Research Article

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Montgomery identity and Ostrowski-type inequalities via quantum calculus

Abstract: In this paper, we prove a quantum version of Montgomery identity and prove some new Ostrowski-type inequalities for convex functions in the setting of quantum calculus. Moreover, we discuss several special cases of newly established inequalities and obtain different new and existing inequalities in the field of integral inequalities.

Keywords: Ostrowski inequalities, q-integral, quantum calculus, integral inequalities, difference operators, convex functions

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1 Introduction

Over the past few decades, different kinds of integral inequalities have drawn the attention of many mathematicians. In the study of various types of equations, such as integro-differential equations and impulsive differential equations, these inequalities play an important role (see [1–6] and references therein). Therefore, a large number of research activities on this subject are being carried out.

The classical integral inequality associated with the differentiable mappings is as follows:

Theorem 1. [7] *If the mapping* $F : [\mu, \nu] \to \mathbb{R}$ *is differentiable on* (μ, ν) *and integrable on* $[\mu, \nu]$, *then the following inequality holds*:

$$\left| F(\varkappa) - \frac{1}{\nu - \mu} \int_{\mu}^{\nu} F(s) \mathrm{d}s \right| \leq \left[\frac{1}{4} + \frac{\left(\varkappa - \frac{\mu + \nu}{2} \right)^2}{(\nu - \mu)^2} \right] (\nu - \mu) \|F'\|_{\infty},$$

for all $\varkappa \in [\mu, \nu]$, *where* $\|F'\|_{\infty} = \sup_{s \in (\mu, \nu)} |F'(s)| < +\infty$. *Moreover,* $\frac{1}{4}$ *is the best possible constant.*

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Theorem 2. [8] *Suppose that $F : [\mu, \nu] \rightarrow \mathbb{R}$ is a differentiable on (μ, ν) and integrable on $[\mu, \nu]$. If $|F'(\varkappa)| \leq M$, for every $\varkappa \in [\mu, \nu]$, then the following inequality*

$$\left| F(\varkappa) - \frac{1}{\nu - \mu} \int_\mu^\nu F(s)ds \right| \leq \frac{M}{\nu - \mu} \left[\frac{(\varkappa - \mu)^2 + (\nu - \varkappa)^2}{2} \right] \tag{1}$$

holds.

Following is the well-known Montgomery identity:

Lemma 1. [9] *If the mapping $F : [\mu, \nu] \rightarrow \mathbb{R}$ is differentiable on (μ, ν) and integrable on $[\mu, \nu]$. then the following identity holds*:

$$F(\varkappa) - \frac{1}{\nu - \mu} \int_\mu^\nu F(s)ds = \int_\mu^\varkappa \frac{s - \mu}{\nu - \mu} F'(s)ds + \int_\varkappa^\nu \frac{s - \nu}{\nu - \mu} F'(s)ds. \tag{2}$$

By changing the variables, we can rewrite (2) in the following way:

Lemma 2. [10, Lemma 1] *Suppose that $F : [\mu, \nu] \rightarrow \mathbb{R}$ is differentiable on (μ, ν) and integrable on $[\mu, \nu]$, then the following equality holds*:

$$F(\varkappa) - \frac{1}{\nu - \mu} \int_\mu^\nu F(s)ds = (\nu - \mu) \left[\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} sF'(s\mu + (1 - s)\nu)ds + \int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 (s - 1)F'(s\mu + (1 - s)\nu)ds \right]. \tag{3}$$

On the other side, in the domain of q-analysis, many works are being carried out initiating from Euler in order to attain adeptness in mathematics that constructs quantum computing q-calculus considered as a relationship between physics and mathematics. In different areas of mathematics, it has numerous applications such as combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other sciences, mechanics, the theory of relativity, and quantum theory [11,12]. Quantum calculus also has many applications in quantum information theory, which is an interdisciplinary area that encompasses computer science, information theory, philosophy, and cryptography, among other areas [13,14]. Apparently, Euler invented this important mathematics branch. He used the q parameter in Newton's work on infinite series. Later, in a methodical manner, the q-calculus that knew without limits calculus was first given by Jackson [15,16]. In 1966, Al-Salam [17] introduced a q-analogue of the q-fractional integral and q-Riemann-Liouville fractional. Since then, the related research has gradually increased. In particular, in 2013, Tariboon and Ntouyas introduced $_\mu D_q$-difference operator and q_μ-integral in [18]. In 2020, Bermudo et al. introduced the notion of $^\nu D_q$ derivative and q^ν-integral in [19].

