

Moral Hazard, Collateral and Liquidity¹

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Abstract

We consider a moral hazard setup wherein leveraged firms have incentives to take on excessive risks and are thus rationed when they attempt to roll over debt. Firms can optimally pledge cash as collateral to reduce rationing, but in the process must liquidate some of their assets. Liquidated assets are purchased by non-rationed firms but their borrowing capacity is also limited by the risk-taking moral hazard. The market-clearing price exhibits cash-in-the-market pricing and depends on the entire distribution of leverage (debt to be rolled over) in the economy. This distribution of leverage, and indeed its very form as roll-over debt, are derived as endogenous outcomes with each firm's choice of leverage anticipating the difficulty for all firms in rolling over debt in future. The model provides a natural linkage between market liquidity and funding liquidity, shows that optimally designed collateral requirements have a stabilizing effect on prices, and illustrates the possible role of leverage in generating deep discounts in prices when adverse asset-quality shocks materialize in good times.

Keywords: leverage, risk-shifting, credit rationing, market liquidity, funding liquidity, fire sales, financial crises.

JEL Classification: G12, G20, D45, D52, D53

1 Introduction

"Where did all the liquidity go? Six months ago, everybody was talking about boundless global liquidity supporting risky assets, driving risk premiums to virtually nothing, and now everybody is talking about a global liquidity crunch, driving risk premiums half the distance to the moon. Tell me, Mac, where did all the liquidity go?" - Paul McCulley, PIMCO Investment Outlook, Summer 2007

Since the seminal contribution of Amihud and Mendelson (1986), the literature on asset pricing with trading frictions has burgeoned. Indeed, many would regard asset pricing with frictions as a new branch of financial economics. On the one hand, the literature on market microstructure, starting with Glosten and Milgrom (1985) and Kyle (1985), has provided a foundation for trading frictions such as bid-ask spread and price impact by appealing to information asymmetry problem between traders and specialists or market-makers. On the other hand, a recent strand of literature (Gromb and Vayanos, 2002 and Brunnermeier and Pedersen, 2005) has recognized that the balance-sheet liquidity of traders is limited due to constraints such as collateral and margin requirements imposed by counterparties and financiers. This limited funding liquidity affects and is affected by the trading liquidity in markets.

While this latter strand of literature has taken an important stride forward in linking the corporate-finance idea of funding liquidity and the asset-pricing idea of market liquidity, it has not yet modeled explicitly the micro-foundations underpinning funding liquidity.¹ We fill this important gap in the literature. We recognize that constraints such as collateral and margin requirements are themselves an *endogenous* response to mitigate underlying agency problems between those who provide finance and those who receive finance, or more generally, between any two parties engaging in trade. We show that collateral constraints and market illiquidity are both manifestations of underlying agency problems between borrowers and financiers, and market liquidity would be in fact far worse if collateral requirements were not in place to ameliorate agency problems. In the same vein, our model provides an agency-theoretic explanation for some features of financial crises such as the linkage between market liquidity and funding liquidity, and deep discounts observed in prices when adverse asset-quality shocks materialize in good times.

Since the backdrop we have in mind is one of trading-based financial institutions which are typically highly levered, we focus on the agency problem of asset substitution or risk-shifting by borrowers (Jensen and Meckling, 1976).² Related to the work of Stiglitz and Weiss (1981)

¹This literature has recognized that the micro-foundation for funding illiquidity stems from principal-agent problems affecting borrower-financier relationships. However, the reduced-form modeling of margin or collateral constraints has often given the impression that such constraints are the source of drying up of liquidity in capital markets.

²In this regard, we differ from Holmstrom and Tirole (1998) who consider the rent-seeking moral hazard

and Diamond (1989, 1991), this risk-shifting problem rations potential borrowers in that it limits the maximum amount of financing they can raise from lenders. In this setting, we show that a collateral requirement – the pledging of cash that can be seized by financiers in case of borrower default – relaxes the extent of rationing. This simple agency-theoretic set-up forms the building block of our benchmark model.

To analyze asset-pricing implications, we cast this building block in an industry equilibrium. Specifically, there is a continuum of firms which have undertaken some ex-ante financing (exogenous initially in the paper, endogenized later). The need to repay or roll over this ex-ante financing gives rise to liquidity shocks faced by firms since asset liquidations may not be feasible on demand to meet these shocks. Thus, firms attempt to meet liquidity shocks by raising roll-over financing, but its extent is limited due to the risk-shifting problem. Firms which are rationed by this agency problem attempt to relax the problem by pledging cash as collateral, which requires them to liquidate some or all of their assets. These liquidated assets are acquired by the set of remaining firms in the economy (as in Shleifer and Vishny, 1992). Though these remaining firms are able to meet their own liquidity shocks, they also potentially face the moral hazard problem, which limits their financing for asset purchase.³ Thus, the liquidation price, determined by the market-clearing condition, reflects the so-called “cash-in-the-market” pricing (a term introduced by Allen and Gale, 1994): When a large number of firms are liquidating assets, market price is below the expected discounted cash flow and is affected by the distribution of liquidity in the economy.

Crucially, the entire general equilibrium is characterized by a single parameter of the economy which measures the (inverse) moral-hazard intensity, namely the extent of financing that can be raised by a firm per unit asset: (1) The moral-hazard intensity divides the set of firms into three categories – those that are rationed and fully liquidated, those that pledge collateral and are partially liquidated, and those that provide liquidity (“arbitrageurs”) and purchase assets at fire-sale prices; (2) Through this division of firms, the moral-hazard intensity determines the equilibrium price at which assets are liquidated; and, finally (3) By determining the cost of liquidating an asset relative to the cost of funding it with external finance, the moral-hazard intensity determines the optimal level of collateral requirement for rationed firms.

An interesting result that stems from this characterization is the following. As moral-hazard intensity *increases* (formally, the spread between the return on the good asset and the risk-shifting asset declines), firms’ ability to raise financing is lowered and equilibrium levels of liquidity in the

problem to motivate limited funding liquidity. The rent-seeking problem is perhaps more relevant or appropriate for management of real assets. In case of financial assets, leverage and induced risk-shifting are more pertinent, as evidenced by the excessive borrowing and doubling-up strategies involved in a large number of events involving significant trading losses in derivatives and bond markets.

³We allow the buying firms to pledge the assets that they buy. The financing they receive is constrained by the risk-shifting problem.

economy fall. In turn, the market for assets clears at *lower prices*. This is simply the result that funding liquidity, measured by (inverse) moral-hazard intensity, affects market liquidity.⁴ Simultaneously, to relax rationing, the optimal *collateral requirement increases*. To summarize, in the *cross-section of states, ranked by moral-hazard intensity*, the level of prices and the tightness of collateral requirement are *negatively* related. This, however, should not be construed as the causal effect of collateral requirement on prices. In fact, we show that if collateral requirements could not be imposed (for example, due to inability to pledge cash reserves or to prevent their diversion), then equilibrium prices would be *lower* for any given level of moral-hazard intensity, compared to the case when collateral requirements are present.

Thus, when constraints such as collateral and margin requirements are designed endogenously to address underlying agency problems related to external finance, such constraints stabilize prices rather than being the cause for drying up of liquidity.⁵

Furthermore, we show that this price-stabilizing role of collateral requirements has important welfare implications for ex-ante debt capacity of firms. In the preceding discussion, the ex-ante structure of liabilities undertaken by firms was treated as being given. We endogenize this structure by assuming that ex ante, firms are ranked by their initial wealth or capital levels and must raise incremental financing up to some fixed (identical) level in order to trade. The incremental financing is raised through short-term debt contracts that give lenders the ability to liquidate ex post in case promised payments are not met. While not critical to the overall thrust of our results, we show that this form of financing – which grants control to lenders in case of default (as in collateral and margin requirements) – is optimal from the standpoint of raising maximum ex-ante finance.

This augmentation of our benchmark model leads to an intriguing, even if somewhat involved, fixed-point problem: On the one hand, the promised payment for a given amount of financing is decreasing in the level of liquidation prices in case of default;⁶ on the other hand, the liquidation price is itself determined by the distribution of promised debt payments since these are the ex-post liquidity shocks faced by firms. We show that there is a unique solution to this fixed-point

⁴Note however that unlike in the recent literature on funding and market liquidity, our measure of funding liquidity is based on the amount of financing that can be raised given an agency problem tied to external finance, and *not* by tightness of an exogenously specified constraint (such as collateral requirement) which is just a response to the agency problem.

⁵This is a counterpoint to the existing literature on asset pricing with frictions, wherein as mentioned before, collateral constraints and margin requirements are generally perceived to exacerbate liquidity problems and price drops. For example, the negative association between prices and level of collateral requirements is reminiscent of the “anti-cushioning” effect of collateral in Brunnermeier and Pedersen (2005), but in our model this is an endogenous outcome rather than being an “effect” of collateral on prices.

⁶This argument is naturally reminiscent of the debt-capacity argument of Shleifer and Vishny (1992) since our model can be considered a general equilibrium variant of their analysis with endogenously modeled financing constraints.

problem, characterized by the fraction of firms that are ex-ante rationed and by the mapping from moral-hazard intensity to price. In fact, the fixed-point is a contraction mapping and enables us to provide a recursive, constructive algorithm for the solution.

While the ex-ante rationing of firms renders analytical results on comparative statics difficult, numerical examples provide valuable insights. First, as the distribution of wealth or capital levels of firms declines in a first-order stochastic dominance (FOSD) sense, firms have to pledge more payment to raise given amount of financing, and, in turn, equilibrium price is lower in each future state of the world. Second, as the distribution of quality of assets worsens in a FOSD sense, the distribution of moral-hazard intensity worsens too, firms face greater financing friction in future, and, in turn, equilibrium ex-ante financing requires higher debt payments. Interestingly, and somewhat counter-intuitively, better ex-ante distribution of quality of assets can in fact be associated with *lower* prices in adverse realizations to asset quality. The reason is that there is endogenous entry in our model: good times in terms of expectations about the future enable even highly levered institutions to be funded ex ante. Even though bad times are less likely to follow, in case they do materialize, the higher leverage of firms set up in the economy implies greater proportion of firms with funding liquidity problems, greater quantity of asset liquidations, and deeper discounts in prices.

This second effect matches well the often-observed “puzzle” in financial markets that when there is a sudden, adverse asset-quality shock in a period of high expectations of fundamentals, the drop in prices seems rather severe. This was highlighted in the introductory quote by Paul McCulley in PIMCO’s Investment Outlook of Summer 2007 following the sub-prime crisis which seems to have switched the financial system from one with abundance of global liquidity to one with a severe glut. While there are many elements at work in explaining this phenomenon, our model clarifies that financial structure, in particular, the extent of highly leveraged institutions in the system, is endogenous to the expectations leading up to the crisis. This endogeneity is crucial to understanding the severity of fire sales that hit asset markets when levered institutions attempt to meet their financial liabilities.

The feedback between promised (ex-ante) debt payments and (ex-post) liquidation price produces an amplification effect of collateral in our model, but unlike the extant literature, this amplification effect is positive for liquidity and efficiency: The presence of collateral requirements stabilizes liquidation prices as well as lowers promised debt payments, and these two effects feed on each other to produce an amplified reduction in ex-ante rationing. To conclude, in our augmented model too, endogenously designed collateral requirements enhance efficiency and trading by reducing the severity of agency problems tied to external finance. More generally, we conjecture that rather than being the source of illiquidity in markets, collateral is in fact the life-blood that sustains high levels of trading amongst financial institutions.

We would like to stress the important role played by financial leverage in our model. First,

the ex-ante structure of leverage undertaken by firms in our model is endogenously derived and determines the nature of ex-post liability-side shocks faced by these firms. These liability shocks constitute the liquidity shocks that necessitate asset sales and pledging of collateral. Second, leverage-based financing raises the possibility of risk-shifting agency problem, which limits funding liquidity, the most critical ingredient of our model. Finally, a combination of these two effects leads to the feature that illiquidity states in our model coincide with states in which quality of firms' assets has deteriorated and leverage on balance-sheets of institutions is high. These states are associated with significant agency problems between borrowers and lenders and lead to low market and funding liquidity.⁷ The distribution and level of leverage is thus the central force driving our results and conclusions.

The paper is organized as follows. Section 2 discusses the related literature. Section 3 sets up the benchmark model and analyzes it to illustrate the price-stabilizing role of collateral. Section 4 augments the benchmark model to introduce the feedback between collateral and ex-ante debt capacity of firms. Section 5 discusses robustness issues. Section 6 concludes. All proofs not contained in the text are provided in Appendix 1. Appendix 2 presents the constructive algorithm to solve the fixed-point problem introduced in Section 4.

2 Related literature

The idea that asset prices may contain liquidity discounts when potential buyers are financially constrained dates back to Williamson (1988) and Shleifer and Vishny (1992).⁸ Since then, fire sales have been employed in finance models regularly, perhaps most notably by Allen and Gale (1994, 1998) to examine the link between limited market participation, volatility, and fragility observed in banking and asset markets. At its roots, our model is closely linked to this literature on fire sales and industry equilibrium view of asset sales. The industry view makes clear that market prices depend on funding liquidity of potential buyers. More broadly, the overall approach and ambition of our paper in relating the distribution of liquidity shocks in an economy to

⁷Acharya and Pedersen (2005) document in Figure 1 of their paper and the related discussion that all significant (more than three standard deviation) illiquidity episodes in the US stock market during the period 1964-1999 have been preceded by significant asset-side shocks: 5/1970 (Penn Central commercial paper crisis), 11/1973 (oil crisis), 10/1987 (stock market crash), 8/1990 (Iraqi invasion of Kuwait), 4-12/1997 (Asian crisis) and 610/1998 (Russian default, LTCM crisis).

⁸Empirically, the idea of fire sales has now found ample empirical evidence in a variety of different settings: in distressed sales of aircrafts in Pulvino (1998), in cash auctions in bankruptcies in Stromberg (2000), in creditor recoveries during industry-wide distress especially for industries with high asset-specificity in Acharya, Bharath and Srinivasan (2007), in equity markets when mutual funds engage in sales of similar stocks in Coval and Stafford (2006), and, finally, in an international setting where foreign direct investment increases during emerging market crises to acquire assets at steep discounts in the evidence by Krugman (1998), Aguiar and Gopinath (2005), and Acharya, Shin and Yorulmazer (2007).

equilibrium outcomes is closest to the seminal paper of Holmstrom and Tirole (1998). However, there are important differences with both these sets of papers.

In Allen and Gale (1994, 1998), the liquidity shocks arise as preference shocks to depositors or investors as in Diamond and Dybvig (1983). In Holmstrom and Tirole (1998), the liquidity shocks arise as production shocks to firms' technologies. In either case, they are not endogenous outcomes. We derive liquidity shocks as being determined in equilibrium by asset-liability mismatch of firms, where the level and distribution of liabilities in the economy is an outcome of model primitives such as the distribution of asset quality and moral hazard problems in future. The liabilities become liquidity "shocks" in our model in the sense that liabilities are known in advance but they take the form of "hard" debt contracts and asset quality is uncertain in future. The optimality of hard debt contract in our model with control rights given to lenders in case of default mirrors closely the work of Aghion and Bolton (1992), Hart and Moore (1994), Hart (1995), and Diamond and Rajan (2001a).

In terms of modeling details, we derive limited funding liquidity as arising due to credit rationing from a risk-shifting moral hazard problem. Our specific modeling technology is closely related to the earlier models in Diamond (1989, 1991). In contrast, Holmstrom and Tirole's model of limited funding liquidity is based on rent-seeking moral hazard. It is our belief that rent-seeking is a more appropriate metaphor for agency problems affecting real or technological choices, whereas risk-substitution fits financial investment choices (typically by highly levered institutions) better. In the risk-shifting set-up, we introduce collateral as a means to relax rationing. This finds exact parallel in an asymmetric information context in the rationing models of Stiglitz and Weiss (1981) and the corresponding signaling model with collateral of Bester (1985). Given this agency-theoretic foundation, our primary goal is to consider the implications of endogenously derived collateral requirements on market prices and liquidity. In this sense, our objectives are the financial markets counterpart to those of Bernanke and Gertler (1989) who considered the role of real collateral, its role in ameliorating agency problems linked to real investments, and its implications for business cycle.

Our work is also related to the seminal work of Kiyotaki and Moore (1997) on credit cycles. In Kiyotaki and Moore (1997) and Krishnamurthy (2003), the underlying asset cannot be pledged because of inalienable human capital.⁹ However, land can be pledged and has value both as a productive asset and as collateral. Caballero and Krishnamurthy (2001) employ a Holmstrom-Tirole approach to liquidity shocks (these are exogenous) and allow firms to post collateral in a manner similar to Kiyotaki and Moore. In contrast, the underlying asset in our model can be pledged but the amount that can be pledged is endogenously determined by the risk-shifting moral hazard constraint and the equilibrium distribution of liquidity shocks.

⁹Krishnamurthy (2003) differs from Kiyotaki and Moore (1997) in that all contingent claims on aggregate variables are allowed subject to collateral constraints.

Finally, our research has implications for the recent work on *exogenous* collateral requirements due to Gromb and Vayanos (2002), Brunnermeier and Pedersen (2005), Plantin and Shin (2006), and Anshuman and Viswanathan (2006). In Gromb and Vayanos (2002), agents can only borrow if each asset is separately and fully collateralized, i.e., borrowing is essentially riskless. In Brunnermeier and Pedersen (2005), the collateral requirement is similarly exogenous: a shock to prices (or volatility) leads to liquidity shocks, that, in turn, leads to liquidation by financial intermediaries who engage in risk management. These models, however, do not explicitly model why lenders engage in risk management and why collateral requirements are imposed (even though they do recognize that agency problems must be at play). Plantin and Shin (2006) consider a dynamic variant of this feedback effect focusing on application to the unwinding of carry trades and their precipitous effect on exchange rates.¹⁰ Anshuman and Viswanathan (2006) point out that the ability to renegotiate constraints can eliminate liquidity crises of the nature analyzed in these papers, unless some other frictions are present. Our paper presents one such friction in terms of the ability of financial intermediaries to substitute risks, which limits their borrowing capacity.¹¹

3 Model

3.1 Informal description

Our model is set up as follows. At date 0, there is a continuum of agents who have access to identical, valuable trading technology (“asset”) of limited size. Agents do not have all of the financing required to incur the fixed costs for setting up firms that will invest in this asset. Agents differ in the amount of personal initial capital. They raise external financing from a set of financiers. Assets are specific in that financiers cannot redeploy them. In fact, we will assume assets are rendered worthless in hands of financiers (unless they sell them right away to those who can deploy them). Conversely, firms are not in the business of providing external finance to each other. Some examples of this setup would be traders setting up hedge funds and borrowing from prime brokers, or specialist firms being set up with reliance on short-term commercial paper based financing, even though some of our assumptions make our caricature of these settings somewhat extreme.

