MORE ON THE DIFFERENTIABILITY OF THE RIEMANN FUNCTION.

By Joseph Gerver.

1. Introduction. This paper is a continuation of a previous paper [1] by the author, in which it was shown that the function $\sum_{k=1}^{\infty} \frac{\sin k^2 x}{k^2}$, which Riemann [2, 3] had believed to be nowhere differentiable, has in fact a derivative of $\frac{-1}{2}$ at all points $(2A+1)\pi/(2B+1)$, where A and B are integers.

Here we will prove that the Riemann function is not differentiable at any points of the form $2A\pi/(2B+1)$ or $(2A+1)\pi/2B$. Together with Hardy's result [4] that the function is not differentiable at any irrational multiple of π, this completely solves the problem of differentiability.

As part of the proof, we will demonstrate a simple necessary condition for the differentiability of the Riemann function at rational multiples of π, a condition which, it seems likely, can be shown to hold for a large class of functions of the form $\sum_{k=1}^{\infty} \frac{\sin f(k) x}{f(k)}$; probably f can be any polynomial of degree $\geqq 2$.

Unfortunately , it is not clear whether or not the condition is sufficient for differentiability, even in the case of the Riemann function. The condition is as follows:

Theorem 1. *$\sum_{k=1}^{\infty} \frac{\sin k^2 x}{k^2}$ is differentiable at $\alpha\pi/\beta$ only if $\sum_{m=1}^{2\beta} \sin m^2 x$ and $\sum_{m=1}^{2\beta} \cos m^2 x$ are both equal to 0 at $\alpha\pi/\beta$; α, β integers.*

If $\cos m^2 x$ and $\sin m^2 x$ are viewed as the real and imaginary parts respectively of a complex number, we find that the function is differentiable only if $\sum_{m=1}^{2\beta} e^{\alpha m^2 \pi i/\beta} = 0$. This can be simplified to $\sum_{m=1}^{2\beta} e^{\kappa \pi i/\beta}$, where $\kappa \equiv \alpha m^2 \bmod 2\beta$; $0 \leqq \kappa < 2\beta$. Since $e^{\pi i/\beta}$ is a primitive 2β-th root of unity, it is sufficient to prove that the 2β-th cyclotomic polynomial (i.e. the irreducible polynomial of the primitive 2β-th roots of unity) does not divide the polynomial

Received October 3, 1969.

$\sum_{m=1}^{2\beta} X^{\kappa}$. This is relatively easy to show in the case where $\alpha/\beta = 2A/(2B+1)$ or $(2A+1)/4(2B+1)$. The second case, and the case $(2A+1)/2(2B+1)$ which was taken care of by Hardy [4], can then be extended to

$$(2A+1)/4^N(2B+1) \text{ and } (2A+1)/2^{2N+1}(2B+1)$$

respectively by induction on N.

To prove Theorem 1, we need two lemmas, both of which are very similar to lemmas proved in the author's last paper [1]. Since they can be proved using methods described in [1], we state them without proof here:

LEMMA 1. *Let μ, λ be any integers such that $0 < \mu \leqq \lambda$. Then, for all sufficiently large n,*

$$|x| < \frac{1}{n^{15}} \Rightarrow \left| \sum_{k=b}^{\infty} \frac{\sin(\lambda k + \mu)^2 x}{(\lambda k + \mu)^2} \right| < \left| \frac{\gamma x}{n} \right|$$

with an appropriate constant γ, where b is the least integer greater than $\left| \frac{\pi}{n\lambda^2 x} \right|$; $x \neq 0$.

LEMMA 2. *Let λ be as above and let τ be any real number. Then, for all sufficiently large n, the following hold:*

If $\sin\tau\cos\tau > 0$, *let* $\frac{-1}{n^3} < x < 0$

If $\sin\tau\cos\tau < 0$, *let* $0 < x < \frac{1}{n^3}$

If $\sin\tau = 0$, *let* $|x| < \frac{1}{n^3}$.

Then $\frac{1}{x\cos\tau} \sum_{k=1}^{z} \frac{\sin[(\lambda k)^2 x + \tau] - \sin\tau}{(\lambda k)^2} > n$ *where z is any integer greater than* $\sqrt{\frac{\pi}{|x|}}$; $x\cos\tau \neq 0$.

