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MORE-OR-LESS-UNIFORM SAMPLING AND LENGTHS OF CURVES

By

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Abstract. More-or-less-uniform samples are introduced and used to estimate lengths of smooth regular strictly convex curves in $\mathbb{R}^2$. Quartic convergence is proved and illustrated by examples.

1. Introduction. The problem of measuring the length of a curve has a long history in mathematics, dating back to ancient geometry. In particular, Archimedes and Liu Hui [11] estimated the length of a circular curve. Jordan, Peano and others introduced digitizations of sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ for the purpose of different feature measurements such as perimeters (see, e.g., [4]). Related historical and contemporary work can be found in [2], [3], [6], [7], [9], [10], and [12].

Let $\gamma\colon [0,T] \to \mathbb{R}^n$ be a smooth regular curve; namely, γ is C^k for some $k \geq 1$ and $\dot\gamma(t) \neq \mathbf{0}$ for all $t \in [0,T]$. The *length* of γ is defined to be

$$d(\gamma) = \int_0^T \|\dot\gamma(t)\| dt \ ,$$

where $\dot\gamma$ is the derivative of γ, and $\|\cdot\|$ is the Euclidean norm. Consider the problem of estimating $d(\gamma)$ from an ordered $(m+1)$-tuple

$$\mathcal{Q} = (q_0, q_1, \ldots, q_m)$$

of points in $\mathbb{R}^n$, where $q_i = \gamma(t_i)$, and $0 = t_0 < t_1 < \cdots < t_i < \cdots < t_m = T$. Depending on what is known about the t_i, the problem may be straightforward or unsolvable.

EXAMPLE 1. Let γ be C^{r+2}, where r is a positive integer, and take m to be a multiple of r. Then $\mathcal{Q}$ gives $\frac{m}{r}$ $(r+1)$-tuples of the form

$$(q_0, q_1, \ldots, q_r), \quad (q_r, q_{r+1}, \ldots, q_{2r}), \ldots \quad , (q_{m-r}, q_{m-r+1}, \ldots, q_m).$$

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The jth $(r+1)$-tuple can be interpolated by a polynomial $\hat{\gamma}_j\colon [t_{(j-1)r}, t_{jr}] \to \mathbb{R}^n$ of degree r, and the track-sum $\hat{\gamma}$ of the $\hat{\gamma}_j$ is everywhere continuous and C^∞ except at the knot points $t_r, t_{2r}, \ldots, t_{m-r}$. Suppose that sampling is uniform: $t_i = \frac{iT}{m}$ for $0 \le i \le m$. The errors in Lagrange interpolation are best studied using Lemma 2.1 in Sec. 2 of Part I of [5]. We find that $\hat{\gamma}(t) = \gamma(t) + O(\frac{1}{m^{r+1}})$ for $t \in [0,T]$, and $\dot{\hat{\gamma}} = \dot{\gamma}(t) + O(\frac{1}{m^r})$ for $t \neq t_r, t_{2r}, \ldots, t_{m-r}$. Consequently, $d(\hat{\gamma}) - d(\gamma) = O(\frac{1}{m^r})$. This error can be shown to be $O(\frac{1}{m^{r+2}})$ or $O(\frac{1}{m^{r+1}})$ according as r is even or odd [8].

EXAMPLE 2. Let $t_1 = \frac{T}{2}$ and $t_i = t_1 + \frac{iT}{2m}$ for $2 \le i \le m$. $\mathcal{Q}$ gives only endpoint information for γ over $[0, \frac{T}{2}]$, and therefore does not even determine an upper bound on $d(\gamma)$ as $m \to \infty$.

An intermediate situation is where the t_i are not given, but sampled *more-or-less uniformly* in the following sense.