¹⁰Morris and Shin (2004) present a model where traders are liquidated when an exogenous trigger price is reached and this trigger is different for each trader.

¹¹Our argument that financing constraints are endogenous reflection of agency problems rather than the source of capital flight and liquidity problems, echoes well with a similar hypothesis put forth by Diamond and Rajan (2001b) in the context of financial crises. They argue that while the literature has found a positive relationship between the extent of short-term borrowing and incidence of crises, they may both be manifestations of the fact that the underlying economy has illiquid investments to start with.

Each asset produces an uncertain cash flow at date 2. Agents (non-financiers) have the option of switching from their asset to an alternate, riskier asset (e.g., through poor risk management of a trade) that is less valuable but may be attractive once external financing is raised. Such a switch will never occur in equilibrium but its possibility will affect the nature and extent of external financing.

At date 1, an observable but non-verifiable public signal concerning the common quality of the valuable assets becomes available. If the optimal contract at date 0 so specifies, financiers may demand repayments at date 1, or they may effectively roll over their financing to date 2. Firms can pledge a part of their asset portfolio to financiers as “collateral”, thereby agreeing to convert it into cash and giving up their option to alter its risk profile. An asset sale market exists where assets can be sold to other firms at market-clearing prices to liquidate assets in exchange for obtaining such cash. Firms acquiring assets may raise financing at date 1 against their existing assets as well as assets to be acquired.

We formally specify and solve the model backwards starting with the second period between date 1 and date 2. To this end, we first assume and later prove in date-0 analysis the following equilibrium outcomes: (1) The optimal date-0 contract takes the form of debt that is due at date 2, but it is hard in the sense that it gives financiers (lenders) the control at date 1 to demand early repayment if it is optimal for them to do so; (2) No excess cash is held at date 0 by an agent who invests and there is no shifting to riskier assets at date 0. Given these assumptions, we solve the second-period model for a particular realization of public information about asset quality.

3.2 Benchmark second-period model

Consider a continuum of firms that have all undertaken some borrowing at date 0. At date 1, these firms face financial liabilities such that firm i is required to pay back ρ_i to its existing creditors. The contract for borrowing is hard and if the promised payment ρ_i is not made at date 1, then creditors take charge and force the firm to liquidate assets. We assume that assets can be liquidated only at date $1\frac{1}{2}$; their liquidation value at date 1 is zero. Thus, there is a timing mismatch between when liabilities are due and when assets can be used to generate liquidity. In this sense, firms’ asset-liability mismatch in duration leads to liquidity shocks.¹² Firms have no internal liquidity and must raise new external finance at date 1 to pay off existing debt. Alternatively, existing creditors can simply roll over their debt provided they are guaranteed an expected repayment of ρ_i at date 2.

¹²Depending on the specific nature of liabilities and assets, the delay could be intra-day (for example, in case of inter-bank borrowing and liquid assets) or several days (for example, in case of public debt and relatively illiquid assets).

As mentioned earlier, we focus attention first on the date 1 aspects of the model, deriving the exact size and distribution of date 0 borrowing of firms in Section 4 (wherein we also provide a justification for the hard nature of borrowing we assume throughout). The time-line for the model, starting at date 1, is specified in Figure 1. All firm owners and creditors are risk-neutral and the risk-free rate of interest is zero.

After raising (new or rolled-over) external finance at date 1, there is the possibility of moral hazard at the level of each firm. In particular, we consider asset-substitution moral hazard along the lines of Jensen and Meckling (1976). Firm's existing investment is in an asset which is a positive net present value investment. However, after raising external financing at date 1 and also after asset sales at date $1\frac{1}{2}$, each firm can switch its investment to another asset. We denote the assets as j , $j \in \{1, 2\}$, yielding a cash flow at date 2 of $y_j > 0$ with probability $\theta_j \in (0, 1)$, and no cash flow otherwise. We assume that $\theta_1 < \theta_2$, $y_1 > y_2$, $\theta_1 y_1 \leq \theta_2 y_2$, and $\theta_1 y_1 \leq \rho_i$. In words, the first asset is riskier and has a higher payoff than the second asset, but the second asset has a greater expected value. Also, taking account of the financial liability at date 1, investing in the first asset is a negative net present value investment for all firms. We assume the shift between assets is at zero cost. The simplest interpretation could be a deterioration in the risk-management function of the financial intermediary, one that for example may allow a trader to engage in riskier strategies with the same underlying asset. We discuss some other possibilities in Section 5.

The external finance at date 1 is raised in the form of debt with face value of f to be repaid at date 2. In our benchmark model, we suppress the market for asset sales at date $1\frac{1}{2}$. Then, the incentive compatibility condition to ensure that firm owners invest in asset $j = 2$ (that is, do not risk-shift to asset $j = 1$) requires that

$$\theta_2(y_2 - f) > \theta_1(y_1 - f). \quad (1)$$

This condition simplifies to an upper bound on the face value of new debt:

$$f < f^* \equiv \frac{(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)}. \quad (2)$$

The funding constraint for firm i requires that

$$\rho_i = \theta(f_i) f_i, \quad (3)$$

where f_i is the face value charged to firm i and $\theta(f_i)$ is the probability of high cash flow from the asset invested in when face value of debt issued is f_i . Note that this condition takes its specific form above because the lender cannot be paid any sum with probability $(1 - \theta_i)$.

Since (IC1) bounds the face value of debt that can provide incentives to invest in the better asset, we obtain credit rationing as formalized in the following Proposition. We stress that this

result is by itself not new (see, for example, Stiglitz and Weiss, 1981). We elevate it to the level of a proposition as it forms the basis of our analysis that follows.

Proposition 1 *Firms with liquidity need ρ at date 0 that is greater than $\rho^* \equiv \theta_2 f^*$ cannot borrow, that is, they are credit-rationed in equilibrium.*

To see this result, note first that $f^* < y_2$ so that borrowing up to face value f^* is indeed feasible in equilibrium provided it enables the borrowing firm to meet its funding needs. In other words, firms with $\rho \leq \rho^* \equiv \theta_2 f^*$ borrow, invest in the better asset, and simultaneously meet their funding constraint. Second, note that for $\rho > \rho^*$, investment is in the first, riskier asset. However, in this case funding constraint requires that the face value be $\hat{f} = \frac{\rho}{\theta_1}$ which is greater than y_1 for all $\rho > \rho^*$. That is, firms with liability ρ exceeding ρ^* cannot borrow and are rationed.

We assume in what follows that the continuum of firms is ranked by liquidity shocks ρ such that $\rho \sim g(\rho)$ over $[\rho_{\min}, \rho_{\max}]$, where $\rho_{\min} \equiv \theta_1 y_1 < \theta_2 y_2 \leq \rho_{\max}$ and $\rho^* \in [\rho_{\min}, \rho_{\max}]$. Thus, Proposition 1 implies that firms in the range $(\rho^*, \rho_{\max}]$ are credit-rationed in our benchmark model.

3.3 Collateral

We extend this benchmark model to consider the market for asset sales and a role for collateral at date 1. Since firms have no internal liquidity, collateral can be provided only by offering to sell (at least) some assets at date $1\frac{1}{2}$. The analysis to follow is a natural counterpart to that of Bester (1985) who demonstrated the role of collateral in relaxing rationing (by screening) in the asymmetric information model of credit rationing by Stiglitz and Weiss (1981).

In particular, the borrowing contract at date 1 now takes the form (f_i, k_i) for firm i where f_i is the face value of debt to be paid at date 2 in return for the funding provided ρ_i , and k_i is the amount of collateral that the firm must put up after meeting its liquidity shock ρ_i . The sequence of events is as follows. First, the firm attempts to raise financing against the contract (f_i, k_i) and if this can enable the firm to meet the liquidity shock ρ_i , it does so. After the liquidity shock has been met, collateral is provided at date $1\frac{1}{2}$ in the amount as per the borrowing terms agreed at date 1. This can be visualized as the firm depositing a collateral of sufficient quantity of its asset to creditors, who have no use or expertise with it and sell it in asset-sale market at date $1\frac{1}{2}$, converting it to cash. Effectively, the cash collateral is generated by liquidating a portion of the firm's asset. For now, we assume this liquidation occurs at an exogenously given price p per unit of the asset. Recall that the risk-shifting problem with respect to firm's assets (other than cash) arises after borrowing and after liquidations have taken place, so that p is the per unit liquidation price of asset $j = 2$ at date $1\frac{1}{2}$. We endogenize this liquidation price in the next subsection. The

cash collateral is assumed to be invested in a storage technology between dates $1\frac{1}{2}$ and 2, whose rate of return is assumed to be zero (no liquidity or quality premium). Finally, assets pay off at date 2.

In essence, the above sequence of events captures the “(il)liquidity” aspect of the liability side of firm’s balance-sheet. The firm is unable to liquidate its assets to meet the liquidity shock at the very instant (intra-day, for example) the shock arises. The firm can however borrow and agree to provide collateral as part of the borrowing contract (by end of day, for example). Lenders rationally anticipate the price at which the firm can liquidate its assets. In other words, they set the collateral requirement taking account of the liquidation value of firm’s assets. Once collateral is provided, the firm continues its operations (beyond the day, for example) and now the likelihood of asset substitution arises.

Reverting to the model, to generate k_i units of collateral, the firm must sell a proportion α_i of its investment, given by $\alpha_i = k_i/p$. We focus on firms rationed in the benchmark model, that is, we consider here $\rho_i > \rho^*$. Dropping the subscripts i , in the presence of collateral, the firm’s total cash position is $[k + (1 - \alpha)y_j]$ at date 2 if asset j is chosen and this happens with probability θ_j , and simply k otherwise. Note that the payoff in the good state is declining in collateral k as long as the liquidation price p is lower than the cash flow y_j . We assume this now to be the case, so that putting up collateral is costly for the borrowing firm, and verify it later when we model the market for asset liquidations.

With collateral, the incentive compatibility constraint takes the form

$$\theta_2 [k + (1 - \alpha)y_2 - f] > \theta_1 [k + (1 - \alpha)y_1 - f]. \quad (4)$$

This simplifies to the condition

$$f < f^{**}(k) \equiv \left[k + f^* \left(1 - \frac{k}{p} \right) \right]. \quad (5)$$

An intuitively more appealing form of $f^{**}(k)$ is

$$f^{**}(k) = [\alpha p + (1 - \alpha)f^*]. \quad (6)$$

This, in turn, yields

$$\rho^{**}(k) = [\theta_2 f^{**}(k) + (1 - \theta_2)k], \quad (7)$$

which simplifies to

$$\rho^{**}(k) = [\alpha p + (1 - \alpha)\rho^*], \quad (8)$$

which can be interpreted as asset sales yielding p per unit to fund the firm, and borrowing yielding ρ^* per unit of assets remaining after asset sales. As is clear from this expression, collateral will relax rationing only in the case where the unit price of asset sale p exceeds the borrowing capacity per unit of asset ρ^* .

Finally, the funding constraint for the firm is given by

$$\rho = \theta_2 f + (1 - \theta_2)k, \quad (9)$$

and the conditions for the collateral requirement to be feasible are that (i) $k \geq 0$, and (ii) $\alpha = \frac{k}{p} \leq 1$.

Combining the incentive compatibility condition, funding constraint, and the two feasibility conditions yields the following proposition on optimal collateral requirement and the extent of its effect in relaxing credit rationing.

Proposition 2 *If the liquidation price p is lower than ρ^* , then no collateral requirement can relax credit rationing for firms with $\rho \in (\rho^*, \rho_{\max}]$.*

If the liquidation price p is greater than ρ^ , then collateral requirement relaxes credit rationing for firms with $\rho \in (\rho^*, p]$, and the collateral requirement takes the form*

$$k(\rho) = \frac{(\rho - \rho^*)}{\left(1 - \frac{\rho^*}{p}\right)}. \quad (10)$$

Furthermore, the collateral requirement $k(\rho)$ is increasing in liquidity shock ρ and decreasing in liquidation price p , and the proportion of firms for which credit rationing is relaxed, $[p - \rho^]$, is increasing in liquidation price p .*

Essentially, the incentive compatibility and funding constraints yield the collateral requirement of the form stated in the proposition. Imposing the two feasibility constraints then yields that the liquidity shock ρ should be lower than liquidation price p for collateral to relax credit rationing.

The liquidation price p plays a crucial role in determining the size of collateral requirement. In particular, if liquidation price is low, then firms have to liquidate a large part of their existing investment. This lowers the cash flows of the firm and exacerbates the risk-substitution problem. To limit this, a lower face value of debt is required, and, then in turn, the funding constraint implies that collateral requirement must be raised. Finally, if liquidation price is higher then more firms that were otherwise rationed can be funded in equilibrium with collateral requirement.

Next, we introduce a market for liquidation of the asset at date $1\frac{1}{2}$ and study how it influences and is influenced by the equilibrium level of collateral requirement. Also, we assumed in the analysis above that $p \leq \rho_{\max}$. We verify below that this will indeed be the case under our maintained assumption $\theta_2 y_2 \leq \rho_{\max}$.

3.4 Market for asset sales

We assume that assets liquidated by firms that face rationing ($\rho > \rho^*$) are acquired by those that are not rationed ($\rho < \rho^*$). We consider standard market clearing for asset liquidation. An important consideration is that asset purchasers, by virtue of their lower liquidity shocks, may be able to raise liquidity not only against their existing assets but also against to-be-purchased assets.

Formally, suppose that a non-rationed firm with liquidity shock ρ acquires α additional units of assets. Then, the incentive-compatibility condition for the non-rationed firm (with rational expectation of its acquisition of assets) takes the form

$$\theta_2[(1 + \alpha)y_2 - f] > \theta_1[(1 + \alpha)y_1 - f], \quad (11)$$

which requires that the interest rate f satisfy the condition:

$$f < f^*(\alpha) = \frac{(1 + \alpha)(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)} = \frac{(1 + \alpha)\rho^*}{\theta_2}. \quad (12)$$

The total amount of liquidity available for asset purchase with such a non-rationed firm is thus given by¹³

$$l(\alpha, \rho) = [\theta_2 f^*(\alpha) - \rho] = [(1 + \alpha)\rho^* - \rho]. \quad (13)$$

That is, the funding ability of a non-rationed firm consists of its spare debt capacity from existing assets, $(\rho^* - \rho)$, plus the liquidity that can be raised against assets to be acquired, $\alpha\rho^*$. The latter term arises because each unit of asset can command ρ^* of borrowing without incidence of moral hazard problem.

The pertinent question is: How many units of assets would this firm be prepared to buy as a function of the price p ? Note that no firm would acquire assets at a price higher than their expected payoff under the better asset. Denoting this price as $\bar{p} = \theta_2 y_2$, we obtain the following demand function $\hat{\alpha}(p, \rho)$ for the firm. For $p > \bar{p}$, $\hat{\alpha} = 0$. For $p < \bar{p}$, $\hat{\alpha}$ is set to its highest feasible value given the liquidity constraint:

$$p \hat{\alpha} = l(\hat{\alpha}, \rho), \quad (14)$$

which simplifies to

$$\hat{\alpha}(p, \rho) = \frac{(\rho^* - \rho)}{(p - \rho^*)}. \quad (15)$$

¹³Note that if non-rationed firms want any additional liquidity beyond $f^*(\alpha)$, these firms would themselves have to pledge collateral and liquidate assets.

Finally, for $p = \bar{p}$, buyers' demand is indifferent between 0 and $\hat{\alpha}$ (evaluated at \bar{p}).

Thus, the total demand for assets for $p < \bar{p}$ is given by

$$D(p, \rho^*) = \int_{\rho_{\min}}^{\rho^*} \hat{\alpha}(p, \rho) g(\rho) d\rho = \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^* - \rho)}{(p - \rho^*)} g(\rho) d\rho, \quad (16)$$

where we have stressed the dependence on (inverse) moral hazard intensity ρ^* .

Given this demand function for non-rationed firms, we can specify the market-clearing condition. Note that the total supply of assets up for liquidation is given by

$$S(p, \rho^*) = \int_{\rho^*}^p \frac{(\rho - \rho^*)}{(p - \rho^*)} g(\rho) d\rho + \int_p^{\rho_{\max}} g(\rho) d\rho. \quad (17)$$

The two terms correspond respectively to (i) partial asset liquidations by firms with $\rho \in (\rho^*, p]$ to meet the collateral requirement, and (ii) complete liquidation of firms with $\rho \in (p, \rho_{\max}]$.

Then, the equilibrium price p^* satisfies the market-clearing condition

$$D(p, \rho^*) = S(p, \rho^*). \quad (18)$$

In particular, if excess demand is positive for all $p < \bar{p}$, then $p^* = \bar{p}$ (since the buyers are indifferent at this price between buying and not buying, and hence their demand can be set to be equal to the supply).

Before characterizing the behavior of the equilibrium price, it is useful to consider properties of the demand and supply functions. First, both demand and supply functions decline in price p . This is because as price increases, asset purchasers can only buy fewer assets given their limited liquidity. Simultaneously, as price increases, the collateral requirement also requires rationed firms to liquidate a smaller quantity of their assets. Hence, what is important is the behavior of excess demand function, $E(p, \rho^*) \equiv [D(p, \rho^*) - S(p, \rho^*)]$, as a function of price p . We focus below on the case where $p < \bar{p}$, relegating the details of the case where $p = \bar{p}$ to the Appendix (in Proof of Proposition 3).

The excess demand function can be rewritten as:

$$E(p, \rho^*) = D(p, \rho^*) - S(p, \rho^*) \quad (19)$$

$$= \int_{\rho_{\min}}^p \frac{(\rho^* - \rho)}{(p - \rho^*)} g(\rho) d\rho - \int_p^{\rho_{\max}} g(\rho) d\rho. \quad (20)$$

Integrating this equation by parts yields that

$$E(p, \rho^*) = -1 + \frac{1}{(p - \rho^*)} \int_{\rho_{\min}}^p G(\rho) d\rho \quad (21)$$

where $G(\rho) = \int_{\rho_{\min}}^{\rho} g(\rho)d\rho$ and $G(\rho_{\min}) = 0$.

The condition that excess demand be zero, i.e., $E(p, \rho^*) = 0$, leads to the relationship

$$p = \rho^* + \int_{\rho_{\min}}^{\rho^*} G(\rho)d\rho. \quad (22)$$

If the solution to this equation exceeds \bar{p} , then we have $p^* = \bar{p}$.

