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More on a Holographic Dual of QCD

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We investigate the interactions among the pion, vector mesons and external gauge fields in the holographic dual of massless QCD proposed in a previous paper [T. Sakai and S. Sugimoto, Prog. Theor. Phys. **113** (2005), 843; hep-th/0412141] on the basis of probe D8-branes embedded in a D4-brane background in type IIA string theory. We obtain the coupling constants by performing both analytic and numerical calculations, and compare them with experimental data. It is found that the vector meson dominance in the pion form factor as well as in the Wess-Zumino-Witten term holds in an intriguing manner. We also study the $\omega \to \pi\gamma$ and $\omega \to 3\pi$ decay amplitudes. It is shown that the interactions relevant to these decay amplitudes have the same structure as that proposed by Fujiwara et al. [T. Fujiwara, T. Kugo, H. Terao, S. Uehara and K. Yamawaki, Prog. Theor. Phys. **73** (1985), 926]. Various relations among the masses and the coupling constants of an infinite tower of mesons are derived. These relations play crucial roles in the analysis. We find that most of the results are consistent with experiments.

§1. Introduction

In a previous paper, Ref. 1), we proposed a holographic dual of $U(N_c)$ QCD with N_f massless flavors, which is constructed by putting probe D8-branes in the D4-brane background. It was shown there that various phenomena that are expected to occur in low energy QCD can be reproduced in this framework. For instance, we showed that the chiral $U(N_f)_L \times U(N_f)_R$ symmetry is spontaneously broken to the diagonal subgroup $U(N_f)_V$. The associated Nambu-Goldstone (NG) bosons were found and identified with the pion. Moreover, we found vector mesons in the spectrum, and the masses and some of the coupling constants among them turn out to be reasonably close to the experimental values.

The purpose of this paper is to study the D4/D8 model in more detail in order to explore the low-energy phenomena involving the mesons. The effective action of our model consists of two parts. One is the five-dimensional Yang-Mills (YM) action on a curved background, which originates from the non-abelian Dirac-Born-Infeld (DBI) action on the probe. The other is the integral of the Chern-Simons (CS) five-form, which results from the CS term on the probe D8-brane. From these, we compute the cubic and some quartic interaction terms among the pion, the vector mesons and the external gauge fields associated with the chiral $U(N_f)_L \times U(N_f)_R$ symmetry.

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The results are compared with the experimental data in order to quantitatively test the conjectured duality. For recent developments toward holographic descriptions of QCD, see also Refs. 2)–19).

In particular, we are interested in the coupling to the external photon field. We examine whether the vector meson dominance hypothesis[20),21)] is satisfied in this model. This hypothesis states that the exchange of vector mesons dominates the electromagnetic interactions of hadrons. For example, the electromagnetic form factor of the pion is dominated by the ρ meson pole as

$$F_\pi(k^2) \simeq \frac{g_\rho g_{\rho\pi\pi}}{k^2 + m_\rho^2} \ , \tag{1·1}$$

where g_ρ is the ρ meson decay constant, m_ρ is the ρ meson mass and $g_{\rho\pi\pi}$ is the $\rho\,\pi\pi$ coupling. In other words, the direct couplings between the photon and the pion are small compared with the indirect interactions resulting from the ρ meson exchange. It has been shown in Refs. 2), 9), 15) and 16) that the pion form factor exhibits vector meson dominance in generic holographic models of QCD, where the contributions from infinitely many vector mesons are important. We reexamine this feature in our model and present a numerical estimation of the dominant terms. Furthermore, we analyze the Wess-Zumino-Witten (WZW) term that includes an infinite tower of vector mesons and demonstrate the complete vector meson dominance in this sector.

The subjects considered in this paper also include the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relations,[22),23)] the pion charge radius, $a_1 \to \pi\,\gamma$ and $a_1 \to \pi\rho$ decay, $\pi\pi$ scattering, the Weinberg sum rules,[24)] and $\omega \to \pi^0\gamma$ and $\omega \to \pi^0\pi^+\pi^-$ decay. For most of the cases, we obtain considerably good agreement with the experimental data.

This paper is organized as follows. In §2, we review the D4/D8 model to the extent needed in this paper. Note that the notation used in this paper is slightly different from that used in Ref. 1). We define our notation in this section. In §3, we investigate the DBI part of the model. Section 4 is devoted to analyzing the WZW term. In §5, we reanalyze the effective action using a different gauge, which simplifies the treatment of the vector meson dominance. We end this paper with summary and discussion in §6. The two appendices summarize some technical computations.

§2. The model

In this section, we review the D4/D8 model proposed in Ref. 1) and define the notation used in this paper.

The D4/D8 model is formulated by placing probe D8-branes into the D4-brane background proposed in Ref. 25) as a supergravity dual of four-dimensional $U(N_c)$ Yang-Mills theory. The metric, dilaton ϕ, and the RR three-form field C_3 in the D4-brane background are given as

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} \left(\eta_{\mu\nu}dx^\mu dx^\nu + f(U)d\tau^2\right) + \left(\frac{R}{U}\right)^{3/2} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right) ,$$

$$e^{\phi} = g_s \left(\frac{U}{R}\right)^{3/4} , \quad F_4 \equiv dC_3 = \frac{2\pi N_c}{V_4}\epsilon_4 \ , \quad f(U) \equiv 1 - \frac{U_{\rm KK}^3}{U^3} \ . \tag{2·1}$$

Here the coordinates x^μ ($\mu = 0,1,2,3$) and τ parameterize the directions along which the D4-brane is extended, and U corresponds to the radial direction transverse to the D4-brane. From the definition of the function $f(U)$, we see that U is bounded from below as $U \geq U_{\rm KK}$. The quantities $d\Omega_4^2$, ϵ_4 and $V_4 = 8\pi^2/3$ are the line element, the volume form, and the volume of a unit S^4 surrounding the D4-brane, respectively, and R and $U_{\rm KK}$ are constant parameters. The constant R is related to the string coupling g_s and the string length l_s as $R^3 = \pi g_s N_c l_s^3$. This background represents N_c D4-branes wrapped on a supersymmetry breaking S^1 parameterized by the parameter τ, whose period is chosen as

$$\tau \sim \tau + 2\pi M_{\rm KK}^{-1} \ , \qquad M_{\rm KK} \equiv \frac{3}{2}\frac{U_{\rm KK}^{1/2}}{R^{3/2}} \ , \tag{2·2}$$

in order to avoid a conical singularity at $U = U_{\rm KK}$. Along this S^1, fermions are taken to be anti-periodic, and they become massive as four-dimensional fields. Adjoint scalar fields on the D4-brane are also expected to acquire a mass via quantum effects, since the supersymmetry is completely broken. Thus, the world-volume theory on the D4-brane effectively becomes the four-dimensional Yang-Mills theory below the Kaluza-Klein mass scale $M_{\rm KK}$. The Yang-Mills coupling $g_{\rm YM}$ (at the scale $M_{\rm KK}$) is given by $g_{\rm YM}^2 = 2\pi M_{\rm KK} g_s l_s$, which is read off of the DBI action of the D4-brane compactified on S^1. The parameters R, $U_{\rm KK}$ and g_s are expressed in terms of $M_{\rm KK}$, $g_{\rm YM}$ and l_s. One can easily show that l_s does not appear in the effective action if it is written in terms of $M_{\rm KK}$ and $g_{\rm YM}$. Therefore, without loss of generality, we can set

$$\frac{2}{9}\, M_{\rm KK}^2 l_s^2 = (g_{\rm YM}^2 N_c)^{-1} \equiv \lambda^{-1} \ , \tag{2·3}$$

which makes R and $U_{\rm KK}$ independent of $g_{\rm YM}$ and N_c. Furthermore, because the $M_{\rm KK}$ dependence is easily recovered through dimensional analysis, it is convenient to work in units in which $M_{\rm KK} = 1$. Then, we have the relations

$$M_{\rm KK} = 1 \ , \quad R^3 = \frac{9}{4} \ , \quad U_{\rm KK} = 1 \ , \quad \frac{1}{g_s l_s^3} = \frac{4\pi}{9} N_c \ . \tag{2·4}$$

The relations (2·3) and (2·4) make it clear that the α' expansion and the loop expansion in string theory correspond to the expansion with respect to $1/\lambda$ and $\lambda^{3/2}/N_c$ in Yang-Mills theory, respectively. In this paper, we consider only the leading terms in this expansion by taking N_c and λ to be sufficiently large.

In order to add N_f flavors of quarks to the supergravity dual of the Yang-Mills theory described by the background (2·1), we place N_f probe D8-branes extended along x^μ ($\mu = 0,1,2,3$), the S^4 directions, and one of the directions in the (U,τ) plane. Here we adopt the probe approximation, assuming $N_c \gg N_f$, and ignore the backreaction from the D8-branes to the D4-brane background. To describe the

D8-branes, it is convenient to introduce new coordinates (y, z) defined by

$$(y, z) \equiv \left(\sqrt{U^3 - 1} \cos\tau \, , \, \sqrt{U^3 - 1} \sin\tau \right) \ . \tag{2·5}$$

It is easy to show that the metric written in (y, z) is smooth everywhere. We consider the probe D8-branes placed at $y = 0$ and extended along the z direction. As discussed in Ref. 1), this brane configuration corresponds to a D4/D8/$\overline{\text{D8}}$ system that represents $U(N_c)$ QCD with N_f massless flavors.

Note that this system possesses $SO(5)$ symmetry corresponding to the rotations of S^4. In this paper, we concentrate on the states that are invariant under $SO(5)$ rotations for simplicity. Because QCD does not have such an $SO(5)$ symmetry, the meson in realistic QCD can only be found in this sector.[*)] Therefore, we can reduce the nine-dimensional gauge theory on the D8-brane to a five-dimensional theory with a five-dimensional $U(N_f)$ gauge field denoted by $A_\mu(x^\mu, z)$ and $A_z(x^\mu, z)$.[**)]

The effective action on the probe D8-brane embedded in the background (2·1) consists of two parts. One is the (non-Abelian) DBI action, and the other is the CS term. After the Kaluza-Klein reduction on S^4, the leading terms in the $1/\lambda$ expansion of the DBI action read

$$S_{\text{D8}}^{\text{DBI}} = \kappa \int d^4x dz \ \mathrm{tr} \left[\frac{1}{2} K^{-1/3} F_{\mu\nu}^2 + K F_{\mu z}^2 \right] \ , \tag{2·6}$$

where

$$\kappa \equiv \frac{\lambda N_c}{108\pi^3} \ , \quad K(z) \equiv 1 + z^2 \ . \tag{2·7}$$

The CS term is

$$S_{\text{D8}}^{\text{CS}} = \frac{N_c}{24\pi^2} \int_{M^4 \times \mathbb{R}} \omega_5(A) \ , \tag{2·8}$$

where $\omega_5(A)$ is the Chern-Simons five-form written in terms of the five-dimensional differential form $A = A_\mu dx^\mu + A_z dz$ as

$$\omega_5(A) = \mathrm{tr} \left(A F^2 - \frac{1}{2} A^3 F + \frac{1}{10} A^5 \right) \ , \tag{2·9}$$

and $M^4 \times \mathbb{R}$ is the five-dimensional space-time parameterized by (x^μ, z).

In order to extract four-dimensional meson fields from the five-dimensional gauge field, we expand the gauge field as

$$A_\mu(x^\mu, z) = \sum_{n=1}^{\infty} B_\mu^{(n)}(x^\mu) \psi_n(z) \ , \tag{2·10}$$

$$A_z(x^\mu, z) = \varphi^{(0)}(x^\mu) \phi_0(z) + \sum_{n=1}^{\infty} \varphi^{(n)}(x^\mu) \phi_n(z) \ , \tag{2·11}$$

[*)] It is believed somewhat optimistically that states charged under $SO(5)$ decouple in the $M_{\text{KK}} \to \infty$ limit. (See Ref. 26) and references therein.) See also Ref. 27) for a recent analysis.

[**)] Here we omit the scalar field $y(x^\mu, z)$, which corresponds to fluctuations of the D8-brane along the transverse direction, for simplicity.

using the complete sets $\{\psi_n(z)\}_{n\geq 1}$ and $\{\phi_n(z)\}_{n\geq 0}$ of functions of z. In order to diagonalize the kinetic terms and the mass terms of the four-dimensional fields $B_\mu^{(n)}(x^\mu)$ and $\varphi^{(n)}(x^\mu)$, we choose the functions $\psi_n(z)$ to be eigenfunctions satisfying the equation

$$-K^{1/3}\,\partial_z\left(K\,\partial_z\psi_n\right) = \lambda_n\psi_n \ , \tag{2·12}$$

where λ_n is the eigenvalue, and the normalization condition is taken to be

$$\kappa\int dz\,K^{-1/3}\psi_n\psi_m = \delta_{nm} \ . \tag{2·13}$$

The functions $\phi_n(z)$ are chosen to satisfy $\phi_n(z) \propto \partial_z\psi_n(z)$ $(n \geq 1)$ and $\phi_0(z) = 1/(\sqrt{\pi\kappa}K(z))$, with the normalization condition

$$\kappa\int dz\,K\phi_n\phi_m = \delta_{nm} \ , \tag{2·14}$$

which is compatible with (2·12) and (2·13).

Inserting the expansion consisting of (2·10) and (2·11) into the action (2·6) and integrating over z, we obtain

$$\begin{aligned} S_{\rm D8}^{\rm DBI} \simeq \int d^4x\ \mathrm{tr}\Bigg[&(\partial_\mu\varphi^{(0)})^2 \\ &+\sum_{n=1}^{\infty}\left(\frac{1}{2}(\partial_\mu B_\nu^{(n)} - \partial_\nu B_\mu^{(n)})^2 + \lambda_n(B_\mu^{(n)} - \lambda_n^{-1/2}\partial_\mu\varphi^{(n)})^2\right)\Bigg] \\ &+ (\text{interaction terms}) \ . \end{aligned} \tag{2·15}$$

From this, we see that we have one massless scalar field, $\varphi^{(0)}$, and a tower of massive vector fields, $B_\mu^{(n)}$, of mass squared λ_n. The scalar fields $\varphi^{(n)}$ with $n \geq 1$ are eaten by the vector fields $B_\mu^{(n)}$. We interpret $\varphi^{(0)}$ as the massless pion field and $B_\mu^{(n)}$ as vector meson fields.

In the expansion given in (2·10) and (2·11), we have implicitly assumed that the gauge field asymptotically vanishes $A_M(x^\mu, z) \to 0$ as $z \to \pm\infty$. The residual gauge transformation that does not violate this condition is obtained with a gauge function $g(x^\mu, z)$ that asymptotically becomes constant: $g(x^\mu, z) \to g_\pm$ as $z \to \pm\infty$. We interpret (g_+, g_-) as an element of the chiral symmetry group $U(N_f)_L \times U(N_f)_R$ in QCD with N_f massless flavors.

In the following sections, we study the interaction of the mesons with the external gauge fields $(A_{L\mu}, A_{R\mu})$ introduced by weakly gauging the $U(N_f)_L \times U(N_f)_R$ chiral symmetry. Of particular interest are the couplings of the mesons to the photon field $A_\mu^{\rm em}$, which can be extracted by setting

$$A_{L\mu} = A_{R\mu} = eQA_\mu^{\rm em} \ , \tag{2·16}$$

where e is the electromagnetic coupling constant and Q is the electric charge matrix given, for example, by

$$Q = \frac{1}{3}\begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix} \ , \tag{2·17}$$

for the $N_f = 3$ case. It is also necessary to introduce the external gauge fields in the calculation of correlation functions among the currents associated with the chiral symmetry following the prescription used in the AdS/CFT correspondence.[28),29)] In order to turn on the external gauge fields, we impose the asymptotic values of the gauge field A_μ on the D8-brane as

$$\lim_{z\to+\infty} A_\mu(x^\mu, z) = A_{L\mu}(x^\mu) \ , \qquad \lim_{z\to-\infty} A_\mu(x^\mu, z) = A_{R\mu}(x^\mu) \ . \tag{2·18}$$

This is implemented by modifying the mode expansion (2·10) as

$$A_\mu(x^\mu, z) = A_{L\mu}(x^\mu)\psi_+(z) + A_{R\mu}(x^\mu)\psi_-(z) + \sum_{n=1}^{\infty} B_\mu^{(n)}(x^\mu)\psi_n(z) \ , \tag{2·19}$$

where the functions $\psi_\pm(z)$ are defined as

$$\psi_\pm(z) \equiv \frac{1}{2}(1 \pm \psi_0(z)) \ , \quad \psi_0(z) \equiv \frac{2}{\pi}\arctan z \ , \tag{2·20}$$

which are the non-normalizable zero modes of (2·12) satisfying $\partial_z\psi_\pm(z) \propto \phi_0(z)$.

