

Morita theorems for categories of comodules

By Mitsuhiro TAKEUCHI

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Introduction

We show that the well-known Morita theorems on equivalences of categories of modules hold true of categories of *comodules* over a field k . We go parallel with [H. Bass, Algebraic K-Theory, Chap. II Categories of Modules and their Equivalences, W. A. Benjamin, Inc., New York, 1968].

Let \mathbf{Com}_Γ and \mathbf{Com}_A denote the categories of right Γ -comodules and left A -comodules, where Γ and A are k -coalgebras.

If ${}_A P_\Gamma$ is a A - Γ -bicomodule, the co-tensor product \square_A determines a “ k -linear” functor: $X_{A^*} \rightarrow X \square_A P_\Gamma$ from \mathbf{Com}_A to \mathbf{Com}_Γ .

Every “ k -linear” equivalence from \mathbf{Com}_A onto \mathbf{Com}_Γ is of the form $?\square_A P_\Gamma$ for some bicomodule ${}_A P_\Gamma$.

We must describe the “co-hom” and “co-end” functors.

A right Γ -comodule X_Γ is *quasi-finite*, if $\mathbf{Com}_\Gamma(F, X)$, the space of Γ -colinear maps from F to X , is finite dimensional for all finite dimensional comodule F_Γ .

Let X_Γ and Y_Γ be right Γ -comodules, where X is quasi-finite. There are a k -vector space $h_\Gamma(X, Y)$ and a Γ -colinear map $\theta: Y \rightarrow h_\Gamma(X, Y) \otimes X$ satisfying the following universal property: If W is a k -vector space and $F: Y_\Gamma \rightarrow W \otimes X_\Gamma$ a Γ -colinear map, there is a *unique* k -linear map $f: h_\Gamma(X, Y) \rightarrow W$ such that $F = (f \otimes I) \circ \theta$.

The “co-hom” $h_\Gamma(X, Y)$ is a contra-variant functor of X_Γ and a covariant functor of Y_Γ .

The “co-end” $e_\Gamma(X) = h_\Gamma(X, X)$ has the following coalgebra structure: There are unique linear maps $\Delta: e_\Gamma(X) \rightarrow e_\Gamma(X) \otimes e_\Gamma(X)$ and $\eta: e_\Gamma(X) \rightarrow k$ such that $(\Delta \otimes I) \circ \theta = (I \otimes \theta) \circ \theta: X \rightarrow e_\Gamma(X) \otimes e_\Gamma(X) \otimes X$ and $(\eta \otimes I) \circ \theta = I: X \rightarrow k \otimes X = X$. Then $(e_\Gamma(X), \Delta, \eta)$ is a k -coalgebra and X is an $e_\Gamma(X)$ - Γ -bicomodule, where $\theta: X \rightarrow e_\Gamma(X) \otimes X$ is the left $e_\Gamma(X)$ -comodule structure map.

By symmetry, for left A -comodules ${}_A X$ and ${}_A Y$, where ${}_A X$ is quasi-finite, the “co-hom” $h_A(X, Y)$ with the canonical A -colinear map $\theta: Y \rightarrow X \otimes h_A(X, Y)$ exists. The “co-end” $e_A(X) = h_A(X, X)$ has a unique k -coalgebra structure making $\theta: X \rightarrow X \otimes e_A(X)$ into a right comodule structure map.

The co-hom and co-end have many properties similar to the usual hom and end. In particular, if ${}_A X_\Gamma$ is a A - Γ -bicomodule and ${}_E Y_\Gamma$ a E - Γ -bicomodule, where E is a k -coalgebra and X_Γ is quasi-finite, then $h_{-\Gamma}(X, Y)$ has a natural E - A -bicomodule structure.

THEOREM. *Let ${}_A P_\Gamma$ be a A - Γ -bicomodule.*

a) *The following are equivalent.*

(i) *The functor $? \square_A P_\Gamma: \mathbf{Com}_A \rightarrow \mathbf{Com}_\Gamma$ is an equivalence.*

(ii) *The functor ${}_A P \square_\Gamma ? : \mathbf{Com}_\Gamma \rightarrow \mathbf{Com}_A$ is an equivalence.*

(iii) *The right comodule P_Γ is a quasi-finite injective cogenerator and there is a canonical isomorphism of k -coalgebras $e_{-\Gamma}(P) \simeq A$.*

(iv) *The left comodule ${}_A P$ is a quasi-finite injective cogenerator and there is a canonical isomorphism of k -coalgebras $e_{A-}(P) \simeq \Gamma$.*

b) *Suppose the above equivalent conditions hold. The Γ - A -bicomodules $Q = h_{-\Gamma}({}_A P_\Gamma, {}_\Gamma \Gamma_\Gamma)$ and $Q' = h_{A-}({}_A P_\Gamma, {}_A A_A)$ are canonically isomorphic. The functor $? \square_\Gamma Q_A$ (resp. ${}_A Q \square_\Gamma ?$) is a quasi-inverse of $? \square_A P_\Gamma$ (resp. ${}_A P \square_\Gamma ?$).*

The bicomodules ${}_A P_\Gamma$ satisfying the conditions of a) can be called “invertible”. Construction of the “inverse” bicomodule ${}_A Q_\Gamma$ is given in b). Two coalgebras A and Γ may be called “Morita equivalent” if there is an invertible bicomodule ${}_A P_\Gamma$.