From this representation of market-clearing condition, we observe that the price can never fall below the threshold level of ρ^* . This is because non-rationed firms can always raise ρ^* of liquidity against each additional unit of asset they purchase. Hence, at $p = \rho^*$, their demand for asset purchase is infinitely high. The second term captures the effect of spare liquidity in the system. Intuitively, if this spare liquidity is high, then the price is at its frictionless value of \bar{p} , else it reflects a fire-sale discount.

Second, the price can never be higher than \bar{p} as above this price, demand is zero and there can be no market clearing. Together, these facts guarantee an interior market-clearing price $p^* \in [\rho^*, \bar{p}]$.

Third, as intuition would suggest, the excess demand function is decreasing in price p , which gives us that p^* is in fact unique.

And, finally, the key determinant of the market-clearing price is the extent of (inverse) moral hazard intensity ρ^* . This is the central parameter that drives all action in the model: It determines the partition of firms into rationed firms and non-rationed firms, the extent of buying power of non-rationed firms, and, also, the level of collateral requirement and thereby the size of asset liquidations.

The resulting equilibrium price satisfies the following proposition:

Proposition 3 *The market-clearing price for asset sales, p^* , is unique and weakly increasing in the (inverse) moral hazard intensity ρ^* in the following manner:*

- (i) *There exists a critical threshold $\hat{\rho}^* < \bar{p}$ such that $p^* = \bar{p}, \forall \rho^* \geq \hat{\rho}^*$; and,*
- (ii) *For $\rho^* < \hat{\rho}^*$, $p^* \in [\rho^*, \bar{p}]$, p^* is strictly increasing in ρ^* , and $p^* = \rho^*$ only when $\rho^* = \rho_{\min}$. Therefore, in this region, there is an illiquidity discount, $[\bar{p} - p^*]$, whose size is declining in ρ^* .*

When ρ^* is above a critical value $\hat{\rho}^* > \rho_{\min}$, assets are liquidated at their highest valuation: Few firms are rationed, buyers (non-rationed firms) have lot of liquidity and sellers (rationed firms) face the weakest possible collateral requirement. As moral hazard becomes worse, that is, ρ^* declines, there is not enough liquidity in the system to absorb the pool of assets being put up for liquidation at the highest price. Hence, the market-clearing price is lower than \bar{p} . Since assets are “cheap”, non-rationed firms demand as much as possible of the liquidated assets with

their entire available liquidity. On the supply side, as price falls, more firms are rationed, and rationed firms face tighter collateral requirements. As moral hazard keeps worsening (ρ^* becomes smaller), prices fall until they hit ρ^* eventually, and this happens when in fact ρ^* equals ρ_{\min} .

Note that the liquidation price exhibits “cash-in-the-market pricing” as in Allen and Gale (1994, 1998) since it depends on the overall amount of liquidity available in the system for asset purchase, which, in turn, is determined by the extent of moral hazard problem. The important message from this analysis is that whether a rationed firm can relax its own borrowing constraint or not by pledging collateral depends upon the liquidity of the potential purchasers of its assets (through the liquidation price) and on the liquidation of assets by other such rationed firms. The moral hazard parameter ρ^* partitions firms endogenously into liquidity providers and takers, based on the magnitude of their liquidity shocks, and one can think of the excess demand for the asset, $E(p, \rho^*) \equiv [D(p, \rho^*) - S(p, \rho^*)]$, given by equation (20), as an inverse measure of the excess financial leverage in the system.¹⁴

Another important observation is that part (ii) of Proposition 3 implies a natural link between funding liquidity of firms and liquidity of asset markets. Funding liquidity in our model is measured by ρ^* , the amount of financing that can be raised per unit of asset. Market illiquidity in our model can be measured as the fire-sale discount in prices, $[\bar{p} - p^*]$. The Proposition formally shows that funding liquidity and market illiquidity are negatively related. While the link here is only from funding liquidity to market liquidity, our augmented model of Section 4 will also formalize the reverse link from market liquidity to (ex-ante) funding liquidity. Unlike the extant literature where funding liquidity is modeled through exogenously specified margin or collateral requirements, our measure of funding liquidity is linked to the amount of financing that can be raised given the risk-shifting problem tied to leverage. Formally, it is given by ρ^*). This linkage is quite important in the analysis that follows.

Reverting to our current model, we combine Proposition 3 with Proposition 2 to obtain the following natural result that collateral required of a rationed firm is higher when the perceived moral hazard is greater.

Proposition 4 *The collateral requirement $k(\rho)$ for a firm with liquidity shock ρ is decreasing in the (inverse) moral hazard intensity ρ^* .*

The following example which assumes a uniform distribution on liquidity shocks helps us illustrate these equilibrium relationships graphically.

¹⁴These features of our model are essentially variants of the industry-equilibrium effects in Shleifer and Vishny (1992)’s model wherein borrowing involves collateral, and collateral induces asset liquidations. Crucially, however, the determinant of rationing and of the limited ability of buyers to purchase are both tied to the same underlying state variable, the extent of moral hazard problem.

Example: Suppose that $\rho \sim Unif[\rho_{\min}, \rho_{\max}]$ and $\bar{p} = \theta_2 y_2 = \rho_{\max}$. Then, solving the market-clearing condition $E(p, \rho^*) = 0$, yields the following equilibrium relationships:

1. If $\rho^* \geq \hat{\rho}^* \equiv \frac{1}{2}(\rho_{\min} + \rho_{\max})$, then the price for asset sales is $p^* = \rho_{\max}$;
2. Otherwise, that is, if $\rho^* < \frac{1}{2}(\rho_{\min} + \rho_{\max})$, then there is cash-in-the-market pricing and the price for asset sales is

$$p^* = \rho_{\max} - \sqrt{(\rho_{\max} - \rho_{\min})} \sqrt{(\rho_{\max} + \rho_{\min} - 2\rho^*)}.$$

3. In the cash-in-the-market pricing region, the equilibrium price p^* is increasing and convex in (inverse) moral hazard intensity ρ^* . In particular,

$$\frac{dp^*}{d\rho^*} = \frac{\sqrt{(\rho_{\max} - \rho_{\min})}}{\sqrt{(\rho_{\max} + \rho_{\min} - 2\rho^*)}} > 0,$$

and

$$\frac{d^2 p^*}{d\rho^{*2}} = \sqrt{(\rho_{\max} - \rho_{\min})} (\rho_{\max} + \rho_{\min} - 2\rho^*)^{-\frac{3}{2}} > 0.$$

4. The collateral requirement $k(\rho)$ is given accordingly by Proposition 2 and the expressions for liquidation price p^* in the two regions (Points 1 and 2 above).

The price p^* and the collateral requirement $k(\rho)$ are illustrated in Figures 2 and 3. Figure 2 shows the cash-in-the-market pricing in asset market when funding liquidity is below $\hat{\rho}^{ast}$. Figure 3 in particular is striking.¹⁵ As the moral hazard problem worsens (ρ^* falls), a smaller range of firms is able to relax rationing and at the same time these firms face increasingly steeper collateral requirement. Finally, Figure 4 plots market illiquidity, measured as the fire-sale discount in asset price, $[\bar{p} - p^*]$, as a function of the funding liquidity per unit of asset, ρ^* , for the example with uniform distribution of liquidity shocks. It illustrates that when funding liquidity is high, market liquidity is at its maximal level. As funding liquidity deteriorates and falls below $\hat{\rho}^*$, market becomes illiquid and increasingly so as funding liquidity deteriorates.

Interpretation of moral hazard intensity: What does it mean to vary the moral hazard parameter ρ^* ? Recall that $\rho^* = \frac{\theta_2(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)}$, so that ρ^* is increasing in θ_2 , the quality of the better asset. Thus, a decrease in ρ^* can be given the economically interesting interpretation of a deterioration in the quality of assets, for example, over the business cycle. Note that we are

¹⁵The parameters in Figure 3 are: $\theta_2 = 0.8$, $y_2 = 12.5$, giving $\rho_{\max} = 10$, and $\theta_1 = 0.2$, $y_1 = 20$, giving $\rho_{\min} = \theta_1 y_1 = 4$.

holding constant the quality of bad asset θ_1 . So strictly speaking, if the better asset deteriorates in quality in a relative sense compared to the other asset during a business-cycle downturn, then the moral hazard problem gets aggravated. Thus, our model entertains a natural interpretation that during economic downturns and following negative shocks to the quality of assets, there is greater credit rationing and tighter collateral requirement in the economy. Accompanying these are lower prices for asset liquidations due to the deterioration in asset quality and the coincident deterioration in funding liquidity.

Interpretation of liquidity shocks: In our analysis, we assumed the liquidity shocks and their distribution were unrelated to the quality of assets. If a deterioration in the quality of assets is in fact associated with a worsening in the distribution of liquidity shocks, then the effects in our model are exacerbated.¹⁶ Formally, this would mean a relationship between θ_2 and the distribution of liquidity shocks $g(\rho)$. We explore and build this link in Section 4 where we introduce and analyze the ex-ante (that is, date 0) structure of the model. Before we do so, however, we prove an important result which casts doubt over the recent claim in asset-pricing and liquidity literature that lack of liquidity and fall in prices in asset markets are attributable to constraints that financial intermediaries face, including collateral requirements. We show below that once collateral is recognized as an endogenous response to relax borrowing constraints, exactly the reverse is in fact true: All else equal, there would be more asset liquidations and lower prices if collateral requirements were not in place.

3.5 Price-stabilizing role of collateral

Suppose that there were no possibility of pledging collateral at all. In this case, firms with liquidity shocks below ρ^* would still be the candidate buyers of assets, but *all* firms with liquidity shocks above ρ^* would be forced to sell *all* of their assets. This is in contrast to the case with collateral where firms with liquidity shocks over the range $[\rho^*, p]$ engage only in partial asset sales, as they are able to meet their liquidity shock by borrowing against collateral. Thus, without collateral, the demand and supply functions for assets are given respectively as:

$$D(p, \rho^*) = \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^* - \rho)}{(p - \rho^*)} g(\rho) d\rho, \text{ and} \quad (23)$$

$$S_{nc}(p, \rho^*) = \int_{\rho^*}^{\rho_{\max}} g(\rho) d\rho, \quad (24)$$

¹⁶On this point, the preceding of significant market illiquidity shocks by asset-side shocks, see the empirical evidence presented in footnote 7 from Acharya and Pedersen (2005).

where we have subscripted the supply to signify that it is different from its form under the case with collateral.

The equilibrium price p_{nc}^* then is the price p that clears the market, that is, sets the excess demand

$$E_{nc}(p, \rho^*) = [D(p, \rho^*) - S_{nc}(p, \rho^*)]$$

to less than or equal to zero.

In this case, the excess demand function is given by

$$E_{nc}(p, \rho^*) = \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^* - \rho)}{(p - \rho^*)} g(\rho) d\rho - \int_{\rho^*}^{\rho_{\max}} g(\rho) d\rho. \quad (25)$$

Integrating this equation by parts yields that

$$E_{nc}(p, \rho^*) = -[1 - G(\rho^*)] + \frac{1}{(p - \rho^*)} \int_{\rho_{\min}}^{\rho^*} G(\rho) d\rho. \quad (26)$$

Thus, the excess demand being zero (for $p < \bar{p}$) leads to the variant of the condition (22) for the case with collateral:

$$p = \rho^* + \frac{1}{[1 - G(\rho^*)]} \int_{\rho_{\min}}^{\rho^*} G(\rho) d\rho. \quad (27)$$

If the solution to this equation exceeds \bar{p} , then we have $p^* = \bar{p}$.

Then, we obtain the following characterization of the price in absence of collateral and its comparison to the price with collateral.

Proposition 5 *The market-clearing price for asset sales in absence of collateral, p_{nc}^* , depends on the (inverse) moral hazard intensity ρ^* in the following manner:*

- (i) *There exists a critical threshold $\hat{\rho}_{nc}^* < \bar{p}$ such that $p_{nc}^* = \bar{p}, \forall \rho^* \geq \hat{\rho}_{nc}^*$; and,*
- (ii) *For $\rho^* < \hat{\rho}_{nc}^*$, $p_{nc}^* = \rho^* + \frac{1}{[1 - G(\rho^*)]} \int_{\rho_{\min}}^{\rho^*} G(\rho) d\rho$, so that $p_{nc}^* \in [\rho^*, \bar{p}]$, p_{nc}^* is increasing in ρ^* , and $p_{nc}^* = \rho^*$ only when $\rho^* = \rho_{\min}$.*

In comparison to the price p^ for asset sales with collateral characterized in Proposition 3:*

- (iii) *$p^* \geq p_{nc}^*$ for all values of ρ^* ; and,*
- (iv) *$\hat{\rho}^* < \hat{\rho}_{nc}^*$, so that the inequality in (iii) above is strict over at least some values of ρ^* below $\hat{\rho}_{nc}^*$.*

The important message from this proposition is the following. Proposition 3 and Proposition 4 together imply that across states of the world, that is, as the moral hazard intensity ρ^* varies, equilibrium price and the level of collateral requirement are negatively correlated: Higher the collateral requirement, lower the prices. This result, however, should not be interpreted as being causal. Proposition 5 makes it clear that were collateral not pledgeable by some of the rationed firms, they would in fact be liquidated completely (as against partially) and equilibrium price would be even lower. The source of the negative correlation implied by Proposition 3 and Proposition 4 is thus not due to collateral requirements per se. Collateral, instead, is itself an endogenous response to the root cause of low prices and illiquidity in the market, namely the moral hazard problem (risk substitution by financial institutions, in our set-up) that limits the funding capacity of firms (borrowing capacity of financial institutions).¹⁷

Below, we revisit our earlier example which assumed a uniform distribution on liquidity shocks. We show explicitly that the price in absence of collateral is lower than the price in its presence.

Example (continued): Assume again that $\rho \sim Unif[\rho_{\min}, \rho_{\max}]$ and $\bar{p} = \theta_2 y_2 = \rho_{\max}$. Then, solving the market-clearing condition $E_{nc}(p, \rho^*) = 0$, yields the following equilibrium relationships:

1. If $\rho^* \geq \hat{\rho}_{nc}^* \equiv \frac{1}{(1+\sqrt{2})}(\rho_{\min} + \sqrt{2}\rho_{\max})$, then the price for asset sales in absence of collateral is $p_{nc}^* = \rho_{\max}$;
2. Otherwise, that is, if $\rho^* < \frac{1}{(1+\sqrt{2})}(\rho_{\min} + \sqrt{2}\rho_{\max})$, then there is cash-in-the-market pricing and the price for asset sales is

$$p_{nc}^* = \rho^* + \frac{(\rho^* - \rho_{\min})^2}{2(\rho_{\max} - \rho^*)}.$$

3. In the cash-in-the-market pricing region, the equilibrium price p_{nc}^* is increasing and convex in (inverse) moral hazard intensity ρ^* .
4. Importantly, the price with collateral p^* is (weakly) greater than the price in absence of collateral p_{nc}^* . In particular,

- $p^*(\rho_{\min}) = p_{nc}^*(\rho_{\min}) = \rho_{\min}$;
- $p^*(\rho) > p_{nc}^*(\rho)$ for all $\rho \in (\rho_{\min}, \hat{\rho}_{nc}^*)$, and $\hat{\rho}^* < \hat{\rho}_{nc}^*$.¹⁸

¹⁷The implication of this endogeneity problem is particularly strong when one thinks about welfare costs that might arise from asset sales that occur at illiquidity discounts and what policy interventions, if any, might be desirable to minimize these costs. In our current set-up, asset sales merely result in transfers across firms and there are no other pecuniary externalities from fire-sale prices. We discuss in Section 5 the implications of introducing such allocation inefficiency from asset sales in our model.

¹⁸This latter result is apparent from the expressions for $\hat{\rho}^*$ and $\hat{\rho}_{nc}^*$: $\hat{\rho}^*$ puts a weight of half each on ρ_{\max} and ρ_{\min} , whereas $\hat{\rho}_{nc}^*$ puts a weight on ρ_{\max} of $\frac{\sqrt{2}}{(1+\sqrt{2})}$, which is greater than half.

- Finally, $p^*(\rho) = p_{nc}^*(\rho) = \rho_{\max}$ for all $\rho \geq \hat{\rho}_{nc}^*$.

This relationship between the price with collateral, p^* , and the price without collateral, p_{nc}^* , is illustrated in Figure 5. Interestingly, p^* and p_{nc}^* both start at value of ρ_{\min} when $\rho^* = \rho_{\min}$, but since p^* hits ρ_{\max} before p_{nc}^* does, the price with collateral is more sensitive to ρ^* than the price without collateral, in the range where they are both below ρ_{\max} .

4 Ex-ante debt capacity and liquidity shocks

In this section, we provide an equilibrium setting that gives rise to the structure of liquidity shocks ρ_i assumed in our model so far. Before we move to modeling details, we provide a summary of what this section achieves.

We endogenize the structure of liquidity shocks in Section 4.1 by assuming that ex ante (at date 0), firms are ranked by their initial wealth or capital levels and must raise incremental financing up to some fixed (identical) level in order to trade. The incremental financing is raised through short-term debt contracts (payable at date 1) that give lenders the ability to liquidate ex post in case promised payments are not met. We show in Section 4.4 that this form of financing – which grants control to lenders in case of default (as in collateral and margin requirements) – is optimal from the standpoint of raising maximum ex-ante finance.

This augmentation of the benchmark model leads to an interesting, even if somewhat involved, fixed-point problem: On the one hand, the promised payment for a given amount of financing is decreasing in the level of liquidation prices in case of default; on the other hand, the liquidation price is itself determined by the distribution of promised debt payments since these are the ex-post liquidity shocks faced by firms. We show in Section 4.2 that there is a unique solution to this fixed-point problem, characterized by the fraction of firms that are ex-ante rationed and the mapping from moral-hazard intensity to price. In fact, the fixed-point is a contraction mapping and enables us to provide a recursive, constructive algorithm for the solution (provided in Appendix 2). While the ex-ante rationing of firms renders analytical results on comparative statics difficult, numerical examples in Section 4.3 confirm some conjectures that follow naturally from our analysis.

4.1 The set-up

The augmented time-line is specified in Figure 6.