Note that if we insert the expansion (2·19) into the action (2·6) and perform the integration over z, the coefficients of the kinetic terms of the gauge fields $A_{L\mu}$ and $A_{R\mu}$ diverge, because $\psi_\pm$ are non-normalizable. This divergence simply reflects the fact that the gauge coupling corresponding to the chiral $U(N_f)_L \times U(N_f)_R$ symmetry is zero. One way to regularize the divergence is to cut off the integration over z at some large but finite value. Another possibility is to simply ignore the divergent kinetic terms of the external gauge field, since we are interested only in the structure of the interactions.

In this paper, we work mainly in the $A_z = 0$ gauge, which can be realized by applying the gauge transformation $A_M \to gA_Mg^{-1} + g\partial_M g^{-1}$ with the gauge function

$$g^{-1}(x^\mu, z) = P\exp\left\{-\int_0^z dz'\, A_z(x^\mu, z')\right\} \ . \tag{2·21}$$

Then, the asymptotic values (2·18) change to

$$\lim_{z\to+\infty} A_\mu(x^\mu, z) = A_{L\mu}^{\xi_+}(x^\mu) \ , \qquad \lim_{z\to-\infty} A_\mu(x^\mu, z) = A_{R\mu}^{\xi_-}(x^\mu) \ , \tag{2·22}$$

where $\xi_\pm(x^\mu) \equiv \lim_{z\to\pm\infty} g(x^\mu, z)$ and

$$A_{L\mu}^{\xi_+}(x^\mu) \equiv \xi_+(x^\mu)A_{L\mu}(x^\mu)\xi_+^{-1}(x^\mu) + \xi_+(x^\mu)\partial_\mu\xi_+^{-1}(x^\mu) \ , \tag{2·23}$$

$$A_{R\mu}^{\xi_-}(x^\mu) \equiv \xi_-(x^\mu)A_{R\mu}(x^\mu)\xi_-^{-1}(x^\mu) + \xi_-(x^\mu)\partial_\mu\xi_-^{-1}(x^\mu) \ . \tag{2·24}$$

Then, the gauge field in the $A_z = 0$ gauge can be expanded as

$$A_\mu(x^\mu, z) = A_{L\mu}^{\xi_+}(x^\mu)\psi_+(z) + A_{R\mu}^{\xi_-}(x^\mu)\psi_-(z) + \sum_{n=1}^{\infty} B_\mu^{(n)}(x^\mu)\psi_n(z) \ . \tag{2·25}$$

The residual gauge symmetry in the $A_z = 0$ gauge is given by the z-independent gauge transformation. The residual gauge symmetry $h(x^\mu) \in U(N_f)$ and the weakly gauged chiral symmetry $(g_+(x^\mu), g_-(x^\mu)) \in U(N_f)_L \times U(N_f)_R$ act on these fields as

$$A_{L\mu} \to g_+ A_{L\mu} g_+^{-1} + g_+ \partial_\mu g_+^{-1} \ , \tag{2·26}$$

$$A_{R\mu} \to g_- A_{R\mu} g_-^{-1} + g_- \partial_\mu g_-^{-1} \ , \tag{2·27}$$

$$\xi_\pm \to h\, \xi_\pm\, g_\pm^{-1} \ , \tag{2·28}$$

$$B_\mu^{(n)} \to h\, B_\mu^{(n)}\, h^{-1} \ . \tag{2·29}$$

Here, the functions $\xi_\pm(x^\mu)$ are interpreted as the $U(N_f)$ valued fields $\xi_{L,R}(x^\mu)$ that carry the pion degrees of freedom in the hidden local symmetry approach.[30),31)] Actually, the transformation property (2·28) is the same as that for $\xi_{L,R}(x^\mu)$ if we interpret $h(x^\mu) \in U(N_f)$ as the hidden local symmetry. These fields are related to the $U(N_f)$ valued pion field $U(x^\mu)$ in the chiral Lagrangian by

$$\xi_+^{-1}(x^\mu)\xi_-(x^\mu) = U(x^\mu) \equiv e^{2i\Pi(x^\mu)/f_\pi} \ . \tag{2·30}$$

The pion field $\Pi(x^\mu)$ is identical to $\varphi^{(0)}(x^\mu)$ in (2·11) up to linear order.[*)]

Choosing $h(x^\mu)$ in (2·28) appropriately, we can choose the gauge such that

$$\xi_+^{-1}(x^\mu) = \xi_-(x^\mu) = e^{i\Pi(x^\mu)/f_\pi} \ . \tag{2·31}$$

In this gauge, the gauge potential in (2·25) can be expanded up to quadratic order in the fields as

$$\begin{aligned} A_\mu = {} & \left(\mathcal{V}_\mu + \frac{1}{2f_\pi^2}[\,\Pi, \partial_\mu \Pi\,] - \frac{i}{f_\pi}[\,\Pi, \mathcal{A}_\mu\,]\right) + \left(\mathcal{A}_\mu + \frac{i}{f_\pi}\partial_\mu \Pi - \frac{i}{f_\pi}[\,\Pi, \mathcal{V}_\mu\,]\right)\psi_0 \\ & + \sum_{n=1}^{\infty} v_\mu^n\, \psi_{2n-1} + \sum_{n=1}^{\infty} a_\mu^n\, \psi_{2n} + \cdots \ , \end{aligned} \tag{2·32}$$

with

$$\mathcal{V}_\mu \equiv \frac{1}{2}(A_{L\mu} + A_{R\mu}) \ , \ \mathcal{A}_\mu \equiv \frac{1}{2}(A_{L\mu} - A_{R\mu}) \ , \ v_\mu^n \equiv B_\mu^{(2n-1)} \ , \ a_\mu^n \equiv B_\mu^{(2n)} \ . \tag{2·33}$$

Note that the functions $\psi_n(z)$ are even and odd functions of z for odd and even values of n, respectively. This implies that v^n and a^n are vector and axial-vector mesons, respectively. As discussed in Ref. 1), the lightest vector meson, v^1, is interpreted as the ρ meson [$\rho(770)$] and the lightest axial vector meson, a^1, is interpreted as the a_1 meson [$a_1(1260)$]. The fields $v^2, v^3, \cdots$ and $a^2, a^3, \cdots$ represent the heavier vector and axial-vector mesons with the same quantum numbers: $\rho(1450), \rho(1700), \cdots$ and $a_1(1640), \cdots$, respectively.

In the $A_z = 0$ gauge, the CS term (2·8) becomes

$$\begin{aligned} S_{\rm D8}^{\rm CS} = {} & -\frac{N_c}{24\pi^2}\int_{M^4} \left(\alpha_4(d\xi_+^{-1}\xi_+, A_L) - \alpha_4(d\xi_-^{-1}\xi_-, A_R)\right) \\ & + \frac{N_c}{24\pi^2}\int_{M^4\times\mathbb{R}} \left(\omega_5(A) - \frac{1}{10}\,{\rm tr}(gdg^{-1})^5\right) \ , \end{aligned} \tag{2·34}$$

*) Here we take the pion field $\Pi(x^\mu)$ to be a Hermitian matrix, while $\varphi^{(0)}(x^\mu)$ and the vector meson fields $B_\mu^{(n)}(x^\mu)$ are anti-Hermitian.

where g is the gauge function given in (2·21), and α_4 reads

$$\alpha_4(V,A) \equiv -\frac{1}{2}\,\mathrm{tr}\left(V(AdA + dAA + A^3) - \frac{1}{2}VAVA - V^3A\right) . \qquad (2\cdot35)$$

The four-dimensional effective action of the mesons written in terms of $\xi_\pm$ (or U) and $B_\mu^{(n)}$, including the external gauge fields ($A_{L\mu}$, $A_{R\mu}$), can be obtained by substituting the gauge potential (2·25) or (2·32) into the five-dimensional Yang-Mills action (2·6) and the CS-term (2·34). This action is automatically consistent with the symmetry expressed by (2·26)–(2·29). The explicit calculation of this effective action is partly given in Ref. 1). It has been shown that the effective action of the pion is given by the Skyrme model[32] as

$$S_{\mathrm{D8}}^{\mathrm{DBI}}\Big|_{v_\mu^n=a_\mu^n=\mathcal{V}_\mu=\mathcal{A}_\mu=0} = \int d^4x \left(\frac{f_\pi^2}{4}\,\mathrm{tr}\left(U^{-1}\partial_\mu U\right)^2 + \frac{1}{32e_S^2}\,\mathrm{tr}\left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U\right]^2\right) , \qquad (2\cdot36)$$

where the pion decay constant f_π and the dimensionless parameter e_S are given by

$$f_\pi^2 \equiv \frac{4}{\pi}\kappa = \frac{1}{27\pi^4}\lambda N_c , \qquad (2\cdot37)$$

$$e_S^{-2} \equiv \kappa \int dz\, K^{-1/3}(1-\psi_0^2)^2 . \qquad (2\cdot38)$$

Also, the CS term (2·34) is identical to the WZW term in QCD that includes the pion field as well as the external gauge fields when we omit the vector meson fields $B_\mu^{(n)}$:

$$S_{\mathrm{D8}}^{\mathrm{CS}}\Big|_{v_\mu^n=a_\mu^n=0} = -\frac{N_c}{48\pi^2}\int_{M^4} Z - \frac{N_c}{240\pi^2}\int_{M^4\times\mathbb{R}} \mathrm{tr}(gdg^{-1})^5 , \qquad (2\cdot39)$$

where

$$\begin{aligned} Z = {} & \mathrm{tr}[(A_RdA_R + dA_RA_R + A_R^3)(U^{-1}A_LU + U^{-1}dU) - \mathrm{p.c.}] \\ & + \mathrm{tr}[dA_RdU^{-1}A_LU - \mathrm{p.c.}] + \mathrm{tr}[A_R(dU^{-1}U)^3 - \mathrm{p.c.}] \\ & + \frac{1}{2}\,\mathrm{tr}[(A_RdU^{-1}U)^2 - \mathrm{p.c.}] + \mathrm{tr}[UA_RU^{-1}A_LdUdU^{-1} - \mathrm{p.c.}] \\ & - \mathrm{tr}[A_RdU^{-1}UA_RU^{-1}A_LU - \mathrm{p.c.}] + \frac{1}{2}\,\mathrm{tr}[(A_RU^{-1}A_LU)^2] . \end{aligned} \qquad (2\cdot40)$$

Here "p.c." represents the terms obtained by making the exchange $A_L \leftrightarrow A_R$ and $U \leftrightarrow U^{-1}$.

In this paper, we analyze the couplings among the pions and vector mesons, including the external gauge fields in more detail. In particular, we examine whether the vector meson dominance hypothesis holds for both the DBI part and the WZW term. We analyze the DBI part in §3 and the WZW term in §4.

§3. DBI part

3.1. *The effective action*

In this subsection, we analyze the effective action obtained by inserting the mode expansion (2·32) into the action (2·6). The effective action written in terms of v_μ^n, a_μ^n and $\xi_\pm$, including the external gauge fields ($A_{L\mu}$, $A_{R\mu}$), is given in Appendix A. Here we consider some of the couplings read off of the action.

It is useful to write the action as

$$S_{\rm D8}^{\rm DBI} = \int d^4x\, \mathcal{L}_{\rm div} + \sum_{j\geq 2} \int d^4x\, \mathcal{L}_j \ , \tag{3·1}$$

where $\mathcal{L}_j$ contains the terms of order j in the fields $\Pi, v_\mu^n, a_\mu^n, \mathcal{V}_\mu$ and $\mathcal{A}_\mu$. The quantity $\mathcal{L}_{\rm div}$ contains the divergent terms that result from the non-normalizable modes $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$. The explicit form of $\mathcal{L}_{\rm div}$ is given in (A·16).

For the quadratic terms, we find

$$\begin{aligned}\mathcal{L}_2 = &\frac{1}{2}\,{\rm tr}\left(\partial_\mu v_\nu^n - \partial_\nu v_\mu^n\right)^2 + \frac{1}{2}\,{\rm tr}\left(\partial_\mu a_\nu^n - \partial_\nu a_\mu^n\right)^2 \\ &+ a_{\mathcal{V}v^n}\,{\rm tr}\left(\partial^\mu\mathcal{V}^\nu - \partial^\nu\mathcal{V}^\mu\right)\left(\partial_\mu v_\nu^n - \partial_\nu v_\mu^n\right) + a_{\mathcal{A}a^n}\,{\rm tr}\left(\partial^\mu\mathcal{A}^\nu - \partial^\nu\mathcal{A}^\mu\right)\left(\partial_\mu a_\nu^n - \partial_\nu a_\mu^n\right) \\ &+ {\rm tr}\left(i\partial_\mu\Pi + f_\pi\mathcal{A}_\mu\right)^2 + m_{v^n}^2\,{\rm tr}\left(v_\mu^n\right)^2 + m_{a^n}^2\,{\rm tr}\left(a_\mu^n\right)^2 \ ,\end{aligned} \tag{3·2}$$

where

$$m_{v^n}^2 \equiv \lambda_{2n-1} \ , \quad m_{a^n}^2 \equiv \lambda_{2n} \ , \tag{3·3}$$

$$a_{\mathcal{V}v^n} \equiv \kappa \int dz\, K^{-1/3}\psi_{2n-1} \ , \quad a_{\mathcal{A}a^n} \equiv \kappa \int dz\, K^{-1/3}\psi_{2n}\psi_0 \ , \tag{3·4}$$

and we have used the fact that the pion decay constant f_π is given by (2·37). Here and in the following, the summation symbol "$\sum_{n=1}^\infty$" is often omitted for notational simplicity.

In order to diagonalize the kinetic term, we define

$$\widetilde{v}_\mu^n \equiv v_\mu^n + a_{\mathcal{V}v^n}\mathcal{V}_\mu \ , \tag{3·5}$$
$$\widetilde{a}_\mu^n \equiv a_\mu^n + a_{\mathcal{A}a^n}\mathcal{A}_\mu \ . \tag{3·6}$$

Then, (3·2) becomes

$$\begin{aligned}\mathcal{L}_2 = &\frac{1}{2}\,{\rm tr}\left(\partial_\mu \widetilde{v}_\nu^n - \partial_\nu \widetilde{v}_\mu^n\right)^2 + \frac{1}{2}\,{\rm tr}\left(\partial_\mu \widetilde{a}_\nu^n - \partial_\nu \widetilde{a}_\mu^n\right)^2 + {\rm tr}\left(i\partial_\mu\Pi + f_\pi\mathcal{A}_\mu\right)^2 \\ &+ m_{v^n}^2\,{\rm tr}\left(\widetilde{v}_\mu^n - a_{\mathcal{V}v^n}\mathcal{V}_\mu\right)^2 + m_{a^n}^2\,{\rm tr}\left(\widetilde{a}_\mu^n - a_{\mathcal{A}a^n}\mathcal{A}_\mu\right)^2 \ .\end{aligned} \tag{3·7}$$

Here, corrections to the kinetic terms of $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ in $\mathcal{L}_{\rm div}$ are omitted.

We segregate the cubic terms $\mathcal{L}_3$ into terms of equal orders in the pion field $\Pi(x^\mu)$:

$$\mathcal{L}_3 = \mathcal{L}_3|_{\pi^0} + \mathcal{L}_3|_{\pi^1} + \mathcal{L}_3|_{\pi^2} \ . \tag{3·8}$$

Note that $\mathcal{L}_3|_{\pi^3}$ does not exist because of parity symmetry.