The above theorem implies that, if X_Γ is a quasi-finite injective cogenerator right Γ -comodule, then the bicomodule $e_{-\Gamma(X)} X_\Gamma$ is invertible with inverse $h_{-\Gamma}(X, \Gamma) \simeq h_{e_{-\Gamma(X)}}(X, e_{-\Gamma}(X))$ and there is a canonical isomorphism of k -coalgebras $e_{e_{-\Gamma(X)}(X)} \simeq \Gamma$.

Similar results are valid with quasi-finite injective cogenerator left A -comodules.

The categories of comodules can be characterized as follow:

THEOREM. *Let \mathbf{A} be a k -abelian category. \mathbf{A} is k -linearly equivalent to \mathbf{Com}_Γ for some k -coalgebra Γ if and only if \mathbf{A} is locally finite in the sense of [2, p. 356] and the space $\mathbf{A}(M, N)$ is finite dimensional over k for each objects M and N of finite length of \mathbf{A} .*

§ 0. Conventions

k is a fixed ground field.

All vector spaces and linear maps are k -vector spaces and k -linear maps. Unadorned \otimes and Hom mean \otimes_k and Hom_k .

If V is a vector space, $V^* = \text{Hom}(V, k)$.

Mod denotes the category of vector spaces.

A *coalgebra* is a triple (C, Δ, η) where C is a vector space, $\Delta: C \rightarrow C \otimes C$ and $\eta: C \rightarrow k$ are linear maps such that $(I \otimes \Delta) \circ \Delta = (\Delta \otimes I) \circ \Delta: C \rightarrow C \otimes C \otimes C$ and $(\eta \otimes I) \circ \Delta = I = (I \otimes \eta) \circ \Delta: C \rightarrow k \otimes C = C = C \otimes k$.

Throughout the paper A, Γ, Θ and \mathcal{E} are coalgebras.

A *right Γ -comodule* is a pair (X, ρ) where X is a vector space and $\rho: X \rightarrow X \otimes \Gamma$ a linear map such that $(I \otimes \Delta) \circ \rho = (\rho \otimes I) \circ \rho: X \rightarrow X \otimes \Gamma \otimes \Gamma$ and $(I \otimes \eta) \circ \rho = I: X \rightarrow X \otimes k = X$.

A comodule is *finite dimensional* if it is as a vector space. Every comodule is the union of finite dimensional subcomodules.

A *Γ -colinear map* $f: X \rightarrow Y$ of right Γ -comodules is a linear map such that $\rho_Y \circ f = (f \otimes I) \circ \rho_X$ where ρ_X and ρ_Y denote the structure maps of X and Y .

Com- Γ denotes the category of right Γ -comodules and Γ -colinear maps. This is abelian, and has direct sums and direct products. (See Note.) The forgetful functor **Com- Γ** \rightarrow **Mod** is exact and preserves direct sums.

If $W \in \mathbf{Mod}$ and $X \in \mathbf{Com-}\Gamma$, $W \otimes X$ has the right Γ -comodule structure $I \otimes \rho_X$, where $\rho_X: X \rightarrow X \otimes \Gamma$ denotes the structure map of X . We then have canonically

$$\mathbf{Com-}\Gamma(W \otimes X, Y) \simeq \mathbf{Hom}(W, \mathbf{Com-}\Gamma(X, Y))$$

for all $Y \in \mathbf{Com-}\Gamma$.

Here and later $\mathbf{A}(X, Y)$ denotes the \mathbf{A} -morphisms from X to Y , where \mathbf{A} is a category and X and Y are objects in \mathbf{A} .

Γ is a right Γ -comodule with structure map $\Delta: \Gamma \rightarrow \Gamma \otimes \Gamma$. We have canonically $\mathbf{Com-}\Gamma(X, W \otimes \Gamma) \simeq \mathbf{Hom}(X, W)$ for all $X \in \mathbf{Com-}\Gamma$ and $W \in \mathbf{Mod}$. Hence $W \otimes \Gamma$ is injective in **Com- Γ** . In particular Γ is an injective cogenerator of **Com- Γ** .

By symmetry left A -comodules and A -colinear maps are defined. **Com- A** denotes the category of left A -comodules. For $X \in \mathbf{Com-}A$ and $W \in \mathbf{Mod}$, $X \otimes W$ has the canonical left A -comodule structure.

A *A - Γ -bicomodule* is a left A -comodule and a right Γ -comodule P such that the A -comodule structure map $\rho_A: P \rightarrow A \otimes P$ is Γ -colinear or equivalently that the Γ -comodule structure map $\rho_\Gamma: P \rightarrow P \otimes \Gamma$ is A -colinear.

Γ is a Γ - Γ -bicomodule, where $\Delta: \Gamma \rightarrow \Gamma \otimes \Gamma$ is the left and right Γ -comodule structure map.