DEFINITION 1. Sampling is *more-or-less uniform* when there are constants $0 < K_l < K_u$ such that, for any sufficiently large integer m, and any $1 \le i \le m$,

$$\frac{K_l}{m} \le t_i - t_{i-1} \le \frac{K_u}{m}.$$

The uniform sampling of Example 1 is more-or-less uniform, and the sampling in Example 2 is not. With more-or-less uniform sampling, increments between successive parameters are neither large nor small in proportion to $\frac{T}{m}$. Then, just as for the uniform sampling of Example 1, piecewise-linear interpolation between sample points approximates the image of γ to $O(\frac{1}{m^2})$, and $d(\gamma)$ to $O(\frac{1}{m^2})$. However, use of piecewise-quadratic instead of piecewise-linear can lead to unfortunate results, because of the need to estimate[1] the parameters t_i for $0 \le i \le m$. If we guess $t_i = \frac{i}{m}$, then the resulting piecewise-quadratic $\hat{\gamma}\colon [0,1] \to \mathbb{R}^n$ is sometimes informative [7], [8], and sometimes not.

EXAMPLE 3. For $0 < i < m$, set $t_i = \frac{(3i+(-1)^i)T}{3m}$. Then sampling is more-or-less uniform, with $K_l = \frac{T}{3}$, $K_u = \frac{5T}{3}$. Let $\gamma\colon [0,\pi] \to \mathbb{R}^2$ be the parametrization $\gamma(t) = (\cos t, \sin t)$ of the unit semicircle in the upper half-plane. When m is small, the image of $\hat{\gamma}$ does not much resemble a semicircle, as in Fig. 1 where $m = 3$ and $d(\hat{\gamma}) - d(\gamma) = 0.0601$. The error in length estimate with piecewise-linear interpolation is -0.0712. When m is large the image of $\hat{\gamma}$ looks semicircular, as in Fig. 2 where $m = 30$. In this case, however, $d(\hat{\gamma}) - d(\gamma) = 0.1194$, an error nearly twice as large as for $m = 6$. Even piecewise-linear interpolation with 31 points gives a better estimate, with error -0.0033. Indeed, as m increases (at least for $m \le 100$), piecewise-quadratic interpolation tends to *increase* errors of length estimates. Linear interpolation is better, but not impressive.

EXAMPLE 4. For $0 < i < m$, let t_i be a random number (according to some distribution) in the interval $[\frac{(3i-1)T}{3m}, \frac{(3i+1)T}{3m}]$. Then sampling is more-or-less uniform, with K_u, K_l as in Example 3.

EXAMPLE 5. Choose $\theta > 0$ and $0 < L_l < L_u$. Set $s_0 = 0$. For $1 \le i \le m$, choose $\delta_i \in [\frac{L_l}{m}, \frac{L_u}{m}]$ independently from (say) the uniform distribution. Define $s_i = s_{i-1} + \delta_i$ for $i = 1, 2, \ldots, m$. The expectation of s_m is $\frac{L_u+L_l}{2}$ and the standard deviation $\frac{L_u-L_l}{2\sqrt{3m}}$.

[1] In Example 1 these were assumed to be given.

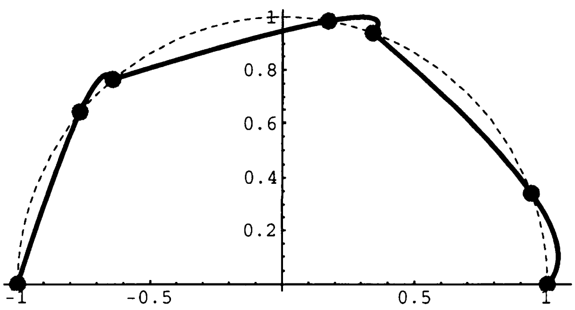

FIG. 1. 7 data points, with 3 successive triples interpolated by piecewise-quadratics, giving length estimate $\pi + 0.0601035$ for the semicircle (shown dashed).

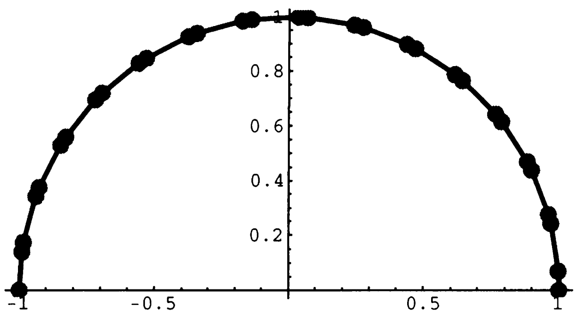

FIG. 2. 31 data points, with 15 successive triples interpolated by piecewise-quadratics, giving length estimate $\pi + 0.119407$ for the semicircle.