Let us first examine $\mathcal{L}_3|_{\pi^2}$, which is relevant to the electromagnetic form factor of the pion:

$$\begin{aligned}\mathcal{L}_3|_{\pi^2} &= \frac{b_{\mathcal{V}\pi\pi}}{f_\pi^2}\,\mathrm{tr}\,((\partial_\mu\mathcal{V}_\nu-\partial_\nu\mathcal{V}_\mu)[\partial^\mu\Pi,\partial^\nu\Pi\,]) \\ &\quad+\frac{b_{v^n\pi\pi}}{f_\pi^2}\,\mathrm{tr}\,\big((\partial_\mu v_\nu^n-\partial_\nu v_\mu^n)[\partial^\mu\Pi,\partial^\nu\Pi\,]\big)-2\,\mathrm{tr}\,(\mathcal{V}_\mu[\,\Pi,\partial^\mu\Pi\,]) \qquad (3\cdot9)\\ &=\frac{1}{f_\pi^2}\,(b_{\mathcal{V}\pi\pi}-a_{\mathcal{V}v^n}b_{v^n\pi\pi})\,\mathrm{tr}\,((\partial_\mu\mathcal{V}_\nu-\partial_\nu\mathcal{V}_\mu)[\partial^\mu\Pi,\partial^\nu\Pi\,]) \\ &\quad+\frac{b_{v^n\pi\pi}}{f_\pi^2}\,\mathrm{tr}\,\big((\partial_\mu\widetilde{v}_\nu^n-\partial_\nu\widetilde{v}_\mu^n)[\partial^\mu\Pi,\partial^\nu\Pi\,]\big)-2\,\mathrm{tr}\,(\mathcal{V}_\mu[\,\Pi,\partial^\mu\Pi\,])\ , \qquad (3\cdot10)\end{aligned}$$

where

$$b_{\mathcal{V}\pi\pi}\equiv\kappa\int dz\,K^{-1/3}\left(1-\psi_0^2\right)\ , \qquad (3\cdot11)$$

$$b_{v^n\pi\pi}\equiv\kappa\int dz\,K^{-1/3}\psi_{2n-1}\left(1-\psi_0^2\right)\ . \qquad (3\cdot12)$$

Note here that the coefficient of the first term in (3·10) is zero. Actually, using the completeness relation

$$\kappa\sum_{n=1}^{\infty}K^{-1/3}(z')\psi_n(z)\psi_n(z')=\delta(z-z')\ , \qquad (3\cdot13)$$

we can verify that

$$\sum_{n=1}^{\infty}a_{\mathcal{V}v^n}b_{v^n\pi\pi}=b_{\mathcal{V}\pi\pi}\ . \qquad (3\cdot14)$$

Therefore, (3·10) becomes

$$\mathcal{L}_3|_{\pi^2}=\frac{b_{v^n\pi\pi}}{f_\pi^2}\,\mathrm{tr}\,\big((\partial_\mu\widetilde{v}_\nu^n-\partial_\nu\widetilde{v}_\mu^n)[\partial^\mu\Pi,\partial^\nu\Pi\,]\big)-2\,\mathrm{tr}\,(\mathcal{V}_\mu[\,\Pi,\partial^\mu\Pi\,])\ . \qquad (3\cdot15)$$

In order to compare the effective action with that given in the literature (e.g. Ref. 31)) we rewrite the Lagrangian using

$$\widehat{v}_\mu^n\equiv\widetilde{v}_\mu^n+\frac{b_{v^n\pi\pi}}{2f_\pi^2}\,[\Pi,\partial_\mu\Pi]\ , \qquad (3\cdot16)$$

and remove the term of the form $\mathrm{tr}\,\big((\partial_\mu\widetilde{v}_\nu^n-\partial_\nu\widetilde{v}_\mu^n)[\partial^\mu\Pi,\partial^\nu\Pi]\big)$. Then, we obtain

$$\begin{aligned}\mathcal{L}_2+\mathcal{L}_3|_{\pi^2} &= \frac12\,\mathrm{tr}\,\big(\partial_\mu\widehat{v}_\nu^n-\partial_\nu\widehat{v}_\mu^n\big)^2+\frac12\,\mathrm{tr}\,\big(\partial_\mu\widetilde{a}_\nu^n-\partial_\nu\widetilde{a}_\mu^n\big)^2+\mathrm{tr}\,(i\partial_\mu\Pi+f_\pi\mathcal{A}_\mu)^2 \\ &\quad+m_{a^n}^2\,\mathrm{tr}\,\big(\widetilde{a}_\mu^n-a_{\mathcal{A}a^n}\mathcal{A}_\mu\big)^2+m_{v^n}^2\,\mathrm{tr}\,\big(\widehat{v}_\mu^n-a_{\mathcal{V}v^n}\mathcal{V}_\mu\big)^2 \\ &\quad-\frac{m_{v^n}^2b_{v^n\pi\pi}}{f_\pi^2}\,\mathrm{tr}\,\big(\widehat{v}_\mu^n[\Pi,\partial^\mu\Pi]\big)+\left(\frac{m_{v^n}^2b_{v^n\pi\pi}a_{\mathcal{V}v^n}}{f_\pi^2}-2\right)\mathrm{tr}\,(\mathcal{V}_\mu[\Pi,\partial^\mu\Pi]) \\ &\quad+\frac{m_{v^n}^2b_{v^n\pi\pi}^2}{4f_\pi^4}\,\mathrm{tr}\,[\Pi,\partial_\mu\Pi]^2-\frac{b_{v^n\pi\pi}^2}{2f_\pi^4}\,\mathrm{tr}\,[\partial_\mu\Pi,\partial_\nu\Pi]^2\ . \qquad (3\cdot17)\end{aligned}$$

Here, it is very important to note the relation

$$\sum_{n=1}^{\infty} m_{v^n}^2 b_{v^n\pi\pi} a_{\mathcal{V}v^n} = 2f_\pi^2 \ , \tag{3·18}$$

which follows straightforwardly from the completeness condition (3·13) and the equation (2·12). This shows that the $\pi\pi\mathcal{V}$ coupling in (3·17) vanishes. As shown in §3.5, this fact is important with regard to the vector meson dominance in the electromagnetic form factor of the pion. Similarly, we can also show the following relations:*)

$$\sum_{n=1}^{\infty} m_{v^n}^2 b_{v^n\pi\pi}^2 = \frac{4}{3} f_\pi^2 \ , \quad \sum_{n=1}^{\infty} b_{v^n\pi\pi}^2 = e_S^{-2} \ . \tag{3·19}$$

Using (3·18) and (3·19), the Lagrangian (3·17) can be rewritten as

$$\begin{aligned}\mathcal{L}_2 + \mathcal{L}_3|_{\pi^2} = &\frac{1}{2}\,\mathrm{tr}\left(\partial_\mu \widehat{v}_\nu^n - \partial_\nu \widehat{v}_\mu^n\right)^2 + \frac{1}{2}\,\mathrm{tr}\left(\partial_\mu \widetilde{a}_\nu^n - \partial_\nu \widetilde{a}_\mu^n\right)^2 + \mathrm{tr}\left(i\partial_\mu \varPi + f_\pi \mathcal{A}_\mu\right)^2 \\ &+ m_{a^n}^2\,\mathrm{tr}\left(\widetilde{a}_\mu^n\right)^2 - 2g_{a^n}\,\mathrm{tr}\left(\widetilde{a}_\mu^n \mathcal{A}^\mu\right) + m_{a^n}^2 a_{\mathcal{A}a^n}^2\,\mathrm{tr}\left(\mathcal{A}_\mu\right)^2 \\ &+ m_{v^n}^2\,\mathrm{tr}\left(\widehat{v}_\mu^n\right)^2 - 2g_{v^n}\,\mathrm{tr}\left(\widehat{v}_\mu^n \mathcal{V}^\mu\right) + m_{v^n}^2 a_{\mathcal{V}v^n}^2\,\mathrm{tr}\left(\mathcal{V}_\mu\right)^2 \\ &- 2g_{v^n\pi\pi}\,\mathrm{tr}\left(\widehat{v}_\mu^n [\varPi, \partial^\mu \varPi]\right) \\ &+ \frac{1}{3f_\pi^2}\,\mathrm{tr}\,[\varPi, \partial_\mu \varPi]^2 - \frac{1}{2e_S^2 f_\pi^4}\,\mathrm{tr}\,[\partial_\mu \varPi, \partial_\nu \varPi]^2 \ ,\end{aligned} \tag{3·20}$$

where

$$g_{a^n} \equiv m_{a^n}^2 a_{\mathcal{A}a^n} \ , \quad g_{v^n} \equiv m_{v^n}^2 a_{\mathcal{V}v^n} \ , \quad g_{v^n\pi\pi} \equiv \frac{b_{v^n\pi\pi} m_{v^n}^2}{2f_\pi^2} \ . \tag{3·21}$$

Here, we can verify that g_{v^n} and g_{a^n} are equal to the decay constants of the vector meson v^n and the axial-vector meson a^n, respectively, by showing that

$$\langle 0|J_\mu^{(V)a}(0)|v^{nb}\rangle = g_{v^n}\delta^{ab}\epsilon_\mu \ , \quad \langle 0|J_\mu^{(A)a}(0)|a^{nb}\rangle = g_{a^n}\delta^{ab}\epsilon_\mu \ , \tag{3·22}$$

where the quantities $J_\mu^{(V,A)}$ are the conserved vector and axial-vector currents coupled with $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$, respectively, ϵ_μ are the polarizations of the vector mesons, and the indices a and b are associated with the generators T^a of $U(N_f)$ as

$$v_\mu^n = i v_\mu^{na} T^a \ , \quad a_\mu^n = i a_\mu^{na} T^a \ , \quad \mathcal{V}_\mu = i\mathcal{V}_\mu^a T^a \ , \quad \mathcal{A}_\mu = i\mathcal{A}_\mu^a T^a \ . \tag{3·23}$$

Note that the decay constants can be recast as

$$\begin{aligned} g_{v^n} &= -\kappa \int dz\, \partial_z \left(K\partial_z \psi_{2n-1}\right) = -2\kappa (K\partial_z\psi_{2n-1})\Big|_{z=+\infty} \ , \\ g_{a^n} &= -\kappa \int dz\, \psi_0\, \partial_z \left(K\partial_z \psi_{2n}\right) = -2\kappa (K\partial_z\psi_{2n})\Big|_{z=+\infty} \ , \end{aligned} \tag{3·24}$$

*) The sum rules (3·14), (3·18) and (3·19) for a closely related five-dimensional model are also derived in Ref. 19) using a similar method.

where we have used (2·12). This shows that the decay constants g_{v^n} and g_{a^n} are fixed uniquely by the asymptotic behavior of the mode functions ψ_{2n-1} and ψ_{2n}, respectively. (See Ref. 2) for analogous formulas.)

The terms linear in Π are

$$\begin{aligned}\mathcal{L}_3|_{\pi^1} = \frac{2i}{f_\pi}\,\mathrm{tr}\Big[&\partial^\mu(\partial_\mu\mathcal{V}_\nu-\partial_\nu\mathcal{V}_\mu)[\Pi,\mathcal{A}^\nu]\,b_{\mathcal{V}\pi\pi}+\partial^\mu(\partial_\mu v^n_\nu-\partial_\nu v^n_\mu)[\Pi,\mathcal{A}^\nu]\,b_{v^n\pi\pi}\\ &+\partial^\mu(\partial_\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu)[\Pi,v^{n\nu}]\,(b_{v^n\pi\pi}-a_{\mathcal{V}v^n})\\ &+\partial^\mu(\partial_\mu\mathcal{V}_\nu-\partial_\nu\mathcal{V}_\mu)[\Pi,a^{m\nu}]\,(-a_{\mathcal{A}a^m})+\partial^\mu(\partial_\mu v^n_\nu-\partial_\nu v^n_\mu)[\Pi,a^{m\nu}]\,(-c_{v^na^m\pi})\\ &+\partial^\mu(\partial_\mu a^m_\nu-\partial_\nu a^m_\mu)[\Pi,v^{n\nu}]\,(-c_{v^na^m\pi})\Big]\\ &-2if_\pi\,\mathrm{tr}\,(\mathcal{A}_\mu[\Pi,\mathcal{V}^\mu])\ ,\end{aligned}\tag{3·25}$$

up to total derivative terms. Here, we have defined

$$c_{v^na^m\pi}\equiv\kappa\int dz\,K^{-1/3}\psi_0\psi_{2n-1}\psi_{2m}\ .\tag{3·26}$$

Using the sum rule (3·14) and the relations

$$\sum_{n=1}^{\infty}a_{\mathcal{V}v^n}c_{v^na^m\pi}=a_{\mathcal{A}a^m}\ ,\tag{3·27}$$

$$\sum_{m=1}^{\infty}a_{\mathcal{A}a^m}c_{v^na^m\pi}=a_{\mathcal{V}v^n}-b_{v^n\pi\pi}\ ,\tag{3·28}$$

which also follow from the completeness condition (3·13), the Lagrangian (3·25) can be rewritten as

$$\begin{aligned}\mathcal{L}_3|_{\pi^1}= \frac{2i}{f_\pi}\,\mathrm{tr}\Big[&\partial^\mu(\partial_\mu\widehat{v}^n_\nu-\partial_\nu\widehat{v}^n_\mu)[\Pi,\mathcal{A}^\nu]\,a_{\mathcal{V}v^n}+\partial^\mu(\partial_\mu\widetilde{a}^m_\nu-\partial_\nu\widetilde{a}^m_\mu)[\Pi,\mathcal{V}^\nu]\,a_{\mathcal{A}a^m}\\ &+\partial^\mu(\partial_\mu\widehat{v}^n_\nu-\partial_\nu\widehat{v}^n_\mu)[\Pi,\widetilde{a}^{m\nu}]\,(-c_{v^na^m\pi})+\partial^\mu(\partial_\mu\widetilde{a}^m_\nu-\partial_\nu\widetilde{a}^m_\mu)[\Pi,\widehat{v}^{n\nu}]\,(-c_{v^na^m\pi})\Big]\\ &-2if_\pi\,\mathrm{tr}\,(\mathcal{A}_\mu[\Pi,\mathcal{V}^\mu])+(\text{quartic terms})\ .\end{aligned}\tag{3·29}$$

Finally, we consider the rest of $\mathcal{L}_3$, which contains no pion field $\Pi(x^\mu)$. It can be shown that

$$\begin{aligned}&\mathcal{L}_3|_{\pi^0}\\ &=\mathrm{tr}\Bigg((\partial^\mu\mathcal{V}^\nu-\partial^\nu\mathcal{V}^\mu)\\ &\qquad\times\Big\{([\mathcal{V}_\mu,v^n_\nu]-[\mathcal{V}_\nu,v^n_\mu])\,a_{\mathcal{V}v^n}+([\mathcal{A}_\mu,a^n_\nu]-[\mathcal{A}_\nu,a^n_\mu])\,a_{\mathcal{A}a^n}+[v^n_\mu,v^n_\nu]+[a^n_\nu,a^n_\mu]\Big\}\\ &\qquad+(\partial^\mu v^{l\nu}-\partial^\nu v^{l\mu})\Big\{[\mathcal{V}_\mu,\mathcal{V}_\nu]\,a_{\mathcal{V}v^l}+[\mathcal{A}_\mu,\mathcal{A}_\nu](a_{\mathcal{V}v^l}-b_{v^l\pi\pi})+([\mathcal{V}_\mu,v^l_\nu]-[\mathcal{V}_\nu,v^l_\mu])\end{aligned}$$

$$
+\left([\mathcal{A}_\mu, a_\nu^m]-[\mathcal{A}_\nu, a_\mu^m]\right) c_{v^n a^m \pi}+[v_\mu^m, v_\nu^n]\, g_{v^l v^m v^n}+[a_\mu^m, a_\nu^n]\, g_{v^l a^m a^n}\Big\}
$$
$$
+\left(\partial^\mu \mathcal{A}^\nu-\partial^\nu \mathcal{A}^\mu\right)\Big\{\left([\mathcal{V}_\mu, a_\nu^n]-[\mathcal{V}_\nu, a_\mu^n]\right) a_{\mathcal{A} a^n}
$$
$$
+\left([\mathcal{A}_\mu, v_\nu^n]-[\mathcal{A}_\nu, v_\mu^n]\right)(a_{\mathcal{V} v^n}-b_{v^n\pi\pi})+\left([v_\mu^n, a_\nu^m]-[v_\nu^n, a_\mu^m]\right) c_{v^n a^m \pi}\Big\}
$$
$$
+\left(\partial^\mu a^{n\nu}-\partial^\nu a^{n\mu}\right)\Big\{\left([\mathcal{V}_\mu, \mathcal{A}_\nu]-[\mathcal{V}_\nu, \mathcal{A}_\mu]\right) a_{\mathcal{A} a^n}+\left([\mathcal{V}_\mu, a_\nu^n]-[\mathcal{V}_\nu, a_\mu^n]\right)
$$
$$
\left.+\left([\mathcal{A}_\mu, v_\nu^m]-[\mathcal{A}_\nu, v_\mu^m]\right) c_{v^m a^n \pi}+\left([v_\mu^l, a_\nu^m]-[v_\nu^l, a_\mu^m]\right) g_{v^l a^m a^n}\Big\}\right)\ , \qquad (3\cdot30)
$$

where

$$
\begin{aligned}
g_{v^l v^m v^n} &\equiv \kappa \int dz\, K^{-1/3} \psi_{2l-1}\psi_{2m-1}\psi_{2n-1}\ ,\\
g_{v^l a^m a^n} &\equiv \kappa \int dz\, K^{-1/3} \psi_{2l-1}\psi_{2m}\psi_{2n}\ .
\end{aligned} \qquad (3\cdot31)
$$

If we rewrite (3·30) in terms of $\widetilde{v}_\mu^n$ and $\widetilde{a}_\mu^n$ defined in (3·5) and (3·6), we obtain

$$
\mathcal{L}_3|_{\pi^0} = \mathrm{tr}\left(g_{v^l v^m v^n}\left(\partial^\mu \widetilde{v}^{l\nu}-\partial^\nu \widetilde{v}^{l\mu}\right)\left[\widetilde{v}_\mu^m, \widetilde{v}_\nu^n\right]+g_{v^l a^m a^n}\left(\partial^\mu \widetilde{v}^{l\nu}-\partial^\nu \widetilde{v}^{l\mu}\right)\left[\widetilde{a}_\mu^m, \widetilde{a}_\nu^n\right]\right.
$$
$$
\left.+g_{v^l a^m a^n}\left(\partial^\mu \widetilde{a}^{n\nu}-\partial^\nu \widetilde{a}^{n\mu}\right)\left\{\left[\widetilde{v}_\mu^l, \widetilde{a}_\nu^m\right]-\left[\widetilde{v}_\nu^l, \widetilde{a}_\mu^m\right]\right\}\right)\ . \qquad (3\cdot32)
$$

Here, corrections to the cubic term in $\mathcal{L}_{\rm div}$ are omitted. From (3·32), we see that the direct cubic couplings of the vector mesons v_μ^n and a_μ^n to the external gauge fields $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ disappear.

