infty$,*

there exists a constant $C > 0$ such that for $u(\rho) - v > 0$ small enough, $\frac{f(v)}{u(\rho)-v} > -C$. Then Hopf's Lemma applied to $-\Delta(u(\rho)-u) - C(u(\rho)-u) < 0$ in $B((\rho+\epsilon)\frac{\overline{x}}{|\overline{x}|}, \epsilon)$ at $\overline{x}$, for any $\overline{x} \in B(0,1)$ such that $|\overline{x}| = \rho$ and $\epsilon > 0$ small enough, is in contradiction with $\nabla u(\overline{x}) = \frac{\overline{x}}{|\overline{x}|} \cdot \frac{du}{dr}(\rho) = 0$.

Under assumption (B) or (C), either (f3) or (f3') are sufficient to prove that a solution which is locally radially symmetric has a global radial symmetry or at least a monotonicity property. This global radial symmetry / monotonicity property is also probably true even without assumption (f2). See [6] for a proof in dimension $N = 2$.

4 Further results

We will not try to give the most general possible results, but just quote some remarks and directions in which our results can be extended.

4.1 Cores can only "go up"

To start with, we may notice that our local symmetry results hold for non-negative solutions of (2) and the solutions may eventually be identically equal to 0 on a non empty subdomain of D. We may also notice that on a radially symmetric core, the minimum of the function is reached on the boundary of the core. One may therefore wonder why a nonnegative solution of (1) when D is, for instance, a ball cannot have cores on which the solution reaches its minimum inside the core (if we forget the nonnegativity condition, such solutions are easy to build). The answer is given by the following result which has been announced in [7].

Proposition 6 *Assume that f satisfies (f1). Let u be a solution of (1) on the unit ball $B(0,1)$ which is radially symmetric up to cores. Assume that $N \geq 2$. With the same notations as in Theorem 5, if $B_i \cap B_j \neq \emptyset \Longrightarrow B_i \subset B_j$ (u is radially symmetric on B_i) and if $\underline{u} = \min_{x \in B_i} u(x) < \min_{x \in \partial B_i} u(x)$, then $\underline{u} < 0$.*

Proof. Consider u^+ and u^- two solutions of $\frac{d^2u}{dr^2} + \frac{N-1}{r} \cdot \frac{du}{dr}' + f(u(r)) = 0$ defined respectively on the intervals $I^- =]r_1^-, r_0^-)$ and $I^+ =]r_0^+, r_1^+)$ (with $0 < r_1^- < r_0^- \leq r_0^+ < r_1^+$), such that $u(r_0^\pm) = a$, $\frac{du^\pm}{dr}(r_0^\pm) = 0$, $u^\pm(r) < a$ where $a > 0$ is chosen in order that $f(a) = 0$. The functions u^- and u^+ are respectively increasing on I^- and decreasing on I^+, at least as long as $\frac{du^\pm}{dr}$ does not vanish. According to the method introduced by L.A. Peletier and J. Serrin in [19], it is possible to extend these solutions uniquely if $\frac{du^\pm}{dr} \neq 0$. Eventually, decreasing r_1^- and increasing r_1^+, we may assume that I^- and I^+ are the maximal intervals in $\mathbb{R}^+$ on which the property is satisfied. Then for any $r \in I^-$, $\frac{du^-}{dr}(r) = 0$ is impossible unless $u^-(r) < \inf_{s \in I^+} u^+(s)$. The functions $r^\pm(t)$ are indeed such that $t = u^\pm(r^\pm(t))$ are solutions of $\frac{(r^\pm)''}{((r^\pm)')^3} = f(t) + (N-1)\frac{1}{r^\pm (r^\pm)'}$. Multiplying by $(r^\pm)'(t)$ and integrating between $u^+(r)$ and a, we obtain for any $r \in I^+$

$$
\begin{aligned}
0 \leq \frac{1}{2}\Big(\frac{du^+}{dr}(r)\Big)^2 &= \int_{u^+(r)}^{a} f(s)\, ds + (N-1) \int_{u^+(r)}^{a} \frac{ds}{r^+(s)\,(r^+)'(s)} \\
&< \int_{u^+(r)}^{a} f(s)\, ds + (N-1) \int_{u^+(r)}^{a} \frac{ds}{r^-(s)\,(r^-)'(s)} \\
&= \frac{1}{2}\Big(\frac{du^-}{dr}(r^-(u^+(r)))\Big)^2 ,
\end{aligned}
$$

since $(r^+)' < 0 < (r^-)'$. This computation is still valid if $\frac{du^-}{dr}(r_0^-) > 0$, $\frac{du^+}{dr}(r_0^+) = 0$ and one can easily extend the argument to the case where $\frac{du^+}{dr} \leq 0$ takes the value 0 in I^+ if we define r^+ by $r^+(t) = \inf\{s > r_0^+ \ : \ u^+(s) = t\}$.