So if m is large, $s_m \approx \frac{L_u+L_l}{2}$ with high probability. For $0 \leq i \leq m$. define $t_i = \frac{s_i T}{s_m}$. Set

$$K_l = \frac{2L_l T}{L_u + L_l} - \theta \ , \qquad K_u = \frac{2L_u T}{L_u + L_l} + \theta.$$

Then with high probability for m large, the sampling $(t_0, t_1, t_2, \ldots, t_m)$ from $[0, T]$ is more-or-less uniform with constants K_l, K_u.

More-or-less uniform sampling is invariant with respect to reparameterizations; namely, if $\phi\colon [0,T] \to [0,T]$ is an order-preserving C^1 diffeomorphism, and if $(t_0, t_1, \ldots, t_m)$ are sampled more-or-less uniformly, then so are $(\phi(t_0), \phi(t_1), \ldots, \phi(t_m))$. So reparameterizations lead to further examples from the ones already given. To state our main result, first take $n = 2$ and suppose that γ is C^4 and (without loss) parameterized by arc-length; namely, $\|\dot{\gamma}\|$ is identically 1. The *curvature* of γ is defined as

$$k(t) = \det(M(t)) \ ,$$

where $M(t)$ is the 2×2 matrix with columns $\dot{\gamma}(t), \ddot{\gamma}(t)$. When $k(t) \neq 0$ for all $t \in [0, T]$, γ is said to be *strictly convex*.

THEOREM 1. Let $\gamma\colon [0,T] \to \mathbb{R}^2$ be strictly convex and suppose that sampling is more-or-less uniform. Then, for some $\tilde{d}(\mathcal{Q})$, calculable in terms of $\mathcal{Q}$,

$$\tilde{d}(\mathcal{Q}) = d(\gamma) + O\left(\frac{1}{m^4}\right).$$

In Sections 2, 3 we prove Theorem 1, constructing $\tilde{d}(\mathcal{Q})$ as a sum of lengths of quadratic arcs interpolating *quadruples* of sample points. In Sec. 4, some examples are given, showing that the quartic convergence of Theorem 1 is the best possible for our construction.

Note added in proof. The authors have recently become aware of [13] which contains work on closely related problems and other interesting references.

2. Quadratics interpolating quadruples. Let $\mathcal{Q}$ be sampled more-or-less uniformly from γ, and suppose (without loss) that m is a positive integer multiple of 3. For each quadruple $(q_i, q_{i+1}, q_{i+2}, q_{i+3})$, where $0 \le i \le m-3$, define $a_0, a_1, a_2 \in \mathbb{R}^2$ and $Q^i(s) = a_0 + a_1 s + a_2 s^2$ by

$$Q^i(0) = q_i\ , \quad Q^i(1) = q_{i+1}\ , \quad Q^i(\alpha) = q_{i+2}\ , \quad \text{and} \quad Q^i(\beta) = q_{i+3}.$$

Then $a_0 = q_i, a_2 = q_{i+1} - a_0 - a_1$, and we obtain two vector equations:

$$a_1\alpha + (p_1 - a_1)\alpha^2 = p_\alpha \quad \text{and} \quad a_1\beta + (p_1 - a_1)\beta^2 = p_\beta\ , \tag{1}$$

where $(p_1, p_\alpha, p_\beta) \equiv (q_{i+1} - q_i, q_{i+2} - q_i, q_{i+3} - q_i)$. Then (1) amounts to four quadratic scalar equations in four scalar unknowns $a_1 = (a_{11}, a_{12}), \alpha, \beta$. Set

$$c = -\det(p_\alpha, p_\beta)\ , \quad d = -\det(p_\beta, p_1)/c\ , \quad e = -\det(p_\alpha, p_1)/c\ ,$$