For $\mathcal{L}_4$, we focus on the quartic terms in the pion field. As explained in §2, the low energy effective action of the pion is given by the Skyrme model (2·36). Then, it follows from (2·36) that

$$
\mathcal{L}_4|_{\pi^4} = -\frac{1}{3f_\pi^2}\,\mathrm{tr}\,[\Pi, \partial_\mu \Pi]^2+\frac{1}{2e_S^2 f_\pi^4}\,\mathrm{tr}\,[\partial_\mu \Pi, \partial_\nu \Pi]^2\ . \qquad (3\cdot33)
$$

Note that (3·33) exactly cancels the $\mathcal{O}(\Pi^4)$ terms (the last two terms) in (3·20).

3.2. *Numerical results*

Here, we summarize the numerically obtained values of the coupling constants to provide a rough estimate of the physical quantities. Listed below are the numerical estimates for some of the masses and coupling constants defined in (3·3) and (3·21) in units for which $M_{\rm KK}=1$.

n	$m_{v^n}^2$	$\kappa^{-1/2} g_{v^n}$	$\kappa^{1/2} g_{v^n\pi\pi}$	$m_{a^n}^2$	$\kappa^{-1/2} g_{a^n}$
1	0.669	2.11	0.415	1.57	5.02
2	2.87	9.10	−0.109	4.55	14.4
3	6.59	20.8	0.0160	9.01	28.3
4	11.8	37.1	−0.00408	15.0	46.9

(3·34)

These are obtained by solving the equation (2·12) numerically using the shooting method, as in Ref. 1).

The coupling constants given in (2·37), (2·38) and (3·11) are easily calculated as

$$f_\pi^2 \simeq 1.27 \cdot \kappa \ , \quad e_S^{-2} \simeq 2.51 \cdot \kappa \ , \quad b_{\mathcal{V}\pi\pi} \simeq 4.69 \cdot \kappa \ . \tag{3·35}$$

It is, however, important to keep in mind that we should not take these numerical values too seriously, because the approximation made in our analysis is very crude. As discussed in Ref. 1), the present model deviates from realistic QCD above the energy scale of $M_{\rm KK}$, which is the same as the mass scale of the vector mesons. Furthermore, all the quarks are assumed to be massless, and the supergravity description and the probe approximation are valid only when $N_c \gg N_f$ and $\lambda \gg 1$. In the following subsections, we compare the numerical values of the coupling constants obtained in our model with the experimental values in order to get some idea of whether or not we are on the right track. It would be interesting to improve the approximation in order to make more accurate predictions.

3.3. *The Skyrme term*

The second term in (2·36), which is called the Skyrme term, can be written as

$$\frac{1}{32e_S^2}\,\mathrm{tr}\left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U\right]^2 = L_1 P_1 + L_2 P_2 + L_3 P_3 \tag{3·36}$$

for $N_f = 3$, where

$$\begin{aligned} &P_1 \equiv \left[\mathrm{tr}(\partial_\mu U^{-1}\partial^\mu U)\right]^2 \ , \quad P_2 \equiv \mathrm{tr}(\partial_\mu U^{-1}\partial_\nu U)\,\mathrm{tr}(\partial^\mu U^{-1}\partial^\nu U) \ , \\ &P_3 \equiv \mathrm{tr}(\partial_\mu U^{-1}\partial^\mu U\partial_\nu U^{-1}\partial^\nu U) \ , \end{aligned} \tag{3·37}$$

and

$$L_1 = \frac{1}{32e_S^2} \ , \quad L_2 = \frac{1}{16e_S^2} \ , \quad L_3 = -\frac{3}{16e_S^2} \ . \tag{3·38}$$

For the case $N_f = 2$, we have the additional relation $P_3 = \frac{1}{2}P_1$. The experimental values for the coefficients L_i $(i = 1, 2, 3)$ (at the scale of the ρ meson mass) are given in Ref. 33) as

$$\begin{aligned} L_1|_{\rm exp} &\simeq (0.4 \pm 0.3) \times 10^{-3}, \\ L_2|_{\rm exp} &\simeq (1.4 \pm 0.3) \times 10^{-3}, \\ L_3|_{\rm exp} &\simeq (-3.5 \pm 1.1) \times 10^{-3}. \end{aligned} \tag{3·39}$$

Our result (3·38) is roughly consistent with experimental results in the case $\kappa \simeq (7-9) \times 10^{-3}$. Note that this value of κ is also consistent with that obtained by combining the experimental values $f_\pi|_{\rm exp} \simeq 92.4$ MeV and $m_\rho|_{\rm exp} \simeq 776$ MeV with the numerical results (3·34) and (3·35):

$$M_{\rm KK} \simeq 949 \text{ MeV} \ , \quad \kappa \simeq 7.45 \times 10^{-3} \ . \tag{3·40}$$

3.4. *KSRF relations*

Here we examine the KSRF relations,[22),23)]

$$g_\rho = 2 g_{\rho\pi\pi} f_\pi^2 \qquad \text{(KSRF(I))} \ , \tag{3·41}$$

$$m_\rho^2 = 2 g_{\rho\pi\pi}^2 f_\pi^2 \qquad \text{(KSRF(II))} \ . \tag{3·42}$$

These two relations lead to

$$g_\rho g_{\rho\pi\pi} = m_\rho^2 \ . \tag{3·43}$$

Then, using the experimental values $g_{\rho\pi\pi}|_{\rm exp} \simeq 5.99$ and $g_\rho|_{\rm exp} \simeq 0.121\ \text{GeV}^2$,[34)] we obtain

$$\left.\frac{4 g_{\rho\pi\pi}^2 f_\pi^2}{m_\rho^2}\right|_{\rm exp} \simeq 2.03 \ , \qquad \left.\frac{g_\rho g_{\rho\pi\pi}}{m_\rho^2}\right|_{\rm exp} \simeq 1.20 \ , \tag{3·44}$$

which show that the relations (3·42) and (3·43) are satisfied to within 20%.

The corresponding values in our model can be estimated by using the numerical values listed in §3.2. The result is

$$\frac{4 g_{v^1\pi\pi}^2 f_\pi^2}{m_{v^1}^2} \simeq 1.31 \ , \qquad \frac{g_{v^1} g_{v^1\pi\pi}}{m_{v^1}^2} \simeq 1.31 \ . \tag{3·45}$$

Note that these values are independent of the parameters in the model. If we use the values of $M_{\rm KK}$ and κ in (3·40), we obtain

$$g_{v^1\pi\pi} \simeq 4.81 \ , \quad g_{v^1} \simeq 0.164\ \text{GeV}^2 \ . \tag{3·46}$$

Remarkably, it is found that the relations given in (3·45) are equivalent to picking out the dominant contribution from the following sum rules:

$$\sum_{n=1}^{\infty} \frac{4 g_{v^n\pi\pi}^2 f_\pi^2}{m_{v^n}^2} = \frac{4}{3} \ , \tag{3·47}$$

$$\sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n\pi\pi}}{m_{v^n}^2} = 1 \ . \tag{3·48}$$

The relation (3·47) is equivalent to the first relation in (3·19), and (3·48) follows from (3·18). The sum rules given in (3·47) and (3·48) were first reported in Refs. 16) and 2), respectively, and have been shown to be satisfied in general five-dimensional models. As a check, using the numerical results for $n = 1, 2, 3$ and 4 given in (3·34), the left-hand sides of (3·47) and (3·48) are evaluated as

$$\sum_{n=1}^{\infty} \frac{4 g_{v^n\pi\pi}^2 f_\pi^2}{m_{v^n}^2} \simeq 1.31 + 0.0210 + 0.000197 + 0.00000717 + \cdots \simeq 1.33 \ , \tag{3·49}$$

$$\sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n\pi\pi}}{m_{v^n}^2} \simeq 1.31 - 0.346 + 0.0505 - 0.0128 + \cdots \simeq 1.00 \ , \tag{3·50}$$

from which we see that the contribution of the lightest vector meson $\widehat{v}^1$ (the ρ meson) dominates the sum.

3.5. *Electromagnetic form factors*

Let us consider the pion form factor $F_\pi(p^2)$ defined by

$$\langle \pi^a(p)|J_\mu^{(V)c}(0)|\pi^b(p')\rangle = f^{abc}(p+p')_\mu F_\pi((p-p')^2) \ , \tag{3·51}$$

where f^{abc} is the structure constant of $U(N_f)$.

Combining the $\widehat{v}^n\mathcal{V}$ and $\widehat{v}^n\pi\pi$ vertices in (3·20) as well as the $\widehat{v}^n$ propagators, as depicted in Fig. 1, we obtain

$$F_\pi(k^2) = \sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n\pi\pi}}{k^2 + m_{v^n}^2} \ . \tag{3·52}$$

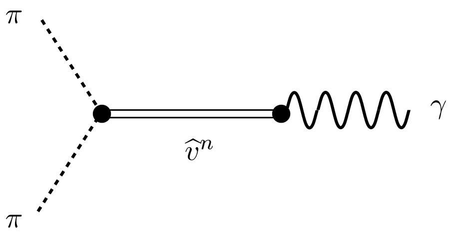

Fig. 1. Pion form factor.

Crucial in this computation is the relation (3·18), which ensures that the direct $\mathcal{V}\pi\pi$ coupling in (3·17) vanishes. As a consistency check of (3·52), we note that

$$F_\pi(0) = \sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n\pi\pi}}{m_{v^n}^2} = 1 \ , \tag{3·53}$$

due to (3·48).

We thus see that our model possesses vector meson dominance for the pion form factor (3·52), because the form factor is saturated by the exchange of vector mesons. This fact was first pointed out in Ref. 2), in which (3·52) is derived by taking the continuum limit in the discretized version of the five-dimensional model. General analyses of the vector meson dominance in various holographic models are also given in Refs. 9), 15) and 16). As we have seen in §3.4, the sum (3·48) is dominated by the ρ meson. Hence, our model exhibits ρ meson dominance to a good approximation in the form factor $F_\pi(k^2)$. Manifestation of the vector meson dominance in the WZW term is examined in the next section, and more general consideration is given in §5.

By expanding the form factor in k^2 as

$$F_\pi(k^2) = 1 - \sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n\pi\pi}}{m_{v^n}^4} k^2 + \mathcal{O}(k^4) \ , \tag{3·54}$$

we can extract the charge radius of the pion as

$$\langle r^2 \rangle^{\pi^\pm} = 6 \sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n\pi\pi}}{m_{v^n}^4} \ . \tag{3·55}$$

Using the sum rule (3·14), we can show

$$\sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n\pi\pi}}{m_{v^n}^4} = \frac{1}{2f_\pi^2} \sum_{n=1}^{\infty} a_{\mathcal{V}v^n} b_{v^n\pi\pi} = \frac{\pi}{8}\kappa^{-1} b_{\mathcal{V}\pi\pi} \simeq 1.84 \cdot M_{\rm KK}^{-2} \ , \tag{3·56}$$

and hence

$$\langle r^2 \rangle^{\pi^\pm} = \frac{3\pi}{4}\kappa^{-1} b_{\mathcal{V}\pi\pi} \simeq 11.0 \cdot M_{\rm KK}^{-2} \ , \tag{3·57}$$

where we have recovered $M_{\rm KK}$ in the last expression. If we use the value of $M_{\rm KK}$ given in (3·40), we have

$$\langle r^2 \rangle^{\pi^\pm} \simeq (0.690 \text{ fm})^2 \ . \tag{3·58}$$

The experimental value for this is[34)]

$$\langle r^2 \rangle^{\pi^\pm} \Big|_{\rm exp} \simeq (0.672 \text{ fm})^2 \ . \tag{3·59}$$

It is also interesting to note the sum rules

$$\sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n v^k v^l}}{m_{v^n}^2} = \delta_{kl} \ , \quad \sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n a^k a^l}}{m_{v^n}^2} = \delta_{kl} \ , \tag{3·60}$$

which are the analogs of (3·53) for (axial-)vector mesons. If the sums in both (3·53) and (3·60) (for $k = l$) are dominated by the contribution of the ρ meson ($n = 1$), we obtain the approximate relation

$$g_{\rho\pi\pi} \simeq g_{\rho v^m v^m} \simeq g_{\rho a^m a^m} \simeq \frac{m_\rho^2}{g_\rho} \ , \tag{3·61}$$

which leads to the universality of the ρ meson couplings,

$$g_{\rho HH} \simeq \frac{m_\rho^2}{g_\rho} \ , \quad (H = \pi, v^m, a^m) \tag{3·62}$$

as discussed in Ref. 15).

In order to determine the extent to which the relation (3·62) is valid, we list some numerical results for $g_{\rho v^n v^n}$ and $g_{\rho a^n a^n}$:

n	$\kappa^{1/2} g_{\rho v^n v^n}$	$\kappa^{1/2} g_{\rho a^n a^n}$
1	0.447	0.286
2	0.269	0.257
3	0.252	0.249
4	0.247	0.246

(3·63)

As argued in Ref. 1), $g_{\rho\pi\pi}$ and $g_{\rho\rho\rho}$ are nearly equal. However, these two values are not in good agreement with those of $g_{\rho v^n v^n}$ $(n \geq 2)$ and $g_{\rho v^n v^n}$ $(n \geq 1)$, among which the universality holds to a good approximation. The contributions from the first five terms in the summations in (3·60) for $k = l = 1, 2$ are estimated as

$$\sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n v^1 v^1}}{m_{v^n}^2} \simeq 1.41 - 0.464 + 0.0581 - 0.00116 + 0.000845 + \cdots \simeq 1.00 \ , \tag{3·64}$$

$$\sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n v^2 v^2}}{m_{v^n}^2} \simeq 0.846 + 0.135 + 0.381 - 0.466 + 0.0993 + \cdots \simeq 0.995 \ , \tag{3·65}$$

$$\sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n a^1 a^1}}{m_{v^n}^2} \simeq 0.902 + 0.467 - 0.453 - 0.0822 + 0.00273 + \cdots \simeq 1.00 \ , \tag{3·66}$$

$$\sum_{n=1}^{\infty} \frac{g_{v^n} g_{v^n a^2 a^2}}{m_{v^n}^2} \simeq 0.810 + 0.119 + 0.104 + 0.316 - 0.468 + \cdots \simeq 0.882 \ . \tag{3·67}$$

3.6. $a_1 \to \pi\gamma$ *and* $a_1 \to \pi\rho$ *decay*

The a_1 meson is the lightest axial-vector meson of $J^{PC} = 1^{++}$. In our model, it is identified with $B_\mu^{(n=2)} = a_\mu^1$. Here we discuss the decay amplitudes of $a_1 \to \pi\gamma$ and $a_1 \to \pi\rho$.

First we show that the decay amplitude of $a_1 \to \pi\gamma$ computed from the effective action in §3.1 vanishes. More generally, we can show that the decay amplitude of $a^m \to \pi\gamma$ vanishes for every $m \geq 1$, where a^m is the axial-vector meson $a_\mu^m = B_\mu^{(2m)}$. The relevant diagrams for this decay amplitude, depicted in Fig. 2, are (1) the direct coupling of $\widetilde{a}^m \pi \mathcal{V}$ in (3·29), which yields an amplitude proportional to $a_{\mathcal{A}a^m}$, and (2) the $\widetilde{a}^m \pi \widehat{v}^n$ vertex in (3·29) accompanied by the $\widehat{v}^n$-$\mathcal{V}$ transition in (3·17). This amplitude is found to be proportional to

$$-\sum_{n\geq 1} c_{v^n a^m \pi} \frac{g_{v^n}}{m_{v^n}^2} = -\sum_{n\geq 1} c_{v^n a^m \pi} a_{\mathcal{V} v^n} = -a_{\mathcal{A} a^m} \ , \tag{3·68}$$

where the sum rule (3·27) is used. Therefore, the two diagrams sum to zero. This fact can be understood more easily from the Lagrangian (3·2) and (3·25).

