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SHORT COMMUNICATION

# MORE WITH THE LEMKE COMPLEMENTARITY ALGORITHM*

Joseph J.M. EVERS

*Twente University of Technology, Netherlands*

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In the case that the matrix of a linear complementarity problem consists of the sum of a positive semi-definite matrix and a co-positive matrix a general condition is deduced implying that the Lemke algorithm will terminate with a complementarity solution. Applications are presented on bi-matrix games, convex quadratic programming and multi-period programs.

*Key words*: Linear Complementarity, Bi-matrix Games, Multi-period Programs.

## 1. Introduction

We consider a linear complementarity problem where, given an $n$-vector $c$ and an $n \times n$-matrix $A$, $m$-vectors $\hat{z}$, $\hat{w}$ are to be determined satisfying:

$$Az - w = c, \qquad z, w \geqq 0, \quad \langle z, w \rangle = 0. \tag{1}$$

($\geqq$ refers to the natural ordering on $\mathbf{R}^n$ and $\langle z, w \rangle$ is the inner product of $z$ and $w$). Such a pair $(\hat{z}, \hat{w})$ is called a complementary solution. Solving the problem with the Lemke-algorithm, a positive auxiliary vector is introduced, transforming the system into:

$$Az + \theta h - w = c, \qquad z, w, \theta \geqq 0, \quad \langle z, w \rangle = 0, \tag{2}$$

$h$ being any fixed positive $n$-vector and $\theta$ being a scalar. A combination $(\tilde{z}, \tilde{w}, \tilde{\theta})$ satisfying (2) is called an almost-complementary solution, abbreviated ac-solution.

Clearly, defining $\bar{\theta} := \max_i\{c_i/h_i \mid c_i > 0\}$, an almost-complementary basic solution is available by $(z^0, w^0, \theta^0) := (0, \bar{\theta}h - c, \bar{\theta})$, together with a ray of ac-solutions $(z^0, w^0, \theta^0) + \lambda(0, h, 1) \mid \lambda \geqq 0\}$. Starting from this particular basic solution $(z^0, w^0, \theta^0)$ the Lemke-algorithm constructs a series of pairwise adjacent basic solutions of the system $Az + \theta h - w = c$, $z, w, \theta \geqq 0$, which are all ac-solutions (cf. [11], [2]).

Concerning the termination of the algorithm there are three possibilities:

(a) because of cycling the algorithm will not stop,

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(b) the algorithm stops at a basic ac-solution $(z^*, w^*, \theta^*)$ with $\theta^* > 0$, or,

(c) stops with a basic ac-solution with $\theta^* = 0$;

clearly, in the latter case a complementarity solution is identified. If system (2) is non-degenerate, cycling is impossible; otherwise, it is possible to endow the Lemke-algorithm with an anti-cycling procedure. Further, the standard theory concerning the Lemke-algorithm shows that stopping at basic ac-solution $(z^*, w^*, \theta^*)$ with $\theta^* > 0$ implies the existence of a ray of ac-solutions

$$\{(z^*, w^*, \theta^*) + \lambda(\underline{z}, \underline{w}, \underline{\theta}) \mid \lambda \geqq 0\}, \quad \text{with } \underline{z} \neq 0.$$

Evidently, any condition imposed on the linear complementarity problem which rules out the existence of such a ray of ac-solutions, implies that the Lemke-algorithm will terminate with a complementary solution and proves the existence of a complementary solution in a constructive manner.

In the main theorem such a general condition is deduced with respect to complementarity problems where the matrix can be written as the sum of a symmetric positive semi-definite matrix and a co-positive matrix (note: a square matrix B is called co-positive if for every non-negative vector $x$: $\langle x, Bx\rangle \geqq 0$). Accordingly, (2) is written:

$$(M+N)z + \theta h - w = c, \qquad z, w, \theta \geqq 0, \quad \langle z, w\rangle = 0, \tag{3}$$

where $M$ is a symmetric positive semi-definite $n \times n$-matrix, $N$ a co-positive matrix, $c$ an $n$-vector, and where $h$ is any positive auxiliary vector with dimension $n$.

## 2. The main theorem

**Theorem 2.0.** *If there exist vectors* $x, y \in \mathbf{R}^n$, $y \geqq 0$, *satisfying* $Mx - N'y \geqq c$ ($N'$ *being the transpose of* $N$), *then, with respect to complementarity problem* (3), *there is no ray of ac-solutions* $\{(z^*, w^*, \theta^*) + \lambda(\underline{z}, \underline{w}, \underline{\theta}) \mid \lambda \geqq 0\}$ *with simultaneously* $\theta^* > 0$ *and* $\underline{z} \neq 0$.

