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More Prolongation Structures

C. HOENSELAERS*)

Research Institute for Fundamental Physics, Kyoto University, Kyoto 606

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We examine the prolongation structures for the modified Korteweg-de Vries equation, the nonlinear Schrödinger equation and the Korteweg-de Vries equation. The former equation yields a subalgebra of $SL(3, R)\otimes R(\lambda)$, while the latter two give $SL(2, R)\otimes R(\lambda)$. It is shown how to reconstruct the equation from the incomplete Lie algebra. This process is not unique and leads to transformations between equations, the Miura transformation being one of them. A new equation arising in connection with the nonlinear Schrödinger equation is discussed.

§1. Introduction

This paper is a sequel to an earlier one in which we studied the prolongation structure for the sine-Gordon equation. Prolongation structures have been introduced by Wahlquist and Estabrook[1] in the context of the Korteweg-de Vries equation and have been described as incomplete Lie algebras.[2] By this we mean an algebra for which not all of the commutators are known. Briefly explained, they arise in the following manner.

Suppose that we are dealing with a system of differential equations for n dependent variables, collectively denoted by u, and two independent variables, the coordinates x and t. Higher order equations can, of course, be written as such a system; u are then elements of the local jet bundle. All our considerations will be purely local and we are dealing only with equations in two dimensions. Generalizations to more than two dimensions are possible,[2)~4)] but there our understanding is even more limited than in the two dimensional case.

Define two-forms F, again denoted so collectively, on an $(n+2)$ dimensional manifold with coordinates (u, x, t) such that the restriction of F to the submanifold labelled by the coordinates (x, t) yields when annulled the original equations. In this process, often called sectioning, u become functions of (x, t) and F are then evaluated on the submanifold. Restriction will be denoted by a solidus and the original equations read

$$F_| = 0\,. \tag{1·1}$$

The integrability conditions of the equations now become the condition that F are closed under exterior differentiation; i.e., there should exist a matrix of one-forms η such that (here and in what follows we omit the wedge which normally appears between differential forms)

$$dF=\eta F=0 \quad \mathrm{mod}\ F\,. \tag{1·2}$$

Such a set of forms F is called a closed ideal.

Prolongation variables, or pseudopotentials, denoted by y, are introduced as follows. Suppose that the one forms

*) Permanent address: Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Str. 1, 8046 Garching, Federal Republic of Germany.

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$$\omega = -dy + A(u, x, t, y)\,dx + B(u, x, t, y)\,dt\,, \tag{1·3}$$

when restricted and annulled are to give equations for y which are integrable if u is a solution of the original equations. The condition for this is

$$d\omega = 0 \mod \omega, F\,. \tag{1·4}$$

The number of pseudopotentials and consequently the length of the vectors A and B is as yet undetermined as we have no a priori way of knowing how many prolongation variables exist. By introducing them we have enlarged the dimension of the manifold on which the various forms live to an unspecified number.

Equations (1·4) yield terms in $dudx$, $dudt$ which can partly be absorbed by F and those in $dydx$, $dydt$ for which one uses (1·3) giving a term of the form $((\partial A/\partial y)B - (\partial B/\partial y)A)\,dxdt$. The terms which cannot be absorbed by F together with the one just mentioned yield a number of equations specifying some of the u derivatives of A and B; y derivatives appear only in the term mentioned above in the form of the Lie bracket with respect to the y coordinates of the two vector fields A and B. If these equations, and it may not always be possible to do so, can be used to determine the u dependence of A and B completely, we are left with several vector fields depending only on the y's and having to satisfy commutator equations like those for a Lie algebra. Not all commutators will be given a priori, but the Jacobi identities will have to be satisfied.

There are two possibilities. The Jacobi identities could determine the commutators completely. On the other hand, some of the commutators could be left undetermined. One then introduces new generators, vector fields, equal to those undetermined commutators and repeats the process of going through the Jacobi identities. If it appears that the algebra can be enlarged, new generators introduced, ad infinitum one can attempt to deduce the structure of the emerging infinite dimensional Lie algebra.

Bäcklund transformations arise in the present context as follows: Consider a map ϕ: $(u, y, x, t) \to (\tilde{u}, y, x, t)$ such that

$$\phi^* F = 0 \mod F, \omega\,. \tag{1·5}$$

This guarantees that $\tilde{u}(u, y, x, t)$ will be a solution of the equations if u is a solution and y a corresponding pseudopotential.