Without loss of generality, we may assume that B_i is the unique core of u (if not, apply the procedure defined in Theorem 5). Up to a translation, we can then identify u^+ and u^- with $\tilde{u}$ and $u_{|B_i}$ respectively and get $0 = u^+(1) > u^-(0) = \underline{u}$. □

In case $N = 1$, the above proof shows that $u^-(r) = u^+(r_0 - r)$ for $r_0 = (r_0^+ + r_0^-)/2$. It is however possible to decompose u in such a way that all cores "go up". Details are left to the reader.

In assumption (f2), the condition that the range of u on which f is locally either Lipschitz + increasing, or decreasing, is open means that there is always an overlapping of these conditions. This is actually crucial to prove that the number of cores is finite, in any bounded subdomain in D.

However, it is clear at least in the case of a ball (see [3,6]) that the right condition to avoid the existence of cores is assumption (f3) on the regularity of f in a neighborhood of $\overline{u}$ whenever $f(\overline{u}) = 0$, $\overline{u} > 0$. Thus the overlapping is unnecessary to obtain symmetry results, as we shall see on the following example. For simplicity, assume that $N = 2$ and replace (f2) and (f3) by the assumption

(f2') There exists an $a > 0$ such that f is decreasing in $[0, a]$, f is locally Lipschitz + increasing on $[a, \infty)$ and $f(a) < 0$.

This condition could of course be extended to each point a such that $f(a) < 0$, f is decreasing on a neighborhood of a_- and Lipschitz + increasing on a neighborhood of a_+, and even also to each point b such that $f(b) > 0$, f is decreasing on a neighborhood of b_+ and Lipschitz + increasing on a neighborhood of b_-, as soon as one controls the number of possible cores. However a statement with such assumptions would be unnecessarily technical. For simplicity again, we shall consider the case of a ball $B = B(0, 1)$. Note that controlling the number of cores is important in our method but does not seem to be required in the continuous rearrangements approach, in the case of a ball [3].

Proposition 7 *Let $N = 2$. Assume that f satisfies (f1) and (f2') and consider a solution $u \in C^2(B) \cap C^0(\bar{B})$ of (1) on the unit ball $D = B$. Then u is radially symmetric and $\frac{du}{dr}(r) < 0$ for any $r \in (0, 1)$.*

Proof. We proceed exactly as in the proof of Lemma 3 with $\Omega = B$. Assume that $\lim_{k\to\infty} \lambda_k =: \overline{\lambda} > \lambda_\Omega$ and consider $\overline{x} \in B$ such that $u(\overline{x}) \geq a$, $\overline{x}$ is the limit of $x_k \in \Sigma_{\lambda_k}$ such that $w_{\lambda_k}(x_k) < 0$, $\nabla w_{\lambda_k}(x_k) = 0$: $w_{\overline{\lambda}}(\overline{x}) = 0$ and $\nabla w_{\overline{\lambda}}(\overline{x}) = 0$. If $u(\overline{x}) > a$, the proof goes as before. The only case one has to consider is the case $u(\overline{x}) = a$, $\overline{x} \in \partial\omega$, $\omega := u^{-1}(a, +\infty) \cap \Sigma_{\overline{\lambda}} \neq \emptyset$. Note that the number of possible cores is finite because at a maximum, $-\Delta u = f(u) > 0$, so that we can give an estimate of $\mathcal{N}$ using the Lipschitz norm of u.

We may first notice that on ω either $w_{\overline{\lambda}} \equiv 0$ or $w_{\overline{\lambda}} > 0$ by the maximum principle. Assume that $w_{\overline{\lambda}}$ is positive and let us look for a contradiction. We will distinguish two cases, depending whether $\overline{x} \in \Sigma_{\overline{\lambda}}$ (Case (1)) or $\overline{x} \in T_{\overline{\lambda}}$ (Case (2)).