The vanishing of the $a_1 \to \pi\gamma$ decay amplitude has been observed in the HLS model[31),35)] and closely related five-dimensional models.[2),16),19)] From the phenomenological point of view, this is not in serious conflict with experiments. The experimental value of the partial width of the $a_1 \to \pi\gamma$ decay mode is approximately 640 KeV, while the total width of a_1 is $250 - 600$ MeV.[34)] Hence, it seems plausible that the $a_1 \to \pi\gamma$ decay process is due to the order $1/N_c$ subleading terms or higher derivative terms, as suggested in Refs. 35) and 31).

Let us next consider the decay mode $a_1 \to \pi\rho$, or, more generally, $a^m \to \pi v^n$. The decay amplitude can be read from the second line of (3·29). Using the equations of motion,

$$\partial^\mu(\partial_\mu \widehat{v}_\nu^n - \partial_\nu \widehat{v}_\mu^n) = m_{v^n}^2 \widehat{v}_\nu^n + \cdots \ , \tag{3·69}$$

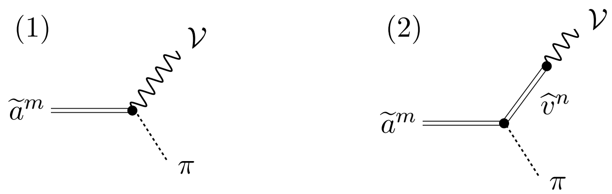

Fig. 2. The relevant diagrams for the $a_1 \to \pi\gamma$ decay amplitude.

$$\partial^\mu(\partial_\mu \widetilde{a}^n_\nu - \partial_\nu \widetilde{a}^n_\mu) = m^2_{a^n} \widetilde{a}^n_\nu + \cdots \ , \tag{3·70}$$

the relevant couplings are extracted as

$$\mathcal{L}_3|_{\pi^1} \sim \cdots - 2i\, g_{a^m v^n \pi}\, \mathrm{tr}\left(\widetilde{a}^m_\mu [\mathit{\Pi}, \widehat{v}^{n\mu}]\right) + \cdots \ , \tag{3·71}$$

with

$$g_{a^m v^n \pi} \equiv \frac{1}{f_\pi}(m^2_{v^n} - m^2_{a^m})\, c_{v^n a^m \pi} \ . \tag{3·72}$$

Then the decay width is given by[35)]

$$\mathit{\Gamma}(a^m \to \pi v^n) = \frac{g^2_{a^m v^n \pi}}{4\pi m^2_{a^m}} |\boldsymbol{p}_{v^n}| \left(1 + \frac{|\boldsymbol{p}_{v^n}|^2}{3m^2_{v^n}}\right) \ , \tag{3·73}$$

where $\boldsymbol{p}_{v^n}$ is the momentum carried by the v^n meson. From the experimental value,[34)]

$$m_{a^1}|_{\rm exp} \simeq 1230 \text{ MeV} \ , \quad \mathit{\Gamma}(a_1 \to \pi\rho)|_{\rm exp} \simeq 150 \sim 360 \text{ MeV} \ , \tag{3·74}$$

the coupling is estimated as $g^2_{a_1\rho\pi}|_{\rm exp} \simeq 7.6 - 18\ \mathrm{GeV}^2$, and the experimental value of the dimensionless combination $c_{\rho a_1 \pi}$, which is related to $g_{a_1\rho\pi}$ through (3·72), is

$$c_{\rho a_1 \pi} \equiv \left. \frac{f_\pi g_{a_1\rho\pi}}{m^2_\rho - m^2_{a_1}} \right|_{\rm exp} \simeq \ \ 0.28 - 0.43 \ . \tag{3·75}$$

On the other hand, the numerical analysis of $c_{v^1 a^1 \pi}$, defined in (3·26), yields

$$c_{v^1 a^1 \pi} \simeq 0.528 \ . \tag{3·76}$$

3.7. $\pi\pi$ *scattering*

It is known that in the chiral limit, the low energy behavior of the $\pi\pi$ scattering amplitude is governed by only the π^4 vertex in the lowest derivative term of the chiral Lagrangian (2·36). However, because the π^4 interaction in (3·33) is canceled by that in (3·20), one might think that the low energy theorem is somehow violated in our model. This, of course, is not true. Here we argue that taking account of the vector meson exchange diagrams yields a $\pi\pi$ scattering amplitude that is consistent with the low energy theorem.

The vertices needed to derive the $\pi\pi$ scattering amplitude, depicted in Fig. 3, consist of (1) the π^4 couplings in (3·33), (2) the direct π^4 couplings in (3·20), and (3) the $\pi\pi\widehat{v}^n$ couplings in (3·20), two of which are contracted by the vector meson exchanges. As we have seen in §3.1, the contributions from (1) and (2) cancel. Also, the effective π^4 vertex obtained from the exchange of the vector mesons (3) is computed as

$$-\sum_{n=1}^{\infty} \frac{g^2_{v^n\pi\pi}}{m^2_{v^n}}\, \mathrm{tr}[\mathit{\Pi}, \partial_\mu \mathit{\Pi}]^2 \ . \tag{3·77}$$

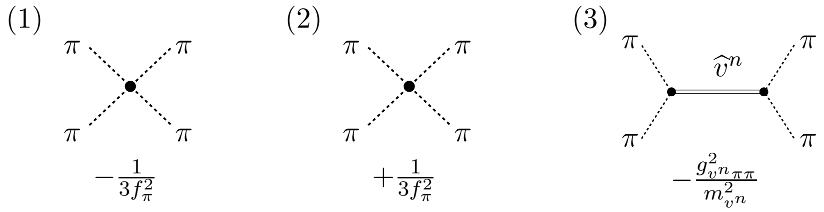

Fig. 3. The relevant diagrams for the $\pi\pi$ scattering.

Then, using the sum rule (3·47), we end up with a term that is identical to the first term (3·33), and we thus conclude that the contribution from (3) is the same as that from (1). In other words, the contributions from (2) and (3) cancel, and the low energy $\pi\pi$ scattering amplitude is governed by the chiral Lagrangian. This fact is trivial if we use the effective action given in Appendix A. Note that the situation here is very similar to that in the HLS model with $a = 4/3$, though vector meson dominance does not hold in that case. (See p. 35 of Ref. 36))

3.8. *Weinberg sum rules*

Before closing this section, let us make a few comments on the Weinberg sum rules, which turn out to be problematic in our model. In our notation, the Weinberg sum rules[24)] state

$$\sum_{n=1}^{\infty}\left(\frac{g_{v^n}^2}{m_{v^n}^2}-\frac{g_{a^n}^2}{m_{a^n}^2}\right)=f_\pi^2 \quad \text{[Weinberg sum rule (I)]} \ , \tag{3·78}$$

$$\sum_{n=1}^{\infty}\left(g_{v^n}^2-g_{a^n}^2\right)=0 \quad \text{[Weinberg sum rule (II)]} \ . \tag{3·79}$$

It was shown in Ref. 2) that both (3·78) and (3·79) are satisfied in the discretized version of the five-dimensional model. On phenomenological grounds, it is often assumed that the sum rules are almost completely dominated by the contributions from the ρ and a_1 mesons alone, that is, that we have

$$\frac{g_\rho^2}{m_\rho^2}-\frac{g_{a_1}^2}{m_{a_1}^2}\simeq f_\pi^2 \ , \quad g_\rho^2\simeq g_{a_1}^2 \ . \tag{3·80}$$

In our case, however, the infinite sums in (3·78) and (3·79) do not converge, as one can guess from the behavior of the numerical data (3·34). Even if this divergence can be removed by appropriately regularizing the infinite sum, as in Refs. 2) and 19), the sums (3·78) and (3·79) are not dominated by ρ and a_1.*) In fact, the ratios of the left-hand sides to the right-hand sides of the relations given in (3·80) are estimated in our model as

$$\frac{1}{f_\pi^2}\left(\frac{g_{v^1}^2}{m_{v^1}^2}-\frac{g_{a^1}^2}{m_{a^1}^2}\right)\simeq\ -7.38 \ , \quad \frac{g_{v^1}^2}{g_{a^1}^2}\simeq 0.177 \ , \tag{3·81}$$

*) This problem was pointed out in Ref. 37) in the context of the discretized model proposed in Ref. 2).

which are both far from 1.

Note that the experimental value of g_{a_1} estimated using τ decay[38)] is $g_{a_1}|_{\rm exp} \simeq 0.177 \pm 0.014$ GeV^2, and the lattice measurement[39)] gives $g_{a_1}|_{\rm lat} \simeq 0.21 \pm 0.02$ GeV^2. Both of these values suggest that g_{a_1} is larger than $g_\rho|_{\rm exp} \simeq 0.12$ GeV^2, though they are still inconsistent with our numerical result, $g_{a_1}/g_\rho \simeq 2.38$. It would be interesting to calculate the corrections in our model to see if this discrepancy is reconciled.

§4. WZW term

In this section, we study the WZW term (2·34) to obtain some interaction terms that involve vector mesons. Here again we work in the $A_z = 0$ gauge and write the five-dimensional gauge field in terms of the differential one-form, $A = A_\mu dx^\mu + A_z dz = A_\mu dx^\mu$. It is useful to first denote the one-form gauge field in (2·25) or (2·32) as

$$A = v + a \ , \tag{4·1}$$

where

$$v \equiv \frac{1}{2}(A_+ + A_-) + \sum_{n=1}^{\infty} v^n \, \psi_{2n-1} \ , \tag{4·2}$$

$$a \equiv \frac{1}{2}(A_+ - A_-)\, \psi_0 + \sum_{n=1}^{\infty} a^n \, \psi_{2n} \ , \tag{4·3}$$

and

$$A_+ \equiv A_L^{\xi_+} = \xi_+ A_L \xi_+^{-1} + \xi_+ d\xi_+^{-1} \ , \quad A_- \equiv A_R^{\xi_-} = \xi_- A_R \xi_-^{-1} + \xi_- d\xi_-^{-1} \ . \tag{4·4}$$

Inserting (4·1) into the Chern-Simons 5-form $\omega_5(A)$ in (2·34), we obtain

$$\begin{aligned}
&\int_{M^4\times\mathbb{R}} \omega_5(A) \\
&= \frac{1}{2}\int_{M^4} \mathrm{tr}\left[(A_+A_- - A_-A_+)\, d(A_+ + A_-) + \frac{1}{2}A_+A_-A_+A_- + (A_+^3 A_- - A_-^3 A_+)\right] \\
&\quad + \int_{M^4\times\mathbb{R}} \mathrm{tr}\left(3\, a\, dv\, dv + a\, da\, da + 3\,(v^2 a + a\, v^2 + a^3)\, dv + 3\left[a\, v\, a\, da\right]_{\text{non-zero}}\right) \ .
\end{aligned} \tag{4·5}$$

(See Appendix B for details.) Here, $[\cdots]_{\text{non-zero}}$ denotes the contribution from the non-zero modes that contain terms with at least one vector meson. It is shown in Ref. 1) that the first line in (4·5), together with the other terms in (2·34), gives the well-known expression of the WZW term (2·39) that depends only on the pion field U and the external gauge fields $A_{L,R}$. The terms in the second line of (4·5) are the new terms that include the interaction with the vector mesons. As a result, we obtain

$$S_{\rm CS}^{\rm D8} = -\frac{N_c}{48\pi^2}\int_{M^4} Z - \frac{N_c}{240\pi^2}\int_{M^4\times\mathbb{R}} \mathrm{tr}(g d g^{-1})^5$$

$$+\frac{N_c}{24\pi^2}\int_{M^4\times\mathbb{R}}\mathrm{tr}\bigg(3\,a\,dv\,dv+a\,da\,da$$
$$+3\,(v^2a+a\,v^2+a^3)\,dv+3\Big[a\,v\,a\,da\Big]_{\text{non-zero}}\bigg)\,. \tag{4·6}$$

In the earlier works concerning the incorporation of vector mesons into the WZW term, there are several adjustable parameters that cannot be fixed by the symmetry in QCD.[40] (See also Refs. 31) and 36).) Contrastingly, the couplings including infinitely many vector mesons in (4·6) are completely fixed, and there is no adjustable parameter. It would therefore be very interesting to determine whether a WZW term of the form (4·6) is consistent with the experimental data. In the following subsections, we examine the phenomenology concerning the πvv, $\pi v\mathcal{V}$, $\pi\mathcal{V}\mathcal{V}$, $v\pi^3$ and $\mathcal{V}\pi^3$ vertices.

4.1. *πvv, $\pi v\mathcal{V}$ and $\pi\mathcal{V}\mathcal{V}$ vertices*

Note that πvv and $\pi v\mathcal{V}$ couplings appear only in the first term of the second line in (4·5).

We set $\mathcal{A}=0$ for simplicity. Then, from the expansion (2·32), we have

$$v=\mathcal{V}+\frac{1}{2f_\pi^2}[\Pi,d\Pi]+\sum_{n=1}^{\infty}v^n\psi_{2n-1}+\cdots\ , \tag{4·7}$$

$$a=\frac{i}{f_\pi}(d\Pi+[\mathcal{V},\Pi])\,\psi_0+\sum_{n=1}^{\infty}a^n\psi_{2n}+\cdots\ . \tag{4·8}$$

Inserting these forms into (4·5), we obtain

$$\int_{M^4\times\mathbb{R}}\omega_5(A)\Big|_{\pi vv,\,\pi v\mathcal{V}}=\int_{M^4\times\mathbb{R}}\mathrm{tr}\,(3\,a\,dv\,dv)\Big|_{\pi vv,\,\pi v\mathcal{V}}$$
$$=-\frac{6i}{f_\pi}\int_{M^4}\mathrm{tr}\,(\,\Pi\,(dv^nd\mathcal{V}+d\mathcal{V}dv^n))\ c_{v^n}$$
$$-\frac{6i}{f_\pi}\int_{M^4}\mathrm{tr}\,(\,\Pi\,dv^ndv^m)\ c_{v^nv^m}\ , \tag{4·9}$$

where

$$c_{v^n}\equiv\frac{1}{2}\int dz\,\partial_z\psi_0\,\psi_{2n-1}=\frac{1}{\pi}\int dz\,K^{-1}\psi_{2n-1}\ , \tag{4·10}$$

$$c_{v^nv^m}\equiv\frac{1}{2}\int dz\,\partial_z\psi_0\,\psi_{2n-1}\psi_{2m-1}=\frac{1}{\pi}\int dz\,K^{-1}\psi_{2n-1}\psi_{2m-1}\ . \tag{4·11}$$

Then, using $\widetilde{v}^n$ defined in (3·5) and the sum rules

$$\sum_{m=1}^{\infty}a_{\mathcal{V}v^m}c_{v^nv^m}=c_{v^n}\ ,\quad \sum_{n=1}^{\infty}a_{\mathcal{V}v^n}c_{v^n}=1\ , \tag{4·12}$$

(4·9) can be written

$$\int_{M^4\times\mathbb{R}}\omega_5(A)\Big|_{\pi vv,\,\pi v\mathcal{V}}=\frac{6i}{f_\pi}\int_{M^4}\mathrm{tr}\,(\,\Pi\,d\mathcal{V}d\mathcal{V}\,)-\frac{6i}{f_\pi}\int_{M^4}\mathrm{tr}\,(\,\Pi\,d\widetilde{v}^nd\widetilde{v}^m)\ c_{v^nv^m}\ . \tag{4·13}$$

It is easy to check that the contributions to the $\pi^0 \to \gamma\gamma$ decay amplitude from the two terms in (4·13) cancel. This is obvious from the fact that there is no $\pi\mathcal{V}\mathcal{V}$ vertex in (4·9) that is written in terms of v^n and $\mathcal{V}$.