Finding the prolongation structure for the sine-Gordon equation has been the subject of the previous paper.[5] Section 2 will deal with the prolongation structure of the modified Korteweg-de Vries equation, the Korteweg-de Vries equation and the nonlinear Schrödinger equation. The Korteweg-de Vries equation has already been the subject of intensive studies;[2),6)~14)] the prolongation algebra shown in the next section on the other hand will be somewhat different from the one discussed in the previous works. The nonlinear Schrödinger equation with its Bäcklund transformation has also been studied.[15)~17)] We shall derive its infinite dimensional prolongation algebra and study it further in § 3. The results of this section have been announced in a brief report.[18]

Is it possible to go in the opposite direction; i.e., given an incomplete Lie algebra, is it possible to find the differential equations whose prolongation structure it is? This has been done in some instances,[12),13)] and we shall see in § 3 that the way back, from the prolongation structure to the equation, is not a unique one. In the case of the Korteweg-de Vries equation we shall rediscover the Miura transformation,[6] and for the nonlinear

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Schrödinger equation we shall find a hitherto unknown equation.

In § 4 we shall show that this new equation has only a finite dimensional prolongation structure, nevertheless it admits a Bäcklund transformation. We shall also derive another new equation related to the modified Korteweg-de Vries equation in the same manner as the latter is related to the Korteweg-de Vries equation.

§ 2. From equations to prolongation structures

The first example we wish to discuss in this section is the modified Korteweg-de Vries equation, which we take in the form

$$v_t + v_{xxx} + \frac{1}{2} v_x{}^3 = 0 . \tag{2·1}$$

The more known form results from differentiating this equation with respect to x and using v_x as variable.

The three forms

$$F_1 = -dvdt + udxdt ,$$

$$F_2 = -dudt + adxdt ,$$

$$F_3 = -dvdx + dadt + \frac{1}{2} u^3 dxdt , \tag{2·2}$$

on a manifold with coordinates (v, u, a, x, t) give when restricted and annulled

$$F| = 0 \Longleftrightarrow -v_x + u = -u_x + a = v_t + a_x + \frac{1}{2} u^3 = 0 ,$$

which is the same as (2·1). It is easy to check that the integrability conditions (1·2) are satisfied.

Pseudopotentials are now introduced by

$$\omega = -dy + A(v, u, a, y) dx + B(v, u, a, y) dt . \tag{2·3}$$

This is a slight specialization of (1·3) as A and B do not depend on the coordinates. Coordinate dependent prolongations have been constructed for the Korteweg-de Vries equation,[14] but, as the original equation does not depend on x and t, they can be reformulated such that they lead to the same algebra.

The terms in (1·4) which cannot be expressed by F or ω yield

$$dadt: \quad B_a + A_v = 0 ,$$

$$dudx: \quad A_u = 0 ,$$

$$dadx: \quad A_a = 0 ,$$

$$dxdt: \quad [AB] = B_u a + B_v u + \frac{1}{2} A_v u^3 .$$

The commutator in the last expression is the ordinary Lie bracket of two vector fields with respect to the prolongation variables y. Remember that we have suppressed the vector

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indices on A and B.

Differentiating the last equation twice with respect to a, we get equations for B which can be solved immediately. We find

$$B=-A_v a+\frac{1}{2}A_{vv}u^2+B_1 u+B_2\,,$$

where A, B_1, B_2 depend only on v and the y's. Inserting this expression into the commutator and equating terms we can proceed with the integration to find

$$A=A_3\sin v+A_2\cos v+A_1\,,$$

$$B=-A_3\left(a\cos v+\frac{1}{2}u^2\sin v\right)+A_2\left(a\sin v-\frac{1}{2}u^2\cos v\right)$$

$$+A_4 u+A_5\cos v+A_6\sin v+A_7\,. \tag{2·4}$$

The A_i depend only on the prolongation variables. The commutators which have to be satisfied by the generators A_i are

$$[A_1A_2]=[A_1A_3]=[A_1A_4]=0\,,$$

$$[A_2A_3]=-A_4\,,\quad [A_3A_4]=-A_5\,,\quad [A_2A_4]=A_6\,,$$

$$[A_1A_7]+[A_3A_6]=0\,,\quad [A_2A_5]-[A_3A_6]=0\,,$$

$$[A_3A_7]+[A_1A_6]=0\,,\quad [A_2A_7]+[A_1A_5]=0\,,$$

$$[A_3A_5]+[A_2A_6]=0\,.$$

We should mention that the integration gives one more term in B. This term proportional to v, however, has to vanish by virtue of the Jacobi identities.

Introducing more generators such that, for instance, $[A_2A_6]=-A_8$ and working through the 161 Jacobi identities we get the incomplete algebra of Table I.

First we note that A_1 and A_7 do not appear as the result of a commutator; furthermore

Table I.

$[A_iA_k]$ $k\rightarrow$ / $i\downarrow$	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0
2		$-A_4$	A_6	0	$-A_8$	0	$-A_9$	$-A_{11}$	0	?
3			$-A_5$	A_8	0	0	$-A_{10}$	0	$-A_{11}$	?
4				A_9	A_{10}	0	0	?	?	0
5					A_{11}	0	?	?	0	?
6						0	?	0	?	?
7							0	0	0	0
8								?	?	0
9									?	?
10										?