Case (1) : We have to prove that ω satisfies an interior sphere condition at $\overline{x}$.

a) If $\nabla u(\overline{x}) \neq 0$, $\partial\omega$ is locally of class C^2 in a neighborhood of $\overline{x} \in \omega \cap \partial B(\tilde{x}, \epsilon)$ for some ball $B(\tilde{x}, \epsilon) \subset \omega$, and $u(x) > a$ for any $x \in B(\tilde{x}, \epsilon)$. According for instance to [12], Hopf's lemma applied to $w_{\overline{\lambda}} > 0$ in ω at $\overline{x}$ provides: $\nabla w_{\overline{\lambda}}(\overline{x}) \cdot (\tilde{x} - \overline{x}) < 0$, a contradiction with $\nabla w_{\overline{\lambda}}(\overline{x}) = 0$.

b) Assume that $\nabla u(\overline{x}) = 0$. The monotonicity of $x_1 \mapsto u(x_1, x')$ gives:

$$\frac{\partial^2 u}{\partial x_1 \partial x_2}(\overline{x}) = \frac{\partial^2 u}{\partial x_1^2}(\overline{x}) = 0$$

because of the following Taylor development:

$$u(x) - a = \sum_{\substack{i,j=1,2 \\ (i,j)\neq(1,1)}} \frac{\partial^2 u}{\partial x_i \partial x_j}(\overline{x}) \cdot (x - \overline{x})_i (x - \overline{x})_j + o(|x - \overline{x}|^2)\ .$$

Since $-\frac{\partial^2 u}{\partial x_2^2}(\overline{x}) = -\Delta u(\overline{x}) = f(a) < 0$, ω again satisfies an interior sphere condition at $\overline{x}$. For the same reason as in case a), we get a contradiction.

Case (2) : Assume now that $\overline{x} \in T_{\overline{\lambda}}$. Because of the definition of $\overline{x}$,

$$\frac{\partial u}{\partial x_1}(\overline{x}) = -\frac{1}{2} \lim_{k\to+\infty} e_1 \cdot \nabla w_{\lambda_k}(x_k) = 0\ .$$

If $\frac{\partial u}{\partial x_2} \neq 0$, we may apply Serrin's lemma, see [12,20].

Lemma 8 *Let $\mathcal{O}$ be a domain in $\mathbb{R}^N$ and assume that near $\overline{x} \in \mathcal{O}$, the boundary of $\mathcal{O}$ consists of two transversally intersecting hypersurfaces $\rho = 0$ and $\sigma = 0$. Suppose that $\rho, \sigma > 0$ in $\mathcal{O}$. Let $w > 0$ be a function in $C^2(\overline{\mathcal{O}})$ with $w > 0$ in $\mathcal{O}$, $w(\overline{x}) = 0$, satisfying the differential inequality $-\Delta w - c(x)\, w \geq 0$ for some function c in $L^\infty(\mathcal{O})$. Assume that $\sum_{i=1}^N \frac{\partial\rho}{\partial x_i}(\overline{x}) \cdot \frac{\partial\sigma}{\partial x_i}(\overline{x}) = 0$ and $D\left(\sum_{i=1}^N \frac{\partial\rho}{\partial x_i} \cdot \frac{\partial\sigma}{\partial x_i}\right)(\overline{x}) = 0$ for any derivative tangent at $\overline{x}$ to the submanifold $\{\rho = 0\} \cap \{\sigma = 0\}$. Then for any direction s which enters $\mathcal{O}$ at $\overline{x}$ transversally to both hypersurfaces, $\frac{\partial w}{\partial s} > 0$ and $\frac{\partial^2 w}{\partial s^2} > 0$.*

This is clearly in contradiction with $\nabla w_{\overline{\lambda}}(\overline{x}) = 0$.

If $\frac{\partial u}{\partial x_2} = 0$, using again that $-\Delta u(\overline{x}) = f(a) < 0$, we may still find a cone $\mathcal{O}$ of summit $\overline{x}$ such that on $\omega \cap \mathcal{O}$, $w_{\overline{\lambda}} > 0$, and as above we get a contradiction with Serrin's lemma. $\square$

4.3 Whole space results

In the case the domain D is the whole space $\mathbb{R}^N$, the method can still be adapted as soon as the moving plane technique can be started from ∞ in any direction. One of the main features of the local moving plane method we develop in this paper is that we do not need to assume a strict positivity of w_λ as soon as f is decreasing in a neighborhood of 0 and can therefore handle in a unified framework the positive solutions as well as the nonnegative solutions that are compactly supported.

Theorem 9 *Assume f satisfies (f1)-(f2). Let u be a C^2 nonnegative solution of (1) satisfying $\lim_{|x|\to+\infty} u(x) = 0$. Then u is radially symmetric up to cores. If it is compactly supported, the support of u is a union of balls with disjoint interiors.*

The assertion on the support is a consequence of Proposition 6. Of course, with a further assumption on the positive critical levels of f, we may get a strict monotonicity on each component of the support.