Suppose that at date 0, there is a continuum of firms with the starting wealth for firm i being w_i . The cumulative distribution function (cdf) of w_i is $Q(w_i)$ over $[0, w_{\max}]$ where $w_{\max} = I - \theta_1 y_1$ (this assumption ensures that we borrow at least the value of the alternative asset so

that there is a risk shifting problem). Each firm has access to an identical investment opportunity that pays off at date 2 and requires at date 0 an investment of I . We assume that this investment shortfall is externally financed via a debt contract with a fixed, promised payment of ρ_i at date 1, against which creditors provide financing of $s_i = (I - w_i)$. Firms can attempt to meet the promised payment ρ_i through rolling over of existing debt or issuance of new debt. In attempting to do so, they can pledge collateral (or what is equivalently in our model, sell assets). If the payment ρ_i cannot be made at date 1, then there is a transfer of control to creditors who liquidate the assets and collect the proceeds. Since it is easier to work with the financing shortfall $s_i = I - w_i$, we note that the cdf of s_i is $R(s_i) = 1 - Q(I - s_i)$ over the support $[\theta_1 y_1, I]$.

The investment opportunity can yield in two periods (date 2) a cash flow y_2 with probability θ_2 . However, at date 1, there is the possibility of moral hazard: Firm owners, if optimal to do so, may switch from the existing safer asset to the riskier asset, which yields a cash flow y_1 with probability θ_1 , where we assume as in our benchmark model that $\theta_1 < \theta_2$, $y_1 > y_2$, and $\theta_1 y_1 < \rho_i < \theta_2 y_2$. Viewed from date 0, θ_2 is uncertain: θ_2 has cdf $H(\theta_2)$ and probability density function (pdf) $h(\theta_2)$ over $[\theta_{\min}, \theta_{\max}]$, where we assume for simplicity that $\theta_{\min} y_2 \geq \theta_1 y_1$, that is, the worst-case expected outcome for the safer asset is no worse than that for the riskier asset. In fact we impose that

$$\theta_{\min} = \frac{\theta_1 y_1}{y_2} \left[1 + \sqrt{1 - \frac{y_2}{y_1}} \right], \quad (28)$$

this assumption ensures that maximum amount that can be borrowed is determined by ρ^* (which is always higher than $\theta_1 y_1$).

Conditional on the realization of θ_2 at date 1, firms may raise financing again in the form of debt in order to fund their “liquidity shock” - namely, the outstanding debt payment ρ_i . Note that ρ_i is fixed in that it is not contingent on the realization of θ , which we assume is observable but not verifiable. Thus, the date-0 structure of this augmented model maps one for one into the date-1 structure in our benchmark model where liquidity shocks were taken as given. In particular, the lower the realization of θ , the lower is $\rho^*(\theta_2)$ and hence the greater is the moral hazard problem; thus θ_2 indexes fundamental information that is related to the severity of the moral hazard problem. We show below that with this additional date 0 structure to the model, the initial distribution of wealth of firms w_i (or shortfall s_i) translates into an equilibrium distribution of their promised debt payments ρ_i .

Consider a particular realization of the quality of investment opportunity, say θ_2 , at date 1. As shown in Proposition 1, firms with liquidity shocks up to $\rho^*(\theta_2) = \theta_2 f^*(\theta_2) = \frac{\theta_2(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)}$ are not rationed. That is, these firms can meet their outstanding debt payments at date 1, continue their investments, and possibly, also acquire more assets. Next, as shown in Proposition 2, firms with liquidity shocks over the range $[\rho^*(\theta_2), p^*(\theta_2)]$ are able to meet their liquidity shocks but only

by pledging collateral from asset-sale proceeds. In other words, these firms can also meet their outstanding debt payments at date 1 and continue their investments, but do not have liquidity to acquire more assets. Finally, firms with liquidity shocks higher than $p^*(\theta_2)$ cannot meet their outstanding debt payments, and creditors liquidate these firms' assets.

Then, assuming that date 0 creditors are risk-neutral (like all other agents in our model), the amount of financing s_i that firm i can raise at date 0, satisfies creditors' individual rationality constraint:

$$s_i = (I - w_i) = \int_{\theta_{\min}}^{p^{*-1}(\rho_i)} p^*(\theta_2)h(\theta_2)d\theta_2 + \int_{p^{*-1}(\rho_i)}^{\theta_{\max}} \rho_i h(\theta_2)d\theta_2, \quad (29)$$

which captures the fact that for low realizations of θ_2 , the moral hazard is severe and firms end up being rationed, unable to meet their debt payments, and thus, liquidated, whereas for high realizations of θ_2 , debt payments are met. The critical threshold determining whether θ_2 realization is "low" or "high" for firm i is given implicitly by the relation: $\rho_i = p^*(\theta_2)$. Also implicit in Equation (29) is the fact that some low wealth borrowers may be excluded as the amount owed $s_i = (I - w_i)$ may not be covered by the maximum amount available for payment the next period.

Note that given a price function $p^*(\theta_2)$ and financing s_i , equation (29) gives the face value ρ_i directly. However, we need to take account of Proposition 3 and recognize that the market-clearing price $p^*(\theta_2)$ itself depends upon the entire distribution of liquidity shocks ρ_i across firms. In case of their firm's default, creditors recover an amount that depends upon the asset liquidation price, and, thus on the liquidity shocks of *other* firms; in turn, each firm's debt capacity, and thereby their future liquidity shocks, depend on the (expectation over the) amount recovered. Thus the model can be viewed as a general equilibrium version of Shleifer and Vishny (1992) with ex-ante contracting and endogenous borrowing capacity determined by moral hazard problems.

With this background, we define the equilibrium of the ex-ante game. An important notational issue to bear in mind is that in the benchmark model, we assumed as exogenously given the distribution of liquidity shocks, $G(\rho)$, but in the augmented model, this distribution is induced by the distribution of financing needs, $R(s)$.

Definition: An equilibrium of the ex-ante borrowing game is (i) a pair of functions $\rho(s_i)$ and $p^*(\theta_2)$, which respectively give the promised face-value for raising financing s_i and equilibrium price given quality of assets θ_2 , and (ii) a truncation point \hat{s} , which is the maximum amount of financing that a firm can raise in equilibrium, such that $\rho(s_i)$, $p^*(\theta_2)$ and \hat{s} satisfy the following fixed-point problem.

1. For every θ_2 , prices are determined by the industry equilibrium condition of Proposition 3:

$$p^*(\theta_2) \leq \rho^*(\theta_2) + \int_{\rho_{\min}}^{p^*(\theta_2)} \hat{G}(u) du, \quad (30)$$

where compared to equation (22), we have replaced distribution of liquidity shocks $G(\cdot)$ with the induced distribution $\hat{G}(\cdot)$ and also substituted the variable of integration ρ with u to avoid confusion with the function $\rho(s_i)$. In particular, $\hat{G}(u)$ is the truncated equilibrium distribution of liquidity shocks given by $\hat{G}(u) = \frac{R(\rho^{-1}(u))}{R(\hat{s})}$. Formally, the distribution of liquidity shocks, $\hat{G}(u)$, is induced by the distribution of financing amounts, $R(s)$, via the function $\text{Prob}[\rho(s_i) \leq u | s_i \leq \hat{s}]$. Note that as in case of equation (22), a strict ($<$) inequality leads to $p^*(\theta_2) = \bar{p}(\theta_2) = \theta_2 y_2$.

2. Given the price function $p^*(\theta_2)$, for every $s_i \in [0, \hat{s}]$, the face value ρ is determined by the requirement that lenders receive in expectation the amount that is lent:

$$s_i = (I - w_i) = \int_{\theta_{\min}}^{p^{*-1}(\rho)} p^*(\theta_2) h(\theta_2) d\theta_2 + \int_{p^{*-1}(\rho)}^{\theta_{\max}} \rho h(\theta_2) d\theta_2. \quad (31)$$

3. The truncation point \hat{s} for maximal financing is determined by the condition

$$\hat{s} \leq \int_{\theta_{\min}}^{\theta_{\max}} p^*(\theta_2) h(\theta_2) d\theta_2, \quad (32)$$

with a strict inequality implying that $\hat{s} = I - \theta_1 y_1$ (all borrowers are financed).

For future reference, we note that differentiating equality versions of Equations (30) and (31) yields alternative but equivalent conditions that

$$\frac{dp}{d\theta_2} = \frac{\frac{d\rho^*(\theta_2)}{d\theta_2}}{1 - \hat{G}(p)} \text{ if } p < \theta_2 y_2, \text{ else } \frac{dp}{d\theta_2} = y_2, \quad (33)$$

and

$$\frac{d\rho}{ds_i} = \frac{1}{1 - H(p^{*-1}(\rho))} \text{ if } \rho \geq p^*(\theta_{\min}), \text{ else } \frac{d\rho}{ds_i} = 1. \quad (34)$$

4.2 The solution

In this section, we show that there is a unique equilibrium to the ex-ante borrowing game given by the solution to the fixed-point problem stated above. We provide an explicit characterization of the solution.

Recall that $s_i = (I - w_i)$ is the financing amount and given a distribution of wealth w_i with cdf $Q(w_i)$; the distribution of s_i given by $R(s_i) = P[(I - w_i) \leq s_i]$ is thus known. As in the definition above, we will work with $R(s_i)$ directly, where $R(s_i)$ has support $[\theta_1 y_1, I]$. In what follows, we suppress the subscript i unless it is necessary. The maximum wealth financed is $\hat{s} = (I - \hat{w})$: in equilibrium, some borrowers with low wealth will be endogenously excluded.

It is easier to analyze the fixed-point problem by working with the inverse functions $s(\rho)$ and $\theta_2(p)$. $s(\rho)$ gives the financing raised ex ante for a given face-value ρ while $\theta_2(p)$ gives the realization of the state θ_2 for the given equilibrium price p . Since these are one-to-one functions, we can follow this approach. Notice that both ρ and p have the domain $[\theta_1 y_1, \theta_{\max} y_2]$ (one cannot have a face value higher than the highest possible price); it is possible that this highest price is not reached in equilibrium and we will account for this.

Our approach to solving the fixed point problem is as follows. Fix a maximal financing \hat{s} . First we invert Equation (30) and solve for $\theta_2(p)$: We show below that since this is an explicit quadratic equation we can solve for this variable, we also impose the constraint that price is always less than $\theta_2 y_2$. Given the solution to $\theta_2(p)$, we can substitute this into the differential equation for $s(\rho)$, Equation (34), this yields an integro-differential equation that has a unique solution. The maximum financing is then uniquely solved by the boundary condition in Equation (32).

Given the cdf of amount financed, $R(s)$, the cdf of face values conditional on financing being over the truncated support of amounts financed $[\theta_1 y_1, \hat{s}]$ is denoted as $\hat{G}(u)$, and is given by $\hat{G}(u) = \frac{R(s(u))}{R(\hat{s})}$, where $\hat{G}(u) = \text{Prob}[\rho \leq u | s \leq \hat{s}] = \text{Prob}[s(\rho) \leq s(u) | s \leq \hat{s}]$.

Define

$$L(p) = p - \int_{\theta_1 y_1}^p \hat{G}(\rho) d\rho, \quad (35)$$

where we have switched back to ρ as being the variable of integration.

Then, setting $L(p) = \rho^*(\theta_2)$ to satisfy equation (30) with equality, we obtain

$$\theta_2 \frac{(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)} = L(p), \quad (36)$$

which yields the following solution for θ_2 (we have a quadratic equation and pick the correct root)

$$\frac{(\theta_1 y_1 + L(p)) + \sqrt{(\theta_1 y_1 + L(p))^2 - 4y_2 L(p)\theta_1}}{2y_2}. \quad (37)$$

Accounting for the fact that prices cannot be above $\theta_2 y_2$ (hence $\theta_2 \geq \frac{p}{y_2}$), we define $\theta_2(p)$

implicitly in terms of $s(\rho)$ as:

$$\theta_2(p) = \max \left\{ \frac{(\theta_1 y_1 + L(p)) + \sqrt{(\theta_1 y_1 + L(p))^2 - 4y_2 L(p)\theta_1}}{2y_2}, \frac{p}{y_2} \right\} \quad (38)$$

on the domain $[\theta_1 y_1, \theta_{\max} y_2]$. Note that this equation defines $\theta_2(p)$ in terms of $s(\rho)$ since $L(p)$ depends on the function $\hat{G}(\rho) = \frac{R(s(\rho))}{R(\hat{s})}$.

Note also that if $p = \theta_1 y_1$, then Equation (38) is determined by Equation (37). In fact, it is the case from Equation (37) that $\theta_2(p) = \theta_{\min}$ as $L(\theta_1 y_1) = \theta_1 y_1$. At $p = \theta_{\max} y_2$, if there is no liquidity crisis, then we have $\theta_2(p) = \theta_{\max}$, the maximum possible price. If this is not true, then we have $\theta_2(\theta_{\max} y_2) > \theta_{\max}$ (there is a liquidity crisis in every possible state).

Next, we solve the differential equation implied by Equation (34) (which is itself equivalent to Equation 31):

$$\frac{ds}{d\rho} = 1 - H(\theta_2(\rho)), \quad (39)$$

where $H(\theta_2)$ is the cdf of θ_2 . Since it is possible that $\theta_2(p) > \theta_{\max}$ in Equation (39), we extend $H(\theta_2)$ by assuming that $H(\theta_2) = 1$ for $\theta_2 > \theta_{\max}$ (this is true and innocuous since $1 - H(\theta_2) = 0$ for such θ_2).

Substituting for $\theta_2(p)$ from equation (38), we obtain that

$$\frac{ds}{d\rho} = 1 - H \left(\max \left\{ \frac{(\theta_1 y_1 + L(\rho)) + \sqrt{(\theta_1 y_1 + L(\rho))^2 - 4y_2 L(\rho)\theta_1}}{2y_2}, \frac{\rho}{y_2} \right\} \right) \quad (40)$$

with the end point constraint that $s(\theta_1 y_1) = \theta_1 y_1$.

This is a standard integro-differential equation of the form

$$\frac{ds}{d\rho} = f \left(\rho, \int_{\theta_1 y_1}^{\rho} \frac{R(s(u))}{R(\hat{s})} du \right) \quad (41)$$

with the endpoint constraint $s(\theta_1 y_1) = \theta_1 y_1$, and it has a unique solution if the function $f(\rho, t)$ is Lipschitz in t and the function $R(s)$ is Lipschitz in s .¹⁹ This is indeed the case in our set-up, technical details of which are relegated to Appendix 1.

¹⁹More details of this proof (we follow Theorem 2.1 from Granas and Dugundji (2003)) are in Appendix 1. Note that the generic function f for expressing the integro-differential equation is not to be confused with the face-value of debt in our benchmark model.

We now solve for the maximal financing \hat{s} , this is given by the condition

$$\hat{s} \leq \int_{\theta_{\min}}^{\theta_{\max}} p(\theta_2)h(\theta_2)d\theta_2 \quad (42)$$

where $p(\theta_2)$ is the inverse function of $\theta_2(p)$ and $h(\theta_2)$ is the density of θ_2 .

The left hand side of Equation (42) is $\theta_1 y_1$ at $\hat{s} = \theta_1 y_1$ and increasing in \hat{s} . The right hand side of Equation (42) is strictly greater than $\theta_1 y_1$ at $\hat{s} = \theta_1 y_1$ and decreasing in \hat{s} .²⁰ Either Equation (42) has a unique solution or no solution with strict inequality at \hat{s} , in that case there is no exclusion and $\hat{s} = I - \theta_1 y_1$.

This completes the proof that the solution exists and is unique.

We state all this in the following proposition.

Proposition 6 *There exists a unique equilibrium to the ex-ante borrowing game defined in Section 4.1. In particular, given a maximal borrowing amount \hat{s} , the borrowing function $s(\rho)$ (financing as a function of face value borrowed) is the unique solution to the integro-differential equation*

$$\frac{ds}{d\rho} = 1 - H \left(\max \left\{ \frac{(\theta_1 y_1 + L(\rho)) + \sqrt{(\theta_1 y_1 + L(\rho))^2 - 4y_2 L(\rho)\theta_1}}{2y_2}, \frac{\rho}{y_2} \right\} \right) \quad (43)$$

with the end point constraint that $s(\theta_1 y_1) = \theta_1 y_1$. Given $s(\rho)$, the inverse equilibrium price function $\theta_2(p)$ is uniquely given by

$$\theta_2(p) = \max \left\{ \frac{(\theta_1 y_1 + L(p)) + \sqrt{(\theta_1 y_1 + L(p))^2 - 4y_2 L(p)\theta_1}}{2y_2}, \frac{p}{y_2} \right\} \quad (44)$$

on the domain $[\theta_1 y_1, \theta_{\max} y_2]$.

The maximal borrowing amount is uniquely given by the boundary condition

$$\hat{s} \leq \int_{\theta_{\min}}^{\theta_{\max}} p(\theta_2)h(\theta_2)d\theta_2 \quad (45)$$

where $p(\theta_2)$ is implicitly a function of \hat{s} .

²⁰To see this note that if we increase \hat{s} , we decrease $\hat{G}(\rho)$, which means we increase $L(p)$ and hence $\theta_2(p)$, therefore $p(\theta_2)$ decreases, therefore the right hand side of the Equation (42) decreases.

In fact, the solution to the fixed-point problem between promised debt payments and liquidation price is a contraction and can be computed using a recursive algorithm that we outline in Appendix 2.

It is also important to note that the feedback between promised debt payments and liquidation price produces an amplification effect of collateral in our model, but unlike the extant literature, this amplification effect is positive for liquidity and efficiency: The presence of collateral requirements stabilizes liquidation prices as well as lowers promised debt payments, and these two effects feed on each other to produce an amplified reduction in ex-ante rationing. To summarize, in our augmented model too, endogenously designed collateral requirements enhance efficiency and trading by reducing the severity of agency problems tied to external finance. More generally, we conjecture that rather than being the source of illiquidity in markets, collateral is in fact the life-blood that sustains high levels of trading amongst financial institutions.

4.3 Numerical examples

The comparative statics with respect to a change in the distribution of wealth w (or financing amount s) and a change in the distribution of fundamentals θ are ambiguous in our model because of the effect of entry (the last marginal project that is financed). If we keep the set of projects that are financed at date 0 fixed, then the comparative statics are easily obtained. However, an improvement in the expectation of fundamentals (for example, a first-order stochastic dominance (FOSD) increase in distribution of θ) has two effects. The first effect is to weakly increase prices at date 1, for a given pool of projects financed at date 0. This increase in prices results in the pool of projects financed at date 0 to be expanded to include higher leverage projects also. We show below that this latter effect means that at low realizations of fundamentals (which are less likely given the FOSD increase), prices can sometimes be lower with better ex-ante expectation of fundamentals.²¹

To understand these effects further, we solve two numerical examples using the recursive algorithm provided in Appendix 2 to compute the equilibrium. In both numerical examples, we consider a situation where the distribution of quality of asset improves in a FOSD sense, and, in turn, so does the moral-hazard intensity.