Many integral inequalities have been studied using quantum and post-quantum integrals for various types of functions. For example, in [19–28], the authors used $_\mu D_q$, $^\nu D_q$-derivatives and q_μ, q^ν-integrals to prove Hermite-Hadamard integral inequalities and their left-right estimates for convex and coordinated convex functions. In [29], Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, Nwaeze and Tameru proved certain parameterized quantum integral inequalities in [30]. Khan et al. proved quantum Hermite-Hadamard inequality using the green function in [31]. Budak et al. [32], Ali et al. [33,34], and Vivas-Cortez et al. [35] developed new quantum Simpson's and quantum Newton-type inequalities for convex and coordinated convex functions. For quantum Ostrowski's inequalities for convex and co-ordinated convex functions one can refer to [36–40].

The purpose of this paper is to study Ostrowski-type inequalities for convex functions by applying newly defined concept of q^ν-integral. In the current literature, we also address the correlation of the results obtained herein with comparable results.

This paper's organization is as follows: We summarize the definition of q-calculus in Section 2, and some related work is given in this setup. The Montgomery identity proof for q^ν-integral is given in Section 3.

Ostrowski-type inequalities are obtained by using the Montgomery identity for q^ν-integral. In Section 4, several special cases of our key findings are presented. The relationship between the findings obtained and the comparable outcomes in the current literature is also discussed. Some findings and further directions for future study are found in Section 5. We assume that the analysis initiated in this paper could provide researchers working on integral inequalities and their applications with a strong source of inspiration.

2 Quantum calculus and some inequalities

First of all, we present some established definitions and related inequalities in q-calculus in this section. Set the notation that follows [12]:

$$[r]_q = \frac{1 - q^r}{1 - q} = 1 + q + q^2 + \ldots + q^{r-1}, \quad q \in (0, 1).$$

Jackson [16] defined the q-Jackson integral of a given function F from 0 to ν as follows:

$$\int_0^\nu F(\varkappa)d_q\varkappa = (1 - q)\nu \sum_{n=0}^\infty q^n F(\nu q^n), \quad \text{where } 0 < q < 1, \tag{4}$$

provided that the sum converges absolutely. Moreover, he defined the q-Jackson integral of a given function over the interval $[\mu, \nu]$ as follows:

$$\int_\mu^\nu F(\varkappa)d_q\varkappa = \int_0^\nu F(\varkappa)d_q\varkappa - \int_0^\mu F(\varkappa)d_q\varkappa. \tag{5}$$

Theorem 3. (Hölder's inequality, [41, p. 604]) *Suppose that* $\varkappa > 0$, $0 < q < 1$, $p_1 > 1$. *If* $\frac{1}{p_1} + \frac{1}{r_1} = 1$, *then*

$$\int_0^\varkappa |F(\varkappa)g(\varkappa)|d_q\varkappa \leq \left(\int_0^\varkappa |F(\varkappa)|^{p_1}d_q\varkappa\right)^{\frac{1}{p_1}}\left(\int_0^\varkappa |g(\varkappa)|^{r_1}d_q\varkappa\right)^{\frac{1}{r_1}}.$$

Definition 1. [18] Let $F : [\mu, \nu] \to \mathbb{R}$ be a continuous function. The q_μ-derivative of F at $\varkappa \in [\mu, \nu]$ is identified by the following expression:

$$_\mu D_q F(\varkappa) = \frac{F(\varkappa) - F(q\varkappa + (1 - q)\mu)}{(1 - q)(\varkappa - \mu)}, \quad \varkappa \neq \mu. \tag{6}$$

If $\varkappa = \mu$, we define $_\mu D_q F(\mu) = \lim_{\varkappa \to \mu} {_\mu D_q F(\varkappa)}$ if it exists and it is finite.

Definition 2. [18] Let $F : [\mu, \nu] \to \mathbb{R}$ be a continuous function. The q_μ-definite integral on $[\mu, \nu]$ is defined by

$$\int_\mu^\nu F(\varkappa)_\mu d_q\varkappa = (1 - q)(\nu - \mu) \sum_{n=0}^\infty q^n F(q^n\nu + (1 - q^n)\mu) = (\nu - \mu)\int_0^1 F((1 - s)\mu + s\nu)d_q s.$$

Kunt et al. [42] obtained the following Montgomery identity for q_μ-definite integrals:

Lemma 3. *If* $F : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ *is a q-differentiable function on* (μ, ν) *such that* $_\mu D_q F$ *is continuous and integrable on* $[\mu, \nu]$, *then we have*:

$$F(\varkappa) - \frac{1}{\nu - \mu}\int_\mu^\nu F(s)_\mu d_q s = (\nu - \mu)\int_0^1 \Lambda_q(s)_\mu D_q F(s\nu + (1 - s)\mu)_0 d_q s,$$

where

$$\Lambda_q(s) = \begin{cases} qs, & s \in \left[0, \dfrac{\varkappa - \mu}{\nu - \mu}\right), \\[3mm] qs - 1, & s \in \left[\dfrac{\varkappa - \mu}{\nu - \mu}, 1\right] \end{cases}$$

and $0 < q < 1$.