In the light of the preceding remarks the consequence of the theorem is obvious:

**Corollary 2.1.** *If the system* $Mx - N'y \geqq c$, $y \geqq 0$, *is solvable* (*M symmetric pos. semi-def., N co-positive*), *then Lemke's algorithm applied to* (3) (*with* $h > 0$) *terminates in a complementary solution.*

The proof of our theorem is based on two auxiliary properties:

**Proposition 2.2.** *Let M, N be* $n \times n$*-matrices, M symmetric positive semi-definite, N co-positive. Let* $c \in \mathbf{R}^n$. *If the system* $(M+N)z \geqq 0$, $\langle c, z\rangle > 0$, $\langle z, (M+N)z\rangle = 0$, $z \in \mathbf{R}^n_+$ *is solvable, then the system* $Mx - N'y \geqq c$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}^n_+$ *is non-solvable.*

**Proof.** If $z \in \mathbf{R}_+^n$ satisfies $\langle z, (M+N)z\rangle = 0$, then the assumptions on $M$ and $N$ imply: $\langle z, Nz\rangle = 0$, $\langle z, Mz\rangle = 0$. The latter implies $Mz = 0$. Consequently, we may conclude that every $z \in \mathbf{R}_+^n$ with $\langle z, (M+N)z\rangle = 0$, $(M+N)z \geqq 0$, satisfies $Nz \geqq 0$, as well. Now, suppose $\bar{z} \in \mathbf{R}_+^n$ and $\bar{x} \in \mathbf{R}^n$, $\bar{y} \in \mathbf{R}_+^n$ are solutions of the first and the second system resp. Then, with $Nz \geqq 0$, $M\bar{z} = 0$, $\bar{z}, \bar{x}, \bar{y} \geqq 0$, we have

$$0 \leqq \langle \bar{y}, N\bar{z}\rangle = -\langle \bar{x}, M\bar{z}\rangle + \langle \bar{y}, N\bar{z}\rangle = -\langle \bar{z}, M\bar{x} - N'\bar{y}\rangle \leqq -\langle \bar{z}, c\rangle < 0.$$

Contradiction: at least one of the systems has to be non-solvable.

**Proposition 2.3.** *If, with respect to* (2), *A being co-positive and h being positive, there is a ray of ac-solutions* $(z^*, w^*, \theta^*) + \lambda(\underline{z}, \underline{w}, \underline{\theta})$, $\lambda \geqq 0$, *with* $\theta^* > 0$, $x \neq 0$, *then* $A\underline{z} \geqq 0$, $\langle c, \underline{z}\rangle > 0$, $\langle \underline{z}, A\underline{z}\rangle = 0$, $\underline{z} \geqq 0$.

**Proof.** With respect to such a ray, we have:

(i) $A\underline{z} + \underline{\theta}h - \underline{w} = 0$, $\underline{z}, \underline{w}, \underline{\theta} \geqq 0$,

(ii) $\langle \underline{z}, \underline{w}\rangle = 0$, $\langle z^*, w^*\rangle = 0$, $\langle \underline{z}, w^*\rangle = 0$, $\langle z^*, \underline{w}\rangle = 0$.

Further, the assumptions imply:

(iii) $\langle \underline{z}, A\underline{z}\rangle \geqq 0$ (by co-positivity of $A$ and $\underline{z} \geqq 0$).

(iv) $\langle \underline{z}, h\rangle > 0$ (by positivity of $h$ and by $\underline{z} \geqq 0$, $\neq 0$).

Multiplying (i) by $\underline{z}$, equality $\langle \underline{z}, \underline{w}\rangle = 0$ implies $\langle \underline{z}, A\underline{z}\rangle + \underline{\theta}\langle \underline{z}, h\rangle = 0$, and hence by (iii) and (iv):

(v) $\underline{\theta} = 0$,

(vi) $\langle \underline{z}, A\underline{z}\rangle = 0$.

Combining (i) and (v), we have:

(vii) $A\underline{z} \geqq 0$.

Multiplying $A(z^* + \lambda\underline{z}) + (\theta^* + \lambda\underline{\theta})h - (w^* + \lambda\underline{w}) = c$ by $(z^* + \lambda\underline{z})$, combining the result with (ii) en (v), we find:

$$\langle z^* + \lambda\underline{z}, A(z^* + \lambda\underline{z})\rangle + \theta^*\langle z^* + \lambda\underline{z}, h\rangle = \langle z^* + \lambda\underline{z}, c\rangle.$$

Since the first term is non-negative, we have for every $\lambda \geqq 0$ the inequality $\theta^*\langle z^* + \lambda\underline{z}, h\rangle \leqq \langle z^* + \lambda\underline{z}, c\rangle$. With $\theta^* > 0$, $z^* \geqq 0$, $h > 0$, $\underline{z} \geqq 0$, the latter implies:

(viii) $\langle c, \underline{z}\rangle > 0$.

Thus, (i), (vi), (vii) and (viii) prove the proposition.