In the following we write $X_\Gamma, {}_A Y$ and ${}_A Z_\Gamma$ to denote that X is a right Γ -comodule, Y a left A -comodule, and Z a A - Γ -bicomodule.

For comodules X_Γ and ${}_\Gamma Y$, the *co-tensor product* $X \square_\Gamma Y$ is the kernel of

$$\rho_X \otimes I, I \otimes \rho_Y: X \otimes Y \longrightarrow X \otimes \Gamma \otimes Y$$

where ρ_X and ρ_Y denote the structure maps of X and Y .

The functors $X \square_\Gamma ?$ and $? \square_\Gamma Y$ are *left exact* and preserve *direct sums*. In particular $X \square_\Gamma (Y \otimes W) \simeq (X \square_\Gamma Y) \otimes W$ and $(W \otimes X) \square_\Gamma Y \simeq W \otimes (X \square_\Gamma Y)$ for $W \in \text{Mod}$.

When ${}_A X_\Gamma$ and ${}_\Gamma Y_\theta$ are bicomodule, the structure maps $\rho_A: X \rightarrow A \otimes X$ and $\rho_\theta: Y \rightarrow Y \otimes \theta$ induce the structure maps $\rho_A \square_\Gamma I: X \square_\Gamma Y \rightarrow (A \otimes X) \square_\Gamma Y = A \otimes (X \square_\Gamma Y)$ and $I \square_\Gamma \rho_\theta: X \square_\Gamma Y \rightarrow X \square_\Gamma (Y \otimes \theta) = (X \square_\Gamma Y) \otimes \theta$ with which $X \square_\Gamma Y$ is a A - θ -bicomodule.

The co-tensor product is *associative*: For comodules and bicomodules $X_\Gamma, {}_\Gamma Y_\theta$ and ${}_\theta Z$, we have

$$(X \square_\Gamma Y) \square_\theta Z = X \square_\Gamma (Y \square_\theta Z)$$

in $X \otimes Y \otimes Z$. This subspace is denoted by $X \square_\Gamma Y \square_\theta Z$.

For comodules X_Γ and ${}_\Gamma Y$, the structure maps ρ_X and ρ_Y induce Γ -colinear isomorphisms $X \simeq X \square_\Gamma \Gamma$ and $Y \simeq \Gamma \square_\Gamma Y$. In particular we have $X \otimes W \simeq X \square_\Gamma (\Gamma \otimes W)$ and $W \otimes Y \simeq (W \otimes \Gamma) \square_\Gamma Y$ for $W \in \text{Mod}$.

Let \mathbf{A} be an abelian category and $I_A: \mathbf{A} \rightarrow \mathbf{A}$ the identity. The natural transformations $\text{End}(I_A)$ from I_A to I_A form a *commutative ring*.

A *k-abelian category* is a pair (\mathbf{A}, σ) where \mathbf{A} is an abelian category and $\sigma: k \rightarrow \text{End}(I_A)$ a ring homomorphism preserving unit. Giving σ is equivalent to making $\mathbf{A}(X, Y)$ into vector spaces for all $X, Y \in \mathbf{A}$ so that the composition maps $\mathbf{A}(Y, Z) \times \mathbf{A}(X, Y) \rightarrow \mathbf{A}(X, Z)$ are bilinear.

Com_Γ and Com_{A-} are *k-abelian categories*.

When \mathbf{A} and \mathbf{B} are *k-abelian categories*, a functor $T: \mathbf{A} \rightarrow \mathbf{B}$ is *linear* if $T: \mathbf{A}(X, Y) \rightarrow \mathbf{B}(T(X), T(Y))$ are linear for all $X, Y \in \mathbf{A}$.

Let $S: \mathbf{B} \rightarrow \mathbf{A}$ and $T: \mathbf{A} \rightarrow \mathbf{B}$ be functors, where \mathbf{A} and \mathbf{B} are categories. If $\mathbf{A}(S(X), Y) \simeq \mathbf{B}(X, T(Y))$ naturally for $X \in \mathbf{B}$ and $Y \in \mathbf{A}$, S is *left adjoint to T* or T is *right adjoint to S*. In this case we write $S \dashv T$.

The left adjoint of T and the right adjoint of S are uniquely determined if they exist.

If \mathbf{A} and \mathbf{B} are *k-abelian* and one of S and T , where $S \dashv T$, is linear, so is the other. The natural isomorphisms $\mathbf{A}(S(X), Y) \simeq \mathbf{B}(X, T(Y))$ then are linear isomorphisms. S is right exact and T is left exact. In fact S preserves *colimits*.

If $S \dashv T$, where S is exact, then T preserves injectives. Indeed if $U \in \mathbf{A}$ is injective, then $\mathbf{B}(X, T(U)) \simeq \mathbf{A}(S(X), U)$ is an exact functor of $X \in \mathbf{B}$.

§ 1. The “co-hom” functor $h_{-\Gamma}(-, -)$

1.1 DEFINITION. A comodule X_Γ is *quasi-finite*, if $\mathbf{Com}_{-\Gamma}(F, X)$ is finite dimensional for all finite dimensional comodule F_Γ .