$[A_4A_9]+[A_3A_{11}]=0,\quad [A_6A_8]+[A_3A_{11}]=0,$
$[A_4A_{10}]-[A_2A_{11}]=0,\quad [A_4A_{10}]+[A_5A_8]=0,$
$[A_5A_9]-[A_6A_{10}]=0,\quad [A_5A_{11}]-[A_8A_{10}]=0,$
$[A_8A_9]+[A_6A_{11}]=0.$

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they commute with all other generators and hence will continue to do so no matter how many new generators we introduce. We note furthermore that there are three "types" of generators, those which commute with A_2, A_3 and A_4 respectively. Commuting two "types" always gives a generator of the third type. All this is very reminiscent of what we found for the sine-Gordon equation.[5] On the other hand, A_2 and A_3 never appear as commutators; thus the emerging structure is

$$[X_i Y_k] = -Z_{i+k}, \quad [Y_i Z_k] = -X_{i+k+1}, \quad [Z_i X_k] = -Y_{i+k+1},$$

$$[X_i X_k] = [Y_i Y_k] = [Z_i Z_k] = 0. \tag{2·5}$$

The identifications are

$$A_2 = X_0, \quad A_5 = X_1, \quad A_{10} = X_2,$$

$$A_3 = Y_0, \quad A_6 = Y_1, \quad A_9 = -Y_2,$$

$$A_4 = Z_0, \quad A_8 = Z_1, \quad A_{11} = -Z_2. \tag{2·6}$$

To compare with the sine-Gordon case: There are no basic generators, i.e., those which appear in (2·4) after integrating out the v, u, a dependence, with negative indices and there are two commutators, the second and third of (2·5), involving a shift of indices. The resulting algebra is a subalgebra of $SL(3, R) \otimes R(\lambda)$ involving only the Taylor part. A representation in terms of infinite matrices and the associated linear eigenvalue problem are easy to find, and we shall not pursue this matter further.

The next example is the nonlinear Schrödinger equation

$$u_t = i(u_{xx} + u^2 u^*). \tag{2·7}$$

$A*$ denotes the complex conjugate. These are actually two equations and we have to consider four forms

$$F_1 = -dudt + adxdt,$$

$$F_2 = -du^* dt + a^* dxdt,$$

$$F_3 = -dudt - i(dadt + u^2 u^* dxdt),$$

$$F_4 = -du^* dt + i(da^* dt + uu^{*2} dxdt). \tag{2·8}$$

The pseudopotentials are introduced in the same manner as before, (2·3). The calculational procedure for integrating out the u, u^*, a, a^* dependence is completely analogous to the one above and we proceed immediately to

$$A = A_4 uu^* + A_3 u + A_2 u^* + A_1,$$

$$B = i(au^* - a^* u)A_4 + i(aA_3 - a^* A_2 + uu^* A_8 + uA_7 + u^* A_6 + A_5), \tag{2·9}$$

where A and B have to satisfy

$$[AB] = B_u a + B_{u^*} a^* - i(u^2 u^* A_u - uu^{*2} A_{u^*}). \tag{2·10}$$

This yields after introducing some new generators the incomplete algebra shown in Table II.

A_4 commutes with every generator and will continue to do so. In fact, it corresponds

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Table II.

$[A_iA_k]$ $i \downarrow$ $k \rightarrow$	2	3	4	5	6	7	8	9	10
1	$-A_6$	A_7	0	0	$-A_9$	A_{10}	0	?	?
2		A_8	0	A_9	0	?	A_2	0	?
3			0	$-A_{10}$	?	0	$-A_3$	?	0
4				0	0	0	0	0	0
5					?	?	?	?	?
6						?	A_6	0	?
7							$-A_7$	?	0
8								$-A_9$	A_{10}
9									?

$[A_1A_9]=[A_5A_6]$, $[A_2A_7]=[A_3A_6]$,
$[A_1A_{10}]=[A_5A_7]$.

to a potential which, unlike the previous example where A_2 and A_3 corresponded to potentials, has no influence on the prolongation structure and can thus be omitted. So far the discussion has been rather parallel to Ref. 16).

There are again three "types" of generators commuting with A_2, A_3 and A_8, respectively. We find immediately the algebra

$$[X_iY_k]=-Y_{i+k}\,,\quad [X_iZ_k]=Z_{i+k}\,,\quad [Y_iZ_k]=X_{i+k}\,,$$
$$[X_iX_k]=[Y_iY_k]=[Z_iZ_k]=0 \tag{2·11}$$

with

$$A_8=X_0\,,\quad A_1=X_1\,,\quad A_5=X_2\,,$$
$$A_2=Y_0\,,\quad A_6=Y_1\,,\quad A_9=Y_2\,,$$
$$A_3=Z_0\,,\quad A_7=Z_1\,,\quad A_{10}=Z_2\,.$$

The identifications of A_1 and A_5 hold, of course, only up to generators which commute with all others.