Clearly, our theorem is a simple consequence of Propositions 2.2 and 2.3. Namely, the sum of a positive semi-definite matrix and a co-positive matrix is a co-positive matrix. Thus, if there is an ac-ray, as mentioned in Theorem 2.0, then (by Proposition 2.3) there is a $z \in \mathbf{R}_+^n$ satisfying $(M+N)z \geqq 0$, $\langle c, z\rangle > 0$, $\langle z, (M+N)z\rangle = 0$, and consequently (by 2.2) the system $Mx + N'y \geqq c$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}_+^n$ is non-solvable.

An interesting consequence of Corollary 2.1 can be found by putting $M := 0$, $c := -N'u - v$, with $u, v \in \mathbf{R}_+^n$.

**Corollary 2.4.** *Let $N$ be a co-positive $n \times n$-matrix. Then, for every $u, v \in \mathbf{R}^n_+$, there is a $z, w \in \mathbf{R}^n_+$ satisfying $Nz - w = -N'u - v$, $\langle z, w\rangle = 0$.*

A simple sufficient condition for matrix $N$ to be co-positive, is the criterion $(N + N') \geqq 0$, being the consequence of the equality $\langle y, Ny\rangle = \frac{1}{2}\langle y, (N + N')y\rangle$, for every $y \in \mathbf{R}^n$. In this context, the result published by Jones [10] might be considered as a special case of Corollary 2.1. Independently, he found in a similar manner that the Lemke algorithm applied on (2) terminates in a complementary solution, provided $A + A' \geqq 0$, $h > 0$, and, in addition, the system $-A'y \geqq c$, $y \in \mathbf{R}^n_+$ is solvable. In order to illustrate the unifying power of our main theorem, we shall discuss some applications.

## 3. Bi-matrix games

We consider a bi-matrix game defined by $m \times n$-matrices $A, B$. Let

$$U := \left\{u \in \mathbf{R}^m_+ \,\middle|\, \sum_{i=1}^m u_i = 1\right\}, \qquad X := \left\{x \in \mathbf{R}^n_+ \,\middle|\, \sum_{j=1}^n x_j = 1\right\}.$$

Then the Nash-equilibrium is defined as a pair $(\hat{u}, \hat{x}) \in U \times X$ such that, for every $u \in U$, $x \in X$: $\langle u, A\hat{x}\rangle \leqq \langle \hat{u}, A\hat{x}\rangle$, $\langle \hat{u}, B\hat{x}\rangle \leqq \langle \hat{u}, Bx\rangle$. It is well known (see [2]) that, in case the matrices are positive, all Nash-equilibria can be deduced from solutions of the complementarity problem: $B'u - v = s^n$, $-Ax - y = -s^m$, $\langle x, v\rangle = 0$, $\langle y, u\rangle = 0$, $x, y, u, v \geqq 0$, where $s^m \in \mathbf{R}^m$, $s^n \in \mathbf{R}^n$ are vectors with all components one. Namely, for $A, B > 0$, a combination $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ is a solution of the complementarity problem if and only if $\hat{u}, \hat{x}$ defined by $\hat{u} := \langle s^m, \bar{u}\rangle^{-1}\bar{u}$, $\hat{x} := \langle s^n, \bar{x}\rangle^{-1}\bar{x}$, is a Nash-equilibrium. Evidently, putting:

$$M := 0, \qquad N := \begin{pmatrix} 0 & B' \\ -A & 0 \end{pmatrix},$$

$$c := (s^n, -s^m), \qquad z := (x, u), \qquad w := (v, y),$$

the problem can be written in our standard form (3). Observing that $N + N'$ is non-negative in the case that $B \geqq A$ (affirming co-positivity), Corollary 2.1 implies that, for $B \geqq A > 0$, the Lemke algorithm will find a complementary solution. Note: in fact no restriction on $A, B$ is needed. Because, defining $\bar{A} := A + \alpha S$, $\bar{b} := B + \beta S$, $S$ being an $m \times n$-matrix all elements one, Nash-equilibria are independent with respect to the scalars $\alpha, \beta$.