where $c, d, e \neq 0$ by strict convexity, and define

$$\rho_1 = \sqrt{e(1+d-e)/d}\ , \qquad \rho_2 = \sqrt{d(1+d-e)/e}.$$

Then (1) has two solutions (as can be verified by substitution):

$$(\alpha_+, \beta_+) = \frac{(1+\rho_1, 1+\rho_2)}{e-d}\ , \qquad (\alpha_-, \beta_-) = \frac{(1-\rho_1, 1-\rho_2)}{e-d}\ , \tag{2}$$

provided ρ_1, ρ_2 are real and $d - e \neq 0$. We now justify these assumptions and show that precisely one of (2) satisfies the additional condition

$$1 < \alpha < \beta. \tag{3}$$

It suffices[2] to deal with the case where $k(t) < 0$ for all $t \in [0,T]$. Then it is rather apparent, for geometrical reasons, that $1 + d - e, -d, -e$, and $e - d$ are all positive asymptotically. Alternatively, these facts can be proved (and sharper estimates obtained) by Mathematica calculations, as in Lemma 1 below. Define

$$l(t) = \frac{\det(\frac{d\gamma}{dt}, \frac{d^3\gamma}{dt^3})}{k(t)}.$$

[2]The other case, where $k(t)$ is everywhere positive, is dealt with by considering the reversed curve $\gamma_r(t) = (\gamma_1(T-t), \gamma_2(T-t))$.

Then, using Taylor's theorem, for $t, u \in [t_i, t_{i+3}]$,

$$\det(\gamma(t) - q_i, \gamma(u) - q_i) = k\frac{(t - t_i)(u - t_i)(u - t)}{2}\left(1 + (u - 2t_i + t)\frac{l}{3}\right) + O\left(\frac{1}{m^5}\right), \quad (4)$$

where k, l are evaluated at t_i.

LEMMA 1.

$$(\alpha_+, \beta_+) = \frac{((t_{i+2} - t_i)(1 + \frac{l(t_{i+2} - t_{i+1})}{6}), (t_{i+3} - t_i)(1 + \frac{l(t_{i+3} - t_{i+1})}{6}))}{t_{i+1} - t_i} + O\left(\frac{1}{m^2}\right). \quad (5)$$

Proof. By (4),

$$c = -k\frac{(t_{i+2} - t_i)(t_{i+3} - t_i)(t_{i+3} - t_{i+2})}{2}\left(1 + (t_{i+3} - 2t_i + t_{i+2})\frac{l}{3}\right) + O\left(\frac{1}{m^5}\right),$$

$$cd = -k\frac{(t_{i+3} - t_i)(t_{i+1} - t_i)(t_{i+1} - t_{i+3})}{2}\left(1 + (t_{i+1} - 2t_i + t_{i+3})\frac{l}{3}\right) + O\left(\frac{1}{m^5}\right),$$

$$ce = -k\frac{(t_{i+2} - t_i)(t_{i+1} - t_i)(t_{i+1} - t_{i+2})}{2}\left(1 + (t_{i+1} - 2t_i + t_{i+2})\frac{l}{3}\right) + O\left(\frac{1}{m^5}\right).$$

Consequently,

$$-d = \frac{t_{i+1} - t_i}{t_{i+3} - t_{i+2}}\frac{t_{i+3} - t_{i+1}}{t_{i+2} - t_i}\left(1 - (t_{i+2} - t_{i+1})\frac{l}{3}\right) + O\left(\frac{1}{m^2}\right),$$

$$-e = \frac{t_{i+1} - t_i}{t_{i+3} - t_{i+2}}\frac{t_{i+2} - t_{i+1}}{t_{i+3} - t_i}\left(1 - (t_{i+3} - t_{i+1})\frac{l}{3}\right) + O\left(\frac{1}{m^2}\right).$$

The lemma follows from these two equations. The detailed calculation can be viewed at the URL address `http://www.cs.uwa.edu.au/~ryszard/4points/`. □