On the other hand, Bermudo et al. [19] gave the following definitions:

Definition 3. [19] Let $F : [\mu, \nu] \to \mathbb{R}$ be a continuous function. The q^ν-derivative of F at $\varkappa \in [\mu, \nu]$ is given by

$${}^\nu D_q F(\varkappa) = \frac{F(q\varkappa + (1 - q)\nu) - F(\varkappa)}{(1 - q)(\nu - \varkappa)}, \quad \varkappa \neq \nu.$$

If $\varkappa = \nu$, we define ${}^\nu D_q F(\nu) = \lim_{\varkappa \to \nu} {}^\nu D_q F(\varkappa)$ if it exists and it is finite.

Definition 4. [19] Let $F : [\mu, \nu] \to \mathbb{R}$ be a continuous function. The q^ν-definite integral on $[\mu, \nu]$ is given by

$$\int_\mu^\nu F(\varkappa)\,{}^\nu d_q\varkappa = (1 - q)(\nu - \mu)\sum_{n=0}^\infty q^n F(q^n\mu + (1 - q^n)\nu) = (\nu - \mu)\int_0^1 F(s\mu + (1 - s)\nu)\,d_qs.$$

3 Quantum Ostrowski-type inequalities

We first demonstrate the quantum Montgomery identity for q^ν-definite integrals in this section. Then we offer some Ostrowski-type inequalities by the use of this identity.

Let us start with the following useful lemma which is a Montgomery identity for q^ν-integral.

Lemma 4. (Quantum Montgomery identity) *If* $F : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ *is a q-differentiable function on* (μ, ν) *such that* ${}^\nu D_q F$ *is continuous and integrable on* $[\mu, \nu]$, *then we have*

$$\frac{1}{\nu - \mu}\int_\mu^\nu F(s)\,{}^\nu d_qs - F(\varkappa) = (\nu - \mu)\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} qs\,{}^\nu D_q F(s\mu + (1 - s)\nu)\,d_qs + \int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 (qs - 1)\,{}^\nu D_q F(s\mu + (1 - s)\nu)\,d_qs, \qquad (7)$$

and $0 < q < 1$.

Proof. It follows from Definition 3 that

$${}^\nu D_q F(s\mu + (1 - s)\nu) = \frac{F(qs\mu + (1 - qs)\nu) - F(s\mu + (1 - s)\nu)}{(1 - q)(\nu - \mu)s}.$$

From the Jackson integral (5), we obtain

$$\left[\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} qs\,{}^\nu D_q F(s\mu + (1 - s)\nu)\,d_qs + \int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 (qs - 1)\,{}^\nu D_q F(s\mu + (1 - s)\nu)\,d_qs\right] \qquad (8)$$

$$= (\nu - \mu) \left[\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} qs \, {}^{\nu}D_q F(s\mu + (1-s)\nu) d_q s + \int_0^1 (qs - 1) {}^{\nu}D_q F(s\mu + (1-s)\nu) d_q s \right.$$

$$\left. - \int_0^{\frac{\nu - \varkappa}{\nu - \mu}} (qs - 1) {}^{\nu}D_q F(s\mu + (1-s)\nu) d_q s \right]$$

$$= (\nu - \mu) \left[\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} {}^{\nu}D_q F(s\mu + (1-s)\nu) d_q s + \int_0^1 (qs - 1) {}^{\nu}D_q F(s\mu + (1-s)\nu) d_q s \right]$$

$$= \int_0^{\frac{\nu - \varkappa}{\nu - \mu}} \frac{F(qs\mu + (1-qs)\nu) - F(s\mu + (1-s)\nu)}{(1-q)s} d_q s + q \int_0^1 \frac{F(qs\mu + (1-qs)\nu) - F(s\mu + (1-s)\nu)}{(1-q)} d_q s$$

$$- \int_0^1 \frac{F(qs\mu + (1-qs)\nu) - F(s\mu + (1-s)\nu)}{(1-q)s} d_q s.$$

By equality (4), we obtain

$$\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} \frac{F(qs\mu + (1-qs)\nu) - F(s\mu + (1-s)\nu)}{(1-q)s} d_q s$$

$$= \sum_{n=0}^{\infty} F\left(q^{n+1}\left(\frac{\nu - \varkappa}{\nu - \mu} \right)\mu + \left(1 - q^{n+1}\left(\frac{\nu - \varkappa}{\nu - \mu} \right) \right)\nu \right) - \sum_{n=0}^{\infty} F\left(q^n\left(\frac{\nu - \varkappa}{\nu - \mu} \right)\mu + \left(1 - q^n\left(\frac{\nu - \varkappa}{\nu - \mu} \right) \right)\nu \right) \qquad (9)$$