## 4. Concave quadratic programming

Let $Q$ be a symmetric positive semi-definite $n \times n$-matrix, let $A$ be an $m \times n$-matrix, let $p \in \mathbf{R}^n$, $r \in \mathbf{R}^m$. Consider the quadratic max-problem: $\hat{\phi} := \sup\langle p, x\rangle - \frac{1}{2}\langle x, Qx\rangle$, over $x \in \mathbf{R}^n_+$, $y \in \mathbf{R}^m_+$, such that $Ax + y = r$. With respect to the standard Lagrangian $\langle p, x\rangle - \frac{1}{2}\langle x, Qx\rangle - \langle u, Ax - r\rangle$, straightforward methods lead to the following properties:

(i) $(x, y)$ is optimal and $(u, v)$ is a Lagrange vector, if and only if $Qx + A'u - v = p$, $Ax + y = r$, $\langle x, v\rangle = 0$, $\langle y, u\rangle = 0$, $x, y, u, v \geqq 0$, and

(ii) the system $Qx + A'u \geqq p$, $Ax \leqq r$, $x, u \geqq 0$ is solvable, if and only if the max-problem is feasible and $\hat{\phi} < +\infty$. Now, writing the complementarity problem of (i) in our standard form (3),

$$M := \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, \qquad N := \begin{pmatrix} 0 & A' \\ -A & 0 \end{pmatrix},$$

$$c := (p, -r), \qquad z := (x, u), \qquad w := (v, y),$$

implying $M$ is symmetric positive semi-definite, $N$ is co-positive (note, $N + N' = 0$), we may conclude:

(iii) there exists an optimal solution $(x, y)$ and a Lagrange vector $(u, v)$, if and only if the max-problem is feasible and $\hat{\phi} < +\infty$; in that case these quantities can be calculated by Lemke's algorithm.

An approach like this is well-known; see for instance [1, 2, 11].

## 5. Invariant optimal solutions in concave quadratic multi-period problems

We consider a multi-period allocation max-problem with a discounted concave quadratic objective function and with a linear valuation on the terminal state

$$\hat{\phi} := \sup(\pi)^h \langle u_{h+1}, Bx_h\rangle + \sum_{t=1}^{w} (\pi)^t (\langle p, x_t\rangle - \tfrac{1}{2}\langle x_t, Qx_t\rangle),$$

over $\{x_t\}_1^h \subset \mathbf{R}_+^n$, $\{y_t\}_1^h \subset \mathbf{R}_+^m$, such that: $Ax_1 + y_1 = Bx_0 + r$, $Ax_t - Bx_{t-1} + y_t = r$, $t = 2, \ldots, h$, where: $0 < \pi < 1$, $p \in \mathbf{R}^n$, $Q$ symmetric positive semi-definite, $A$ and $B$ $m \times n$-matrices, $r \in \mathbf{R}^m$, $h$ the planning horizon, $x_0$ given initial state, and where $u_{h+1} \in \mathbf{R}_+^m$ is the terminal valuation vector. Defining the Lagrangian

$$(\pi)^h \langle u_{h+1}, Bx_h\rangle + \sum_{t=1}^{h} (\pi)^t (\langle p, x_t\rangle - \tfrac{1}{2}\langle x_t, Qx_t\rangle - \langle u_t, Ax_t - Bx_{t-1} - r\rangle + \langle v_t, x_t\rangle),$$

similar properties as (i)–(iii) of Section 4 hold with respect to the complementarity problem: $Qx_t + A'u_t - \pi B'u_{t+1} - v_t = p$, $Ax_b \mp Bx_{t-1} + y_t = r$, $\langle x_t, v_t\rangle = 0$, $\langle y_t, u_t\rangle = 0$, $x_t, y_t, u_t, v_t \geqq 0$, for all $t = 1, \ldots, h$. In that context $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ is called an invariant optimal solution if $Q\hat{x} + (A - \pi B)'\hat{u} - \hat{v} = p$, $-(A - B)\hat{x} - \hat{y} = -r$, $\langle \hat{x}, \hat{v}\rangle = 0$, $\langle \hat{y}, \hat{u}\rangle = 0$, $\hat{x}, \hat{y}, \hat{u}, \hat{v} \geqq 0$; namely, putting $x_0 := \hat{x}$, $u_{n+1} := \hat{u}$, one may verify that $(x_t, y_t) := (\hat{x}, \hat{y})$, $t = 1, \ldots, h$, $(\hat{u}_t, \hat{v}_t) := (u, v)$, $t = 1, \ldots, h$ resp. are an optimal solution and a Lagrange sequence, indeed. Writing the definition of the invariant optimal solution concept in our standard form (3), where

$$M := \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, \qquad N := \begin{pmatrix} 0 & (A - \pi B)' \\ -(A - B) & 0 \end{pmatrix},$$

$$c := (p, -r), \qquad z := (x, u), \qquad w := (v, y),$$

one may verify that the conditions of Corollary 2.1 are satisfied, in the case that $0 < \pi \leqq 1$, $B \geqq 0$ (implying $N + N' \geqq 0$), and, in addition the system $(A - B)'u \geqq p$, $(A - \pi B)x \leqq r$, $u, x \geqq 0$ is solvable. Recently, studies concerning invariant optimal solutions for multi-period problem are published by several authors [3, 4], and [6–10]. We studied the problem independently of Jones [10]. A recent study on linear complementarity and its applications in O.R. is published by Bastian [1]. The author is indebted to J.F. Benders for helpful suggestions.

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