We continue with the assumption that $k(t) < 0$ for all t. Then (3) follows from (5) for m large with $(\alpha, \beta) = (\alpha_+, \beta_+)$. Then, for $0 \leq s \leq \beta$, $Q^i(s) = q_i + a_1 s + a_2 s^2$, where

$$a_1 = \frac{p_\alpha - \alpha^2 p_1}{\alpha - \alpha^2} = \frac{p_\beta - \beta^2 p_1}{\beta - \beta^2} \quad \text{and} \quad a_2 = \frac{\alpha p_1 - p_\alpha}{\alpha - \alpha^2} = \frac{\beta p_1 - p_\beta}{\beta - \beta^2}. \quad (6)$$

LEMMA 2.

$$a_1 = (t_{i+1} - t_i)\dot{\gamma}(t_i)\left(1 + O\left(\frac{1}{m}\right)\right) + O\left(\frac{1}{m^3}\right),$$

$$a_2 = \frac{(t_{i+1} - t_i)^2}{2}\ddot{\gamma}(t_i)\left(1 + O\left(\frac{1}{m}\right)\right) + O\left(\frac{1}{m^2}\right)\dot{\gamma}(t_i) + O\left(\frac{1}{m^3}\right).$$

Proof. From (5),

$$(\alpha^2, \alpha - \alpha^2) = \frac{(t_{i+2} - t_i)(t_{i+2} - t_i, t_{i+1} - t_{i+2})}{(t_{i+1} - t_i)^2} + O\left(\frac{1}{m}\right).$$

Then by (6),

$$\begin{aligned}
a_1 &= \frac{(t_{i+1}-t_i)^2(p_\alpha - \alpha^2 p_1)}{(t_{i+2}-t_i)(t_{i+1}-t_{i+2})}\left(1+O\left(\frac{1}{m}\right)\right)\\
&= \frac{(t_{i+1}-t_i)^2(q_{i+2}-q_i) - ((t_{i+2}-t_i)^2 + O(\frac{1}{m^3}))(q_{i+1}-q_i))}{(t_{i+2}-t_i)(t_{i+1}-t_{i+2})}\left(1+O\left(\frac{1}{m}\right)\right)\\
&= \frac{(t_{i+1}-t_i)^2((t_{i+2}-t_i)\dot\gamma + (t_{i+2}-t_i)^2\frac{\ddot\gamma}{2}) - (t_{i+2}-t_i)^2((t_{i+1}-t_i)\dot\gamma + (t_{i+1}-t_i)^2\frac{\ddot\gamma}{2})}{(t_{i+2}-t_i)(t_{i+1}-t_{i+2})}\\
&\qquad \times\left(1+O\left(\frac{1}{m}\right)\right)\\
&\quad + O\left(\frac{1}{m^3}\right)\frac{(t_{i+1}-t_i)\dot\gamma + (t_{i+1}-t_i)^2\frac{\ddot\gamma}{2} + O(\frac{1}{m^3})}{(t_{i+2}-t_i)(t_{i+1}-t_{i+2})}\\
&= (t_{i+1}-t_i)\dot\gamma\left(1+O\left(\frac{1}{m}\right)\right) + O\left(\frac{1}{m^2}\right)\dot\gamma + O\left(\frac{1}{m^3}\right),
\end{aligned}$$

where $\dot\gamma, \ddot\gamma$ are evaluated at t_i. In similar fashion,

$$\begin{aligned}
a_2 &= \frac{(t_{i+1}-t_i)^2(\alpha p_1 - p_\alpha)}{(t_{i+2}-t_i)(t_{i+1}-t_{i+2})}\left(1+O\left(\frac{1}{m}\right)\right)\\
&= \frac{((t_{i+1}-t_i)(t_{i+2}-t_i) + O(\frac{1}{m^3}))(q_{i+1}-q_i) - (t_{i+1}-t_i)^2(q_{i+2}-q_i)}{(t_{i+2}-t_i)(t_{i+1}-t_{i+2})}\left(1+O\left(\frac{1}{m}\right)\right)\\
&= \frac{(t_{i+1}-t_i)(t_{i+2}-t_i)(q_{i+1}-q_i) - (t_{i+1}-t_i)^2(q_{i+2}-q_i)}{(t_{i+2}-t_i)(t_{i+1}-t_{i+2})}\left(1+O\left(\frac{1}{m}\right)\right)\\
&\qquad + O\left(\frac{1}{m^2}\right)\dot\gamma + O\left(\frac{1}{m^3}\right)\\
&= \frac{(t_{i+1}-t_i)^2((t_{i+1}-t_i)-(t_{i+2}-t_i))\ddot\gamma}{2(t_{i+1}-t_{i+2})}\left(1+O\left(\frac{1}{m}\right)\right) + O\left(\frac{1}{m^2}\right)\dot\gamma + O\left(\frac{1}{m^3}\right).
\end{aligned}$$