The basic generators all carry positive indices and the algebra determined by (2·11) is thus $SL(2, R)\otimes R(\lambda)$ or the Taylor part of $A_1^{(1)}$. Again representations are well known.

The Korteweg-de Vries equation has been discussed quite often and it seems almost superfluous to include a discussion of its prolongation structure. For reasons which will become apparent later we, nevertheless, do so and use it in the form

$$v_t+v_{xxx}+\frac{3}{2}v_x{}^2=0\,. \tag{2·12}$$

The forms F are

$$F_1=-dvdt+udxdt\,,$$

$$F_2=-dudt+adxdt\,,$$

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$$F_3 = -dvdx + dadt + \frac{3}{2}u^2 dxdt\,. \tag{2·13}$$

Pseudopotentials are again introduced by (2·3) and integrating out the v, u, a dependence gives

$$A = \frac{1}{2}v^2 A_3 + vA_2 + A_1\,,$$

$$B = \left(-av + \frac{1}{2}u^2 - \frac{1}{2}uv^2\right)A_3 - (a+uv)A_2 - \left(u + \frac{1}{2}v^2\right)A_4 - \frac{1}{2}v^2 A_7 - vA_6 - A_5\,.$$

A and B have to satisfy

$$[AB] = B_u a + B_v u + \frac{3}{2}u^2 A_v\,.$$

With $A_8 = [A_1 A_6]$ we arrive at the incomplete algebra listed in Table III. Note that A_2, A_3, $A_4 - A_7$ make up the algebra $SL(2, R)$, A_3 commutes with $A_4 + A_7$ and $A_5 - A_8$. Finally, after some trial and error we find that the identifications

$$A_1 = \frac{1}{\sqrt{2}}(Z_0 - Y_1)\,, \qquad A_5 = \frac{1}{2}(Z_1 - Y_2)\,,$$

$$A_2 = -X_0\,, \qquad A_6 = -X_1\,,$$

$$A_3 = \sqrt{2}\,Y_0\,, \qquad A_7 = \frac{1}{\sqrt{2}}(Y_1 - Z_0)\,,$$

$$A_4 = \frac{1}{\sqrt{2}}(Z_0 + Y_1)\,, \qquad A_8 = \frac{1}{\sqrt{2}}(Z_1 + Y_2)\,,$$

A_1 and A_5 being, of course, only defined up to generators which commute with all others, give precisely algebra (2·11).

The prolongation algebra for the Korteweg-de Vries equation in the original formulation[1] is $[SL(2, R) \otimes R(\lambda)] \otimes H$[13] where H is a five dimensional Heisenberg algebra. The appearance of H is presumably due to using $u = v_x$ as variable for the equation.

There exist homomorphisms from the infinite algebras (2·5) and (2·11) into $3n$ dimensional algebras through the recursion relations

Table III.

$[A_i A_k]$ $i \downarrow$ $k \rightarrow$	2	3	4	5	6	7	8
1	A_4	A_2	A_6	0	A_8	0	?
2		A_3	A_7	$-A_8$	0	A_4	?
3			$-A_2$	$-A_6$	$-(A_7+A_4)$	A_2	$-A_6$
4				?	?	A_6	?
5					?	?	?
6						A_8	?
7							?

$[A_4 A_5] = -[A_1 A_8]$, $[A_4 A_6] = -[A_2 A_8]$,
$[A_1 A_8] = -[A_7 A_8]$, $3[A_4 A_8] = -[A_5 A_7]$.

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$$G_{i+n}=\sum_{k=0}^{n-1} a_k G_{i+k}\,.$$

G stands for X, Y and Z and a_k are constant parameters. These relations determine all generators with index $\geq n$ in terms of the first n ones. Inserting them into (2·5) and (2·11) then gives the commutators for the finite algebra.

§ 3. From incomplete algebras to differential equations

There is a dualism between Lie algebras and differential forms. Consider an n dimensional Lie algebra of generating vectors A_i, say, in a space with coordinates y. The dualism is expressed by the vector valued one-forms

$$\omega=-dy+A_i\xi^i\,, \tag{3·1}$$

which define two-forms Φ by

$$\begin{aligned}\Phi&=d\omega \qquad \text{mod } \omega\\ &=A_i d\xi^i-\frac{1}{2}[A_iA_k]\xi^i\xi^k\,.\end{aligned} \tag{3·2}$$

The generators A_i are supposed to be linearly independent and (3·2) thus defines at least n two-forms. If we are dealing with a complete algebra they are just the Maurer-Cartan structure forms of the algebra.