1.2 EXAMPLE. A comodule X_Γ is *finitely cogenerated*, if it is isomorphic to a subcomodule of $W \otimes \Gamma$ for some finite dimensional vector space W . Finitely cogenerated comodules are quasi-finite.

1.3 PROPOSITION. For a comodule X_Γ , the following are equivalent.

- (i) X_Γ is quasi-finite.
- (ii) The functor $\mathbf{Mod} \rightarrow \mathbf{Com}_{-\Gamma}, W \mapsto W \otimes X$ has the left adjoint.

PROOF. Assume (i). If F_Γ is a finite dimensional comodule, $\mathbf{Com}_{-\Gamma}(F, W \otimes X) \simeq W \otimes \mathbf{Com}_{-\Gamma}(F, X) \simeq \text{Hom}(\mathbf{Com}_{-\Gamma}(F, X)^*, W)$ for $W \in \mathbf{Mod}$. When Y_Γ is an arbitrary comodule, let $\{Y_\lambda\}$ be the finite dimensional subcomodules of Y . Then

$$\begin{aligned} \mathbf{Com}_{-\Gamma}(Y, W \otimes X) &= \lim_{\longleftarrow \lambda} \mathbf{Com}_{-\Gamma}(Y_\lambda, W \otimes X) \simeq \lim_{\longleftarrow \lambda} \text{Hom}(\mathbf{Com}_{-\Gamma}(Y_\lambda, X)^*, W) \\ &= \text{Hom}(\lim_{\longrightarrow \lambda} \mathbf{Com}_{-\Gamma}(Y_\lambda, X)^*, W). \end{aligned}$$

Hence (ii) holds.

Conversely if $W \mapsto W \otimes X$ has the left adjoint, then for each finite dimensional comodule F_Γ , the functor $W \mapsto \mathbf{Com}_{-\Gamma}(F, W \otimes X) = W \otimes \mathbf{Com}_{-\Gamma}(F, X)$, $\mathbf{Mod} \rightarrow \mathbf{Mod}$ preserves direct products. Since a vector space V is finite dimensional if and only if the functor $W \mapsto W \otimes V$ preserves direct products, it follows that X_Γ is quasi-finite.

1.4 DEFINITION. For a quasi-finite comodule X_Γ , the left adjoint of $W \mapsto W \otimes X$ is written as $Y_\Gamma \mapsto h_{-\Gamma}(X, Y)$, $\mathbf{Com}_{-\Gamma} \rightarrow \mathbf{Mod}$. We have canonical isomorphisms

$$\mathbf{Com}_{-\Gamma}(Y, W \otimes X) \simeq \text{Hom}(h_{-\Gamma}(X, Y), W).$$

Let $\theta: Y \rightarrow h_{-\Gamma}(X, Y) \otimes X$ denote the Γ -colinear map associated with the identity of $h_{-\Gamma}(X, Y)$. For any $W \in \mathbf{Mod}$ and any Γ -colinear map $f: Y \rightarrow W \otimes X$, there is a unique linear map $u: h_{-\Gamma}(X, Y) \rightarrow W$ such that $f = (u \otimes I) \circ \theta$.

1.5 Let $u: X'_\Gamma \rightarrow X_\Gamma$ and $v: Y_\Gamma \rightarrow Y'_\Gamma$ be Γ -colinear maps of right Γ -comodules, where X_Γ and X'_Γ are quasi-finite. The composite $Y \xrightarrow{v} Y' \xrightarrow{\theta} h_{-\Gamma}(X', Y') \otimes X'$

$\xrightarrow{I \otimes u} h_{-r}(X', Y') \otimes X$ is of the form

$$Y \xrightarrow{\theta} h_{-r}(X, Y) \otimes X \xrightarrow{h_{-r}(u, v) \otimes I} h_{-r}(X', Y') \otimes X$$

with a uniquely determined linear map $h_{-r}(u, v)$. In this way $h_{-r}(X, Y)$ is a “bilinear” functor, covariant in Y_r and contra-variant in X_r .

1.6 For a quasi-finite X_r , the functor $h_{-r}(X, ?)$ is right exact and preserves direct sums, since it has the right adjoint. In particular there is the canonical isomorphism $W \otimes h_{-r}(X, Y) \simeq h_{-r}(X, W \otimes Y)$ for all $W \in \mathbf{Mod}$ and $Y \in \mathbf{Com}_r$. Or equivalently the colinear map

$$I \otimes \theta: W \otimes Y \longrightarrow W \otimes h_{-r}(X, Y) \otimes X$$

satisfies the universal mapping property of (1.4).

1.7 For a quasi-finite comodule X_r and a bicomodule ${}_A Y_r$, the structure map $\rho_A: Y \rightarrow A \otimes Y$ induces the structure map $h_{-r}(I, \rho_A): h_{-r}(X, Y) \rightarrow h_{-r}(X, A \otimes Y) \simeq A \otimes h_{-r}(X, Y)$ with which $h_{-r}(X, Y)$ is a left A -comodule. The canonical map $\theta: Y \rightarrow h_{-r}(X, Y) \otimes X$ is then A - Γ -bilinear.