□

In particular,

$$\frac{dQ^i}{ds} = O\left(\frac{1}{m}\right) \quad \text{and} \quad \frac{d^2Q^i}{ds^2} = O\left(\frac{1}{m^2}\right), \tag{7}$$

for $s \in [0, \beta]$. The quadratics Q^i, determined by $\mathcal{Q}$ and i, need to be reparameterized for comparison with the original curve γ.

3. Proof of Theorem 1.

Let $\psi\colon [t_i, t_{i+3}] \to [0, \beta]$ be the cubic given by

$$\psi(t_i) = 0, \quad \psi(t_{i+1}) = 1, \quad \psi(t_{i+2}) = \alpha, \quad \psi(t_{i+3}) = \beta.$$

Using Lemma 1 (see `http://www.cs.uwa.edu.au/~ryszard/4points/`, especially for treatment of $O(\frac{1}{m^2})$ errors in (5)) yields

$$\frac{d^k\psi}{dt^k} = O(m), \quad \text{for } k = 1, 2, 3. \tag{8}$$

In particular, ψ is a diffeomorphism for m large. Define $\tilde{\gamma}_i = Q^i \circ \psi\colon [t_i, t_{i+3}] \to \mathbb{R}^2$. Then $\tilde{\gamma}_i$ is a polynomial of degree at most 6 and, using (7), (8), it turns out that its derivatives of all orders are $O(1)$. The C^4 function $f = \tilde{\gamma}_i - \gamma$ is $\mathbf{0}$ at $t_i, t_{i+1}, t_{i+2}, t_{i+3}$, and consequently,

$$f(t) = (t - t_i)(t - t_{i+1})(t - t_{i+2})g(t), \quad \text{where } g(t) = (t - t_{i+3})h(t),$$

and $g, h\colon [t_i, t_{i+3}] \to \mathbb{R}^2$ are C^1, C^0 respectively. Here we use Lemma 2.1 of Part I of [5], and to estimate errors. Because $\frac{d^4 f}{dt^4} = O(1), h = O(1)$. Therefore, $g = O(\frac{1}{m})$. Also, $\dot{g} = O(\frac{d^4 f}{dt^4}) = O(1)$. Therefore,

$$\dot{f} = O\left(\frac{1}{m^3}\right) \quad \text{and} \quad f = O\left(\frac{1}{m^4}\right). \tag{9}$$

Write $\dot{\tilde{\gamma}}_i(t)$ in the form $(1 + \langle \dot{f}(t), \dot{\gamma}(t)\rangle)\dot{\gamma}(t) + v(t)$, where $v(t)$ is the projection of $\dot{f}(t)$ onto the line orthogonal to $\dot{\gamma}(t)$. By (9), $v = O(\frac{1}{m^3})$ and, because $\|\dot{\gamma}\| = 1$,

$$\|\dot{\tilde{\gamma}}_i(t)\| = (1 + \langle \dot{f}(t), \dot{\gamma}(t)\rangle)\|\dot{\gamma}(t)\| + O\left(\frac{1}{m^6}\right).$$