Varying the distribution of moral-hazard intensity

Our first numerical example provides some counterintuitive insights and is constructed as

²¹Note that we do not have an explicit role for “volatility” in the model. Since a better distribution of asset quality leads to lower defaults in the model, our second comparative static could be interpreted to some extent as delivering results one would get with low versus high volatility of news about the asset quality. But perhaps a more accurate description of our comparative static exercise is that it is about “downside risk”.

follows:

1. Let $I = 1$, $w_{\min} = 0$, $y_1 = 4$, $y_2 = 1$, $\theta_1 = 0.05$, $w_{\max} = I - \theta_1 y_1 = 1 - 0.2 = 0.8$.
Hence s has support $[0.2, 1]$.
2. Let $t = 1 - 0.2 = 0.8$ (which is also the value of w_{\max}) and suppose that

$$R(s) = \frac{s - 0.2}{t}, \quad (46)$$

which is the uniform distribution. We suppose that $H(\theta)$ is given by the following distribution on $[\theta_{\min}, \theta_{\max}]$:

$$H(\theta) = 1 - \left(1 - \frac{\theta - \theta_{\min}}{\theta_{\max} - \theta_{\min}}\right)^{1/\gamma}, \quad (47)$$

where γ , $\gamma > 0$ (note that $\gamma = 1$ corresponds to the uniform distribution). A higher value of γ implies first-order stochastic dominance (FOSD); in fact for any truncation \hat{s} , a higher value of γ implies FOSD (Hopenhayn (1992) refers to this as monotone conditional dominance or MCD). Also, note that $E[\theta]$ is $\theta_{\min} + \frac{(\theta_{\max} - \theta_{\min})\gamma}{1 + \gamma}$ which is increasing in γ .

We let γ take values in $\{0.5, 5.0\}$. We show for these values the distributions of $\rho(s)$ in one plot (Figure 7) and $p(\theta)$ in another plot (Figure 8). The figures show large variations in prices and hence large variations in liquidity shocks (face value of debt) as we change the distribution of fundamentals. In Figure 9, we employ two panels – Figure 9a shows the cumulative distribution of liquidity shocks (the endogenous $G(\rho)$ function) and Figure 9b shows the (endogenous) cumulative distribution of prices.

There are two countervailing intuitions at play in this example. First, if we keep \hat{s} fixed, an increase in fundamentals in a FOSD sense leads to lower face values for debt and hence lower endogenous liquidity shocks (this is apparent from Figure 7). The lower liquidity shocks, in turn, lead to higher prices state by state. However, as fundamentals improve, the pool of projects financed at date 0 expands (the threshold \hat{s} below which projects are financed moves to the right on the x-axis, as can be seen in Figure 7). This means that more levered firms are set up in the economy. If this leverage effect dominates so that with low realizations of fundamentals at date 1 (low values of θ), more distress and de-leveraging (asset sales) occur, then market-clearing prices are lower (this is apparent in Figure 8).

In the first example discussed above, this second effect dominates, i.e., the pool of projects financed at date 0 is significantly worse. Consequently, an improvement in distribution of fundamentals in a FOSD sense results in worse prices in financial distress (Figure 8). This is consistent with Figure 9a, that shows a higher cumulative distribution (in a FOSD sense) of liquidity shocks

when expectations for the future are better. However, we do note that in an ex-ante sense, the probability of reaching these low fundamental states is much lower with better expectation of distribution of θ in a FOSD sense (see Figure 9b which shows the cumulative distribution function of prices $p(\theta)$ under the two distributions). Hence, in expectation prices are still higher, which is precisely why \hat{s} is higher in Figure 7 and higher leverage is sustained at date 0.

This example makes it clear that good times in terms of expectations about the future enable even poorly capitalized institutions to be funded ex ante and the resulting distribution of leverage in the economy can potentially lead to (il)liquidity effects in prices that are worse during crises that follow better times. Put another way, downside risk or negative skewness of future prices can be higher in good times.

This counterintuitive effect arises due to the effect of distribution of fundamentals on endogenous entry of firms at date 0. If this entry effect is weak, then prices are in fact higher state by state at date 1 when the distribution of fundamentals at date 0 is better. To see this, we repeat the example above with a different distribution for borrowing shocks, we now use:

$$R(s) = 1 - \left(1 - \frac{s - 0.2}{t}\right)^{1/\zeta}, \quad (48)$$

with $\zeta = 0.05$. In our prior example, the uniform distribution corresponds to $\zeta = 1$. A higher ζ implies lower capital levels and more borrowing at date 0 in a FOSD sense. This distribution has much thinner density in the right tail, reducing the effect of entry. Figures 10, 11 and 12 show the relevant equilibrium outcomes for this example.

In Figure 10, we see again that $\rho(s)$ is lower when we move to better fundamentals and that \hat{s} is higher. But now in Figure 11, we see that state by state, it is the low fundamentals case ($\gamma = 0.5$) that has the lower price (though the difference is quite small). Here the entry effect (the change in \hat{s}) has been muted because of the thinness of the left (right) tail in the distribution of initial capital (borrowing) levels.

Figure 12a now shows that the endogenous distribution of liquidity shocks $G(\rho)$ is higher in a FOSD sense for the lower fundamentals case. This explains why prices are lower state by state for weaker fundamentals. Finally, Figure 12b shows that higher fundamentals lead to higher expected prices in an FOSD sense.

We note that we actually found it rather hard to construct this second example in that the right tail of the borrowing distribution had to be thinned considerably. This suggests that our first example is potentially quite robust. Indeed, it seems reasonable that high expectations lead to more leveraged players being financed, and hence lower prices if the really adverse asset-quality states materialize.

4.4 Optimality of debt contracts with lender control

A key aspect of our model is the use of debt contracts where the lender has control. In particular, the lender calls the loan at time 1, inducing a liquidity shock. While this seems to correspond well to the nature of margins in financial contracts, we are unaware of any literature directly rationalizing such margin contracts. One may wonder why this makes sense and why renegotiation does not occur.²²

We argue in this subsection that in a model of incomplete contracts that follows Aghion and Bolton (1992) (see also Hart and Moore (1994), Hart (1995) and Diamond and Rajan (2001)), the contract with lender control maximizes the ex-ante financing available to investors. Our proof consists of two steps. First, we show that debt is the optimal contract. Second, we show that borrower control at date 1 is dominated by lender control at date 1.

Consider any particular realization of asset quality θ_2 . Intuitively, in absence of lender control, the borrower can always invoke the moral hazard problem and strategically renegotiate the lender down to $\rho^*(\theta_2)$. This would lower the payoffs available to lenders. In contrast, with lender control, the maximum amount available to lenders via collateralization and asset sales is $p^*(\theta_2) \geq \rho^*(\theta_2)$. Hence, lender control yields much higher payoffs to the lender ex post. Ex-ante, it is thus in the borrower's interest to give control rights to the lender and raise as much ex-ante debt financing as possible.²³ In effect, this rationalizes the use of margin contracts which give the lender the power to call the loan (as is common in financial markets). We formalize this intuition next.

To prove our results, we make two assumptions in the spirit of Aghion and Bolton (1992) and Hart and Moore (1994).

Assumption C1: Courts can verify whether the state 0 occurs or whether $\{y_1, y_2\}$ occurs, however they cannot distinguish between states $\{y_1, y_2\}$.

This assumption essentially states that there is some coarseness in the enforcement ability of courts. While contracts can distinguish between low and high states, they cannot discriminate between different high states.

Assumption C2: While the interim state θ_2 is observable, it is not contractible.

This assumption is similar to Aghion and Bolton (1992) and forces the contract designer to

²²Diamond (2004) in his Presidential address also discusses why short-term debt may resolve incentive problems. He focuses on an environment where the collective action problem makes it hard to renegotiate short-term debt and leads to a run on the firm. This is better for the borrower in an ex-ante sense. Diamond and Rajan (2001) present a similar argument to Diamond (2004).

²³Note that our model differs from the standard Aghion and Bolton (1992) model in that borrower's ability to invoke the moral hazard problem gives the borrower too much power ex post. The only way to limit this is to give the lender the ex-post control rights.

give control conditional on the state θ_2 to either the lender or the borrower. We believe that this assumption is justifiable because hedge funds have complex portfolio strategies with many liquid and illiquid positions – while the lender and hedge fund may agree on a valuation, courts may find it difficult to verify this.

Assumption C3: Payments at date 1 (ex-post states) cannot be bigger than the maximum payoff in that state or smaller than 0.

This is a standard assumption that limits liability and does not allow payments in excess of what is available.

These three assumptions essentially deliver the result that we want. From Assumption C1, the optimal contract must be a pair $\{0, \rho_i\}$ that pays off the same amount whether states y_1 or y_2 occur (we do not formally prove this).

Assumption C2 implies that we have to compare borrower control or lender control in every state. With borrower control, if $\rho_i \leq f^*(\theta_2)$, the borrower will honor the contract and the lender will get $\theta_2 \rho_i$. However, if $\rho_i > f^*(\theta_2)$, then the borrower will credibly threaten to switch to the bad project. Hence, the lender will renegotiate the claim to $\rho_i = f^*(\theta_2)$ and obtain only $\rho^*(\theta_2) = \theta_2 f^*(\theta_2)$. Hence with borrower control, the lender gets $\max[\theta_2 \rho_i, \rho^*(\theta_2)]$.

In contrast, with lender control, the lender can threaten the borrower with liquidation at market prices. Hence, in this case, the lender gets $\max[\rho_i, p^*(\theta_2)]$, where $p^*(\theta_2) \geq \rho^*(\theta_2)$ with $>$ inequality in states with sufficiently high θ_2 and $\rho_i > \theta_2 \rho_i$.

Thus, borrowing with control rights allocated to the lender always generates higher ex-post payoff to the lender and thus greater ex-ante borrowing capacity for the borrower. We state this as a Proposition.

Proposition 7 *Under assumptions (C1)–(C3), the optimal contract is debt and lender control always yields a greater region of financed projects than borrower control.*

Proposition 7 provides a rationalization for the structure of financing contracts for trading intermediaries where the moral hazard of risk-shifting is most pertinent. Lenders lend to borrowers and call the loan on interim information unless collateral is posted. This contract gives strong ex-post control to the lender but reduces the borrower's ability to choose among risky projects and renegotiate.

One may also ask why the borrower cannot buy back debt at market prices (though no renegotiation is allowed). Effectively this is worse than borrower control as the borrower pays $\theta_2 \rho_i$ for a face value of ρ_i if $\rho_i \leq f^*(\theta_2)$ (which is identical to that with borrower control). If $\rho_i > f^*(\theta_2)$, the borrower will shift to the more risky project since the debt cannot be renegotiated,

and the lender will be bought out for $\theta_1 \rho_i$ which is the value of claim. Notice that this regime does not give the lender enough ex post power and is in fact worse than the borrower control as the risk shifting occurs in equilibrium. Hence it is dominated ex ante by lender control. In practice, borrowers cannot buyout lenders at market prices when there is a margin call, they have to post the margin or be liquidated. This commitment feature that occurs in our model through the assignment of bargaining rights to lenders is a critical aspect of our model.

More importantly, in the context of this paper, the Proposition rationalizes the contract structure that we have employed in our analysis and matches features of margin financing closely.

5 Robustness issues

In this section, we discuss some of the important assumptions that have gone into our analysis and attempt to understand how robust the model is to these assumptions.

5.1 Choice of risk-shifting technology

We acknowledge that our choice of risk-shifting technology from asset 2 to asset 1, as merely switching from a stream of risky to even riskier cash flows without incurring any costs or without engaging in any trades, has the flavor of risk-shifting in the context of real assets. Put another way, in the case of financial assets, one would ideally want the shift of assets to arise because of the sale of risky asset and the purchase of even riskier asset, and potentially clear the markets at date 2 from such shifts. Our choice has been based primarily on simplicity and parsimony. Nevertheless, we believe there are at least a few justifications and interpretations that accredit the choice.

First, the shift in assets could represent simply a deterioration in the risk-management function of the financial intermediary, for example, not constraining traders from following doubling-up strategies and allowing (or even encouraging) them to put additional capital at risk so as to “gamble for resurrection.” Second, the riskier technology could in fact be outside of the traditional assets invested by the financial sector. Given the risk-shifting incentive, institutions may be willing to pay positive price for the option-value of an asset that otherwise represents a negative net-present value investment. The sellers of such assets from outside of the traditional financial sector may only be too willing to be the recipient of this benefit. An example here would be the “reaching for yield” behavior attributed in recent times to hedge funds and other players as their alphas from previously successful strategies have been eroded by competition. The growth in markets for alternative risks and the “excess” in funding of sub-prime mortgages are again cases in point.

5.2 Specificity in lending and asset markets

One question that arises is why the non-rationed firms do not lend to the rationed firms. One rationale to believe such lending would occur is that players within the financial sector understand financial assets better and may have superior monitoring technology to lend to other financial firms compared to dispersed lenders (such as purchasers of commercial paper, for example). Such superior peer-monitoring skills have been employed in the literature to provide a micro-foundation for the existence of inter-bank lending (Rochet and Tirole, 1996). Since improved monitoring mitigates the opportunity to engage in asset substitution, such lending would in general improve efficiency. However, equilibrium between the market for lending and the market for acquiring assets ensures that funding illiquidity persists at least when the moral-hazard intensity is sufficiently severe. The reason for this is that if there is limited funding in the system as a whole, then asset markets will clear only at fire-sale prices, and if this is the case, potential lenders – who are also potential asset acquirers – would be willing to provide financing only at rates that ensure them the same return as the purchase of cheap assets. Since the face-value of loans would be constrained by the risk-shifting problem, only limited financing would be possible in equilibrium.²⁴

The converse of this question is why the financiers in our model (assumed to be the dispersed type) do not participate in the market for assets. Again, they might lack the expertise or sophistication to operate complex financial assets.²⁵ On the one hand, liquidation to such inefficient users would result in allocation inefficiencies in the model. On the other hand, such users would not find prices attractive (relative to non-rationed industry insiders) unless fire-sale discount becomes relatively steep. To summarize, as in the original models of Williamson (1988) and Shleifer and Vishny (1992), the idea of asset-specificity is key to ensuring that there is limited participation by financiers in the market for assets.

Importantly, this specificity also justifies why collateral required to be posted by financiers is generally in the form of cash or treasury securities, rather than in the form of risky assets underlying the business activities of borrowers. Financiers may not operate or risk-manage complex assets

²⁴See Acharya and Yorulmazer (2007) for modeling of such linkages between markets for financial and real assets. Further, Acharya, Gromb and Yorulmazer (2007) argue that in cases where a large number of players are liquidity takers and only a handful remain as potential liquidity providers, these providers may act strategically and charge higher than competitive lending rates in order to force greater asset sales and extract further price discounts. This argument has also been made in the context of predatory trading in capital markets by Brunnermeier and Pedersen (2005) and Carlin, Lobo and Viswanathan (2007).

²⁵This has been witnessed painfully during the sub-prime collapse of Summer 2007. The opacity of balance-sheets of financial institutions and the inability of even sophisticated lenders such as prime brokers to value complex products like CDO and CLO tranches (and the lack of any secondary trading platform for the same) seem to have led to a freeze in inter-bank lending, securitization and financing of assets such as leveraged buyouts that rely on such securitization.

well if collateral of such form is held on their own books, and if held on the books of borrowers, financiers may lack the expertise to monitor well and constrain the risk-shifting problem. Thus, hair-cuts or discounts to market valuations charged when posting collateral, are increasing in the complexity and riskiness of assets being posted and become steeper during times of tight liquidity, that is, when borrower-lender agency problems are exacerbated.

5.3 Extending date-0 aspects of the model

5.3.1 Insurance arrangements at date 0

In our model, liquidity shocks at date 1 arise due to liabilities undertaken by firms at date 0. In principle, such liabilities can be foreseen and hence potentially hedged by firms through management of asset duration, holding of liquid assets, and pre-arrangement of lines of credit (as in Holmstrom and Tirole, 1998). Changing asset duration and holding liquidity can be economically expensive. The lines of credit generally contain a Material Adverse Change (MAC) clause, which allows the provider of the line to revoke access in case the borrower's condition has deteriorated sufficiently (see Sufi, 2006 for empirical evidence that this clause is invoked in practice). Indeed, one reason why such clauses feature in optimal contracting of the line of credit is precisely to avoid agency problems tied to borrower-lender relationships. In our view, modeling liquidity shocks as liability shocks is thus a metaphor for the residual asset-liability mis-match on the balance-sheet of firms.

Especially in our context of financial intermediaries, the liquidity shocks can be more broadly interpreted as arising due to change in the mark-to-market valuations of financial securities such as swaps where ex-ante contract values are zero, but ex post, depending upon the realization of underlying price shocks, the valuation may result in liabilities or cash outflows. Such shocks cannot be hedged perfectly as that would be tantamount to completely undoing the position undertaken through the security in the first place.

5.3.2 Risk-shifting and cash management at date 0

Our model considers risk-shifting at date $1\frac{1}{2}$, after new debt has been issued (or initial debt has been rolled over) at date 1 to meet liability shocks from the initial debt issued at date 0. One key question is whether firms could engage in risk-shifting at date 0 or whether firms could save cash at date 0.

In the current model, a firm will not risk shift at date 0. With complete information, it would be common knowledge that such risk-shifting had occurred and the price would then be $\theta_1 y_1 < \rho_i$, i.e., the firm would be liquidated for sure. Intuitively, the option to risk-shift is worth more

alive than dead, i.e., early exercise of the risk-shifting option is never optimal.

A similar answer is obtained for holding cash. A firm that invests at date 0 will never hold any excess cash, i.e., it will not borrow beyond the investment I and hold some cash. This follows from the convexity of the $\rho(s)$ function. An increase in s beyond I by one dollar leads to an increase in $\rho(s)$ of say $x > 1$. At date 1, the firm has an extra dollar of cash but more than extra dollar of margin calls. If the firm has a liquidity shock bigger than $\rho^*(\theta_2)$, it needs to raise an extra $(x - 1)$ units of capital. If it has a liquidity shock lower than $\rho^*(\theta_2)$, it loses debt capacity equivalent to $(x - 1)$ units and hence buys less assets from distressed firms. In either case, it is suboptimal to have borrowed and held cash.