The $\pi\mathcal{V}\mathcal{V}$ vertex comes from Z given in (2·40):

$$\begin{aligned}\int_{M^4} Z\Big|_{\pi\mathcal{V}\mathcal{V}} &= \int_{M^4}\Big(\mathrm{tr}\left[(A_R dA_R + dA_R A_R)U^{-1}dU - \text{p.c.}\right] \\ &\qquad + \mathrm{tr}\left[dA_R dU^{-1} A_L U - \text{p.c.}\right]\Big)\Big|_{\pi\mathcal{V}\mathcal{V}} \\ &= \frac{12i}{f_\pi}\int_{M^4} \mathrm{tr}\,(\,\Pi\, d\mathcal{V} d\mathcal{V}\,)\ . \end{aligned} \tag{4·14}$$

Combining (4·13) and (4·14), and, furthermore, rewriting them in terms of $\widetilde{v}^n$ given in (3·5) or $\widehat{v}^n$ given in (3·16), we obtain

$$\begin{aligned} S_{\rm CS}^{\rm D8}\Big|_{\pi v v,\,\pi v\mathcal{V},\,\pi\mathcal{V}\mathcal{V}} &= -\frac{N_c}{48\pi^2}\int_{M^4} Z\Big|_{\pi\mathcal{V}\mathcal{V}} + \frac{N_c}{24\pi^2}\int \omega_5(A^g)\Big|_{\pi v v,\,\pi v\mathcal{V}} \\ &= -\frac{N_c}{4\pi^2}\frac{i}{f_\pi}\int_{M^4} \mathrm{tr}\,(\Pi\, d\widetilde{v}^n d\widetilde{v}^m)\ c_{v^n v^m} \\ &= -\frac{N_c}{4\pi^2}\frac{i}{f_\pi}\int_{M^4} \mathrm{tr}\,(\Pi\, d\widehat{v}^n d\widehat{v}^m)\ c_{v^n v^m} \\ &\quad - \frac{N_c}{2\pi^2}\frac{i}{f_\pi^3}(c_n - d_n)\int_{M^4} \mathrm{tr}\,(d\Pi\, d\Pi\, d\Pi\, \widehat{v}^n) + \mathcal{O}(\Pi^5)\ , \end{aligned} \tag{4·15}$$

where

$$d_{v^n} \equiv \frac{1}{2}\int dz\, \psi_0^2 \partial_z \psi_0\, \psi_{2n-1} = \frac{1}{\pi}\int dz\, K^{-1}\psi_0^2 \psi_{2n-1}\ , \tag{4·16}$$

and we have used the sum rule

$$\sum_{m=1}^{\infty} b_{v^m\pi\pi} c_{v^n v^m} = c_{v^n} - d_{v^n}\ . \tag{4·17}$$

We thus conclude that there exist no direct three-point couplings including the external photon field. This demonstrates the vector meson dominance in this sector.

4.2. $v\pi^3$ *and* $\mathcal{V}\pi^3$ *vertices*

The $v^n\pi^3$ vertex comes from the $3\,a\,dv\,dv$, $3\,a^3 dv$ and $3[a\,v\,a\,da]_{\text{non-zero}}$ terms in (4·5):

$$\int_{M^4\times\mathbb{R}} \mathrm{tr}(3\,a\,dv\,dv)\Big|_{v\pi^3} = \frac{12i}{f_\pi^3}\int_{M^4} \mathrm{tr}\,(\,d\Pi\, d\Pi\, d\Pi\, v^n)\, c_{v^n}\ , \tag{4·18}$$

$$\int_{M^4\times\mathbb{R}} \mathrm{tr}\left(3\,a^3 dv + 3\Big[a\,v\,a\,da\Big]_{\text{non-zero}}\right)\Bigg|_{v\pi^3} = -\frac{12i}{f_\pi^3}\int_{M^4} \mathrm{tr}\,(d\Pi\, d\Pi\, d\Pi\, v^n)\ d_{v^n}\ . \tag{4·19}$$

From (4·18) and (4·19), we obtain

$$\int_{M^4\times\mathbb{R}} \omega_5(A)\Big|_{v\pi^3} = \frac{12i}{f_\pi^3}(c_{v^n} - d_{v^n}) \int_{M^4} \mathrm{tr}(d\Pi\, d\Pi\, d\Pi\, v^n) \ . \qquad (4·20)$$

Rewriting this relation in terms of $\widetilde{v}^n$ in (3·5), and using the sum rules

$$\sum_{n=1}^{\infty} a_{\mathcal{V}v^n} c_{v^n} = 1 \ , \qquad \sum_{n=1}^{\infty} a_{\mathcal{V}v^n} d_{v^n} = \frac{1}{\pi}\int dz\, K^{-1}\psi_0^2 = \frac{1}{3} \ , \qquad (4·21)$$

we obtain

$$\int_{M^4\times\mathbb{R}} \omega_5(A)\Big|_{v\pi^3} = \frac{12i}{f_\pi^3}(c_{v^n} - d_{v^n}) \int_{M^4} \mathrm{tr}(d\Pi\, d\Pi\, d\Pi\, \widetilde{v}^n) - \frac{8i}{f_\pi^3}\int_{M^4} \mathrm{tr}(d\Pi\, d\Pi\, d\Pi\, \mathcal{V}) \ . \qquad (4·22)$$

The $\mathcal{V}\pi^3$ vertex can be read off of Z in (2·40):

$$\int_{M^4} Z\Big|_{\mathcal{V}\pi^3} = \int_{M^4} \mathrm{tr}\left[A_R(dU^{-1}U)^3 - \mathrm{p.c.}\right]\Big|_{\mathcal{V}\pi^3} = -\frac{16i}{f_\pi^3}\int_{M^4} \mathrm{tr}\left(d\Pi\, d\Pi\, d\Pi\, \mathcal{V}\right) \ . \qquad (4·23)$$

Collecting (4·22) and (4·23), we obtain

$$S_{\mathrm{CS}}^{\mathrm{D8}}\Big|_{v\pi^3,\,\mathcal{V}\pi^3} = -\frac{N_c}{48\pi^2}\int_{M^4} Z\Big|_{\mathcal{V}\pi^3} + \frac{N_c}{24\pi^2}\int_{M^4\times\mathbb{R}} \omega_5(A)\Big|_{v\pi^3,\,\mathcal{V}\pi^3} = \frac{N_c}{2\pi^2}\frac{i}{f_\pi^3}(c_{v^n} - d_{v^n}) \int_{M^4} \mathrm{tr}(d\Pi\, d\Pi\, d\Pi\, \widetilde{v}^n) \ . \qquad (4·24)$$

This again exhibits the vector meson dominance. Moreover, if we write the action in terms of $\widehat{v}^n$, the $\widehat{v}^n\pi^3$ coupling in (4·24) cancels that in (4·15), and we finally obtain

$$S_{\mathrm{CS}}^{\mathrm{D8}}\Big|_{\pi vv,\,\pi v\mathcal{V},\,\pi\mathcal{V}\mathcal{V},v\pi^3,\mathcal{V}\pi^3} = -\frac{N_c}{4\pi^2}\frac{i}{f_\pi}\int_{M^4} \mathrm{tr}\left(\Pi\, d\widehat{v}^n d\widehat{v}^m\right) c_{v^n v^m} + \mathcal{O}(\Pi^5) \ . \ (4·25)$$

4.3. *$\omega \to \pi^0\gamma$ and $\omega \to \pi^0\pi^+\pi^-$ decay*

From the coupling (4·25), we can calculate the $\omega \to \pi^0\gamma$ and $\omega \to \pi^0\pi^+\pi^-$ decay amplitudes. Here, ω is the iso-singlet component of the lightest vector meson, $\widehat{v}^1$. Because of the complete vector meson dominance and the absence of the direct $\widehat{v}^n\pi^3$ coupling, the former is given by the vertex $\omega \to \widehat{v}^n\rho$, followed by the $\widehat{v}^n \to \gamma$ transition, and the latter is given by $\omega \to \pi\widehat{v}^n$, followed by $\widehat{v}^n \to 2\pi$ (see Fig. 4). These diagrams are identical to those in the Gell-Mann - Sharp - Wagner (GSW) model,[41] which is known to be in good agreement with experimental data. Let us examine how it works in our model.

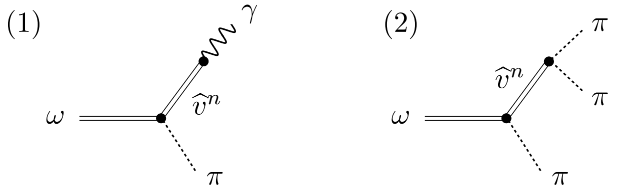

Fig. 4. The relevant diagrams for (1) $\omega \to \pi\gamma$ and (2) $\omega \to \pi\pi\pi$.

The calculation of the $\omega \to \pi^0\gamma$ decay amplitude is analogous to that given in Refs. 40) and 36), and we obtain

$$\Gamma(\omega \to \pi^0\gamma) = \frac{N_c^2}{3}\frac{\alpha}{64\pi^4 f_\pi^2}\left(\sum_{m=1}^{\infty}\frac{c_{v^1 v^m} g_{v^m}}{m_{v^m}^2}\right)^2 |\boldsymbol{p}_\pi|^3 = \frac{N_c^2}{3}\frac{\alpha}{64\pi^4 f_\pi^2} c_{v^1}^2 |\boldsymbol{p}_\pi|^3 \ , \tag{4·26}$$

where $\alpha = e^2/4\pi$, and we have used the sum rule (4·12). Here, c_{v^1} plays the role of the parameter g in Ref. 40) and it is shown there that the decay width is consistent with the experimental value when $g \simeq g_{\rho\pi\pi}$.*) Remarkably, this is exactly what we have in our model. In fact, it can easily be shown that, in general,

$$c_{v^n} = g_{v^n\pi\pi} \ , \tag{4·27}$$

using (2·12) and integrating by parts in the expression for $g_{v^n\pi\pi}$ given by (3·21) and (3·12).

Similarly, the $\omega \to \pi^0\pi^+\pi^-$ decay width is[36),40)]

$$\Gamma(\omega \to \pi^0\pi^+\pi^-) = \frac{m_\omega}{192\pi^3}\iint dE_+ dE_- \left[|\boldsymbol{q}_-|^2|\boldsymbol{q}_+|^2 - (\boldsymbol{q}_+\cdot\boldsymbol{q}_-)^2\right]|F_{\omega\to 3\pi}|^2 \ , \tag{4·28}$$

where $E_\pm$ are the energies and $\boldsymbol{q}_\pm$ are the momenta of $\pi^\pm$ in the rest frame of ω, and

$$F_{\omega\to 3\pi} \equiv -\frac{N_c}{4\pi^2 f_\pi}\sum_{n=1}^{\infty} c_{v^1 v^n} g_{v^n\pi\pi} \times\left(\frac{1}{m_{v^n}^2 + (q_+ + q_-)^2} + \frac{1}{m_{v^n}^2 + (q_- + q_0)^2} + \frac{1}{m_{v^n}^2 + (q_0 + q_+)^2}\right) , \tag{4·29}$$

where q_0 and $q_\pm$ are the four momenta of π^0 and $\pi^\pm$, respectively. The results of our numerical analysis suggest that the $n = 1$ term dominates the sum. If we replace the entire sum with this single term, the expression for the decay width in (4·28)

*) A recent experiment value[34)] is $\Gamma(\omega \to \pi^0\gamma)\big|_{\rm exp} \simeq 0.757$ MeV, which implies $c_\omega|_{\rm exp} \simeq 5.80$.

becomes the same as that in Ref. 36) with the parameters g and c_i ($i = 1, 2, 3$) chosen as

$$g = \frac{2f_\pi^2 g_{v^1\pi\pi}}{m_{v^1}^2} c_{v^1v^1} = b_{v^1\pi\pi} c_{v^1v^1} \ , \tag{4·30}$$

and $c_1 - c_2 = c_3 = 1$. It is shown in Refs. 40) and 36) that the decay width is consistent with the experimental results again with $g \simeq g_{\rho\pi\pi}$. Note that the right-hand side of the relation (4·30) is the main contribution on the left-hand side of (4·17). If we approximate (4·30) with (4·17), the relation (4·30) is replaced by

$$g \simeq c_{v^1} - d_{v^1} = g_{v^1\pi\pi} - d_{v^1} \ , \tag{4·31}$$

where we have used (4·27). We have found numerically that d_{v^1} is much smaller than c_{v^1}, as seen in (4·32). This implies that $g \simeq g_{\rho\pi\pi}$, as desired.

Here we list the results of our numerical estimations of several quantities:

n	$\kappa\ c_{v^1v^n}$	$\kappa^{-1/2} b_{v^n\pi\pi}$	$\kappa^{1/2} c_{v^n}$	$\kappa^{1/2} d_{v^n}$
1	0.202	1.58	0.415	0.0875
2	−0.0992	−0.0964	−0.109	0.0485
3	0.0284	0.00619	0.0160	−0.0367
4	−0.00618	−0.000880	−0.00408	0.00887

(4·32)

As a check, we note that the numerically determined value of (4·30) is $g \simeq 0.319\,\kappa^{-1/2}$, while that of (4·31) is $g \simeq 0.328\,\kappa^{-1/2}$. It is found that the above approximation gives reasonable results, though they are not extremely close to $g_{v^1\pi\pi} \simeq 0.415\,\kappa^{-1/2}$. If we use the value of κ in (3·40), the parameter g in (4·30) is estimated as $g \simeq 3.69$, which is about 64% of the experimental value of $g_{\rho\pi\pi}$.

To determine the extent to which the result is affected by including the contributions from $n > 1$ terms in (4·29), let us estimate the decay width by performing the integration in (4·28). By using (3·40) and the experimental values $N_c = 3$, $m_{\pi^\pm} \simeq 140$ MeV and $m_{\pi^0} \simeq 135$ MeV, we obtain

$$\Gamma_k(\omega \to \pi^+\pi^-\pi^0) \simeq \{2.48,\ 2.58,\ 2.58,\ 2.58, \cdots\} \text{ MeV} \ . \tag{4·33}$$

Here, $\Gamma_k(\omega \to 3\pi)$ denotes the decay width with the exchange of the v^n ($1 \le n \le k$) vector mesons incorporated in (4·29). Therefore, the contribution of the ρ meson exchange dominates the sum. Unfortunately, this value is much smaller than the experimental value, $\Gamma(\omega \to \pi^+\pi^-\pi^0)|_{\text{exp}} \simeq 7.56$ MeV.[34)] This is mainly due to the smallness of the coupling g estimated above.

§5. Vector meson dominance revisited

In the previous sections, we have shown that our model exhibits the vector meson dominance by examining the couplings with the external gauge fields one by one. Here we present a more systematic way to understand why it works.

In this subsection, we work with the expansions given in (2·19) and (2·11). We can gauge away $\varphi^{(n)}$ in (2·11) without changing the asymptotic condition (2·18).

Then, these expansions are written as

$$A_\mu = \mathcal{V}_\mu + \mathcal{A}_\mu \psi_0 + \sum_{n=1}^{\infty} v_\mu^n \, \psi_{2n-1} + \sum_{n=1}^{\infty} a_\mu^n \, \psi_{2n} \ , \tag{5·1}$$

$$A_z = -i \, \Pi \, \phi_0 \ , \tag{5·2}$$

where Π denotes the pion field $\varphi^{(0)}$ in (2·11). Note that v_μ^n, a_μ^n and Π in these expansions are not exactly equal to those appearing in §§3 and 4, but they are related through certain field redefinitions. If we rewrite the expansion (5·1) in terms of $\widetilde{v}_\mu^n$ and $\widetilde{a}_\mu^n$ given in (3·5) and (3·6), we have

$$A_\mu = \mathcal{V}_\mu \, \psi_v + \mathcal{A}_\mu \psi_a + \sum_{n=1}^{\infty} \widetilde{v}_\mu^n \, \psi_{2n-1} + \sum_{n=1}^{\infty} \widetilde{a}_\mu^n \, \psi_{2n} \ , \tag{5·3}$$

where*)

$$\psi_v \equiv 1 - \sum_{n=1}^{\infty} a_{\mathcal{V} v^n} \psi_{2n-1} \ , \quad \psi_a \equiv \psi_0 - \sum_{n=1}^{\infty} a_{\mathcal{A} a^n} \psi_{2n} \ . \tag{5·4}$$

Note that it can be shown using (2·13) and (3·4) that

$$0 = \int dz \, K^{-1/3} \psi_v \psi_m = \int dz \, K^{-1/3} \psi_a \psi_m \tag{5·5}$$

for all m. Equivalently, we have

$$0 = \int dz \, K^{-1/3} \psi_v f = \int dz \, K^{-1/3} \psi_a f \tag{5·6}$$

for an arbitrary normalizable function $f(z)$. From this fact, we immediately see that if we write the action in terms of $\widetilde{v}_\mu^n$ and $\widetilde{a}_\mu^n$ defined in (3·5) and (3·6), many of the couplings that include $\mathcal{V}_\mu$ or $\mathcal{A}_\mu$ vanish. In fact, following the procedure described in Appendix A, we obtain

$$\begin{aligned}
\kappa \int dz \ \mathrm{tr} \left[\frac{1}{2} K^{-1/3} F_{\mu\nu}^2 \right] = \mathrm{tr} \Bigg[& \frac{1}{2e^2} \left((F_{\mu\nu}^{A_L})^2 + (F_{\mu\nu}^{A_R})^2 \right) \\
& + \frac{1}{2} (\partial_\mu \widetilde{v}_\nu^n - \partial_\nu \widetilde{v}_\mu^n)^2 + \frac{1}{2} (\partial_\mu \widetilde{a}_\nu^n - \partial_\nu \widetilde{a}_\mu^n)^2 \\
& + (\partial_\mu \widetilde{v}_\nu^n - \partial_\nu \widetilde{v}_\mu^n) ([\widetilde{v}^{p\mu}, \widetilde{v}^{q\nu}] \, g_{v^n v^p v^q} + [\widetilde{a}^{p\mu}, \widetilde{a}^{q\nu}] \, g_{v^n a^p a^q}) \\
& + (\partial_\mu \widetilde{a}_\nu^n - \partial_\nu \widetilde{a}_\mu^n) ([\widetilde{v}^{p\mu}, \widetilde{a}^{q\nu}] - [\widetilde{v}^{q\nu}, \widetilde{a}^{p\mu}]) \, g_{v^p a^n a^q} \\
& + \frac{1}{2} [\widetilde{v}_\mu^m, \widetilde{v}_\nu^n] [\widetilde{v}^{p\mu}, \widetilde{v}^{q\nu}] \, g_{v^m v^n v^p v^q} \\
& + \frac{1}{2} [\widetilde{a}_\mu^m, \widetilde{a}_\nu^n] [\widetilde{a}^{p\mu}, \widetilde{a}^{q\nu}] \, g_{a^m a^n a^p a^q}
\end{aligned}$$