Given these two lemmas, the proof of Theorem 1 runs roughly as follows: We know that $\frac{\sin m^2(x + \alpha\pi/\beta)}{m^2} = \frac{\sin(m^2 x + \kappa\pi/\beta)}{m^2}$ where $\kappa \equiv \alpha m^2 \bmod 2\beta$.

Fix n and let $b \approx \left| \frac{\pi}{4n\beta^2 x} \right|$. For x sufficiently close to 0, and $k < b$, $\frac{\sin[(2\beta k - m)^2 x + \kappa\pi/\beta]}{(2\beta k - m)^2}$ can be approximated by $\frac{\sin[(2\beta k)^2 x + \kappa\pi/\beta]}{(2\beta k)^2}$.

Then $\sum_{m=1}^{2\beta} \frac{\sin[2\beta k)^2 x + \kappa\pi/\beta]}{(2\beta^2 k)^2}$ can be written in the form

$$\frac{t\sin[(2\beta k)^2 x + \tau] - t\sin\tau}{(2\beta k)^2}$$

with appropriate constants t and τ, where the term $-t\sin\tau$ insures that the function be equal to 0 at 0.

Unless $\sum_{m=1}^{2\beta}\cos m^2(\alpha\pi/\beta)=0$ (in which case $\cos\tau=0$), we can apply Lemma 2 and we find that $\sum_{m=1}^{2\beta(b-1)}\frac{\sin m^2(x+\alpha\pi/\beta)}{m^2}$ becomes very large in absolute value near zero, on one side or the other. Furthermore, the function is positive on that side of zero if $\sum_{m=1}^{2\beta}\sin m^2(\alpha\pi/\beta)\leqq 0$ and negative if $\sum_{m=1}^{2\beta}\sin m^2(\alpha\pi/\beta)\leqq 0$.

For the tail end, we consider $\sum_{k=b}^{\infty}\frac{\sin[(2\beta k+m)^2x+\kappa\pi/\beta]-\sin\kappa\pi/\beta}{(2\beta k+m)^2}$ for each m from 1 to 2β separately and apply Lemma 1. Since, ignoring the term $-\sin\kappa\pi/\beta$, the series goes to 0; when this term is included, the series must approach $-\sin\kappa\pi/\beta\sum_{k=b}^{\infty}\frac{1}{(2\beta k+m)^2}$ which is approximately $-\sin\kappa\pi/\beta\sum_{k=b}^{\infty}\frac{1}{(2\beta k)^2}=\frac{-\sin\kappa\pi/\beta}{2\beta b}=-\left|\frac{2n\beta x}{\pi}\right|\sin\kappa\pi/\beta$, so the series becomes very large in absolute value near zero, on both sides, and positive or negative depending on whether $\sum_{m=1}^{2\beta}\sin m^2(\alpha\pi/\beta)$ is less than or greater than 0 respectively.

2. A necessary condition for differentiability. Let $\kappa\equiv\alpha m^2 \bmod 2\beta$, $0\leqq\kappa<2\beta$ and let $\omega=\kappa\pi/\beta$. Keep in mind that κ and ω are functions of m.

We will assume that $\sum_{m=1}^{2\beta}\sin\omega$ and $\sum_{m=1}^{2\beta}\cos\omega$ are not both equal to 0, and then show that $\sum_{m=1}^{\infty}\frac{\sin(m^2x+\omega)}{m^2}$ is not differentiable at 0. Specifically, we will prove that either the right or left derivative is equal to $+\infty$ or $-\infty$; that is, $\frac{1}{x}\sum_{m=1}^{\infty}\frac{\sin(m^2x+\omega)-\sin\omega}{m^2}$ is greater than n (or less than $-n$) for any n, given x sufficiently close to 0, possibily with the condition that x must be >0 or <0.

Let a be the least integer greater than $\frac{1}{2\beta}\sqrt{\frac{\pi}{|2x|}}$, let c be the least integer greater than $\left|\frac{\pi}{4\beta^2x}\right|$ and let z be any integer such that $\sqrt{\frac{\pi}{|x|}}<z<c$.

We will now show that if x is close to 0, then $\sum_{m=1}^{2\beta z}\frac{\sin(m^2x+\omega)-\sin\omega}{m^2}$ is close to $\sum_{k=1}^{z}\sum_{m=1}^{2\beta}\frac{\sin[(2\beta k)^2x+\omega]-\sin\omega}{(2\beta k)^2}$.

First, consider $k < a$.