1.8 Let ${}_E X_r$ be a bicomodule, where X_r is quasi-finite. For each comodule Y_r , the composite $Y \xrightarrow{\theta} h_{-r}(X, Y) \otimes X \xrightarrow{I \otimes \rho_E} h_{-r}(X, Y) \otimes E \otimes X$, where ρ_E denotes the E -comodule structure of X , is of the form

$$Y \xrightarrow{\theta} h_{-r}(X, Y) \otimes X \xrightarrow{\rho \otimes I} h_{-r}(X, Y) \otimes E \otimes X$$

with a uniquely determined linear map $\rho: h_{-r}(X, Y) \rightarrow h_{-r}(X, Y) \otimes E$. With the structure map ρ , $h_{-r}(X, Y)$ is a right E -comodule. The image of θ is contained in $h_{-r}(X, Y) \square_E X$.

1.9 If ${}_E X_r$ and ${}_A Y_r$ are bicomodules, where X_r is quasi-finite, then $h_{-r}(X, Y)$ is a A - E -bicomodule and the map $\theta: Y \rightarrow h_{-r}(X, Y) \square_E X$ is A - Γ -bilinear.

1.10 PROPOSITION. For a bicomodule ${}_E X_r$, the following are equivalent.

- (i) X_r is quasi-finite.
- (ii) The functor $\mathbf{Com}_E \rightarrow \mathbf{Com}_r, Z_E \mapsto Z \square_E X_r$ has the left adjoint.

In this case the left adjoint of $Z_E \mapsto Z \square_E X_r$ is given by $Y_r \mapsto h_{-r}(X, Y)$.

PROOF. Suppose (i). Then $h_{-r}(X, Y)$ is a right E -comodule for all comodule Y_r . We claim that the map $\theta: Y \rightarrow h_{-r}(X, Y) \square_E X$ satisfies the following universal map-

ping property: For each comodule $Z_{\mathcal{E}}$ and each Γ -colinear map $f: Y \rightarrow Z \square_{\mathcal{E}} X$, there is a unique \mathcal{E} -colinear map $u: h_{-\Gamma}(X, Y) \rightarrow Z$ such that $f = (u \square I) \circ \theta$. Indeed there is a unique linear map $u: h_{-\Gamma}(X, Y) \rightarrow Z$ such that $f = (u \otimes I) \circ \theta: Y \rightarrow Z \otimes X$. The composites $q_1: h_{-\Gamma}(X, Y) \xrightarrow{\rho} h_{-\Gamma}(X, Y) \otimes \mathcal{E} \xrightarrow{u \otimes I} Z \otimes \mathcal{E}$ and $q_2: h_{-\Gamma}(X, Y) \xrightarrow{u} Z \xrightarrow{\rho_Z} Z \otimes \mathcal{E}$ coincide, since $(q_1 \otimes I) \circ \theta = (I \otimes \rho_X) \circ f = (\rho_Z \otimes I) \circ f = (q_2 \otimes I) \circ \theta: Y \rightarrow Z \otimes \mathcal{E} \otimes X$, where ρ, ρ_X and ρ_Z denote the \mathcal{E} -comodule structure maps of $h_{-\Gamma}(X, Y)$, X and Z respectively. Hence the map u is \mathcal{E} -colinear. Thus (i) implies (ii).

Suppose (ii). Since the functor $W \mapsto W \otimes \mathcal{E}, \mathbf{Mod} \rightarrow \mathbf{Com}_{-\mathcal{E}}$ has the left adjoint by (1.2) and (1.3), so has the composite, $\mathbf{Mod} \rightarrow \mathbf{Com}_{-\Gamma}, W \mapsto (W \otimes \mathcal{E}) \square_{\mathcal{E}} X \simeq W \otimes X$. Hence X_{Γ} is quasi-finite by (1.3).

1.11 REMARK. Let ${}_{\mathcal{E}}X_{\Gamma}, {}_A Y_{\Gamma}$ and ${}_A Z_{\mathcal{E}}$ be bicomodules, where X_{Γ} is quasi-finite. If $f: Y \rightarrow Z \square_{\mathcal{E}} X$ is a A - Γ -bilinear map, then it is easy to check that the associated map $u: h_{-\Gamma}(X, Y) \rightarrow Z$ is A - \mathcal{E} -bilinear.

1.12 If the quasi-finite comodule X_{Γ} is injective, then the functor $h_{-\Gamma}(X, ?)$ is exact. Indeed the functor $Y_{\Gamma} \mapsto h_{-\Gamma}(X, Y)^* \simeq \mathbf{Com}_{-\Gamma}(Y, X)$ is exact.