Now assume that we are only given an incomplete algebra. As we have seen, such an algebra consists of some known commutators and some algebraic relations among the unknown ones. Of course, the Jacobi identities have to be checked to ensure that everything is consistent. The two-forms Φ split naturally into two sets, Ω and Σ. There are n forms Ω; they contain the $d\xi^i$ and those products of the ξ^i coming from known commutators. The unknown commutators are supposed to be linearly independent modulo their algebraic relations. The forms Σ which contain only products of the ξ^i are those multiplying the linearly independent unknown commutators. As we have started from a Lie algebra, it follows that $d\Omega=0$ mod Ω, Σ and $d\Sigma=0$ mod Ω, Σ.

Such a set of forms Ω, Σ is called an invariant differential system[12),13)] or a constant coefficient ideal.[2),19)] It can be analysed for its Cartan characters, genus g, etc. When Ω and Σ are restricted to a solution manifold and annulled, i.e.,

$$\Omega_|=\Sigma_|=0 \tag{3·3}$$

there are at least g exact one-forms — the corresponding functions to be used as coordinates — implied by the system (3·3) such that the ξ^i are linear combinations of the coordinate differentials multiplied by functions of the coordinates. In case we find more than g exact one-forms, only g of them will be linearly independent and we have our choice which of them to use as coordinates. The ξ^i are then chosen such that the Σ forms vanish identically and the Ω forms yield a set of partial differential equations.

Given a set of forms Ω, Σ with genus >2 one may ask whether it is possible to reduce the system to a smaller one.[13)] To this end one considers linear combinations

$$\zeta=\alpha_i\xi^i$$

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with constant coefficients a_i such that

$$\zeta d\zeta = 0 \quad \mathrm{mod}\ \Omega, \Sigma .$$

Such a ζ, preferably one which lowers the genus, can be set to zero and one of the Ω forms disappears. Note, however, that reduction turns an independent one-form into a dependent one, i.e., a coordinate into a potential, and that thus the equations implied by the original and the reduced system will be quite different.

Consider the incomplete five dimensional algebra

$$[A_1 A_2] = -A_3 , \qquad [A_2 A_3] = -A_4 ,$$

$$[A_1 A_3] = A_5 , \qquad [A_2 A_5] = 0 ,$$

$$[A_1 A_4] = 0 , \qquad [A_1 A_5] + [A_2 A_4] = 0 .$$

Inserted into (3·3), this gives

$$d\xi^1 = 0 , \qquad \xi^1\xi^5 - \xi^2\xi^4 = 0 ,$$

$$d\xi^2 = 0 , \qquad \xi^3\xi^4 = 0 ,$$

$$d\xi^3 = -\xi^1\xi^2 , \qquad \xi^3\xi^5 = 0 ,$$

$$d\xi^4 = -\xi^2\xi^3 , \qquad \xi^4\xi^5 = 0 ,$$

$$d\xi^5 = \xi^1\xi^3 . \tag{3·4}$$

The genus of this ideal is 2, i.e., the ξ^i are combinations of two exact one-forms. The obvious choice is, of course, to take ξ^1 and ξ^2 as those one-forms. Let us, nevertheless, see whether there are other possibilities. Obviously

$$\xi^4 = \alpha\eta , \qquad \xi^5 = \beta\eta , \qquad \xi^3 = \gamma\eta$$

with some one-form η. Then one derives from the $d\xi^4$ and $d\xi^5$ equations

$$d(\sqrt{\alpha^2+\beta^2}\,\eta) = 0 .$$

Hence set

$$\eta = \frac{dt}{\sqrt{\alpha^2+\beta^2}}$$

and without loss of generality

$$\xi^3 = -u\ dt , \qquad \xi^4 = \sin v\ dt , \qquad \xi^5 = \cos v\ dt .$$

Furthermore

$$\xi^1 = \sin v\ \theta + (b \sin v - a \cos v)\,dt ,$$

$$\xi^2 = \cos v\ \theta + (a \sin v + b \cos v)\,dt$$

with some other one-form θ. Eliminating the $dv\theta$ term in the $d\xi^1$ and $d\xi^2$ equations one finds

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$$d\left(\theta+\left(b+\frac{1}{2}u^2\right)dt\right)=0\,.$$

With

$$\theta=dx-\left(b+\frac{1}{2}u^2\right)dt$$

we have

$$\xi^1=\sin v\ dx-\left(a\ \cos v+\frac{1}{2}u^2\sin v\right)dt=dz\,,$$

$$\xi^2=\cos v\ dx+\left(a\ \sin v-\frac{1}{2}u^2\cos v\right)dt=dw\,,$$

$$\xi^3=-udt\,,$$

$$\xi^4=\sin v\ dt\,,$$

$$\xi^5=\cos v\ dt\,.$$

We have found four exact one-forms, the differentials of x, t, z and w, in our system and we may use any two of them as coordinates. Using x and t we find

$$v_x=u\,,\qquad u_x=a\,,$$

$$v_t+a_x+\frac{1}{2}u^3=0\,.$$

This is hardly surprising as the incomplete algebra with which we started consisted precisely of the five basic generators for the modified Korteweg-de Vries equation.