The derivatives of

$$\frac{\sin[(2\beta k - m)^2 x + \omega] - \sin\omega}{(2\beta k - m)^2} \text{ and } \frac{\sin[(2\beta k)^2 x + \omega] - \sin\omega}{(2\beta k)^2}$$

are $\cos[(2\beta k - m)^2 x + \omega/(2\beta k - m)^2]$ and $\cos[(2\beta k)^2 x + \omega/(2\beta k^2]$ respectively. Since the double derivatives,

$$-(2\beta k - m)^2 \sin[(2\beta k - m)^2 x + \omega/(2\beta k - m)^4]$$
$$\text{and } -(2\beta k)^2 \sin[(2\beta k)^2 x + \omega/(2\beta k)^4],$$

never exceed $4\beta^2 k^2$ in absolute value, the absolute value of the difference of the derivatives cannot exceed $|4\beta^2 k^2 x[1 - (2\beta k - m)^2/(2\beta k)^2]|$, which is less than $|9\beta^2 kx|$, assuming $m \leqq 2\beta$. Therefore, since

$$\frac{\sin[(2\beta k - m)^2 x + \omega] - \sin\omega}{(2\beta k - m)^2} \text{ and } \frac{\sin[(2\beta k)^2 x + \omega] - \sin\omega}{(2\beta k)^2}$$

are both equal to 0 at 0, the absolute value of their difference must be less than $9\beta^2 kx^2$, which is less than $\frac{9\beta}{2} x^2 \sqrt{\frac{\pi}{2|x|}}$, since $k < a$. Summing over all $m \leqq 2\beta(a-1)$, we get

$$\left| \sum_{m=1}^{2\beta(a-1)} \frac{\sin(m^2 x + \omega) - \sin\omega}{m^2} - \sum_{k=1}^{a-1} \sum_{m=1}^{2\beta} \frac{\sin[(2\beta k)^2 x + \omega] - \sin\omega}{(2\beta k)^2} \right| < \left| \frac{9\pi}{4} \beta x \right|.$$

Now we will consider $a \leqq k < z$.

$$\left| \frac{\sin[(2\beta k - m)^2 x + \omega] - \sin\omega}{(2\beta k - m)^2} - \frac{\sin[(2\beta k - m)^2 x + \omega] - \sin\omega}{(2\beta k)^2} \right|$$
$$< \frac{1}{(2\beta k - m)^2} - \frac{1}{(2\beta k)^2} < \frac{1}{\beta^2 k^3}$$

and

$$\left| \frac{\sin[(2\beta k - m)^2 x + \omega] - \sin\omega}{(2\beta k)^2} - \frac{\sin[(2\beta k)^2 x + \omega] - \sin\omega}{(2\beta k)^2} \right|$$
$$< |x| \left(1 - \frac{(2\beta k - m)^2}{(2\beta k)^2}\right) < \left| \frac{3x}{k} \right|,$$

since neither of the derivatives exceed 1. Therefore

$$\left| \frac{\sin[(2\beta k - m)^2 x + \omega] - \sin\omega}{(2\beta k - m)^2} - \frac{\sin[(2\beta k)^2 x + \omega] - \sin\omega}{(2\beta k)^2} \right|$$
$$< \frac{1}{\beta^2 k^3} + \left| \frac{3x}{k} \right| < \left| \frac{4x}{k} \right|, \text{ since } k \geqq a.$$

Finally,

$$\left| \sum_{m=2\beta(a-1)+1}^{2\beta z} \frac{\sin(m^2x+\omega)-\sin\omega}{m^2} - \sum_{k=a}^{z}\sum_{m=1}^{2\beta} \frac{\sin[(2\beta k)^2x+\omega]-\sin\omega}{(2\beta k)^2} \right|$$

$$< 2\beta \sum_{k=a}^{z} \left|\frac{4x}{k}\right| < |8\beta x| \log c < |8\beta x| \log \frac{1}{|x|}.$$

Since $|8\beta x| \log \frac{1}{|x|}$ dominates $\left|\frac{9\pi}{4}\beta x\right|$, we can ignore the latter term.