$$= F(\nu) - F\left(\left(\frac{\nu - \varkappa}{\nu - \mu} \right)\mu + \left(1 - \left(\frac{\nu - \varkappa}{\nu - \mu} \right) \right)\nu \right) = F(\nu) - F(\varkappa)$$

and

$$\int_0^1 \frac{F(qs\mu + (1-qs)\nu) - F(s\mu + (1-s)\nu)}{(1-q)s} d_q s$$

$$(10)$$

$$= \sum_{n=0}^{\infty} F(q^{n+1}\mu + (1-q^{n+1})\nu) - \sum_{n=0}^{\infty} F(q^n\mu + (1-q^n)\nu) = F(\nu) - F(\mu).$$

Now by (4) and Definition 4, we have

$$\int_0^1 \frac{F(qs\mu + (1-qs)\nu) - F(s\mu + (1-s)\nu)}{(1-q)} d_q s$$

$$= \sum_{n=0}^{\infty} q^n F(q^{n+1}\mu + (1-q^{n+1})\nu) - \sum_{n=0}^{\infty} q^n F(q^n\mu + (1-q^n)\nu)$$

$$= \frac{1}{q} \sum_{n=0}^{\infty} q^{n+1} F(q^{n+1}\mu + (1-q^{n+1})\nu) - \sum_{n=0}^{\infty} q^n F(q^n\mu + (1-q^n)\nu)$$

$$= \frac{1}{q} \sum_{n=0}^{\infty} q^n F(q^n\mu + (1-q^n)\nu) - \frac{1}{q}F(\mu) - \sum_{n=0}^{\infty} q^n F(q^n\mu + (1-q^n)\nu) \qquad (11)$$

$$= \left(\frac{1}{q} - 1 \right) \sum_{n=0}^{\infty} q^n F(q^n\mu + (1-q^n)\nu) - \frac{1}{q}F(\mu)$$

$$= \frac{1}{q(\nu - \mu)} \int_\mu^\nu F(s) {}^{\nu}d_q s - \frac{1}{q}F(\mu).$$

Using (9), (10), and (11) in (8), we have the required identity (7). □

Remark 1. On taking limit as $q \to 1^-$ in Lemma 4, identity (7) reduces to (3).

Theorem 4. *Suppose that $F : [\mu, \nu] \subset \mathbb{R} \to \mathbb{R}$ is a q-differentiable function on (μ, ν) and $^\nu D_q F$ is continuous and integrable on $[\mu, \nu]$. If $|^\nu D_q F|^{p_1}$ is convex on $[\mu, \nu]$, where $p_1 \geq 1$, then we have the following inequality*:

$$\left| \frac{1}{\nu - \mu} \int_\mu^\nu F(s)^\nu d_q s - F(\varkappa) \right|$$

$$\leq (\nu - \mu) \left[A_1^{1-\frac{1}{p_1}}(\mu, \nu, q, \varkappa)(|^\nu D_q F(\mu)|^{p_1} A_2(\mu, \nu, q, \varkappa) + |^\nu D_q F(\nu)|^{p_1} A_3(\mu, \nu, q, \varkappa))^{\frac{1}{p_1}} \right.$$

$$\left. + A_4^{1-\frac{1}{p_1}}(\mu, \nu, q, \varkappa)(|^\nu D_q F(\mu)|^{p_1} A_5(\mu, \nu, q, \varkappa) + |^\nu D_q F(\nu)|^{p_1} A_6(\mu, \nu, q, \varkappa))^{\frac{1}{p_1}} \right],$$

where

$$A_1(\mu, \nu, q, \varkappa) = \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} qs\, d_q\varkappa = \frac{q}{[2]_q}\left(\frac{\nu - \varkappa}{\nu - \mu}\right)^2,$$

$$A_2(\mu, \nu, q, \varkappa) = \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} qs^2\, d_q\varkappa = \frac{q}{[3]_q}\left(\frac{\nu - \varkappa}{\nu - \mu}\right)^3,$$

$$A_3(\mu, \nu, q, \varkappa) = \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} qs\, d_q\varkappa - \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} qs^2\, d_q\varkappa = A_1(\mu, \nu, q, \varkappa) - A_2(\mu, \nu, q, \varkappa),$$

$$A_4(\mu, \nu, q, \varkappa) = \int_{\frac{\nu-\varkappa}{\nu-\mu}}^1 (1 - qs)\, d_q s = \int_0^1 (1 - qs)\, d_q s - \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} (1 - qs)\, d_q s$$