On the other hand, we are not compelled to use x and t as coordinates, let us use z and t instead. This gives

$$\xi^1=dz\,,$$

$$\xi^2=\cot v\ dz+a\ \mathrm{cosec}\, v\ dt\,,$$

while the other forms remain unchanged. The resulting equation turns out to be

$$(\cos v)_t+\sin^3 v(\cos v)_{zzz}=0\,. \tag{3·5}$$

It is obvious that quite a number of other equations can be derived by using various combinations as coordinates. The transformations connecting them change coordinates into potentials and vice versa.

The situation changes if we consider the Korteweg-de Vries equation. The basic generators are according to the previous section G_0, X_1, Y_1, Z_1-Y_2. Equation (3·3) for the corresponding one-forms $\xi^1\cdots\xi^6$ becomes

$$\begin{aligned}
&d\xi^1=\xi^2\xi^3\,, && 2\xi^1\xi^6-\xi^3\xi^4-\xi^4\xi^5=0\,,\\
&d\xi^2=-\xi^1\xi^2\,, && \xi^3\xi^6+\xi^5\xi^6=0\,,\\
&d\xi^3=\xi^1\xi^3\,, && \xi^4\xi^6=0\,,\\
&d\xi^4=\xi^2\xi^6-\xi^3\xi^5\,, &&
\end{aligned}$$

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$$d\xi^5 = -\xi^1\xi^5 + \xi^2\xi^4 ,$$

$$d\xi^6 = \xi^1\xi^6 + \xi^3\xi^4 . \tag{3·6}$$

This system has genus 3, even though we have used only the basic generators for the Korteweg-de Vries equation! There is, however, a Cauchy characteristic since ξ^2 does not appear in the Σ forms.

To see what has happened, let us introduce coordinates. There are three exact one-forms and the ξ^i are expressible in terms of them as

$$\xi^1 = -vdx + (a+uv)dt ,$$

$$\xi^2 = \frac{1}{\sqrt{2}}(-v^2dx + (2va + v^2u - u^2)dt + 2dz) ,$$

$$\xi^3 = \frac{1}{\sqrt{2}}(-dx + udt) ,$$

$$\xi^4 = vdt ,$$

$$\xi^5 = \frac{1}{\sqrt{2}}(dx + (u+v^2)dt) ,$$

$$\xi^6 = \frac{1}{\sqrt{2}}dt . \tag{3·7}$$

The system (3·6) then yields the equations

$$v_x = u , \qquad v_z = 1 ,$$

$$u_x = a , \qquad u_z = 0 ,$$

$$v_t + a_x + \frac{3}{2}u^2 = 0 , \qquad a_z = 0 .$$

They are, of course, equivalent to the Korteweg-de Vries equation (2·12) and the z dependence simply reflects the fact that we can add anything independent of x and t to v.

The interesting linear combination of the ξ^i for reducing system (3·6) to a smaller one is

$$\zeta = \xi^2 + a\xi^1 + \frac{a^2}{2}\xi^3 .$$

By setting $\zeta = 0$ or

$$\xi^2 = -a\xi^1 - \frac{a^2}{2}\xi^3 \tag{3·8}$$

in (3·6) we get a system of genus 2. The introduction of coordinates yields the same result as for the unreduced system and we can use (3·7) with v, u, a replaced by $\hat{v}$, $\hat{u}$, $\hat{a}$. z is no longer an independent coordinate but a potential. The equations implied by the reduced system are

$$\hat{u}_x = \hat{v}_x - \frac{1}{2}\left(\hat{v} + \frac{a}{\sqrt{2}}\right)^2 , \qquad \hat{a}_x = \hat{u}$$

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and finally

$$\hat{v}_t + \hat{v}_{xxx} - \frac{3}{2}\left(\hat{v} + \frac{a}{\sqrt{2}}\right)^2 \hat{v}_x = 0 . \tag{3·9}$$

This is the modified Korteweg-de Vries equation. The potential z is determined by (3·8), viz.

$$dz = \frac{1}{2\sqrt{2}}\left[\left(\hat{v} + \frac{a}{\sqrt{2}}\right)^2 dx - \left(2\hat{a}\left(\hat{v} + \frac{a}{\sqrt{2}}\right) + \left(\hat{v} + \frac{a}{\sqrt{2}}\right)^2 \hat{u} - \hat{u}^2\right) dt\right]. \tag{3·10}$$

Now we only have to remember that

$$v = \hat{v} + z$$

is a solution of the Korteweg-de Vries equation (2·12) to rediscover the Miura transformation.[6]