Now $\sum_{m=1}^{2\beta} \frac{\sin[(2\beta k)^2x+\omega]-\sin\omega}{(2\beta k)^2}$ can be expressed in the form

$$\frac{t\sin[(2\beta k)^2x+\tau]-t\sin\tau}{(2\beta k)^2}$$

with appropriate constants t and τ, since all the terms in the series have the same period. It is clear that $t\sin\tau$ and $t\cos\tau$ (the value and derivative respectively at 0) are equal to $\sum_{m=1}^{2\beta}\sin\omega$ and $\sum_{m=1}^{2\beta}\cos\omega$. Therefore $t \neq 0$ unless $\sum_{m=1}^{2\beta}\sin\omega = \sum_{m=1}^{2\beta}\cos\omega = 0$. This means that if $\sum_{m=1}^{2\beta}\cos\omega \neq 0$, we can apply Lemma 2, and conclude that

$$\frac{1}{tx\cos\tau}\sum_{k=1}^{z}\sum_{m=1}^{2\beta}\frac{\sin[(2\beta k)^2x+\omega]-\sin\omega}{(2\beta k)^2} > \sqrt[3]{\frac{1}{|x|}},$$

provided $x\sin\tau\cos\tau < 0$ or $\sin\tau = 0$. Since $tx\cos t\sqrt[3]{\frac{1}{|x|}}$ dominates $8\beta x\log\frac{1}{|x|}$, $\frac{1}{tx\cos\tau}\sum_{m=1}^{2\beta z}\frac{\sin(m^2x+\omega)-\sin\omega}{m^2} > \frac{1}{2}\sqrt[3]{\frac{1}{|x|}}$ for $|x|$ sufficiently small. In other words, $|x| < \frac{1}{8n^3} \Rightarrow \frac{1}{tx\cos\tau}\sum_{m=1}^{2\beta z}\frac{\sin(m^2x+\omega)-\sin\omega}{m^2} > n.$

Note that since $x\sin\tau\cos\tau < 0$, $\sum_{m=1}^{2\beta z}\frac{\sin(m^2x+\omega)-\sin\omega}{m^2}$ must have the opposite sign of $t\sin\tau$, on one side of zero, unless $\sin\tau = 0$.

Furthermore, if $\sum_{m=1}^{2\beta}\cos\omega = 0$, but $\sum_{m=1}^{2\beta}\sin\omega \neq 0$, then

$$\sum_{k=1}^{z}\sum_{m=1}^{2\beta}\frac{\sin[(2\beta k)^2x+\omega]-\sin\omega}{(2\beta k)^2} \leqq 0$$

for all x, so

$$\sum_{m=1}^{2\beta z}\frac{\sin(m^2x+\omega)-\sin\omega}{m^2} < |8\beta x|\log\frac{1}{|x|} \text{ or } > -|8\beta x|\log\frac{1}{|x|},$$

depending on whether $t\sin\tau$ is greater or less than 0 respectively.

Now we will take care of the tail end.

Fix n. Let $z = b - 1$, where b is the least integer greater than $\left| \frac{\pi}{4n\beta^2 x} \right|$. We will prove that as x approaches 0, $\sum_{m=2\beta b}^{\infty} \frac{\sin(m^2 x + \omega) - \sin\omega}{m^2}$ gets much larger in absolute value than $|8\beta x| \log \frac{1}{|x|}$, moving in the same direction as $\sum_{m=1}^{2\beta(b-1)} \frac{\sin(m^2 x + \omega) - \sin\omega}{m^2}$, for values of x approaching 0 from at least one side.

For each m from 1 to 2β, $\left| \sum_{k=b}^{\infty} \frac{\sin[(2\beta k + m)^2 x + \omega]}{(2\beta k + m)^2} \right| < \left| \frac{\gamma x}{n} \right|$, with some constant γ, if $|x| < \frac{1}{n^{15}}$ (Lemma 1). Therefore,

$$\sum_{m=2\beta b+1}^{\infty} \frac{\sin(m^2 x + \omega) - \sin\omega}{m^2}$$

is very close to $-\sum_{m=2\beta b+1}^{\infty} \frac{\sin\omega}{m^2}$.