Then

$$\begin{aligned}\int_{t_i}^{t_{i+3}} \|\dot{\tilde{\gamma}}_i(t)\| - \|\dot{\gamma}(t)\| dt &= \int_{t_i}^{t_{i+3}} \langle \dot{f}(t), \dot{\gamma}(t)\rangle dt + O\left(\frac{1}{m^7}\right) \\ &= -\int_{t_i}^{t_{i+3}} \langle f(t), \ddot{\gamma}(t)\rangle dt + O\left(\frac{1}{m^7}\right),\end{aligned}$$

after integration by parts. By (9), the right-hand side is $O(\frac{1}{m^5})$; namely,

$$\int_{t_i}^{t_{i+3}} \|\dot{\gamma}(t)\| dt - d(Q^i) = O\left(\frac{1}{m^5}\right),$$

and so

$$\tilde{d}(\mathcal{Q}) \equiv \sum_{j=0}^{\frac{m}{3}-1} d(Q^{3j}) = d(\gamma) + O\left(\frac{1}{m^4}\right).$$

Notice that a track-sum $\tilde{\gamma}$ of the arcs swept out by the Q^{3j} gives an $O(\frac{1}{m^4})$ uniformly accurate approximation of the image of γ. Although $\tilde{\gamma}$ is not C^1 at $t_3, t_6, \ldots, t_{m-3}$, the differences in left and right derivatives are $O(\frac{1}{m^3})$, and hardly discernible when m is large.

4. Convergence rates. In Example 3, piecewise 3-point quadratic interpolation gives a poor estimate of the semicircle, in particular of its length. Now we check the performance of the alternative piecewise 4-point quadratic interpolation used in the proof of Theorem 1.

EXAMPLE 6. As in Example 3, take $m = 6$ and use the same more-or-less uniform sampling of parameters t_i. The piecewise 4-point quadratic interpolant in Fig. 4 is more semicircular and the error in the length estimate is reduced from 0.0601 to -0.0072.

Let ϵ_m be the absolute value of the error in the length estimate using piecewise 4-point quadratic interpolation, where values of m not divisible by 3 are accounted for

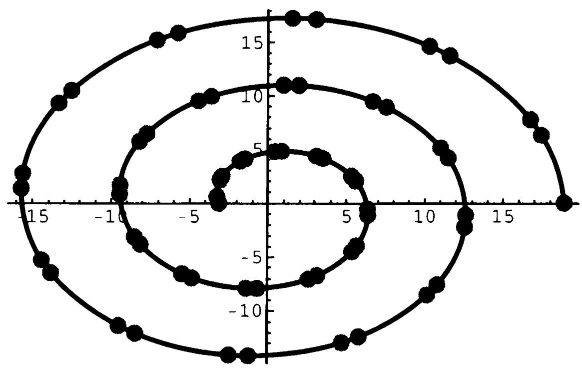

FIG. 3. A piecewise 4-point quadratic approximation to a spiral (singular point excluded), using the more-or-less uniform sampling of Example 3 and 61 data points ($m = 60$). True length: 173.608, estimate: 173.539, piecewise 3-point quadratic estimate: 181.311.

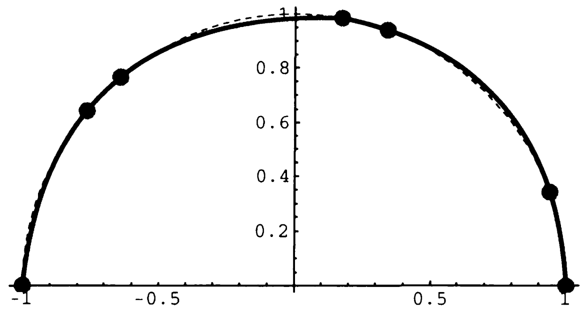

FIG. 4. Piecewise 4-point quadratic using 7 data points ($m = 6$) from a semicircle (shown dashed). Length estimate: $\pi - 0.00723637$.

by a simple modification. The plot in Fig. 5 of $-\log \epsilon_m$ against $\log m$ for $3 \le m \le 100$ appears linear, and the least-squares estimate of slope is approximately 3.83. According to Theorem 1, the limiting slope is at least 4 as $m \to \infty$. So the evidence points to exactly quartic convergence in this example

Our experiences with other curves and other more-or-less uniform samplings are similar to Example 6. We have also mentioned the spiral in connection with the sampling of Example 3. We give one further example, of a cubic. There are not a lot of changes.