Corollaire 10 *Assume f satisfies (f1)-(f3). Let u be a C^2 nonnegative solution of (1) satisfying $\lim_{|x|\to+\infty} u(x) = 0$. Then any connected component of $\{x \in \mathbb{R}^N : u(x) > 0\}$ is a ball (or $\mathbb{R}^N$), and u restricted to each of these components is radially symmetric and decreasing.*

A further assumption on the regularity of f at 0 would provide the result that the solution has to be radially symmetric, positive and decreasing with respect to some point in $\mathbb{R}^N$.

4.4 Fully nonlinear case

It is possible to generalize the results given in Sections 1-3 for the Laplacian to more general fully nonlinear elliptic equations of the type

$$F\Big(u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}\Big) = 0 \quad \text{with} \quad i,j = 1,2,\ldots N\,, \tag{3}$$

when F is only continuous with respect to u, even in the case where the highest order part of the operator cannot be written in divergence form. Since quasilinear and fully nonlinear elliptic equations are out of the general scope of this paper, we shall simply refer to [8] for an up to date list of references and further comments on the connection of the issue of the symmetry of the solutions with their assumed regularity.

Under the assumptions, which are directly inspired from [16,17], consider

$$
\begin{aligned}
F : \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^{N\times N} &\to \mathbb{R} \\
(s,p,Q) &\mapsto F(s,p,Q)
\end{aligned}
$$

with the following properties:

(F1) F is continuous and C^1 with respect to $Q = (Q_{ij})_{i,j=1,2,\ldots N}$,

(F2) F is either Lipschitz + increasing in s, or has the following strict decay property

$$F(u+w,p,Q+R) > F(u,p,Q)$$

for any $N \times N$ nonnegative symmetric matrix R and any $w < 0$, provided $(w,R) \neq (0,0)$,

(F3) For any (s,p,Q) such that $F(s,p,Q) = 0$, there exists a neighborhood of (s,p,Q) in $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^{N\times N}$ on which F is Lipschitz + increasing with respect to s,

(F4) For any $\xi \in \mathbb{R}^N$, $F_{p_ip_j}(s,p,Q)\xi_i\xi_j \geq \overline{\lambda}(s,p,Q)|\xi|^2$ for some $\overline{\lambda}(s,p,Q)$ which is uniformly positive.

(F5) F has the following symmetry property with respect to e_1:

$$F(s,(-p_1,p_2,\ldots p_N),\tilde{Q}) = F(s,p,Q)\ ,$$

$$
\begin{aligned}
\tilde{Q} = (Q_{11}, -Q_{12}, -Q_{13}, \ldots\ - Q_{1N}, -Q_{21}, Q_{22}, Q_{23}, \ldots\ Q_{2N}, \ldots \\
-Q_{N1}, Q_{N2}, \ldots\ Q_{NN})\ ,
\end{aligned}
$$

as well as for any direction $\gamma \in S^{N-1}$ such that $|\gamma - e_1| < \epsilon$ for some given $\epsilon > 0$.

Theorem 11 *Assume that f and D respectively satisfy (F1)-(F5) and (B). Let $u \in C^2(D) \cap C^0(\bar{D})$ be a positive solution of (3) such that $u_{|\partial D} = 0$. Then u is monotone non increasing up to cores on $\tilde{D} = \{x \in D\ :\ x \cdot e_1 \geq 0\}$ in the direction e_1.*

Of course, a similar monotonicity result holds for unbounded domains. Note that assumption (F5) as in [16] is quite restrictive (see [6] for an example). The proofs go exactly as for the Laplacian, but present purely computational technicalities that are unessential and will not be presented here. The main point is that one has to make sure that the local inversion theorem (unique continuation argument in the proof of Theorem 2) preserves the ellipticity of the operator.

4.5 The $|x|$ dependent case

Imposing a dependence in $|x|$ is easy and we can for instance state the following result. Consider in Case (B)

$$\Delta u + f(|x|, u) = 0,\ u > 0 \qquad \text{in } D\ , \tag{4}$$

$$u = 0 \qquad \text{on } \partial D. \tag{5}$$

Theorem 12 *Under the same assumptions as in Theorem 1, provided these assumptions on f are uniform in x, if D satisfies condition (B) and if f is monotone non increasing in $|x|$, then the same results hold for any solution of (4)-(5). If moreover $f(|x|, u)$ is decreasing in $|x|$, then no cores may exist.*

Actually, to obtain the nonexistence of cores, it is sufficient to ask that $f(|x|, u)$ is decreasing in $|x|$ for any u such that f is not decreasing or constant in u. In that case, by Lemma 3, u would be symmetric with respect to some hyperplane $T_{\overline{\lambda}}$ in the range $u_1 < u(\overline{x}) < u_2$, in contradiction with the fact that

$$0 = \Delta u_{\overline{\lambda}} + f(u_{\overline{\lambda}}, |x_{\overline{\lambda}}|) = \Delta u + f(u, |x_{\overline{\lambda}}|) > \Delta u + f(u, |x|) = 0\ ,$$

except if $\overline{\lambda} = 0$.