Since $\left| \frac{\sin\omega}{(2\beta k)^2} - \frac{\sin\omega}{(2\beta k + m)^2} \right| < \left| \frac{nx}{\pi\beta k^2} \right|$ for $1 \leqq m \leqq 2\beta$, $k \geqq b$,

$$\left| \sum_{m=2\beta b+1}^{\infty} \frac{\sin\omega}{m^2} - \sum_{k=b}^{\infty} \sum_{m=1}^{2\beta} \frac{\sin\omega}{(2\beta k)^2} \right| < \frac{8n^2\beta^2 x^2}{\pi^2} < \left| \frac{x}{n} \right|$$

and

$$\left| \sum_{m=2\beta b+1}^{\infty} \frac{\sin(m^2 x + \omega) - \sin\omega}{m^2} + \sum_{k=b}^{\infty} \sum_{m=1}^{2\beta} \frac{\sin\omega}{(2\beta k)^2} \right| < \left| \frac{(\gamma + 1)x}{n} \right|$$

Now $\sum_{k=b}^{\infty} \sum_{m=1}^{2\beta} \frac{\sin\omega}{(2\beta k)^2} = \frac{1}{4\beta^2} \sum_{k=b}^{\infty} \frac{1}{k^2} \cdot \sum_{m=1}^{2\beta} \sin\omega$ and $\sum_{k=b}^{\infty} \frac{1}{k^2} > \frac{1}{b} > \left| \frac{3n\beta^2 x}{\pi} \right|$.

Therefore, using the $t \sin\tau$ notation for $\sum_{m=1}^{2\beta} \sin\omega$,

$$\frac{1}{t \sin\tau} \sum_{m=2\beta b+1}^{\infty} \frac{\sin(m^2 x + \omega) - \sin\omega}{m^2} < -\left| \frac{2n\beta^2 x}{\pi} \right|, \quad t \sin\tau \neq 0.$$

In other words, $\sum_{m=2\beta b+1}^{\infty} \frac{\sin(m^2 x + \omega) - \sin\omega}{m^2}$ must have the opposite sign of $t \sin\tau$ on both sides of 0, unless $\sin\tau = 0$.

This leaves out those values of m from $2\beta(b-1) + 1$ to $2\beta b$, but the sum from all these values cannot exceed $1/2\beta(b-1)^2$, so they can be ignored.

Therefore, if $|x| < \frac{1}{n^{15}}$ and $x \sin\tau \cos\tau < 0$, then

$$\frac{1}{tx \cos\tau} \sum_{m=1}^{\infty} \frac{\sin(m^2 x + \omega) - \sin\omega}{m^2} > n.$$

If $t\cos\tau = 0$ and $t\sin\tau \neq 0$, then

$$\frac{1}{t\sin\tau}\sum_{m=1}^{\infty}\frac{\sin(m^2x+\omega)-\sin\omega}{m^2} < -\left|\frac{n\beta^2 x}{\pi}\right|,$$

since $\dfrac{t(\sin\tau)n\beta^2 x}{\pi}$ dominates $8\beta x\log\dfrac{1}{|x|}$.

Finally, if $t\sin\tau = 0$ and $t\cos\tau \neq 0$, the function has a full derivative of $+\infty$ or $-\infty$.

This completes the proof of Theorem 1.

3. Points of non-differentiability. We will now prove the following theorem:

THEOREM 2. *The function* $\sum_{k=1}^{\infty}\dfrac{\sin k^2 x}{k^2}$ *is not differentiable at any point* $\pi 2A/(2B+1)$ *or* $\pi(2A+1)/2B$; A, B *integers.*

As was pointed out in the introduction, the necessary condition of Theorem 1 is equivalent to the condition that the 2β-th cyclotomic polynomial divides the polynomial $\sum_{m=1}^{2\beta} X^{\kappa}$.

We will prove that this condition does not hold in the following two cases:

1) $\alpha/\beta = 2A/(2B+1)$

2) $\alpha/\beta = (2A+1)/4(2B+1)$.

In addition, we have a third case, which was proved by Hardy [4]:

3) $\alpha/\beta = (2A+1)/2(2B+1)$.

In case 1), we have

$$\begin{aligned}2A(2B+1+m)^2 &= 2A(2B+1)^2+4A(2B+1)m+2Am^2\\ &\equiv 2Am^2 \bmod 2(2B+1)\end{aligned}$$

and

$$\begin{aligned}2A(2B+1-m)^2 &= 2A(2B+1)^2-4A(2B+1)m+2Am^2\\ &\equiv 2Am^2 \bmod 2(2B+1)\end{aligned}$$

so $\sum_{m=1}^{2\beta} X^{\kappa} = 2\sum_{m=1}^{\beta} X^{\kappa}$, and all the coefficients of $\sum_{m=1}^{\beta} X^{\kappa}$ are even except the final one ($\kappa = 0$), which is odd.