5.3.3 Collateralized borrowing at date 0

We do not allow risky collateral in our current model. If at all, risky collateral can only be used in this model at date 0 (because it would have a market at date 1). Allowing for this possibility, however, requires analysis of two markets, the market for the collateral asset and the market for the asset being financed. Further, it requires understanding the financing arrangements in the collateral market. Given the whole set of additional issues involved, we do this in a subsequent paper.

5.3.4 Entry of date-0 rationed borrowers at date 1

We also do not allow original investors to choose between investing and waiting in our model. Rampini and Viswanathan (2007) analyze these tradeoffs in a different model with walk-away constraints and full contingent claims (but exogenous capital prices). In their model, agents with low productivity choose to wait; in our model, this would translate into agents with low wealth. In principle, the staying out of low-capitalized agents at date 0 would weaken the endogenous entry effect that in our numerical examples led to lower prices with better fundamentals. One could argue, however, that there may be learning-by-doing style expertise effects so that the better-capitalized insiders may stand a relative advantage than the poorly-capitalized outsiders, except in the extreme situation where almost all insiders are in distress.

6 Conclusion

In this paper, we have attempted to provide a moral-hazard based, agency-theoretic foundation to collateral constraints in a set-up where asset-pricing implications of such constraints can be studied. We would like to reiterate that our general equilibrium variant of the Shleifer and Vishny (1992) industry-equilibrium model is characterized entirely by a single parameter, the

moral-hazard intensity, which drives the extent of financing friction, the optimal level of collateral requirement designed to ameliorate the moral hazard, and finally, the level of equilibrium prices. This characterization is crucial to understanding the fundamental sources of funding and market illiquidity that are jointly witnessed during financial crises.

In ongoing work, we are examining the case where the quality of assets or fundamentals (formally, θ_2) is only imperfectly observed by different firms, and the price serves to aggregate this information. This feature has the potential to produce a further amplification effect in the model and sharper falls in market and funding liquidity when fundamentals deteriorate (say in a first-order stochastic dominance sense), compared to our benchmark model.

Separately, we are also examining the possibility of contagion across asset markets, when there is uncertainty about portfolio composition of financial institutions and there is a shock in fundamentals to some of the assets. Such uncertainty, resulting from the opaqueness of increasingly complex balance-sheets of trading institutions (and to an extent, necessary for them to prevent erosion of their “alphas”), has been argued to be a significant contributor to market and funding liquidity problems. The spillover of the recent sub-prime mortgage collapse onto broader credit markets presents a case in point.

Finally, we are addressing the welfare question as to whether collateral requirements set bilaterally between counterparties in over-the-counter products are efficient from a system-wide perspective. In particular, collateral has a price-stabilizing effect in our paper, and if there is a pecuniary externality from stable prices that is not fully internalized in the bilateral setting of collateral requirements, then there may be a role for centralized counterparties and exchanges in ensuring that these requirements are not set inefficiently.

We believe that such pursuits represent only the tip of the iceberg and much work remains in integrating agency-theoretic corporate-finance issues into main-stream asset-pricing literature, especially in the context of understanding liquidity issues in truly dynamic set-ups. The simple building blocks of this paper may serve as a useful starting point for such modeling.

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Appendix 1

Proof of Proposition 3: We first prove that the market-clearing price p^* exists and is unique.

Step 1. The demand function for assets is given by

$$D(p, \rho^*) = \begin{cases} \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^* - \rho)}{(p - \rho^*)} g(\rho) d\rho & \text{if } \rho^* \leq p \\ \left[0, \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^* - \rho)}{(p - \rho^*)} g(\rho) d\rho \right] & \text{if } p = \bar{p} \end{cases}$$

where at price \bar{p} , we get an interval of possible demand as buyers are indifferent between not buying and buying up to their maximum liquidity. Hence, the excess demand function is given by

$$E(p, \rho^*) = \begin{cases} -1 + \frac{1}{(p - \rho^*)} \int_{\rho_{\min}}^p G(\rho) d\rho & \text{if } \rho^* \leq p \\ \left[-1 + \frac{1}{(p - \rho^*)} \int_{\rho_{\min}}^p G(\rho) d\rho, -\int_{\rho^*}^p \frac{(\rho - \rho^*)}{(p - \rho^*)} g(\rho) d\rho - \int_p^{\rho_{\max}} g(\rho) d\rho \right] & \text{if } p = \bar{p} \end{cases}$$

where as before we get an interval at \bar{p} .

Step 2. Note that the excess demand for $p = \rho^*$ is positive infinity.

Step 3. If the excess demand is positive for all $p < \bar{p}$, the price must be \bar{p} as at \bar{p} the interval definition of excess demand above includes 0. So, \bar{p} is the only feasible price. Intuitively, if there are more agents willing to buy than sell at the highest possible price, this must be the price.

Step 4. If the excess demand is negative as $p \rightarrow \bar{p}$, we must have at least one solution for p . However, we note that for $\rho^* < p < \bar{p}$, the derivative of the excess demand (when the excess demand is ≥ 0) is given by

$$\frac{\partial E(p, \rho^*)}{\partial p} = -\frac{1}{(p - \rho^*)^2} \int_{\rho_{\min}}^p G(\rho) d\rho + \frac{G(p)}{p - \rho^*} \leq -\frac{1}{(p - \rho^*)} + \frac{G(p)}{p - \rho^*} < 0, \quad (49)$$

where we have used the fact that a positive excess demand implies that $\frac{1}{(p - \rho^*)} \int_{\rho_{\min}}^p G(\rho) d\rho \geq 1$ and that $G(p) < 1$.

Hence when the excess demand is zero, its derivative must also be negative, thus we can only have one price that sets the excess demand to zero and the price \bar{p} is unique.

Step 5. To prove that p^* is increasing in ρ^* , note that the excess demand function has a positive derivative with respect to ρ^* for all $p < \bar{p}$ (as can be verified using the expression for excess demand in Step 1 above). Since the excess demand function is strictly downward sloping for positive excess demand, it immediately follows that p^* is strictly increasing in ρ^* if $p^* < \bar{p}$; otherwise the price just stays at \bar{p} .

Step 6. It follows from Step 5 that there exists a unique critical value $\hat{\rho}^* \in (\rho_{\min}, \bar{p})$ such that the market-clearing price $p^* = \bar{p}, \forall \rho^* \geq \hat{\rho}^*$ and $p^* < \bar{p}$ otherwise, in which case p^* satisfies

equation (22). Note also from equation (22) that we must have $p^* \geq \rho^*$ with equality arising only when $\rho^* = \rho_{\min}$.

This completes the proof. \diamond

Proof of Proposition 5: The proof of parts (i) and (ii) of the Proposition follow quite closely the steps in the proof of Proposition 3. Hence, we prove only parts (iii) and (iv) below.

Using equations (20) and (25), we obtain that for $\rho^* = \rho_{\min}$, $p^* = p_{nc}^* = \rho_{\min}$. We also obtain from these equations that

$$E(p, \rho^*) - E_{nc}(p, \rho^*) = \int_{\rho^*}^p \frac{(\rho^* - \rho)}{(p - \rho^*)} g(\rho) d\rho + \int_{\rho^*}^p g(\rho) d\rho, \quad (50)$$

which simplifies to yield

$$E(p, \rho^*) - E_{nc}(p, \rho^*) = \int_{\rho^*}^p \frac{(p - \rho)}{(p - \rho^*)} g(\rho) d\rho > 0, \text{ for } p > \rho^*. \quad (51)$$

Now, by definition, for $p_{nc}^* < \bar{p}$, $E_{nc}(p_{nc}^*, \rho^*) = 0$. It follows then that $E(p_{nc}^*, \rho^*) > 0$. Since $E(p, \rho^*)$ is decreasing in p (see the proof of Proposition 3), it follows that if $E(p^*, \rho^*) = 0$, then $p^* > p_{nc}^*$, else if $E(p^*, \rho^*) > 0$, then $p^* = \bar{p} > p_{nc}^*$. A similar argument implies that if $p_{nc}^* = \bar{p}$, then $p^* = \bar{p}$ as well, and part (iii) of the proposition is proved.

Finally, part (iv) of the proposition relies on the proof of part (iii) and the additional observation that just as $\frac{\partial E(p, \rho^*)}{\partial p} < 0$, we also have that

$$\frac{\partial E_{nc}(p, \rho^*)}{\partial p} = - \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^* - \rho)}{(p - \rho^*)^2} g(\rho) d\rho < 0. \quad \diamond \quad (52)$$

Completion of Proof of Proposition 6:

We now fill in the details of the contraction mapping theorem that we use to prove existence and uniqueness. Granas and Dugundji (2003), Theorem 2.1, shows a general approach to existence of Volterra integral equations of the second kind, we adapt their proof to our set up.

We first show that if $f(\rho, t)$ is Lipschitz in t with Lipschitz constant L_1 and $G(\rho)$ is Lipschitz in ρ with Lipschitz constant L_2 , we can prove existence and uniqueness, at the end of the proof we provide sufficient conditions of the Lipschitz continuity of these functions.

$$\text{Let } L = \max\left\{L_1, \frac{L_2}{R(\hat{s})}\right\}$$

Let \mathbf{E} be the Banach space of all continuous real valued function on $[\theta_1 y_1, \theta_{\max} y_2]$ equipped with the norm

$$\|s\| = \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} |s(\rho)| \quad (53)$$

This norm is equivalent to the standard sup norm $\|x\|_s$ (a function Lipschitzian in one norm is Lipschitzian in any equivalent norm) because

$$e^{-L\theta_{\max}y_2}\|x\|_s \leq \|x\| \leq \|x\|_s, \quad (54)$$

further it is complete.

Define $M(s)(\rho) = \int_{\theta_1 y_1}^{\rho} \frac{R(s(u))}{R(\hat{s})} du$ where s refers to the function $s(\rho)$ on $[\theta_1 y_1, \theta_{\max} y_2]$. We first note that

$$\begin{aligned} & \|M(s') - M(s)\| \\ & \leq \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} \left| \frac{R(s'(u))}{R(\hat{s})} - \frac{R(s(u))}{R(\hat{s})} \right| du \\ & \leq \frac{L_1}{R(\hat{s})} \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} |s'(u) - s(u)| du \\ & \leq L \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} |s'(u) - s(u)| du \\ & \leq L \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} e^{Lu} e^{-Lu} |s'(u) - s(u)| du \\ & \leq L \|s' - s\| \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} e^{Lu} du \\ & = L \|s' - s\| \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \frac{e^{L\rho} - e^{L\theta_1 y_1}}{L} \\ & \leq (1 - e^{-L(\theta_{\max} y_2 - \theta_1 y_1)}) \|s' - s\| \end{aligned} \quad (55)$$

Next define the map $F: \mathbf{E} \rightarrow \mathbf{E}$ by

$$F(s)(\rho) = \int_{\theta_1 y_1}^{\rho} \rho f(t, M(s)(t)) d\rho \quad (56)$$

where s is the function $s(\rho)$. We wish to show this is a contractive map, hence

$$\begin{aligned} & \|F(s') - F(s)\| \\ & \leq \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} |f(t, M(s')(t)) - f(t, M(s)(t))| dt \\ & \leq L \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} |M(s')(t) - M(s)(t)| dt \\ & \leq L \|M(s') - M(s)\| \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} e^{Lt} dt \end{aligned}$$

$$\begin{aligned}
&\leq L \|M(s') - M(s)\| \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \frac{e^{L\rho} - e^{L\theta_1 y_1}}{L} \\
&\leq (1 - e^{-L(\theta_{\max} y_2 - \theta_1 y_1)}) \|M(s') - M(s)\| \\
&\leq (1 - e^{-L(\theta_{\max} y_2 - \theta_1 y_1)})^2 \|s' - s\|
\end{aligned} \tag{57}$$

which is contractive as $(1 - e^{-L(\theta_{\max} y_2 - \theta_1 y_1)}) < 1$. Hence by the Banach contraction theorem, we have a unique fixed point in \mathbf{E} and the sequence given by successive iterations $F^n(s)$ converges to this unique fixed point uniformly in the norm $\|\cdot\|$ and hence in the standard sup norm $\|\cdot\|_s$.

We now fill in the details of Lipschitz continuity. We know that if f is differentiable with bounded derivative $f'(\rho) \leq L$, the f is Lipschitz with constant $K < L$. It suffices for the cdf R to assume that it has bounded derivative over the interval $[\theta_1 y_1, \theta_{\max} y_2]$.

To prove that the function $f(\rho, t)$ defined by

$$f(\rho, t) = 1 - H \left(\max \left\{ \frac{(\theta_1 y_1 + (\rho - t)) + \sqrt{(\theta_1 y_1 + (\rho - t))^2 - 4y_2(\rho - t)\theta_1}}{2y_2}, \frac{\rho}{y_2} \right\} \right) \tag{58}$$

is Lipschitz in ρ , define the auxiliary function

$$\hat{f}(\rho, t) = 1 - H \left(\frac{(\theta_1 y_1 + (\rho - t)) + \sqrt{(\theta_1 y_1 + (\rho - t))^2 - 4y_2(\rho - t)\theta_1}}{2y_2} \right) \tag{59}$$

which is Lipschitz in ρ provided the cdf $H(\cdot)$ is differentiable with bounded derivatives. But this suffices for function f . Given ρ , let $\hat{t}(\rho)$ be the point where the two terms in the maximum function are equal (the function is not differentiable at this point in t). If $t, t' \geq \hat{t}(\rho)$, the Lipschitz continuity of $\hat{f}(\rho, t)$ in t suffices. If $t > \hat{t}(\rho) > t'$ we note that $|f(\rho, t) - f(\rho, t')| = |f(\rho, t) - f(\rho, \hat{t}(\rho))|$ and we can use the Lipschitz continuity of $\hat{f}(\rho, t)$ in t as follows:

$$\begin{aligned}
&f(\rho, t) - f(\rho, t') \\
&= f(\rho, t) - f(\rho, \hat{t}(\rho)) \\
&= \hat{f}(\rho, t) - \hat{f}(\rho, \hat{t}(\rho)) \\
&= L_1 |\hat{t}(\rho) - t| \\
&\leq L_1 |t' - t|
\end{aligned} \tag{60}$$

which completes the proof of Lipschitz continuity of the function $f(\rho, t)$ in t . \diamond

Appendix 2

Solving the integro-differential equation

We now discuss the numerical method used to solve the integro-differential equation,

$$\frac{ds}{d\rho} = 1 - H \left(\max \left\{ \frac{(\theta_1 y_1 + L(\rho)) + \sqrt{(\theta_1 y_1 + L(\rho))^2 - 4y_2 L(\rho)\theta_1}}{2y_2}, \frac{\rho}{y_2} \right\} \right) \quad (61)$$

with the end point constraint that $s(\theta_1 y_1) = \theta_1 y_1$.

Find the initial value of \hat{s} on $[\theta_1 y_1, \theta_{\max} y_2]$ as follows,

$$\hat{s} = \int_{\theta_{\min}}^{\theta_{\max}} \theta_2 y_2 h(\theta_2) d\theta_2 \quad (62)$$

where we have used the fact that $\theta_2 y_2$ is the highest possible price in each state.

The recursive algorithm works as follows. Start with $s(\rho) = \rho$ on $[\theta_1 y_1, \theta_{\max} y_2]$. Use this to derive a first order Riemann sum numerical approximation to the integral on a discrete grid $[t_0 = \theta_1 y_1, t_1, \dots, t_N = \theta_{\max} y_2]$ as

$$\int_{\theta_1 y_1}^{t_n} \hat{G}(\rho) d\rho = \sum_{k=1}^n (t_k - t_{k-1}) \hat{G}(t_{k-1}). \quad (63)$$

For each t_n ,

$$L(t_n) = t_n - \sum_{k=1}^n (t_k - t_{k-1}) \hat{G}(t_{k-1}). \quad (64)$$

The integro-differential equation is then approximated by the first order Taylor expansion

$$\begin{aligned} s(t_{n+1}) \\ = s_n + (t_{n+1} - t_n) \left(1 - H \left(\max \left\{ \frac{(\theta_1 y_1 + L(t_n)) + \sqrt{(\theta_1 y_1 + L(t_n))^2 - 4y_2 L(t_n)\theta_1}}{2y_2}, \frac{t_n}{y_2} \right\} \right) \right) \end{aligned} \quad (65)$$

This yields a new grid approximation $s(t_n)$, we set the value of \hat{s} as $s(t_N)$. Now repeat the above process until convergence occurs (maximum difference in $s(t_n)$ is 0.001). This ensures that \hat{s} also converges.

| t = 1 | | t = 1 1/2 | t = 2 |
|---|--|---|--|
| LIQUIDITY SHOCKS | DEBT FINANCING | MARKET FOR ASSETS AT PRICE p | MORAL HAZARD PROBLEM |
| <ul style="list-style-type: none"> • Firm i receive liquidity shock ρ_i | <ul style="list-style-type: none"> • Firms with low ρ_i: - Borrow, face value f_i • Firms with moderate ρ_i: - Borrow, face value f_i - Pledge collateral k_i • Firms with high ρ_i: - Credit-rationed | <ul style="list-style-type: none"> • Buy assets • Liquidate some assets to meet collateral requirement • Are entirely liquidated | <ul style="list-style-type: none"> • Choose between safe and risky asset • Assets pay off, debt is due • Choose between safe and risky asset • Assets pay off, debt is due |

Figure 1: Timeline of the benchmark model.