$$= \frac{1 - q}{[2]_q}\left(\frac{\varkappa - \mu}{\nu - \mu}\right) + \frac{q}{[2]_q}\left(\frac{\varkappa - \mu}{\nu - \mu}\right)^2,$$

$$A_5(\mu, \nu, q, \varkappa) = \int_{\frac{\nu-\varkappa}{\nu-\mu}}^1 (s - qs^2)\, d_q s = \int_0^1 (s - qs^2)\, d_q s - \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} (s - qs^2)\, d_q s$$

$$= \frac{1}{[2]_q[3]_q} - \frac{1}{[2]_q}\left(\frac{\nu - \varkappa}{\nu - \mu}\right)^2 + \frac{q}{[3]_q}\left(\frac{\nu - \varkappa}{\nu - \mu}\right)^3,$$

and

$$A_6(\mu, \nu, q, \varkappa) = \int_{\frac{\nu-\varkappa}{\nu-\mu}}^1 (1 - s)(1 - qs)\, d_q s = \int_0^1 (1 - s)(1 - qs)\, d_q s - \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} (1 - s)(1 - qs)\, d_q s$$

$$= \int_0^1 (1 - qs)\, d_q s - \int_0^1 (s - qs^2)\, d_q s - \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} (1 - qs)\, d_q s + \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} (s - qs^2)\, d_q s$$

$$= A_4(\mu, \nu, q, \varkappa) - A_5(\mu, \nu, q, \varkappa),$$

where $0 < q < 1$.

Proof. By Lemma 4 and quantum power mean inequality, we obtain that

$$\left| \frac{1}{\nu - \mu} \int_{\mu}^{\nu} F(s)^{\nu} d_q s - F(\varkappa) \right|$$

$$\leq (\nu - \mu) \int_0^1 |\Psi_q(s)| |^{\nu} D_q F(s\mu + (1-s)\nu)| d_q s$$

$$\leq (\nu - \mu) \left[\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} qs|^{\nu} D_q F(s\mu + (1-s)\nu)| d_q s + \int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 (1 - qs)|^{\nu} D_q F(s\mu + (1-s)\nu)| d_q s \right]$$

$$\leq (\nu - \mu) \left[\left(\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} qs d_q s \right)^{1 - \frac{1}{p_1}} \left(\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} qs \, |^{\nu} D_q F(s\mu + (1-s)\nu)|^{p_1} d_q s \right)^{\frac{1}{p_1}} \right.$$

$$\left. + \left(\int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 (1 - qs) d_q s \right)^{1 - \frac{1}{p_1}} \left(\int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 (1 - qs)|^{\nu} D_q F(s\mu + (1-s)\nu)|^{p_1} d_q s \right)^{\frac{1}{p_1}} \right].$$

As $|^{\nu} D_q F|^{p_1}$ is convex on $[\mu, \nu]$, we have

$$\left| \frac{1}{\nu - \mu} \int_{\mu}^{\nu} F(s)^{\nu} d_q s - F(\varkappa) \right|$$

$$\leq (\nu - \mu) \left[\left(\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} qs d_q s \right)^{1 - \frac{1}{p_1}} \left(\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} qs(|^{\nu} D_q F(\mu)|^{p_1} s + |^{\nu} D_q F(\nu)|^{p_1}(1 - s)) d_q s \right)^{\frac{1}{p_1}} \right.$$

$$\left. + \left(\int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 (1 - qs) \right)^{1 - \frac{1}{p_1}} \left(\int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 (1 - qs)(|^{\nu} D_q F(\mu)|^{p_1} s + |^{\nu} D_q F(\nu)|^{p_1}(1 - s)) d_q s \right)^{\frac{1}{p_1}} \right]$$

$$= (\nu - \mu) \left[\left(\int_0^{\frac{\nu - \varkappa}{\nu - \mu}} qs d_q s \right)^{1 - \frac{1}{p_1}} \left(|^{\nu} D_q F(\mu)|^{p_1} \int_0^{\frac{\nu - \varkappa}{\nu - \mu}} qs^2 d_q s + |^{\nu} D_q F(\nu)|^{p_1} \int_0^{\frac{\nu - \varkappa}{\nu - \mu}} (qs - qs^2) d_q s \right)^{\frac{1}{p_1}} \right.$$

$$\left. + \left(\int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 (1 - qs) d_q s \right)^{1 - \frac{1}{p_1}} \left(|^{\nu} D_q F(\mu)|^{p_1} \int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 s(1 - qs) d_q s + |^{\nu} D_q F(\nu)|^{p_1} \int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 (1 - s)(1 - qs) d_q s \right)^{\frac{1}{p_1}} \right]$$