In case 2), we need only consider odd values of m, since $\alpha(2m)^2\pi/8(2B+1) = \alpha m^2\pi/2(2B+1)$ and we already know that $\sum_{m=1}^{2(2B+1)} e^{\alpha m^2\pi/2(2B+1)} = 0$ [1].

We have

$$
\begin{aligned}
(2A+1)[2(2B+1)+m]^2 &= 4(2A+1)(2B+1)^2 \\
&\quad + 4(2A+1)(2B+1)\mathrm{m} + (2A+1)m^2 \\
&= 4(2A+1)(2B+1)(2B+1+m) + (2A+1)m^2 \\
&\equiv (2A+1)m^2 \bmod 8(2B+1), \text{ since } m \text{ is odd.}
\end{aligned}
$$

Therefore $\sum_{m(\mathrm{odd})=1}^{2\beta-1} X^{\kappa} = 4 \sum_{m(\mathrm{odd})=1}^{2(2B+1)-1} X^{\kappa}$ and all the coefficients of $\sum_{m(\mathrm{odd})=1}^{2(2B+1)} X^{\kappa}$ are even, except the one corresponding to $m = 2B+1$, which is odd.

Now let $P(X)$ be either $\sum_{m=1}^{2B+1} X^{\kappa}$, where $\beta = 2B+1$, or $\sum_{m(\mathrm{odd})=1}^{2(2B+1)-1} X^{\kappa}$, where $\beta = 4(2B+1)$. Also let $F(X)$ be the 2β-th cyclotomic polynomial and let $Q(X) = P(X)/F(X)$. Since $P(X)$ has one odd coefficient, $Q(X)$ must have at least one odd coefficient also.. Let u and v be the greatest and least exponents respectively of $Q(X)$ with odd coefficients. Then, since the initial and final coefficients of $F(X)$ are both equal to 1, $P(X)$ must have at least two distinct odd coefficients, namely those with exponents $\phi(2\beta)+u$ and v, were $\phi(2\beta)$ is the degree of $F(X)$. This contradicts our assumption that $P(X)$ has exactily one odd coefficient. Therefore $F(X)$ does not divide $P(X)$ and the necessary condition of Theorem 1 does not hold in either of the two cases examined.

Now we will extend cases 2) and 3) to $(2A+1)/4^N(2B+1)$ and $(2A+1)/2^{2N+1}(2B+1)$ respectively, by induction on N.

We know that $\sum_{m=1}^{2M(2B+1)} \exp[(2A+1)m^2\pi/2^M(2B+1)]$ is equal to

$$
\sum_{m(\mathrm{even})=2}^{2^{M+2}(2B+1)} \exp[(2A+1)m^2\pi/2^{M+2}(2B+1)]
$$

so we need only consider

$$
\sum_{m(\mathrm{odd})=1}^{2^{M+2}(2B+1)-1} \exp[(2A+1)m^2\pi/2^{M+2}(2B+1)].
$$

Now,

$$
\begin{aligned}
&(2A+1)[2^M(2B+1)+m]^2 \\
&\quad = 2^{2M}(2A+1)(2B+1)^2 + 2^{M+1}(2A+1)(2B+1)m + (2A+1)m^2 \\
&\quad \equiv 2^{M+1}(2B+1) + (2A+1)m^2 \bmod 2^{M+2}(2B+1) \\
&\qquad \text{if } m \text{ is odd and } M \geqq 2.
\end{aligned}
$$

Since

$$
\exp[\kappa\pi i/2^{M+2}(2B+1)] + \exp\{[2^{M+1}(2B+1)+\kappa]\pi i/2^{M+2}(2B+1)\} = 0,
$$

we have $\sum_{m(\text{odd})=1}^{2^{M+2}(2B+1)} \exp[(2A+1)m^2\pi i/2^{M+2}(2B+1)] = 0$, which completes the proof for case 2). But since the κ and $2^{M+1}(2B+1)+\kappa$ terms cancel, we know, from [1], that the sum taken over these two terms, and therefore the sum over all odd terms, is differentiable, so we can also extend case 3).

Added in Proof. The converse of Theorem 1 is true. Indeed, if $\sum_{m=1}^{2\beta} e^{\alpha\pi i f(m)/\beta} = 0$, then the derivative exists and is equal to

$$\frac{1}{2\beta}\sum_{m=1}^{2\beta}(m-1)\cos\alpha f(m)/\beta.$$

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