Something similar happens in the case of the nonlinear Schrödinger equation. The forms $\xi^1 \cdots \xi^7$ dual to the basic generators G_0, G_1, X_2 yield the system

$$\begin{aligned} d\xi^1 &= \xi^2 \xi^3 , & \xi^2\xi^7 - \xi^4\xi^5 &= 0 , \\ d\xi^2 &= -\xi^1 \xi^2 , & \xi^3\xi^7 - \xi^4\xi^6 &= 0 , \\ d\xi^3 &= \xi^1 \xi^3 , & \xi^5\xi^7 &= 0 , \\ d\xi^4 &= \xi^2\xi^6 - \xi^3\xi^5 , & \xi^6\xi^7 &= 0 , \\ d\xi^5 &= -\xi^1\xi^5 + \xi^2\xi^4 , \\ d\xi^6 &= \xi^1\xi^6 - \xi^3\xi^4 , \\ d\xi^7 &= \xi^5\xi^6 . \end{aligned} \tag{3·11}$$

Again the genus is 3. Coordinates and functions can be introduced such that

$$\begin{aligned} &\xi^1 = dz + abdt , \\ &\xi^2 = adx - edt , \qquad \xi^3 = bdx + fdt , \\ &\xi^4 = dx , \quad \xi^5 = adt , \quad \xi^6 = bdt , \quad \xi^7 = dt . \end{aligned} \tag{3·12}$$

Inserting this into (3·11) gives

$$\begin{aligned} a_x &= e , & a_z &= -a , \\ a_t &= -(e_x + a^2 b) , & e_z &= -e , \\ b_x &= f , & b_z &= b , \\ b_t &= (f_x + ab^2) , & f_z &= f . \end{aligned} \tag{3·13}$$

By letting $t \to it$, $z \to iz$ and identifying a and b with u and u^*, respectively, we get the nonlinear Schrödinger equation. The z dependence reflects the arbitrary constant phase by which the wave function can be multiplied.

Reduction of the system proceeds, as before, by looking for a form ζ as a linear combination of ξ^i such that $\zeta d\zeta = 0$ and then setting it to zero. We find

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$$\zeta=\xi^1+\frac{1}{\sqrt{2}}\left(\alpha\xi^2+\frac{1}{\alpha}\xi^3\right)=0\,. \tag{3·14}$$

Introducing coordinates for the reduced ideal gives again (3·12) with a, b, e, f replaced by the corresponding quantities with hat; z becomes a potential.

The equations implied by (3·11) with (3·14) are

$$\hat{a}_t=-\left(\hat{a}_x-\frac{\alpha}{\sqrt{2}}\hat{a}^2-\frac{\sqrt{2}}{\alpha}\hat{a}\hat{b}\right)_x,$$

$$\hat{b}_t=\left(\hat{b}_x+\frac{1}{\sqrt{2}\alpha}\hat{b}^2+\sqrt{2}\alpha\hat{a}\hat{b}\right)_x \tag{3·15}$$

and

$$dz=-\frac{1}{\sqrt{2}}\left(\alpha\hat{a}+\frac{1}{\alpha}\hat{b}\right)dx+\left[\frac{1}{\sqrt{2}}\left(\alpha\hat{e}-\frac{1}{\alpha}\hat{f}\right)-\hat{a}\hat{b}\right]dt\,. \tag{3·16}$$

Now

$$a=\hat{a}e^{-z}\,,\qquad b=\hat{b}e^{z}\,,$$

satisfy (3·13) if $\hat{a}$ and $\hat{b}$ are solutions of (3·15) and z given by (3·16).

By making use of the above-mentioned correspondence between (3·13) and the nonlinear Schrödinger equation, it is easy to show that

$$u=i\sqrt{2}w_x e^{w^*-w} \tag{3·17}$$

satisfies the nonlinear Schrödinger equation if w solves

$$w_t=i(w_{xx}-w_x{}^2+2w_x w_x{}^*)\,. \tag{3·18}$$

The transformation (3·17) between solutions of the new equation (3·18) and the nonlinear Schrödinger equation is analogous to the Miura transformation. We shall study the new equation in the following section.

In closing this section we remark that the system conjugate to the incomplete algebra for the basic four generators of the sine-Gordon equation[5] reproduces the equation exactly.