$$= (\nu - \mu) \left[A_1^{1 - \frac{1}{p_1}}(\mu, \nu, q, \varkappa)(|^{\nu} D_q F(\mu)|^{p_1} A_2(\mu, \nu, q, \varkappa) + |^{\nu} D_q F(\nu)|^{p_1} A_3(\mu, \nu, q, \varkappa))^{\frac{1}{p_1}} \right.$$

$$\left. + A_4^{1 - \frac{1}{p_1}}(\mu, \nu, q, \varkappa)(|^{\nu} D_q F(\mu)|^{p_1} A_5(\mu, \nu, q, \varkappa) + |^{\nu} D_q F(\nu)|^{p_1} A_6(\mu, \nu, q, \varkappa))^{\frac{1}{p_1}} \right],$$

which completes the proof.   □

Theorem 5. *Suppose that* $F : [\mu, v] \subset \mathbb{R} \to \mathbb{R}$ *is a q-differentiable on* (μ, v) *and* $^v D_q F$ *is continuous and integrable on* $[\mu, v]$. *If* $|^v D_q F|^{p_1}$ *is convex on* $[\mu, v]$ *for some* $p_1 > 1$ *with* $\frac{1}{r_1} + \frac{1}{p_1} = 1$, *then we have,*

$$\left| \frac{1}{v - \mu} \int_\mu^v F(s)\, ^v d_q s - F(\varkappa) \right| \tag{12}$$

$$\leq (v - \mu) \left[\left(\frac{v - \varkappa}{v - \mu} \right)^{1 + \frac{1}{r_1}} \left(\frac{q}{[r_1 + 1]_q} \right)^{\frac{1}{r_1}} \left(|^v D_q F(\mu)|^{p_1} \frac{1}{[2]_q} \left(\frac{v - \varkappa}{v - \mu} \right)^2 + |^v D_q F(v)|^{p_1} \left(\frac{v - \varkappa}{v - \mu} - \frac{1}{[2]_q} \left(\frac{v - \varkappa}{v - \mu} \right)^2 \right) \right)^{\frac{1}{p_1}} \right.$$

$$\left. + \left(\int_{\frac{v-\varkappa}{v-\mu}}^1 (1 - qs)^{r_1} d_q s \right)^{\frac{1}{r_1}} \left(|^v D_q F(\mu)|^{p_1} \left(\frac{1}{[2]_q} \left(1 - \left(\frac{v - \varkappa}{v - \mu} \right)^2 \right) \right) + |^v D_q F(v)|^{p_1} \left(\frac{q}{[2]_q} - \frac{v - \varkappa}{v - \mu} + \frac{1}{[2]_q} \left(\frac{v - \varkappa}{v - \mu} \right)^2 \right) \right)^{\frac{1}{p_1}} \right],$$

where $0 < q < 1$.

Proof. On taking the modulus in Lemma 4 and applying the quantum Hölder's inequality, we obtain that

$$\left| \frac{1}{v - \mu} \int_\mu^v F(s)\, ^v d_q s - F(\varkappa) \right|$$

$$\leq (v - \mu) \int_0^1 |\Psi_q(s)| |^v D_q F(s\mu + (1 - s)v)| d_q s$$

$$\leq (v - \mu) \left[\int_0^{\frac{v-\varkappa}{v-\mu}} qs |^v D_q F(s\mu + (1 - s)v)| d_q s + \int_{\frac{v-\varkappa}{v-\mu}}^1 (1 - qs) |^v D_q F(s\mu + (1 - s)v)| d_q s \right]$$

$$\leq (v - \mu) \left[\left(\int_0^{\frac{v-\varkappa}{v-\mu}} (qs)^{r_1} d_q s \right)^{\frac{1}{r_1}} \left(\int_0^{\frac{v-\varkappa}{v-\mu}} |^v D_q F(s\mu + (1 - s)v)|^{p_1} d_q s \right)^{\frac{1}{p_1}} + \left(\int_{\frac{v-\varkappa}{v-\mu}}^1 (1 - qs)^{r_1} d_q s \right)^{\frac{1}{r_1}} \left(\int_{\frac{v-\varkappa}{v-\mu}}^1 |^v D_q F(s\mu + (1 - s)v)|^{p_1} d_q s \right)^{\frac{1}{p_1}} \right].$$