*) The infinite sums in (5·4) should be regarded as formal expressions, since they do not uniformly converge to smooth, normalizable functions. In the following, we take the limit of an infinite sum after performing the integration over z.

$$+\left([\widetilde{v}^m_\mu,\widetilde{v}^n_\nu][\widetilde{a}^{p\mu},\widetilde{a}^{q\nu}]+[\widetilde{v}^m_\mu,\widetilde{a}^p_\nu][\widetilde{v}^{n\mu},\widetilde{a}^{q\nu}]\right.$$
$$\left.\left.-[\widetilde{v}^m_\mu,\widetilde{a}^p_\nu][\widetilde{v}^{n\nu},\widetilde{a}^{q\mu}]\right)g_{v^mv^na^pa^q}\right] . \qquad (5\cdot7)$$

[See (A·24) for the definition of the four-point coupling constants.] This shows that all the couplings between the external gauge fields (A_L, A_R) and the vector meson fields $(\widetilde{v}^n, \widetilde{a}^n)$ vanish in the first term of the effective action (2·6). Here we have used the relations

$$\kappa\int dz\,K^{-1/3}\psi^n_a\psi^m_v=\kappa\int dz\,K^{-1/3}\psi_a\psi^{n-1}_0\ , \qquad (\text{for } n\ge 1,\ m\ge 0) \qquad (5\cdot8)$$

$$\kappa\int dz\,K^{-1/3}\psi^n_a\psi^m_v=\kappa\int dz\,K^{-1/3}\psi^n_0\psi_v\ , \qquad (\text{for } m\ge 1,\ n\ge 0) \qquad (5\cdot9)$$

which can be shown by using (5·6), and we have set

$$e^{-2}\equiv\frac{\kappa}{4}\int dz\,K^{-1/3}\psi_v(1+\psi_0^2)\ .$$

This is divergent (or ill-defined), because ψ_v is a function that approaches 1 at $z\to\pm\infty$.

It is important to note that we cannot conclude that $\psi_v=\psi_a=0$ from the relation (5·6). Actually, using (2·12) and (2·13), one can show

$$\kappa\int dz\,K\,\partial_z\psi_v\,\partial_z\psi_{2n-1}=-\lambda_{2n-1}a_{\mathcal{V}v^n}\ , \qquad (5\cdot10)$$

$$\kappa\int dz\,K\,\partial_z\psi_a\,\partial_z\psi_{2n}=-\lambda_{2n}a_{\mathcal{A}a^n}\ . \qquad (5\cdot11)$$

Then, the second term in the effective action (2·6) is calculated as

$$\kappa\int dz\ \mathrm{tr}\left[KF^2_{z\nu}\right]$$
$$=\mathrm{tr}\Big[m^2_{v^n}(\widetilde{v}^n_\mu-a_{\mathcal{V}v^n}\mathcal{V}_\mu)^2+m^2_{a^n}(\widetilde{a}^n_\mu-a_{\mathcal{A}a^n}\mathcal{A}_\mu)^2+(i\partial_\mu\mathit{\Pi}+f_\pi\mathcal{A}_\mu)^2$$
$$+2ig_{a^mv^n\pi}\,\widetilde{a}^m_\mu[\mathit{\Pi},\widetilde{v}^{n\mu}]-2g_{v^n\pi\pi}\,\widetilde{v}^n_\mu[\mathit{\Pi},\partial^\mu\mathit{\Pi}]$$
$$-c_{a^na^m}[\mathit{\Pi},\widetilde{a}^n_\mu][\mathit{\Pi},\widetilde{a}^{n\mu}]-c_{v^nv^m}[\mathit{\Pi},\widetilde{v}^n_\mu][\mathit{\Pi},\widetilde{v}^{n\mu}]\Big]\ , \qquad (5\cdot12)$$

where

$$c_{a^na^m}\equiv\frac{1}{\pi}\int dz\,K^{-1}\psi_{2n}\psi_{2m}\ . \qquad (5\cdot13)$$

Here, we have used the relation (4·27), as well as the fact that $g_{a^mv^n\pi}$ defined in (3·72) is equal to

$$g_{a^mv^n\pi}=f_\pi\int dz\,\psi_{2m}\partial_z\psi_{2n-1}\ , \qquad (5\cdot14)$$

which can be shown by using (2·12). The mesons couple to the external gauge fields only through the $\widetilde{v}^n \to \mathcal{V}$ and $\widetilde{a}^n \to \mathcal{A}$ transitions in (5·12).

The expression for the WZW term can also be simplified by using the expansions (5·3) and (5·2). It is easy to see that all the terms including $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ vanish because of the relation (5·6). This demonstrates the complete vector meson dominance in the WZW term. Moreover, because the pion field Π appears only in (5·2), the terms with two or more pion fields vanish, as we partly observed in §4.2. Inserting (5·3) and (5·2) into (2·8), we obtain

$$\begin{aligned} S_{\rm D8}^{\rm CS} = &-\frac{N_c}{4\pi^2}\frac{i}{f_\pi}\int_{M^4}\mathrm{tr}\Big[\Pi\, dB^n dB^m\, c_{nm} \\ &\qquad + \Pi\,(dB^m B^n B^p + B^m B^n dB^p)\, c_{mnp} + \Pi\, B^m B^n B^p B^q\, c_{mnpq}\Big] \\ &+ \frac{N_c}{24\pi^2}\int_{M^4}\mathrm{tr}\Big[B^m B^n dB^p\, d_{mn|p} - \frac{3}{2}B^m B^n B^p B^q\, d_{mnp|q}\Big]\ , \end{aligned} \tag{5·15}$$

where $B^{2n-1}\equiv\widetilde{v}^n$, $B^{2n}\equiv\widetilde{a}^n$ and

$$\begin{aligned} &c_{mn}\equiv\frac{1}{\pi}\int dz\, K^{-1}\psi_m\psi_n\ ,\quad c_{mnp}\equiv\frac{1}{\pi}\int dz\, K^{-1}\psi_m\psi_n\psi_p\ , \\ &c_{mnpq}\equiv\frac{1}{\pi}\int dz\, K^{-1}\psi_m\psi_n\psi_p\psi_q\ , \\ &d_{mn|p}\equiv\int dz\,(\psi_n\partial_z\psi_m-\psi_m\partial_z\psi_n)\,\psi_p\ ,\quad d_{mnp|q}\equiv\int dz\,\psi_m\psi_n\psi_p\,\partial_z\psi_q\ . \end{aligned} \tag{5·16}$$

§6. Summary and discussion

In this paper, we computed the effective action including the pion, the vector mesons and the external gauge fields associated with the chiral $U(N_f)_L\times U(N_f)_R$ symmetry, based on the D4/D8 model proposed in Ref. 1), which is conjectured to be a holographic dual of large N_c QCD with N_f massless flavors. We estimated various coupling constants numerically and compared these values with the experimental values. The agreement was, of course, not perfect, but we think that it is good enough to believe that our model nicely captures the expected features of QCD even quantitatively.

One of the major issues addressed in this paper is vector meson dominance. An intuitive explanation for this phenomenon in our model is as follows (see also Ref. 16)). As seen in §5, the external gauge fields appearing in (5·3) have support only at the boundary, corresponding to $z\to\pm\infty$, while the pion and the vector mesons correspond to the normalizable modes, which vanish as $z\to\pm\infty$. Therefore the external gauge fields cannot couple to the mesons, unless the divergent factor K in the second term of (2·6) picks up the contribution in the $z\to\pm\infty$ limit. For the second term of (2·6), we know that the fields v_μ^n are the mass eigenmodes and hence that the fields $\widetilde{v}_\mu^n$ mix with $\mathcal{V}_\mu$.

We have found various useful sum rules among the masses and the coupling constants of an infinite tower of vector mesons. These follow from the completeness condition of the mode functions (3·13). As far as we have determined, the contribution from the ρ meson is always the most dominant term in the sum rules. Approximating these infinite sums with the contribution from only the ρ meson, we obtained KSRF-type relations in §3.4 and, furthermore, the approximate ρ meson dominance and ρ meson coupling universality, as explained in §3.5.

In order to make more reliable predictions, we must take into account the string loop corrections and the α' corrections and also go beyond the probe approximation. It would be quite interesting to see how our results for the KSRF relations, the Weinberg sum rules, the a_1 meson decay amplitudes, etc., are improved by incorporating such corrections. (See a forthcoming paper, Ref. 42) for a discussion along this line.)

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Appendix A
—— *The Effective Action (DBI Part)* ——

Here we calculate the field strength of the gauge field appearing in (2·25) and (2·32):

$$\begin{aligned} A_\mu &= \frac{1}{2}(A^{\xi_+}_{L\mu} + A^{\xi_-}_{R\mu}) + \frac{1}{2}(A^{\xi_+}_{L\mu} - A^{\xi_-}_{R\mu})\psi_0 + \sum_{n=1}^{\infty} v^n_\mu\,\psi_{2n-1} + \sum_{n=1}^{\infty} a^n_\mu\,\psi_{2n} \\ &= \widehat{\mathcal{V}}_\mu + \widehat{\mathcal{A}}_\mu\psi_0 + \sum_{n=1}^{\infty} v^n_\mu\,\psi_{2n-1} + \sum_{n=1}^{\infty} a^n_\mu\,\psi_{2n}\ , \end{aligned} \tag{A·1}$$

where

$$\widehat{\mathcal{V}}_\mu \equiv \frac{1}{2}(A^{\xi_+}_{L\mu} + A^{\xi_-}_{R\mu})\ , \quad \widehat{\mathcal{A}}_\mu \equiv \frac{1}{2}(A^{\xi_+}_{L\mu} - A^{\xi_-}_{R\mu})\ . \tag{A·2}$$

In the $\xi_+^{-1} = \xi_- = e^{i\Pi/f_\pi}$ gauge, we can expand these fields as

$$\widehat{\mathcal{V}}_\mu = \mathcal{V}_\mu + \frac{1}{2f_\pi^2}[\,\Pi, \partial_\mu\Pi\,] - \frac{i}{f_\pi}[\,\Pi, \mathcal{A}_\mu\,] + \cdots\ , \tag{A·3}$$

$$\widehat{\mathcal{A}}_\mu = \mathcal{A}_\mu + \frac{i}{f_\pi}\partial_\mu\Pi - \frac{i}{f_\pi}[\,\Pi, \mathcal{V}_\mu\,] + \cdots\ . \tag{A·4}$$

Then, the field strengths are obtained as

$$F_{z\mu} = \partial_z A_\mu = \widehat{\mathcal{A}}_\mu\frac{2}{\pi K} + \sum_{n=1}^{\infty} v^n_\mu\,\partial_z\psi_{2n-1} + \sum_{n=1}^{\infty} a^n_\mu\,\partial_z\psi_{2n}\ , \tag{A·5}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = F_{\mu\nu}\Big|_{\rm even} + F_{\mu\nu}\Big|_{\rm odd} \ , \tag{A·6}$$

$$\begin{aligned} F_{\mu\nu}\Big|_{\rm even} &= F^{\widehat{\mathcal{V}}}_{\mu\nu} + [\widehat{\mathcal{A}}_\mu, \widehat{\mathcal{A}}_\nu]\psi_0^2 + \sum_{n\geq 1}(D^{\widehat{\mathcal{V}}}_\mu v^n_\nu - D^{\widehat{\mathcal{V}}}_\nu v^n_\mu)\psi_{2n-1} \\ &\quad + \sum_{n\geq 1}([\widehat{\mathcal{A}}_\mu, a^n_\nu] - [\widehat{\mathcal{A}}_\nu, a^n_\mu])\psi_0\psi_{2n} \\ &\quad + \sum_{n,m\geq 1}[v^n_\mu, v^m_\nu]\psi_{2n-1}\psi_{2m-1} + \sum_{n,m\geq 1}[a^n_\mu, a^m_\nu]\psi_{2n}\psi_{2m} \ , \end{aligned} \tag{A·7}$$

$$\begin{aligned} F_{\mu\nu}\Big|_{\rm odd} &= (D^{\widehat{\mathcal{V}}}_\mu \widehat{\mathcal{A}}_\nu - D^{\widehat{\mathcal{V}}}_\nu \widehat{\mathcal{A}}_\mu)\psi_0 + \sum_{n\geq 1}(D^{\widehat{\mathcal{V}}}_\mu a^n_\nu - D^{\widehat{\mathcal{V}}}_\nu a^n_\mu)\psi_{2n} \\ &\quad + \sum_{n\geq 1}([\widehat{\mathcal{A}}_\mu, v^n_\nu] - [\widehat{\mathcal{A}}_\nu, v^n_\mu])\psi_0\psi_{2n-1} \\ &\quad + \sum_{m,n\geq 1}([v^n_\mu, a^m_\nu] - [v^n_\nu, a^m_\mu])\psi_{2m}\psi_{2n-1} \ , \end{aligned} \tag{A·8}$$

where $F_{\mu\nu}\Big|_{\rm even}$ and $F_{\mu\nu}\Big|_{\rm odd}$ denote the parts that are even and odd under $z \to -z$, respectively, and we have defined

$$F^{\widehat{\mathcal{V}}}_{\mu\nu} \equiv \partial_\mu \widehat{\mathcal{V}}_\nu - \partial_\nu \widehat{\mathcal{V}}_\mu + [\widehat{\mathcal{V}}_\mu, \widehat{\mathcal{V}}_\nu] \ , \quad D^{\widehat{\mathcal{V}}}_\mu * \equiv \partial_\mu + [\widehat{\mathcal{V}}_\mu, *] \ . \tag{A·9}$$

The following are useful relations here:

$$F^{\widehat{\mathcal{V}}}_{\mu\nu} + [\widehat{\mathcal{A}}_\mu, \widehat{\mathcal{A}}_\nu]\psi_0^2 = \frac{1}{2}(\xi_+ F^{A_L}_{\mu\nu}\xi_+^{-1} + \xi_- F^{A_R}_{\mu\nu}\xi_-^{-1}) - [\widehat{\mathcal{A}}_\mu, \widehat{\mathcal{A}}_\nu](1-\psi_0^2) \ , \tag{A·10}$$

$$D^{\widehat{\mathcal{V}}}_\mu \widehat{\mathcal{A}}_\nu - D^{\widehat{\mathcal{V}}}_\nu \widehat{\mathcal{A}}_\nu = \frac{1}{2}(\xi_+ F^{A_L}_{\mu\nu}\xi_+^{-1} - \xi_- F^{A_R}_{\mu\nu}\xi_-^{-1}) \ . \tag{A·11}$$