§ 4. Miscellaneous remarks

The equation encountered at the end of the previous section appears to be new. We shall first study its prolongation structure and Bäcklund transformations. It can be expressed by the following forms:

$$F_1=-dwdt+vdxdt\,,$$

$$F_2=-dw^*dt+v^*dxdt\,,$$

$$F_3=-dwdx-i(dvdt+(2vv^*-v^2)dxdt)\,,$$

$$F_4=-dw^*dx+i(dv^*dt+(2vv^*-v^{*2})dxdt)\,. \tag{4·1}$$

Forms ω are again introduced as in (2·3). The equations to be satisfied by A and B are

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$$A_v = A_{v^*} = B_v - iA_w = B_{v^*} + iA_{w^*} = 0\,,$$

$$[AB] = iA_w(v^2 - 2vv^*) - iA_{w^*}(v^{*2} - 2vv^*) + B_w v + B_{w^*} v^*\,.$$

The solution is

$$A = A_1 e^{-(w+w^*)} + A_2\,,$$

$$B = -ie^{-(w+w^*)}(v - v^*)A_1 + A_3$$

and the A_i commute. The algebra is a three dimensional Abelian algebra and the only nontrivial potential is given by

$$dy = e^{-(w+w^*)}dx - ie^{-(w+w^*)}(v - v^*)\,dt\,. \qquad (4\cdot2)$$

There is no infinite number of conservation laws, the algebra is finite, and therefore we did not expect a Bäcklund transformation. It came as a surprise when we checked (1·5) that

$$\tilde{w} = w + \ln y \qquad (4\cdot3)$$

is again a solution. As w enters Eq. (4·1) only up to an additive complex constant and y is also only determined up to a constant, there are three free constants hidden in (4·3).

We can, however, not generate more than one new solution by (4·3). Suppose w_0 is a solution and y_0 the corresponding potential. It then follows from (4·2) by inserting $w_1 = w_0 + \ln y_0$ and using the equation for y_0 that

$$dy_1 = \frac{1}{{y_0}^2}dy_0\,.$$

Hence $w_2 = w_1 + \ln y_1$ is equivalent to w_1 up to the above mentioned constants.

The transformation (4·3) acts on solutions of the modified Schrödinger equation via (3·17). The new solution is

$$\tilde{u} = u + i\sqrt{2}\frac{1}{y}e^{-2w}\,. \qquad (4\cdot4)$$

Aside from the trivial solution we have found the following solutions for our new equation:

$$w = i\alpha(x + 3\alpha t)\,;$$

$$w = \ln g + i\alpha\int g^2 dx\,,$$

$$g = \left(\frac{\alpha}{\beta}\right)^{1/3}(1+\sqrt{3})^{-1/2}\cot\left(\frac{1}{2}\mathrm{sn}\left(-\frac{\sqrt[3]{\alpha\beta^2}}{2^4\sqrt{3}}x,\ \sin\frac{\pi}{12}\right)\right). \qquad (4\cdot5)$$

Let us now turn to the question whether we can find generalizations of the Miura transformation. As we have seen in the previous section, it arose from turning a global symmetry into a local one. Consider the equations

$$v_t + v_{xxx} + \frac{1}{2}(v_x)^n = 0 \qquad (4\cdot6)$$

for positive integers n. They are obviously invariant under $v \to v + \text{const}$. Now turn this

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global symmetry into a local one by setting

$$v=u+y \tag{4·7}$$

giving

$$u_t+y_t+u_{xxx}+y_{xxx}+\frac{1}{2}(u_x+y_x)^n=0\,. \tag{4·8}$$

This is, of course, meaningless unless we require y to be a potential of a yet to be discovered equation for u. Hence we set

$$y_x=k(u)\,,$$

$$y_t=l(u,\ u_x,\ u_{xx})\,. \tag{4·9}$$

The integrability condition $k_t=l_x$ gives, using (4·8) to replace u_t and (4·9) to express the various derivatives of y, an equation which can be solved by sorting out the coefficients of u_{xxx} and u_{xx}. The intermediate result is

$$l=-k_u u_{xx}+\frac{1}{2}k_{uu}u_x{}^2-\frac{1}{2}k^n$$

and

$$\frac{1}{2}k_{uuu}u_x{}^3+\frac{3}{2}k_u k_{uu}u_x{}^2+\frac{1}{2}k_u\sum_{i=2}^{n}\binom{n}{i}u_x{}^i k^{n-i}=0\,.$$

It is obvious that $n>3$ yields $k=\text{const}$; furthermore $n<2$ also gives nothing interesting, the equations are linear, after all.

$n=2$ gives $k=-(1/6)u^2+\alpha u+\beta$ which in turn yields an equation which can be transformed to the modified Korteweg-de Vries equation and (4·7) again is the Miura transformation.

For $n=3$ we find

$$k=\alpha\ \sin(u+\beta)$$

and the equation we get by inserting (4·9) into (4·8) reads

$$u_t+u_{xxx}+\frac{1}{2}u_x{}^3+\frac{3}{2}\alpha^2 u_x\sin^2(u+\beta)=0\,. \tag{4·10}$$

This is the equation we get from "gauging" the global symmetry of the modified Korteweg-de Vries equation.

Hardly necessary to say, we get Eq. (3·15) from gauging the z-dependence in (3·13). Gauging the additive constant in (3·18) gives yet another more complicated equation.

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