The only difficulty that may occur is the case $\lambda = \lambda_\Omega$ in Lemma 3, which can be solved by noticing first that if $\Omega = D$, then $\lambda_\Omega = 0$ and then by applying the iteration method of the proof of Theorem 5 with care. Details are left to the reader. $\square$

References

[1] A.D. Alexandrov, *A characteristic property of the spheres,* Ann. Mat. Pura Appl. **58** (1962) 303-354.

[2] F. Brock, *Continuous Steiner symmetrization,* Math. Nach. **172** (1995) 25-48.

[3] F. Brock, *Continuous rearrangements and symmetry of solutions of elliptic problems,* Proc. Indian Acad. Sci. Math. Sci. **110** (2000) 157-204.

[4] C. Cortázar, M. Elgueta, P. Felmer, *On a semilinear elliptic problem in $\mathbb{R}^N$ with a non-Lipschitzian non-linearity,* Adv. in Differential equations **1** no. 2 (1996) 199-218.

[5] C. Cortázar, M. Elgueta, & P. Felmer, *Symmetry in an elliptic problem and the blow-up set of a quasilinear heat equation,* Comm. Partial Differential equations **21** no. 3-4 (1996) 507-520.

[6] J. Dolbeault, & P. Felmer, *Symmetry and monotonicity properties for positive solutions of semilinear elliptic PDE's,* Comm. Partial Differential equations **25** (2000) 1153-1169.

[7] J. Dolbeault, & P. Felmer, *Symétrie pour des équations semi-linéaires elliptiques [Symmetry of the solutions of semilinear elliptic equations],* C. R. Acad. Sci. Paris Sér. I. **329** (1999) 677-682.

[8] J. Dolbeault, P. Felmer, & R. Monneau, *Local symmetry and non uniformly elliptic operators,* in preparation.

[9] A. Farina, *Propriétés de monotonie et de symétrie unidimensionnelle pour les solutions de $\Delta u + f(u) = 0$ avec des fonctions f éventuellement discontinues,* C. R. Acad. Sc. Paris **330** Sér. I (2000) 973-978.

[10] A. Farina, *Monotonicity and one-dimensional symmetry for the solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^N$ with possibly discontinuous nonlinearity,* Adv. Math. Sciences Appl. **11** no. 2 (2001) 811-834.

[11] B. Franchi, E. Lanconelli, & J. Serrin, *Existence and uniqueness of nonnegative solutions of quasilinear equations in $\mathbb{R}^N$*, Adv. in Math. **118** (1996) 177-243.

[12] B. Gidas, W.-M. Ni, L. Nirenberg, *Symmetry and Related Properties via the Maximum Principle,* Comm. Math. Phys. **68** (1979) 209-243.

[13] B. Gidas, W.-M. Ni, & L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$*, Adv. in Math. Studies **7 A** (1979) 209-243.

[14] L. Hörmander, *The analysis of linear partial differential operators III, Pseudo-differential operators,* Grundlehren der Mathematischen Wissenschaften 274, Springer, Berlin, Heidelberg (1985).

[15] H.G. Kaper, M.K. Kwong, Y. Li, *Symmetry Results for Reaction Diffusion equations,* Differential Integral equations **6** no. 5 (1993) 1045-1056.

[16] C. Li, *Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains,* Comm. Partial Differential equations **16** (1991) 585-615.

[17] Y. Li, W.-M. Ni, *Radial symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$*, Comm. Partial Differential equations **18** no. 5-6 (1993) 1043-1054.

[18] O. Lopes, *Radial Symmetry of Minimizers for some translation and rotation invariant functionals,* J. Differential equations **124** (1996) 378-388.

[19] L.A. Peletier, & J. Serrin, *Uniqueness of nonnegative solutions of semilinear equations in $\mathbb{R}^N$*, J. Differential equations **61** (1986) 380-397.

[20] J. Serrin, *A symmetry problem in potential theory,* Arch. Ration. Mech. Anal. **43** (1971) 304-318.