Using an assumption that $|^v D_q F|^{p_1}$ is convex on $[\mu, v]$, we have

$$\left| \frac{1}{v - \mu} \int_\mu^v F(s)\, ^v d_q s - F(\varkappa) \right|$$

$$\leq (v - \mu) \left[\left(\int_0^{\frac{v-\varkappa}{v-\mu}} (qs)^{r_1} d_q s \right)^{\frac{1}{r_1}} \left(|^v D_q F(\mu)|^{p_1} \int_0^{\frac{v-\varkappa}{v-\mu}} s\, d_q s + |^v D_q F(v)|^{p_1} \int_0^{\frac{v-\varkappa}{v-\mu}} (1 - s) d_q s \right)^{\frac{1}{p_1}} \right.$$

$$\left. + \left(\int_{\frac{v-\varkappa}{v-\mu}}^1 (1 - qs)^{r_1} d_q s \right)^{\frac{1}{r_1}} \left(|^v D_q F(\mu)|^{p_1} \int_{\frac{v-\varkappa}{v-\mu}}^1 s\, d_q s + |^v D_q F(v)|^{p_1} \int_{\frac{v-\varkappa}{v-\mu}}^1 (1 - s) d_q s \right)^{\frac{1}{p_1}} \right]$$

$$= (v - \mu) \left[\left(\frac{v - \varkappa}{v - \mu} \right)^{1 + \frac{1}{r_1}} \left(\frac{q}{[r_1 + 1]_q} \right)^{\frac{1}{r_1}} \left(|^v D_q F(\mu)|^{p_1} \frac{1}{[2]_q} \left(\frac{v - \varkappa}{v - \mu} \right)^2 + |^v D_q F(v)|^{p_1} \left(\frac{v - \varkappa}{v - \mu} - \frac{1}{[2]_q} \left(\frac{v - \varkappa}{v - \mu} \right)^2 \right) \right)^{\frac{1}{p_1}} \right.$$

$$\left. + \left(\int_{\frac{v-\varkappa}{v-\mu}}^1 (1 - qs)^{r_1} d_q s \right)^{\frac{1}{r_1}} \left(|^v D_q F(\mu)|^{p_1} \left(\frac{1}{[2]_q} \left(1 - \left(\frac{v - \varkappa}{v - \mu} \right)^2 \right) \right) + |^v D_q F(v)|^{p_1} \left(\frac{q}{[2]_q} - \frac{v - \varkappa}{v - \mu} + \frac{1}{[2]_q} \left(\frac{v - \varkappa}{v - \mu} \right)^2 \right) \right)^{\frac{1}{p_1}} \right],$$

which completes the proof. $\qquad\square$

4 Some special cases

In this section, some special cases of our main results are discussed and several new results in the field of Ostrowski and midpoint-type inequalities are obtained.

Remark 2. If we consider Theorem 4, then

(i) by using $|^\nu D_q F(\varkappa)| \le M$ and taking $p_1 = 1$ in Theorem 4, we have the following new quantum Ostrowski-type inequality:

$$\left| \frac{1}{\nu - \mu} \int_\mu^\nu F(s)^\nu d_q s - F(\varkappa) \right| \le \frac{M}{[2]_q (\nu - \mu)} [q(\varkappa - \mu)^2 + (1 - q)(\varkappa - \mu)(\nu - \mu) + q(\nu - \varkappa)^2]. \tag{13}$$

Moreover, on taking limit as $q \to 1^-$ inequality (13) reduces to (1);

(ii) by choosing $\varkappa = \frac{\mu + q\nu}{[2]_q}$, we have the following new midpoint-type inequality

$$\left| \frac{1}{\nu - \mu} \int_\mu^\nu F(s)^\nu d_q s - F\left(\frac{\mu + q\nu}{[2]_q} \right) \right|$$

$$\le (\nu - \mu) \left[\left(\frac{q}{[2]_q^3} \right)^{1 - \frac{1}{p_1}} \left(|^\nu D_q F(\mu)|^{p_1} \left(\frac{q}{[2]_q^3 [3]_q} \right) + |^\nu D_q F(\nu)|^{p_1} \left(\frac{q^2 + q^3}{[2]_q^3 [3]_q} \right) \right)^{\frac{1}{p_1}} \right. \tag{14}$$

$$\left. + \left(\frac{q}{[2]_q^3} \right)^{1 - \frac{1}{p_1}} \left(|^\nu D_q F(\mu)|^{p_1} \left(\frac{2q}{[2]_q^3 [3]_q} \right) + |^\nu D_q F(\nu)|^{p_1} \left(\frac{-q + q^2 + q^3}{[2]_q^3 [3]_q} \right) \right)^{\frac{1}{p_1}} \right].$$

Specifically, on applying the limit as $q \to 1^-$ and taking $p_1 = 1$, we have the following midpoint-type inequality in [6, Theorem 2.2]

$$\left| \frac{1}{\nu - \mu} \int_\mu^\nu F(s) ds - F\left(\frac{\mu + \nu}{2} \right) \right| \le (\nu - \mu) \left[\frac{|F'(\mu)| + |F'(\nu)|}{8} \right];$$