EXAMPLE 7. The cubic is given parametrically by $\gamma(t) = (t, -t^3)$ for $t \in [0.1, 0.5]$ and sampled in the random fashion of Example 4 for $3 \le m \le 100$. The plot of $-\log \epsilon_m$ against $\log m$ is shown in Fig. 6. The least-squares estimate of slope is 4.00. So the evidence suggests only quartic convergence.

5. Concluding remarks. The condition that sampling be more-or-less uniform is rather unrestrictive. For $\epsilon > 0$, sampling is said to be *ϵ-uniform* when, in some parameterization,

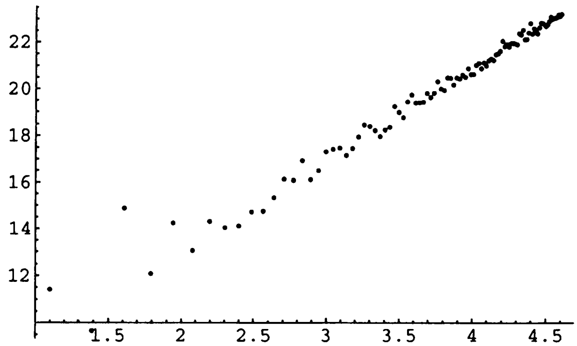

FIG. 5. $-\log \epsilon_m$ against $\log m$, for the semicircle and $3 \leq m \leq 100$. Estimate of slope: 3.82694.

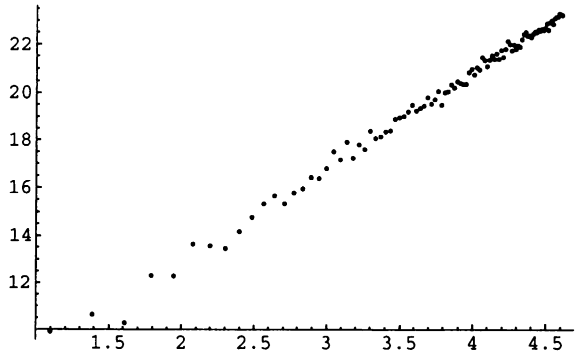

FIG. 6. $-\log \epsilon_m$ against $\log m$, for the cubic and $3 \leq m \leq 100$. Estimate of slope: 4.00347.

$$t_i = \frac{iT}{m} + O\left(\frac{1}{m^{1+\epsilon}}\right).$$

For this more regular kind of sampling, it is possible to get results like Theorem 1 using simpler constructions for $\tilde{d}(\mathcal{Q})$, as in [7], [8].

There is also some analogous work for estimating lengths of digitized curves; indeed, the analysis of digitized curves in $\mathbb{R}^2$ is one of the most intensively studied subjects in image data analysis. A digitized curve is the result of a process (such as contour tracing, $2D$ skeleton extraction, or $2D$ thinning) that maps a curve-like object (such as the boundary of a region) onto a computer-representable curve. As before, $\gamma\colon [0, T] \to \mathbb{R}^2$ is a strictly convex curve parameterized by arc-length. An analytical description of γ is not given, and numerical measurements of points on γ are corrupted by a process of *digitization*: γ is digitized within an orthogonal grid of points $(\frac{i}{m}, \frac{j}{m})$, where i, j are permitted to range over integer values, and m is a fixed positive integer called the *grid resolution*. Depending on the digitization model [2], γ is mapped onto a digital curve and approximated by a polygon $\hat{\gamma}_m$ whose length is an estimator for $d(\gamma)$. Approximating polygons $\hat{\gamma}_m$ based on local configurations of digital curves does not ensure multigrid length convergence, but global approximation techniques yield *linearly* convergent estimates, namely,

$$d(\gamma) - d(\hat{\gamma}_m) = O\left(\frac{1}{m}\right)$$

[3, 9]. In the special case of discrete straight line segments in $\mathbb{R}^2$, a stronger result is proved [1], where superlinear $O(\frac{1}{m^{1,5}})$ orders of asymptotic length estimates are given. In Theorem 1, convergence is of order 4, but $\mathcal{Q}$ arises from more-or-less uniform sampling, as opposed to digitization. So strict comparisons cannot yet be made. These issues will be revisited in the future.

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