1.13 For comodules and bicomodules X_{Γ}, Z_A and ${}_A Y_{\Gamma}$, where X_{Γ} is quasi-finite, the canonical map

$$\partial: h_{-\Gamma}(X, Z \square_A Y) \longrightarrow Z \square_A h_{-\Gamma}(X, Y)$$

is a unique map such that the composite

$$Z \square_A Y \xrightarrow{\theta} h_{-\Gamma}(X, Z \square_A Y) \otimes X \xrightarrow{\partial \otimes I} Z \square_A h_{-\Gamma}(X, Y) \otimes X$$

equals $I \square \theta$, where note that $\theta: Y \rightarrow h_{-\Gamma}(X, Y) \otimes X$ is left A -colinear.

1.14 PROPOSITION. The map ∂ is an isomorphism if either

- a) Z_A is injective, or
- b) X_{Γ} is (quasi-finite and) injective.

PROOF. By definition $\partial: h_{-\Gamma}(X, A \square_A Y) \rightarrow A \square_A h_{-\Gamma}(X, Y)$ is an isomorphism. Consider both hand sides of $\partial: h_{-\Gamma}(X, Z \square_A Y) \rightarrow Z \square_A h_{-\Gamma}(X, Y)$ as functors of Z_A . Since they commute with direct sums, it follows that ∂ is an isomorphism if Z_A is injective. If X_{Γ} is injective, then they are left exact by (1.12). Since each comodule Z_A can be imbedded into an exact sequence of the form $0 \rightarrow Z \rightarrow W_1 \otimes A \rightarrow W_2 \otimes A$ for some $W_i \in \mathbf{Mod}$, ∂ is then isomorphic.

1.15 If ${}_E X_\Gamma, {}_\theta Z_A$ and ${}_A Y_\Gamma$ are bicomodules, where X_Γ is quasi-finite, then the map $\partial: h_{-\Gamma}(X, Z \square_A Y) \rightarrow Z \square_A h_{-\Gamma}(X, Y)$ is θ - \mathcal{E} -bilinear. The proof is similar to (1.10).

1.16 By definition the ∂ map satisfies the following *associativity*: If $X_\Gamma, W_\theta, {}_\theta Z_A$ and ${}_A Y_\Gamma$ are comodules and bicomodules, where X_Γ is quasi-finite, then the following diagram commutes:

$$\begin{array}{ccc} W \square_\theta h_{-\Gamma}(X, Z \square_A Y) & \xrightarrow{I \square \partial} & W \square_\theta Z \square_A h_{-\Gamma}(X, Y) \\ \uparrow \partial & & \uparrow \partial \\ h_{-\Gamma}(X, W \square_\theta (Z \square_A Y)) & = & h_{-\Gamma}(X, (W \square_\theta Z) \square_A Y). \end{array}$$

1.17 Assume X_Γ is a quasi-finite comodule. Put $e_{-\Gamma}(X) = h_{-\Gamma}(X, X)$. Let $\Delta: e_{-\Gamma}(X) \rightarrow e_{-\Gamma}(X) \otimes e_{-\Gamma}(X)$ and $\eta: e_{-\Gamma}(X) \rightarrow k$ be the linear maps such that $(\Delta \otimes I) \circ \theta = (I \otimes \theta) \circ \theta: X \rightarrow e_{-\Gamma}(X) \otimes e_{-\Gamma}(X) \otimes X$ and $I = (\eta \otimes I) \circ \theta: X \rightarrow X = k \otimes X$. Then $(e_{-\Gamma}(X), \Delta, \eta)$ is a coalgebra and X an $e_{-\Gamma}(X)$ - Γ -bicomodule, where $\theta: X \rightarrow e_{-\Gamma}(X) \otimes X$ is the left $e_{-\Gamma}(X)$ -comodule structure map.

The coalgebra $e_{-\Gamma}(X)$ is the coalgebra of “co-endomorphisms” of X .

1.18 If ${}_E X_\Gamma$ is a bicomodule, where X_Γ is quasi-finite, then the structure map $\rho_E: X \rightarrow \mathcal{E} \otimes X$ corresponds to a linear map $c: e_{-\Gamma}(X) \rightarrow \mathcal{E}$ by $\rho_E = (c \otimes I) \circ \theta$. Then c is a coalgebra map.

Conversely a coalgebra map $c: e_{-\Gamma}(X) \rightarrow \mathcal{E}$ makes X_Γ into a bicomodule ${}_E X_\Gamma$.

The coalgebra $e_{-\Gamma}(X)$ is a \mathcal{E} - \mathcal{E} -bicomodule through c . This structure coincides with $e_{-\Gamma}(X) = h_{-\Gamma}({}_E X_\Gamma, {}_E X_\Gamma)$.

1.19 By symmetry, if ${}_E X$ is a quasi-finite comodule, the functor $W \mapsto X \otimes W, \text{Mod} \rightarrow \text{Com}_{E^-}$ has the left adjoint $h_{E^-}(X, ?)$ with adjunction $\theta: Y \rightarrow X \otimes h_{E^-}(X, Y)$ for each comodule ${}_E Y$.

$e_{E^-}(X) = h_{E^-}(X, X)$ has a unique coalgebra structure making X into a \mathcal{E} - $e_{E^-}(X)$ -bicomodule through $\theta: X \rightarrow X \otimes e_{E^-}(X)$.