(iii) by taking $r_1 = 1$ in Theorem 4, we have the following inequality:

$$\left| \frac{1}{\nu - \mu} \int_\mu^\nu F(s)^\nu d_q s - F(\varkappa) \right| \le (\nu - \mu) \{ |^\nu D_q F(\mu)| [A_2(\mu, \nu, q, \varkappa) + A_5(\mu, \nu, q, \varkappa)]$$

$$+ |^\nu D_q F(\nu)| [A_3(\mu, \nu, q, \varkappa) + A_6(\mu, \nu, q, \varkappa)] \}; \tag{15}$$

(iv) by setting $r_1 = 1$, $\varkappa = \frac{\mu + q\nu}{[2]_q}$, we have the following new midpoint-type inequality in [24],

$$\left| \frac{1}{\nu - \mu} \int_\mu^\nu F(s)^\nu d_q s - F\left(\frac{\mu + q\nu}{[2]_q} \right) \right| \le (\nu - \mu) \left[|^\nu D_q F(\mu)| \left(\frac{3q}{[2]_q^3 [3]_q} \right) + |^\nu D_q F(\nu)| \left(\frac{-q + 2q^2 + 2q^3}{[2]_q^3 [3]_q} \right) \right]. \tag{16}$$

Remark 3. If we consider Theorem 5, then

(i) by using $|^\nu D_q F(\varkappa)| \le M$ in Theorem 5, we have the following new quantum Ostrowski-type inequality:

$$\left| \frac{1}{\nu - \mu} \int_\mu^\nu F(s)^\nu d_q s - F(\varkappa) \right| \le M(\nu - \mu) \left[\left(\frac{\nu - \varkappa}{\nu - \mu} \right)^2 \left(\frac{q}{[r_1 + 1]_q} \right)^{\frac{1}{r_1}} + \left(\int_{\frac{\nu - \varkappa}{\nu - \mu}}^1 (1 - qs)^{r_1} d_q s \right)^{\frac{1}{r_1}} \left(\frac{\varkappa - \mu}{\nu - \mu} \right)^{\frac{1}{p_1}} \right]. \tag{17}$$

Specifically on taking limit as $q \to 1^-$, we obtain the following Ostrowski-type inequality given in [38, Theorem 3 with $s = 1$];

$$\left| \frac{1}{v - \mu} \int_{\mu}^{v} F(s) \mathrm{d}s - F(\varkappa) \right| \leq \frac{M(v - \mu)}{(r_1 + 1)^{\frac{1}{r_1}}} \left[\left(\frac{\varkappa - \mu}{v - \mu} \right)^2 + \left(\frac{v - \varkappa}{v - \mu} \right)^2 \right];$$

(ii) by choosing $\varkappa = \frac{\mu + qv}{[2]_q}$, we have the following new midpoint-type inequality:

$$\left| \frac{1}{v - \mu} \int_{\mu}^{v} F(s)^v d_q s - F\left(\frac{\mu + qv}{[2]_q} \right) \right|$$

$$\leq (v - \mu) \left[\left(\frac{1}{[2]_q} \right)^{1 + \frac{1}{r_1}} \left(\frac{q}{[r_1 + 1]_q} \right)^{\frac{1}{r_1}} \left(|^v D_q F(\mu)|^{p_1} \frac{1}{[2]_q^3} + |^v D_q F(v)|^{p_1} \frac{2q + q^2}{[2]_q^3} \right)^{\frac{1}{p_1}} \right. \tag{18}$$

$$\left. + \left(\int_{\frac{1}{[2]_q}}^{1} (1 - qs)^{r_1} d_q s \right)^{\frac{1}{r_1}} \left(|^v D_q F(\mu)|^{p_1} \frac{2q + q^2}{[2]_q^3} + |^v D_q F(v)|^{p_1} \frac{q^3 + q^2 - q}{[2]_q^3} \right) \right].$$

Specifically on taking limit as $q \to 1^-$, we obtain the following midpoint-type inequality:

$$\left| \frac{1}{v - \mu} \int_{\mu}^{v} F(s) \mathrm{d}s - F\left(\frac{\mu + v}{2} \right) \right| \leq \left(\frac{v - \mu}{4} \right) \left(\frac{4}{r_1 + 1} \right)^{\frac{1}{r_1}} [|F'(\mu)| + |F'(v)|],$$

which was proved by Kırmacı in [6].

It is worth mentioning that the deduced results (13)–(18) are also new in the literature on Ostrowski inequalities.

5 Conclusion

In this paper, we proved some new Ostrowski-type inequalities for differentiable functions using the notions of quantum calculus. We also discussed some special cases of newly established inequalities and found different midpoint-type inequalities and Ostrowski-type inequalities. It is an interesting and new problem that the upcoming researchers can develop similar inequalities for different kinds of convexities and integral operators in their future work.

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