The action (2·6) is calculated as follows. The second term in (2·6) is

$$\begin{aligned} \kappa\int dz\, K\, {\rm tr}\, F_{z\mu}^2 &= {\rm tr}\left[\frac{4}{\pi}\kappa\,\widehat{\mathcal{A}}_\mu^2 + m^2_{v^n}(v^n_\mu)^2 + m^2_{a^n}(a^n_\mu)^2\right] \\ &= {\rm tr}\left[\frac{f_\pi^2}{4}(U^{-1}\partial_\mu U)^2 + m^2_{v^n}(v^n_\mu)^2 + m^2_{a^n}(a^n_\mu)^2\right. \\ &\qquad + \frac{f_\pi^2}{4}\Big(A^2_{L\mu} + A^2_{R\mu} - 2U^{-1}A_{L\mu}UA^\mu_R \\ &\qquad\quad \left. - 2A^\mu_L U\partial_\mu U^{-1} - 2A^\mu_R U^{-1}\partial_\mu U\Big)\right] \ , \end{aligned} \tag{A·12}$$

where we have used the relation (2·37). The first term in (2·6) is more complicated. We segregate it into terms of equal orders in the vector meson fields (v^n_μ, a^n_μ) as

$$\begin{aligned} \kappa\int dz\,\frac{1}{2}K^{-1/3}\,{\rm tr}\,F_{\mu\nu}^2 &= \kappa\int dz\,\frac{1}{2}K^{-1/3}\,{\rm tr}\left[F_{\mu\nu}\Big|^2_{\rm even} + F_{\mu\nu}\Big|^2_{\rm odd}\right] \\ &\equiv \mathcal{L}_{(a,v)^0} + \mathcal{L}_{(a,v)^1} + \mathcal{L}_{(a,v)^2} + \mathcal{L}_{(a,v)^3} + \mathcal{L}_{(a,v)^4} \ , \end{aligned} \tag{A·13}$$

where $\mathcal{L}_{(a,v)^m}$ denotes the terms with m vector meson fields v_μ^n and a_μ^n. The order zero terms are

$$\begin{aligned}\mathcal{L}_{(a,v)^0} = \kappa \int dz\, \frac{1}{2} K^{-1/3}\, \mathrm{tr}\Bigg[&\left(\frac{1}{2}(\xi_+ F_{\mu\nu}^{A_L}\xi_+^{-1} + \xi_- F_{\mu\nu}^{A_R}\xi_-^{-1}) - [\widehat{\mathcal{A}}_\mu, \widehat{\mathcal{A}}_\nu](1-\psi_0^2) \right)^2 \\ &+ \left(\frac{1}{2}(\xi_+ F_{\mu\nu}^{A_L}\xi_+^{-1} - \xi_- F_{\mu\nu}^{A_R}\xi_-^{-1}) \right)^2 \psi_0^2 \Bigg] \\ = \mathrm{tr}\Bigg[&\frac{1}{2e^2}\left((F_{\mu\nu}^{A_L})^2 + (F_{\mu\nu}^{A_R})^2 \right) + \frac{b_{\mathcal{V}\pi\pi}}{4} U^{-1} F_{\mu\nu}^{A_L} U F^{A_R\,\mu\nu} \\ &- \frac{b_{\mathcal{V}\pi\pi}}{2}(\xi_+ F_{\mu\nu}^{A_L}\xi_+^{-1} + \xi_- F_{\mu\nu}^{A_R}\xi_-^{-1})[\widehat{\mathcal{A}}^\mu, \widehat{\mathcal{A}}^\nu] + \frac{1}{2e_S^2}[\widehat{\mathcal{A}}_\mu, \widehat{\mathcal{A}}_\nu]^2 \Bigg] , \end{aligned} \tag{A·14}$$

where

$$\begin{aligned} e^{-2} &\equiv \frac{\kappa}{4}\int dz\, K^{-1/3}(1+\psi_0^2) , \\ b_{\mathcal{V}\pi\pi} &\equiv \kappa \int dz\, K^{-1/3}(1-\psi_0^2) , \quad e_S^{-2} \equiv \kappa \int dz\, K^{-1/3}(1-\psi_0^2)^2 . \end{aligned} \tag{A·15}$$

Here, e^{-2} is divergent, and we should cut off the z integral to make it finite. Hence, the divergent part $\mathcal{L}_{\mathrm{div}}$ consists simply of the kinetic terms of $A_{L\mu}$ and $A_{R\mu}$:

$$\mathcal{L}_{\mathrm{div}} = \frac{1}{2e^2}\,\mathrm{tr}\left[(F_{\mu\nu}^{A_L})^2 + (F_{\mu\nu}^{A_R})^2 \right] . \tag{A·16}$$

The terms linear in v_μ^n or a_μ^n are

$$\begin{aligned}\mathcal{L}_{(a,v)^1} = \mathrm{tr}\Bigg[&\left(\frac{1}{2}(\xi_+ F^{A_L\,\mu\nu}\xi_+^{-1} + \xi_- F^{A_R\,\mu\nu}\xi_-^{-1}) \right) \\ &\quad \times \left((D_\mu^{\widehat{\mathcal{V}}} v_\nu^n - D_\nu^{\widehat{\mathcal{V}}} v_\mu^n)\, a_{\mathcal{V}v^n} + ([\widehat{\mathcal{A}}_\mu, a_\nu^n] - [\widehat{\mathcal{A}}_\nu, a_\mu^n])\, a_{\mathcal{A}a^n} \right) \\ &- [\widehat{\mathcal{A}}^\mu, \widehat{\mathcal{A}}^\nu]\left((D_\mu^{\widehat{\mathcal{V}}} v_\nu^n - D_\nu^{\widehat{\mathcal{V}}} v_\mu^n)\, b_{v^n\pi\pi} + ([\widehat{\mathcal{A}}_\mu, a_\nu^n] - [\widehat{\mathcal{A}}_\nu, a_\mu^n])\, b_{a^n\pi\pi\pi} \right) \\ &+ \left(\frac{1}{2}(\xi_+ F^{A_L\,\mu\nu}\xi_+^{-1} - \xi_- F^{A_R\,\mu\nu}\xi_-^{-1}) \right) \\ &\quad \times \left((D_\mu^{\widehat{\mathcal{V}}} a_\nu^n - D_\nu^{\widehat{\mathcal{V}}} a_\mu^n)\, a_{\mathcal{A}a^n} + ([\widehat{\mathcal{A}}_\mu, v_\nu^n] - [\widehat{\mathcal{A}}_\nu, v_\mu^n])\, (a_{\mathcal{V}v^n} - b_{v^n\pi\pi}) \right) \Bigg] , \end{aligned} \tag{A·17}$$

where

$$a_{\mathcal{V}v^n} \equiv \kappa \int dz\, K^{-1/3}\psi_{2n-1} , \quad a_{\mathcal{A}a^n} \equiv \kappa \int dz\, K^{-1/3}\psi_0\psi_{2n} , \tag{A·18}$$

$$b_{v^n\pi\pi} \equiv \kappa \int dz\, K^{-1/3}\psi_{2n-1}(1-\psi_0^2) , \quad b_{a^n\pi\pi\pi} \equiv \kappa \int dz\, K^{-1/3}\psi_0\psi_{2n}(1-\psi_0^2) . \tag{A·19}$$

The terms quadratic in (v_μ^n, a_μ^n) are

$$\begin{aligned}\mathcal{L}_{(a,v)^2} = \mathrm{tr}\Bigg[\, & \frac{1}{2}(D_\mu^{\widehat{\mathcal{V}}} v_\nu^n - D_\nu^{\widehat{\mathcal{V}}} v_\mu^n)^2 \\ & + \frac{1}{2}([\widehat{\mathcal{A}}_\mu, a_\nu^n] - [\widehat{\mathcal{A}}_\nu, a_\mu^n])([\widehat{\mathcal{A}}^\mu, a^{m\,\nu}] - [\widehat{\mathcal{A}}^\nu, a^{m\,\mu}])\, c_{a^n a^m \pi\pi} \\ & + (D_\mu^{\widehat{\mathcal{V}}} v_\nu^n - D_\nu^{\widehat{\mathcal{V}}} v_\mu^n)([\widehat{\mathcal{A}}^\mu, a^{m\,\nu}] - [\widehat{\mathcal{A}}^\nu, a^{m\,\mu}])\, c_{v^n a^m \pi} \\ & + \frac{1}{2}\left(\xi_+ F^{A_L\,\mu\nu}\xi_+^{-1} + \xi_- F^{A_R\,\mu\nu}\xi_-^{-1}\right)\left([v_\mu^n, v_\nu^n] + [a_\mu^n, a_\nu^n]\right) \\ & - [\widehat{\mathcal{A}}^\mu, \widehat{\mathcal{A}}^\nu]\left([v_\mu^n, v_\nu^m](\delta_{nm} - c_{v^n v^m \pi\pi}) + [a_\mu^n, a_\nu^m](\delta_{nm} - c_{a^n a^m \pi\pi})\right) \\ & + \frac{1}{2}(D_\mu^{\widehat{\mathcal{V}}} a_\nu^n - D_\nu^{\widehat{\mathcal{V}}} a_\mu^n)^2 \\ & + \frac{1}{2}([\widehat{\mathcal{A}}_\mu, v_\nu^n] - [\widehat{\mathcal{A}}_\nu, v_\mu^n])([\widehat{\mathcal{A}}^\mu, v^{m\,\nu}] - [\widehat{\mathcal{A}}^\nu, v^{m\,\mu}])\, c_{v^n v^m \pi\pi} \\ & + (D_\mu^{\widehat{\mathcal{V}}} a_\nu^m - D_\nu^{\widehat{\mathcal{V}}} a_\mu^m)([\widehat{\mathcal{A}}^\mu, v^{n\,\nu}] - [\widehat{\mathcal{A}}^\nu, v^{n\,\mu}])\, c_{v^n a^m \pi} \\ & + \frac{1}{2}\left(\xi_+ F^{A_L\,\mu\nu}\xi_+^{-1} - \xi_- F^{A_R\,\mu\nu}\xi_-^{-1}\right)([v_\mu^n, a_\nu^m] - [v_\nu^n, a_\mu^m])\, c_{v^n a^m \pi}\Bigg]\,, \end{aligned} \tag{A·20}$$

where

$$\begin{aligned} c_{v^n a^m \pi} &\equiv \kappa \int dz\, K^{-1/3}\psi_0\psi_{2n-1}\psi_{2m}\ , \\ c_{a^n a^m \pi\pi} &\equiv \kappa \int dz\, K^{-1/3}\psi_0^2\psi_{2n}\psi_{2m}\ , \\ c_{v^n v^m \pi\pi} &\equiv \kappa \int dz\, K^{-1/3}\psi_0^2\psi_{2n-1}\psi_{2m-1}\ . \end{aligned} \tag{A·21}$$

Similarly, $\mathcal{L}_{(a,v)^3}$ and $\mathcal{L}_{(a,v)^4}$ are

$$\begin{aligned}\mathcal{L}_{(a,v)^3} = \mathrm{tr}\Bigg[& (D_\mu^{\widehat{\mathcal{V}}} v_\nu^n - D_\nu^{\widehat{\mathcal{V}}} v_\mu^n)\left([v^{p\,\mu}, v^{q\,\nu}]\, g_{v^n v^p v^q} + [a^{p\,\mu}, a^{q\,\nu}]\, g_{v^n a^p a^q}\right) \\ & + \Big([\widehat{\mathcal{A}}_\mu, a_\nu^n] - [\widehat{\mathcal{A}}_\nu, a_\mu^n]\Big)\left([v^{p\,\mu}, v^{q\,\nu}]\, g_{a^n v^p v^q} + [a^{p\,\mu}, a^{q\,\nu}]\, g_{a^n a^p a^q}\right) \\ & + \Big((D_\mu^{\widehat{\mathcal{V}}} a_\nu^n - D_\nu^{\widehat{\mathcal{V}}} a_\mu^n)\, g_{v^p a^n a^q} + ([\widehat{\mathcal{A}}_\mu, v_\nu^n] - [\widehat{\mathcal{A}}_\nu, v_\mu^n])\, g_{a^q v^p v^n}\Big) \\ & \quad \times ([v^{p\,\mu}, a^{q\,\nu}] - [v^{p\,\nu}, a^{q\,\mu}])\Bigg]\,, \end{aligned} \tag{A·22}$$

and

$$\mathcal{L}_{(a,v)^4} = \mathrm{tr}\Bigg[\frac{1}{2}[v_\mu^m, v_\nu^n][v^{p\,\mu}, v^{q\,\nu}]\, g_{v^m v^n v^p v^q} + \frac{1}{2}[a_\mu^m, a_\nu^n][a^{p\,\mu}, a^{q\,\nu}]\, g_{a^m a^n a^p a^q}$$

$$+\left([v_\mu^m, v_\nu^n][a^{p\,\mu}, a^{q\,\nu}] + [v_\mu^m, a_\nu^p][v^{n\,\mu}, a^{q\,\nu}] - [v_\mu^m, a_\nu^p][v^{n\,\nu}, a^{q\,\mu}]\right) g_{v^m v^n a^p a^q}\Bigg] \ , \tag{A·23}$$

where

$$\begin{aligned}
g_{v^n v^p v^q} &\equiv \kappa \int dz\, K^{-1/3}\psi_{2n-1}\psi_{2p-1}\psi_{2q-1} \ , \\
g_{v^n a^p a^q} &\equiv \kappa \int dz\, K^{-1/3}\psi_{2n-1}\psi_{2p}\psi_{2q} \ , \\
g_{a^n v^p v^q} &\equiv \kappa \int dz\, K^{-1/3}\psi_0\psi_{2n}\psi_{2p-1}\psi_{2q-1} \ , \\
g_{a^n a^p a^q} &\equiv \kappa \int dz\, K^{-1/3}\psi_0\psi_{2n}\psi_{2p}\psi_{2q} \ , \\
g_{v^m v^n v^p v^q} &\equiv \kappa \int dz\, K^{-1/3}\psi_{2m-1}\psi_{2n-1}\psi_{2p-1}\psi_{2q-1} \ , \\
g_{a^m a^n a^p a^q} &\equiv \kappa \int dz\, K^{-1/3}\psi_{2m}\psi_{2n}\psi_{2p}\psi_{2q} \ , \\
g_{v^m v^n a^p a^q} &\equiv \kappa \int dz\, K^{-1/3}\psi_{2m-1}\psi_{2n-1}\psi_{2p}\psi_{2q} \ .
\end{aligned} \tag{A·24}$$

Appendix B
—— *Calculation of* $\int_5 \omega_5(A)$ ——

Here we outline the derivation of (4·5). Because we are working in the $A_z = 0$ gauge, we have

$$\omega_5(A) = \mathrm{tr}\left(AdAdA + \frac{3}{2}A^3 dA\right) \ . \tag{B·1}$$

We then find

$$\begin{aligned}
\int_5 \mathrm{tr}\, AdAdA &= \int_5 \mathrm{tr}\left((v+a)d(v+a)d(v+a)\right) \\
&= \int_5 \mathrm{tr}\left(v\,dv\,da + v\,da\,dv + a\,dv\,dv + a\,da\,da\right) \\
&= \int_5 \mathrm{tr}\left(-d(v\,dv\,a + v\,a\,dv) + 3\,a\,dv\,dv + a\,da\,da\right) \\
&= \frac{1}{2}\int_4 \mathrm{tr}((A_+A_- - A_-A_+)\,d(A_+ + A_-)) + \int_5 (3\,a\,dv\,dv + a\,da\,da) \ ,
\end{aligned} \tag{B·2}$$

$$\begin{aligned}
\int_5 \mathrm{tr}\, A^3 dA &= \int_5 \mathrm{tr}(v+a)^3 d(v+a) \\
&= \int_5 \mathrm{tr}\left((v^3 + v\,a^2 + a\,v\,a + a^2 v)\,da + (a\,v^2 + v\,a\,v + v^2 a + a^3)\,dv\right)
\end{aligned}$$

$$
\begin{aligned}
&= \int_5 \mathrm{tr}\,\big(2(v^2a + a\,v^2 + a^3)\,dv + 2\,a\,v\,a\,da - d(v^3a + v\,a^3 + a\,v\,a^2 + a^2v\,a)\big) \\
&= 2\int_5 \mathrm{tr}\,\big((v^2a + a\,v^2 + a^3)\,dv + a\,v\,a\,da\big) \\
&\qquad - \frac{1}{8}\int_4 \mathrm{tr}\,\big((A_+ + A_-)^3(A_+ - A_-) + (A_+ + A_-)(A_+ - A_-)^3\big)\ . \qquad (\mathrm{B}\cdot 3)
\end{aligned}
$$

The first line on the right-hand side of (B·3) contains some terms that do not include vector mesons:

$$
\begin{aligned}
2\int_5 \mathrm{tr}(a\,v\,a\,da) &= \frac{1}{8}\int_5 \mathrm{tr}\,((A_+ - A_-)\psi_0(A_+ + A_-)(A_+ - A_-)\psi_0 d\,((A_+ - A_-)\psi_0)) \\
&\qquad + 2\int_5 \Big[\mathrm{tr}(a\,v\,a\,da)\Big]_{\text{non-zero}} \\
&= \frac{1}{12}\int_4 \mathrm{tr}\,\big((A_+ + A_-)(A_+ - A_-)^3\big) + 2\int_5 \Big[\mathrm{tr}(a\,v\,a\,da)\Big]_{\text{non-zero}}\ . \\
&\qquad\qquad (\mathrm{B}\cdot 4)
\end{aligned}
$$

As above, $[\cdots]_{\text{non-zero}}$ extracts the terms that include at least one vector meson. Then using (B·4), (B·3) can be rewritten as

$$
\begin{aligned}
\int_5 \mathrm{tr}\,A^3 dA &= 2\int_5 \mathrm{tr}\left((v^2a + a\,v^2 + a^3)\,dv + \Big[a\,v\,a\,da\Big]_{\text{non}-\text{zero}}\right) \\
&\qquad + \frac{1}{3}\int_4 \mathrm{tr}\left[\frac{1}{2}A_+A_-A_+A_- + (A_+^3A_- - A_-^3A_+)\right]\ . \qquad (\mathrm{B}\cdot 5)
\end{aligned}
$$

Finally, combining (B·2) and (B·3), we obtain (4·5).

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