§ 2. Pre-equivalence data

2.1 PROPOSITION. Let $T: \text{Com}_{-A} \rightarrow \text{Com}_{-\Gamma}$ be a “linear” functor. If T is left exact and preserves direct sums, there is a bicomodule ${}_A P_\Gamma$ such that $T(Z_A) \simeq Z \square_A P$ as a functor of $Z \in \text{Com}_{-A}$.

PROOF. Since T preserves direct sums, $W \otimes T(Z_A) \simeq T(W \otimes Z)$ for all $W \in \mathbf{Mod}$ and $Z \in \mathbf{Com}_A$. If we put $P = T(A)$, the exact sequence

$$Z \xrightarrow{\rho} Z \otimes A \xrightarrow[I \otimes A]{\rho \otimes I} Z \otimes A \otimes A,$$

where ρ is the structure map, induces the exact sequence

$$T(Z) \xrightarrow{T(\rho)} Z \otimes P \xrightarrow[I \otimes T(A)]{\rho \otimes I} Z \otimes A \otimes P$$

for all comodule Z_A , since T is left exact.

This means that P is a A - Γ -bicomodule, where $T(A): P \rightarrow A \otimes P$ is the left A -comodule structure map. For each comodule Z_A , $T(\rho)$ induces a natural isomorphism $T(Z) \simeq Z \square_A P$.

2.2 LEMMA. Let ${}_A P_\Gamma$ and ${}_A R_\Gamma$ be bicomodules and let $T = ? \square_A P$ and $U = ? \square_A R$ be the associated functors: $\mathbf{Com}_A \rightarrow \mathbf{Com}_\Gamma$. Let $\alpha: T \rightarrow U$ be a natural transformation. There is a unique bilinear map $f: P \rightarrow R$ such that $\alpha = ? \square_A f$.

PROOF. Put $f = \alpha(A): P = T(A) \rightarrow U(A) = Q$. Then for each $W \in \mathbf{Mod}$, $I \otimes f = \alpha(W \otimes A): W \otimes P = T(W \otimes A) \rightarrow U(W \otimes A) = W \otimes Q$. Since $A: A \rightarrow A \otimes A$ is right A -colinear, the following diagram commutes:

$$\begin{array}{ccccccc} T(A) & = & P & \xrightarrow{f} & Q & = & U(A) \\ T(A) \downarrow & & \downarrow \rho_P & & \downarrow \rho_Q & & \downarrow U(A) \\ T(A \otimes A) & = & A \otimes P & \xrightarrow{I \otimes f} & A \otimes Q & = & U(A \otimes A) \end{array}$$

where ρ_P and ρ_Q denote the A -comodule structure maps of P and Q . Hence f is bilinear. If Z_A is a comodule, $\alpha(Z) = I \square_A f$, since $\alpha(Z \otimes A) = I \square_A f$ and Z is a subcomodule of $Z \otimes A$.

2.3 DEFINITION. A set of pre-equivalence data $(A, \Gamma, {}_A P_\Gamma, {}_\Gamma Q_A, f, g)$ consists of coalgebras A and Γ , bicomodules ${}_A P_\Gamma$ and ${}_\Gamma Q_A$, and bilinear maps $f: A \rightarrow P \square_\Gamma Q$ and $g: \Gamma \rightarrow Q \square_A P$ making the following diagrams commute:

$$\begin{array}{ccc} P & \simeq & P \square_\Gamma \Gamma \\ \parallel & & \downarrow I \square g \\ A \square_A P & \xrightarrow{f \square I} & P \square_\Gamma Q \square_A P \end{array} \quad \begin{array}{ccc} Q & \simeq & Q \square_A A \\ \parallel & & \downarrow I \square f \\ \Gamma \square_\Gamma Q & \xrightarrow{g \square I} & Q \square_A P \square_\Gamma Q. \end{array}$$

If f and g are isomorphisms, (P, Q, f, g) is a set of *equivalence data*.

2.4 REMARK. Let $S = ? \square_{\Gamma} Q: \mathbf{Com}_{-\Gamma} \rightarrow \mathbf{Com}_{-A}$ and $T = ? \square_A P: \mathbf{Com}_{-A} \rightarrow \mathbf{Com}_{-\Gamma}$ be the linear functors determined by Q and P . The bilinear maps f and g can be identified with the natural transformations $f: I \rightarrow ST$ and $g: I \rightarrow TS$ by (2.2). The diagrams of (2.3) commute if and only if $Tf = gT: T \rightarrow TST$ and $fS = Sg: S \rightarrow STS$.

Hence if f is an isomorphism, then the pair $(f^{-1}: ST \rightarrow I, g: I \rightarrow TS)$ gives an adjoint relation $S \dashv T$.

If f and g are isomorphisms, then S and T are equivalence.

2.5 THEOREM. Let (A, Γ, P, Q, f, g) be a set of pre-equivalence data. Assume $f: A \rightarrow P \square_{\Gamma} Q$ is injective.