Figure 2: Equilibrium price p^* as a function of (inverse) moral hazard intensity

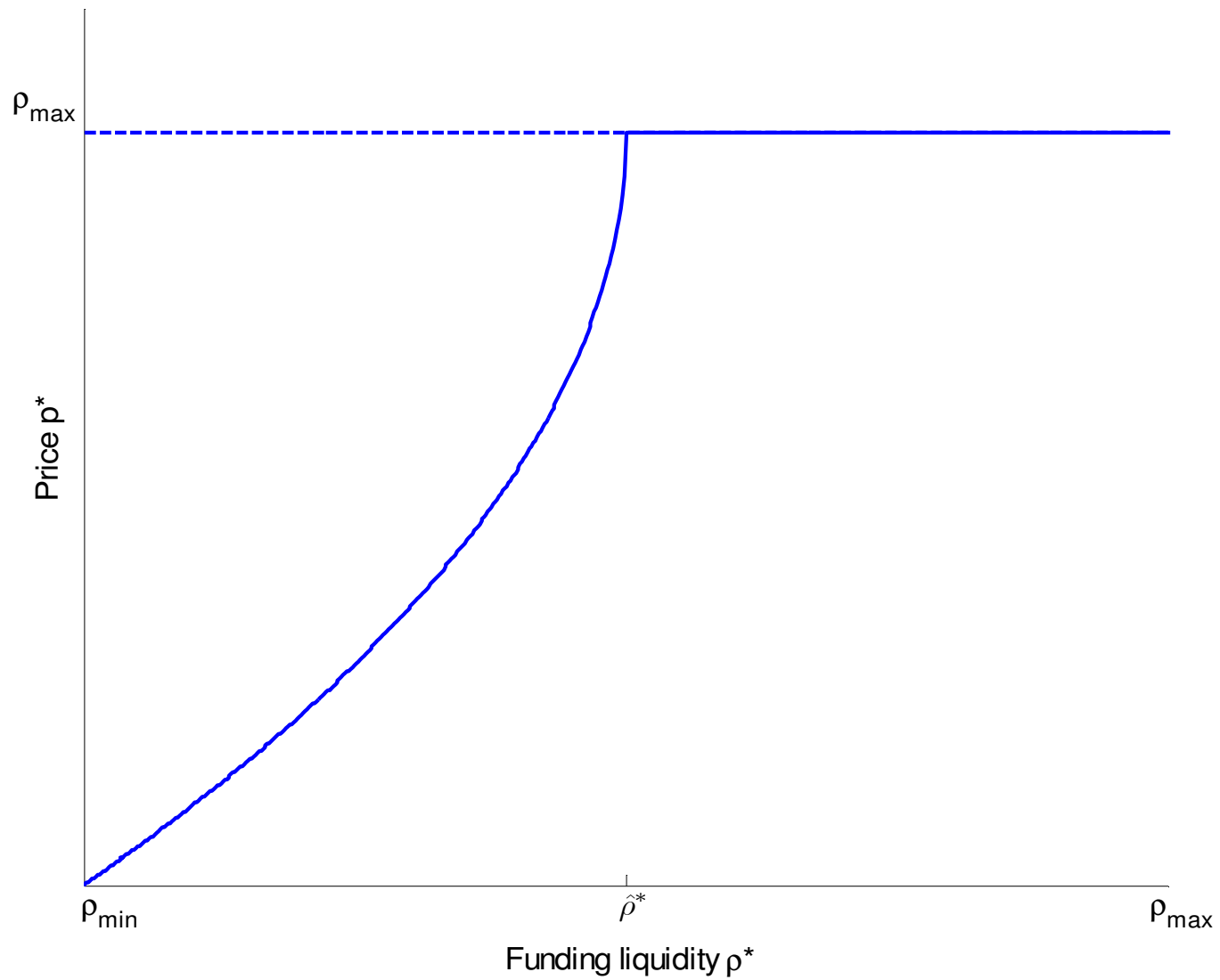


Figure 3: Equilibrium collateral constraint $k(\rho)$ as a function of (inverse) moral hazard intensity ρ

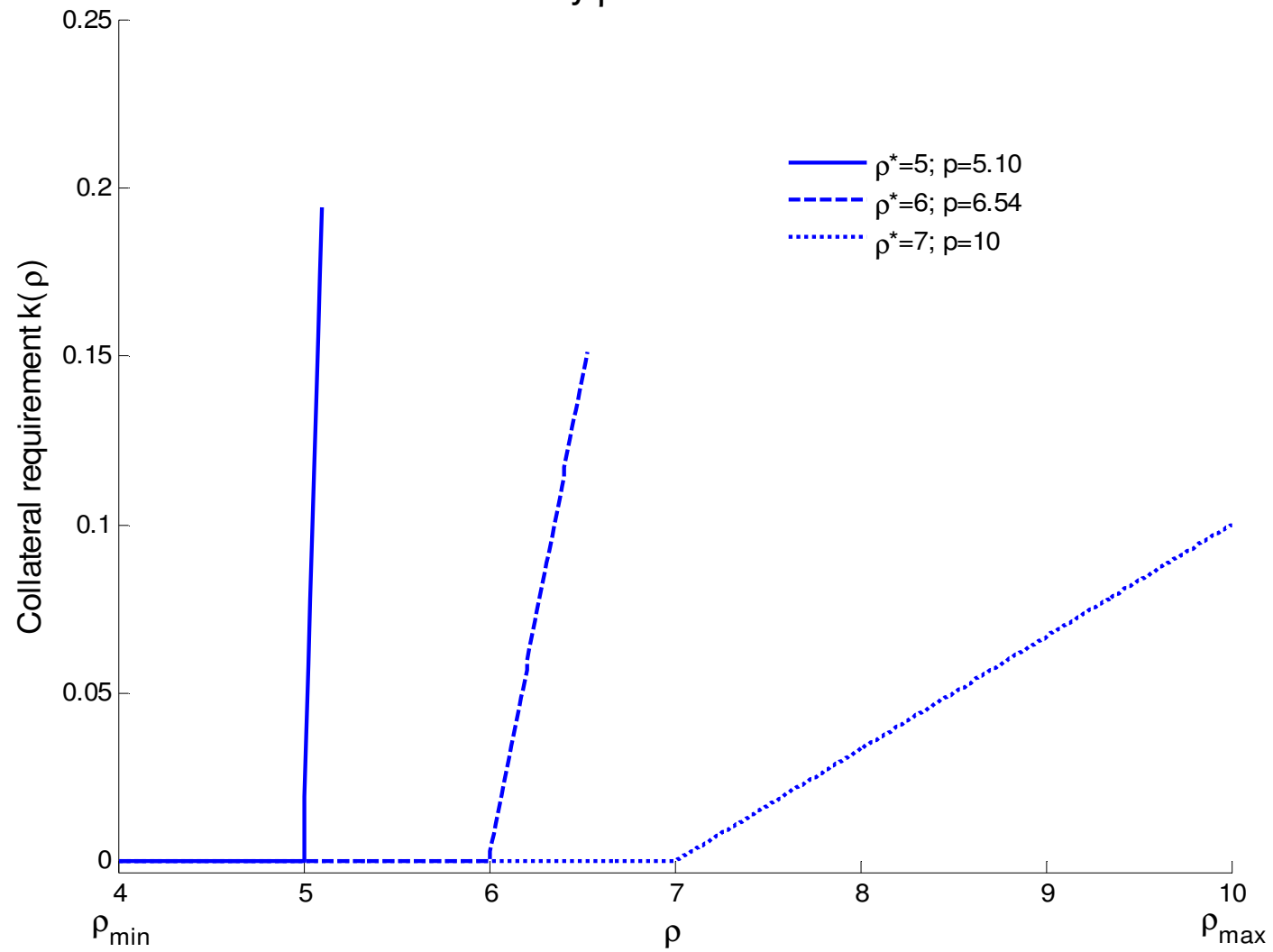


Figure 4: The relationship between market (il)liquidity and funding liquidity

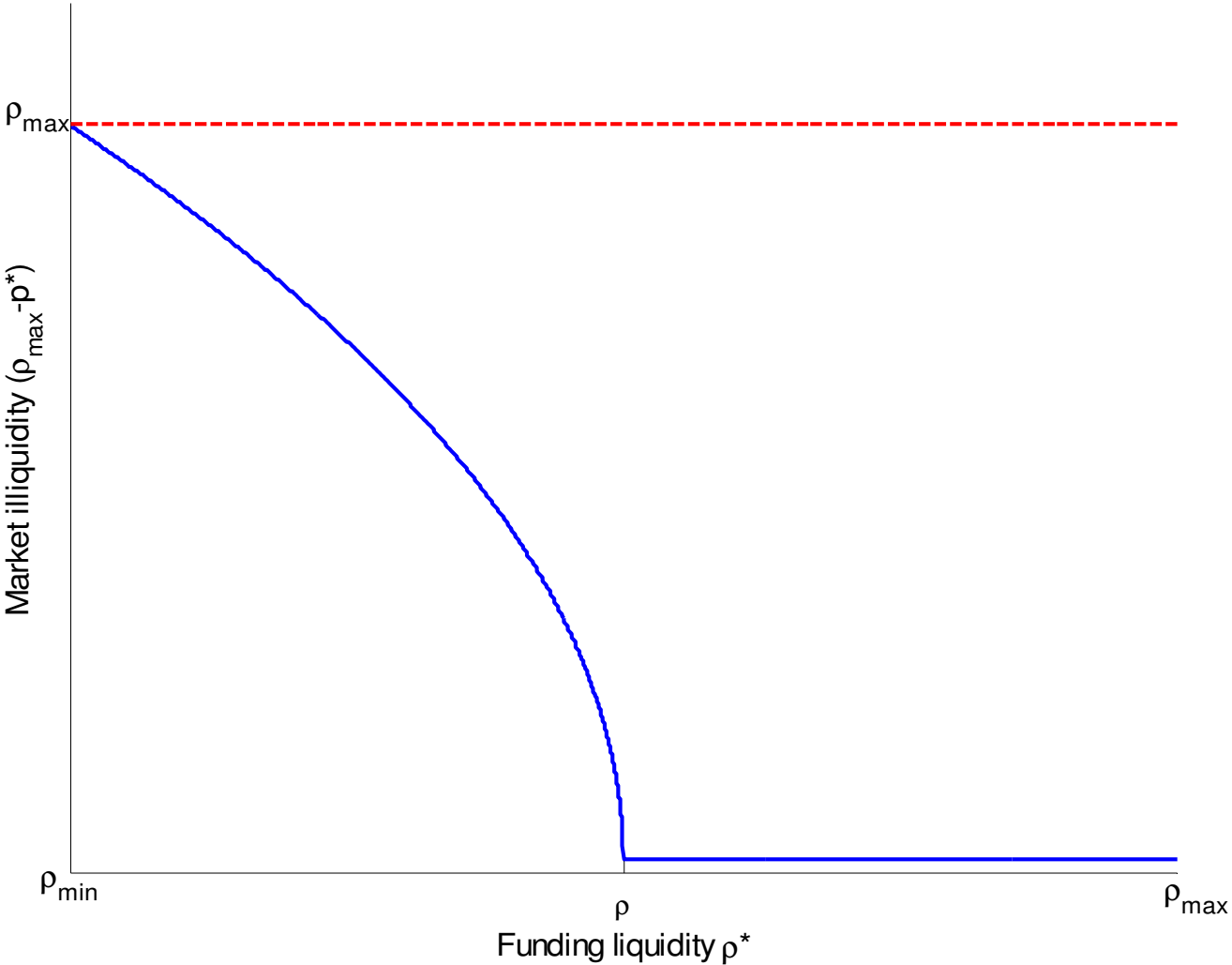
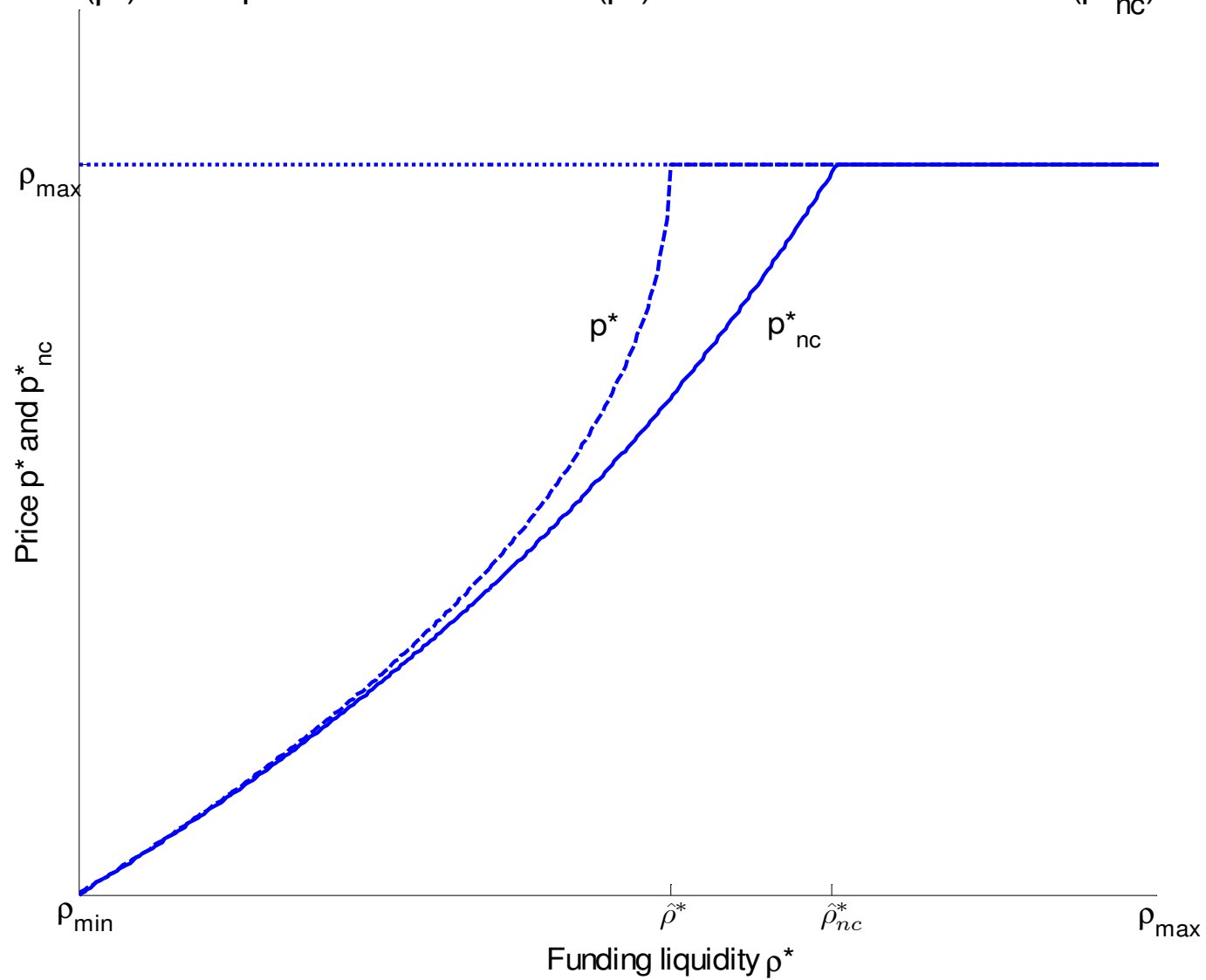


Figure 5: Equilibrium price as a function of (inverse) moral hazard intensity (ρ^*) in the presence of collateral (p^*) and absence of collateral (p_{nc}^*)



| t = 0 | t = 1 | t = 1 1/2 | t = 2 |
|--|--|---|--|
| FIRST ROUND OF BORROWING | DEBT DUE | MARKET FOR ASSETS | MORAL HAZARD PROBLEM |
| | ADDITIONAL DEBT FINANCING | AT PRICE $p(\theta_2)$ | |
| | REALIZATION OF ASSET QUALITY θ_2 | | |
| <ul style="list-style-type: none"> • Firm i with wealth w_i raises $(I - w_i)$ with debt of face value ρ_i | <ul style="list-style-type: none"> • Firms with low ρ_i : - Borrow again • Firms with moderate ρ_i : - Borrow again - Pledge collateral • Firms with high ρ_i : - Credit-rationed | <ul style="list-style-type: none"> • Buy assets • Liquidate some assets to meet collateral requirement • Are entirely liquidated | <ul style="list-style-type: none"> • Choose between safe and risky asset • Assets pay off, debt is due • Choose between safe and risky asset • Assets pay off, debt is due |

Figure 6: Timeline of the augmented model.

Figure 7: $\rho(s)$ for various γ

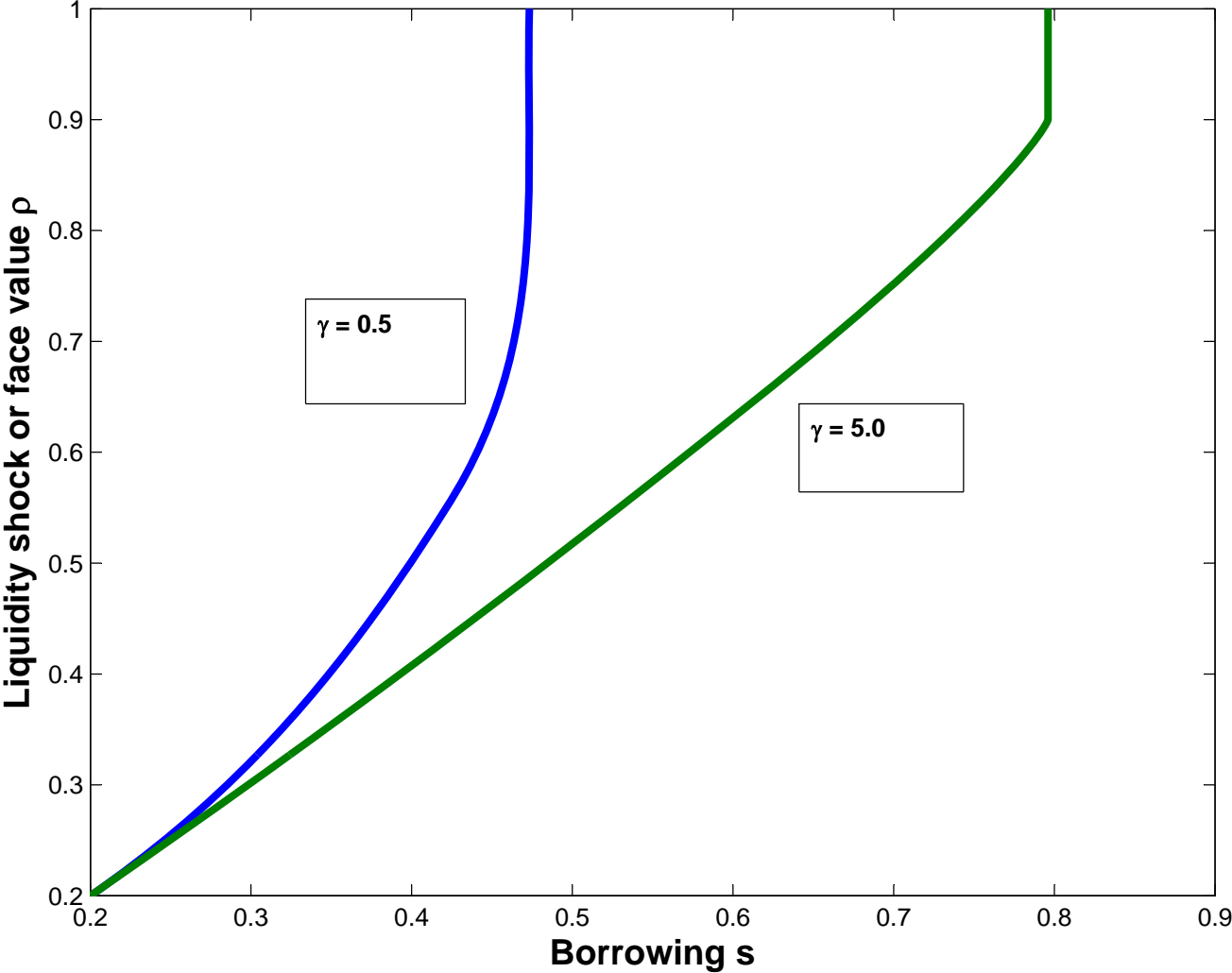


Figure 8: $p(\theta)$ for various γ

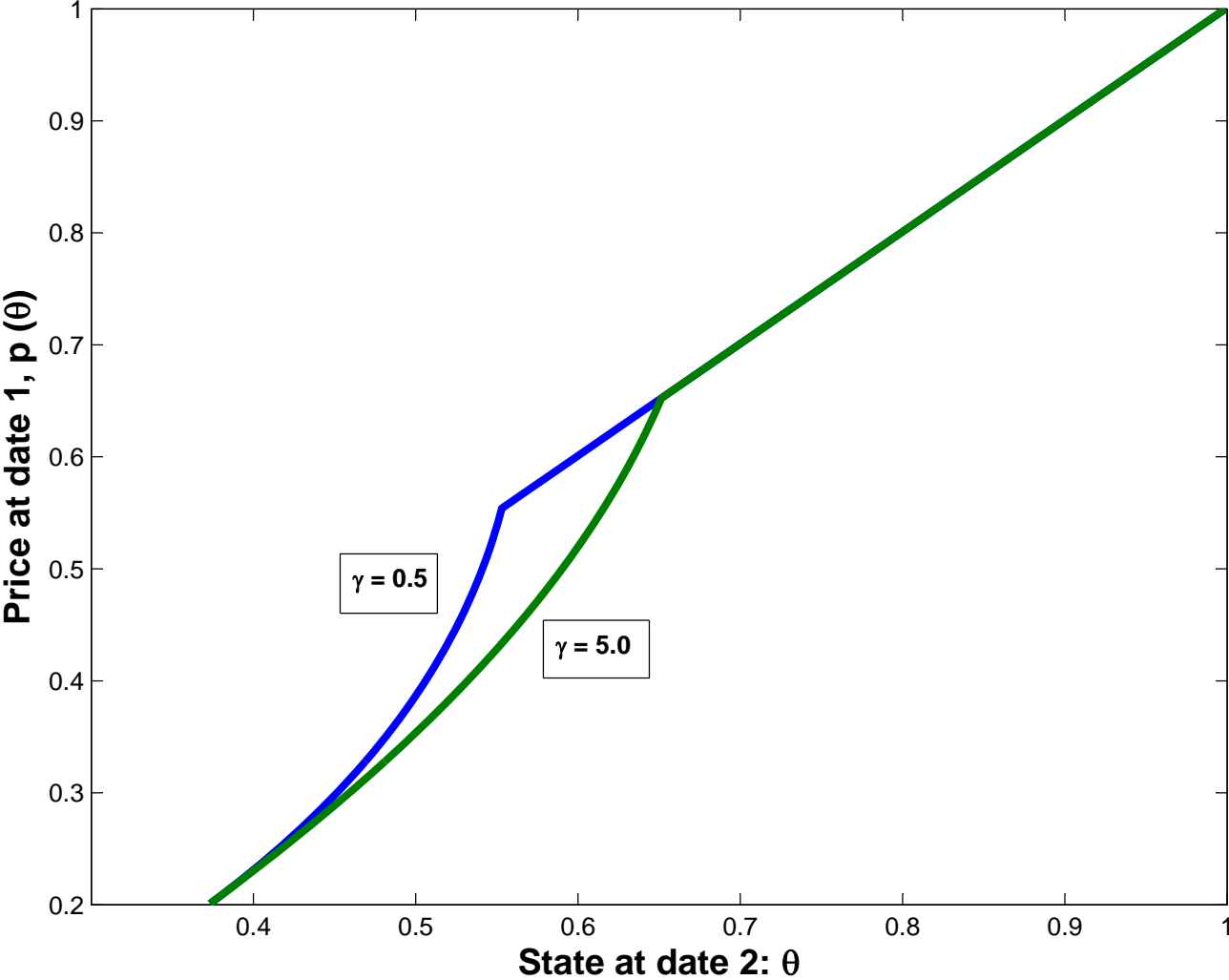


Figure 9a: CDF of $\rho(s)$ in equilibrium

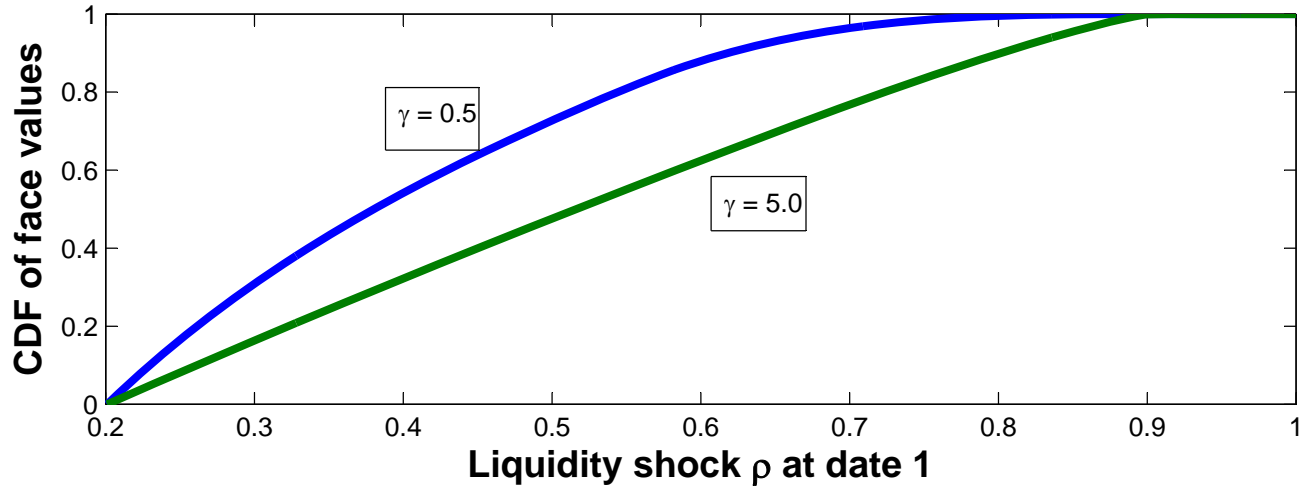


Figure 9b: CDF of prices in equilibrium

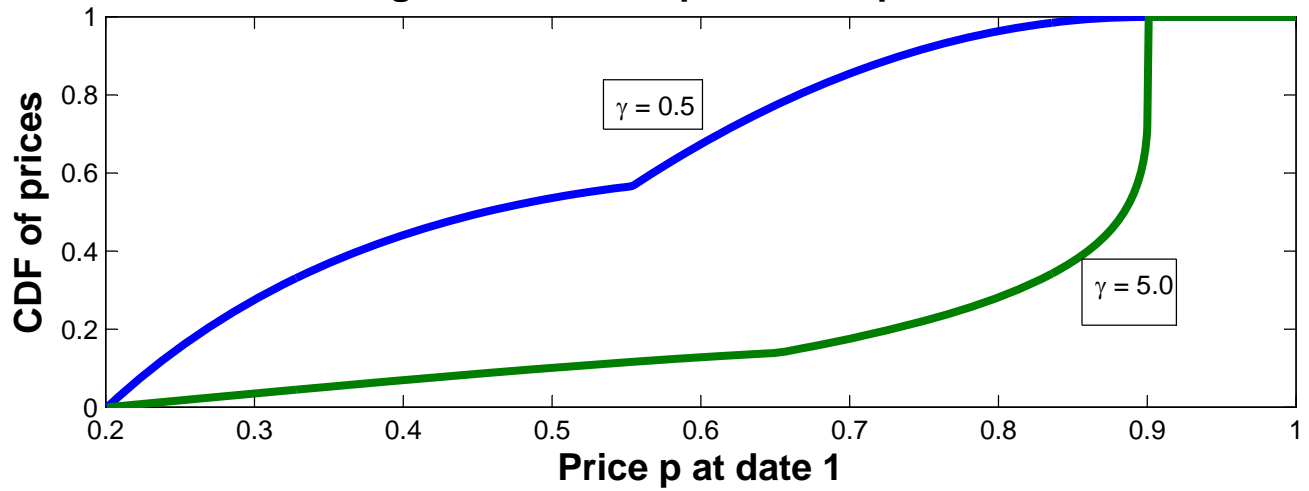


Figure 10: $\rho(s)$ for various γ

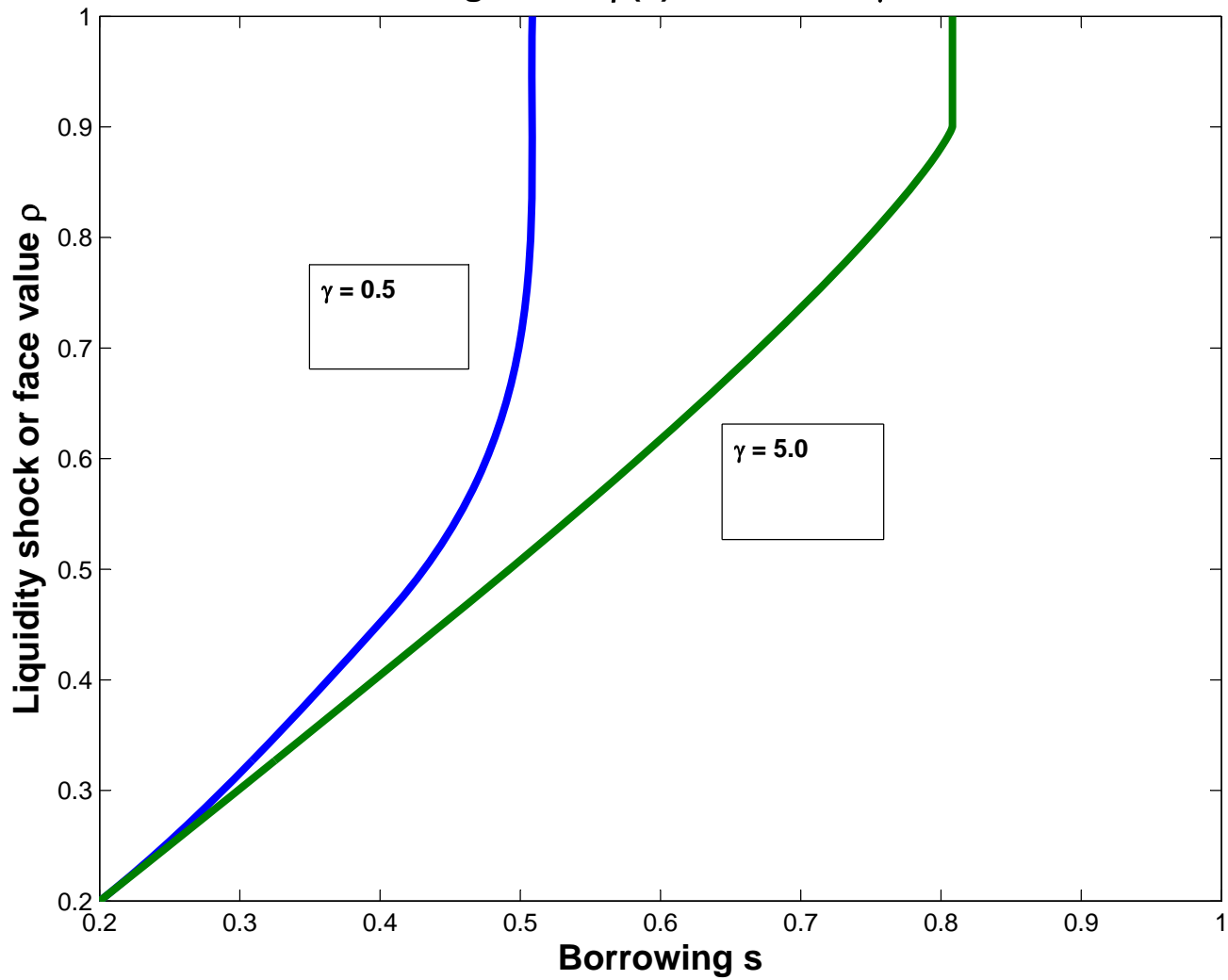


Figure 11: $p(\theta)$ for various γ

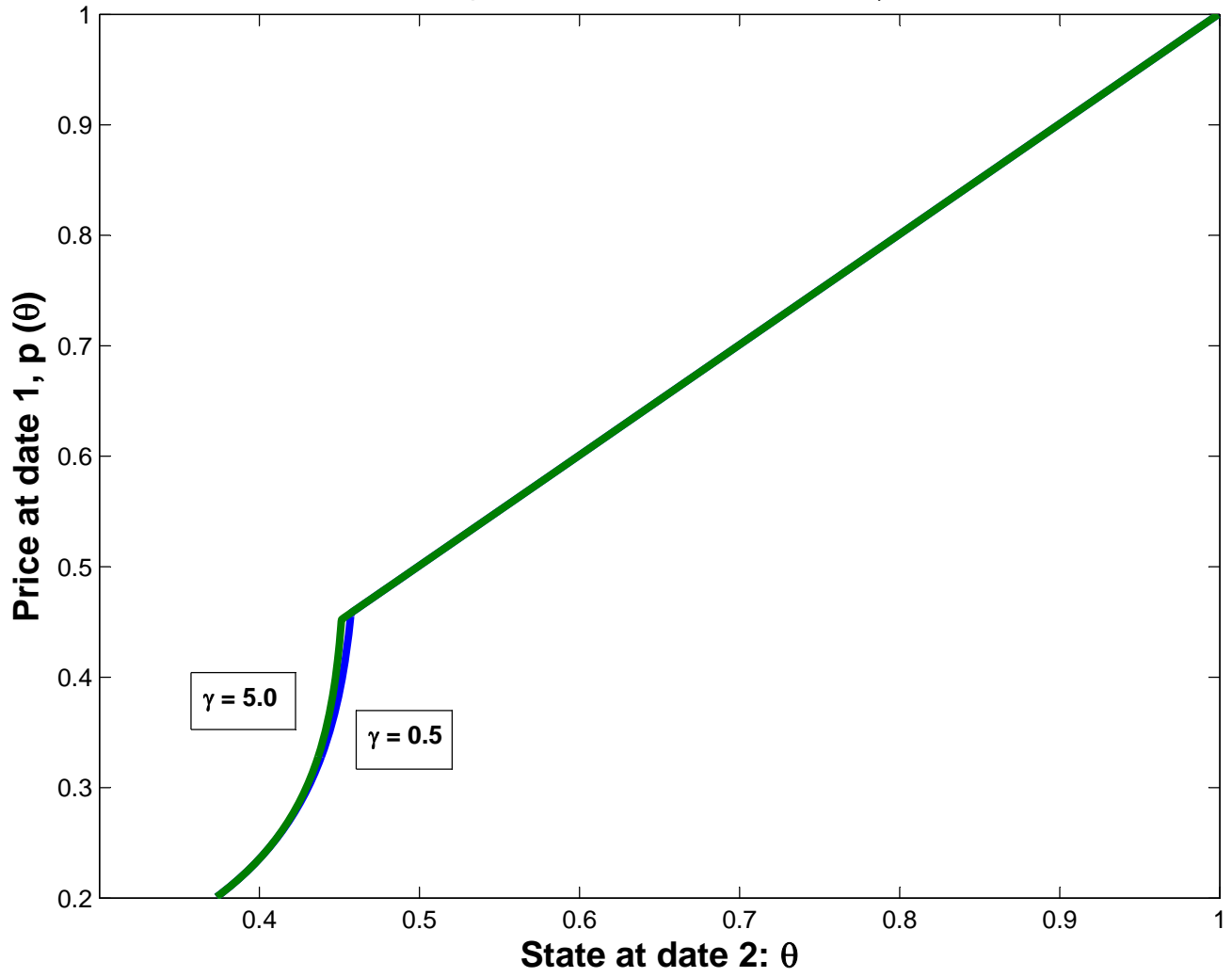


Figure 12a: CDF of $\rho(s)$ in equilibrium

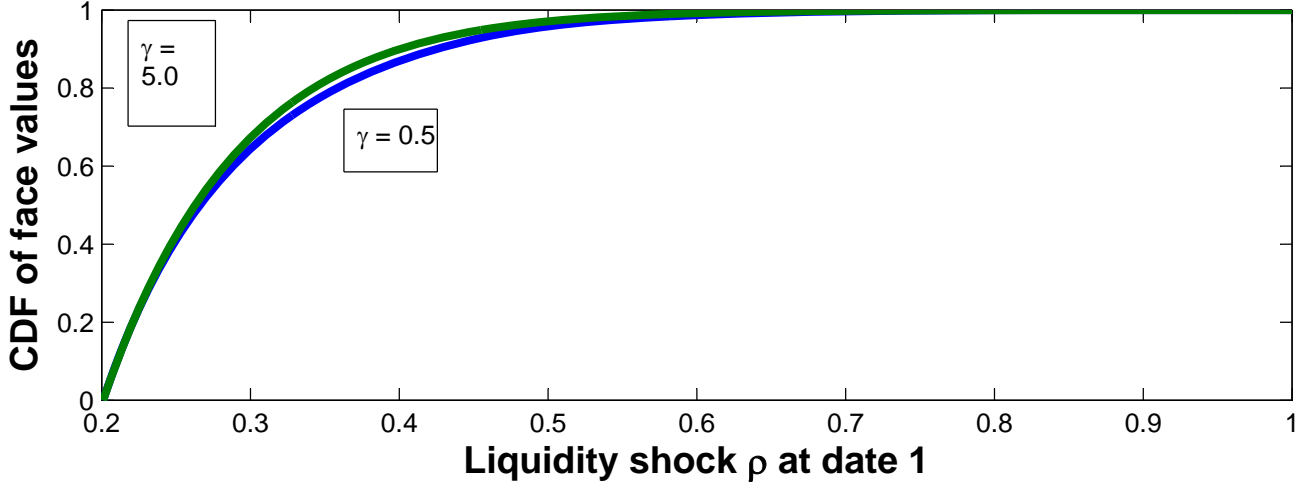


Figure 12b: CDF of prices in equilibrium