- (1) f is an isomorphism.
- (2) The comodules P_{Γ} and ${}_{\Gamma}Q$ are quasi-finite injective.
- (3) The comodules ${}_A P$ and Q_A are cogenerators.
- (4) g induces bicomodule isomorphisms

$$h_{-\Gamma}(P, \Gamma) \simeq Q \quad \text{and} \quad h_{\Gamma-}(Q, \Gamma) \simeq P.$$

- (5) The bicomodule structures ${}_A P_{\Gamma}$ and ${}_{\Gamma} Q_A$ induce coalgebra isomorphisms

$$e_{-\Gamma}(P) \simeq A \quad \text{and} \quad e_{\Gamma-}(Q) \simeq A.$$

PROOF. (1) Put ${}_A V_A = P \square_{\Gamma} Q$. View A as a sub-bicomodule of V via f . The diagram

$$\begin{array}{ccc} V & \simeq & V \square_A A \\ \parallel & & \downarrow \cap \\ A \square_A V & \hookrightarrow & V \square_A V \end{array}$$

commutes, since $I \square_{\Gamma} I \square_A f = I \square_{\Gamma} g \square_A I = f \square_{\Gamma} I \square_A I: P \square_{\Gamma} Q \rightarrow P \square_{\Gamma} Q \square_A P \square_{\Gamma} Q$. But in $V \square_A V$ we have $A \square_A V \cap V \square_A A = A \square_A A$. Hence $A = V$.

(2) Since f is an isomorphism, $S = ? \square_{\Gamma} Q \dashv T = ? \square_A P$. Hence P_{Γ} is quasi-finite by (1.10). Since S is exact, T preserves injectives. Hence $P_{\Gamma} = T(A_A)$ is injective. By symmetry ${}_{\Gamma}Q$ is quasi-finite injective.

- (3) Since $A \simeq P \square_{\Gamma} Q \hookrightarrow P \otimes Q$, ${}_A P$ and Q_A are cogenerators.

(4) Since S and the functor $Y_{\Gamma} \mapsto h_{-\Gamma}(P, Y)$ are the left adjoints of T (1.10), there is a canonical isomorphism of functors $h_{-\Gamma}(P, Y) \simeq Y \square_{\Gamma} Q, \forall Y \in \mathbf{Com}_{-\Gamma}$. Hence

$h_{-\Gamma}(P, \Gamma) \simeq \Gamma \square_{\Gamma} Q = Q$. This equals the bilinear map (1.11) induced by g . By symmetry g induces a bicomodule isomorphism $h_{-\Gamma}(Q, \Gamma) \simeq P$.

(5) The composite isomorphism

$$e_{-\Gamma}(P) = h_{-\Gamma}(P, P) \xrightarrow{\circ} P \square_{\Gamma} h_{-\Gamma}(P, \Gamma) \simeq P \square_{\Gamma} Q \xrightarrow{f} A$$

equals the coalgebra map $e_{-\Gamma}(P) \rightarrow A$ determined by the bicomodule structure ${}_A P_{\Gamma}$. By symmetry the bicomodule ${}_{\Gamma} Q_A$ induces a coalgebra isomorphism $e_{\Gamma}(Q) \simeq A$.

§ 3. Constructing an equivalence from a comodule.

Let P_{Γ} be a quasi-finite comodule and $A = e_{-\Gamma}(P)$. View ${}_A P_{\Gamma}$ as a bicomodule. Let ${}_{\Gamma} Q_A = h_{-\Gamma}(P, \Gamma)$, $g = \theta: \Gamma \rightarrow Q \square_A P$ and $f: A = h_{-\Gamma}(P, P) = h_{-\Gamma}(P, P \square_{\Gamma} \Gamma) \xrightarrow{\circ} P \square_{\Gamma} h_{-\Gamma}(P, \Gamma) = P \square_{\Gamma} Q$.

3.1 PROPOSITION. (P, Q, f, g) is a set of pre-equivalence data.

PROOF. f and g are bilinear by (1.9) and (1.15). The equality $f \square I = I \square g: P \rightarrow P \square_{\Gamma} Q \square_A P$ follows from the defining relation of ∂ (1.13) and the equality $I \square f = g \square I: Q \rightarrow Q \square_A P \square_{\Gamma} Q$ from the associativity of ∂ (1.16).

3.2 PROPOSITION. f is injective if and only if P_{Γ} is injective.

PROOF. The “if” part follows from (1.14) and the “only if” part from (2.5).

3.3 PROPOSITION. g is injective if and only if P_{Γ} is a cogenerator.

PROOF. The “only if” part follows from (2.5). The functor $W \mapsto W \otimes P, \mathbf{Mod} \rightarrow \mathbf{Com}_{-\Gamma}$ preserves direct products, since it has the left adjoint. Hence, if P_{Γ} is a cogenerator, there is an injective right Γ -colinear map $i: \Gamma \rightarrow W \otimes P$ for some $W \in \mathbf{Mod}$. Since there is a linear map $t: Q = h_{-\Gamma}(P, \Gamma) \rightarrow W$ such that $i = (I \otimes t) \circ g$, g is injective.

3.4 COROLLARY. (P, Q, f, g) is a set of a equivalence data if and only if P_{Γ} is a (quasi-finite) injective cogenerator.

3.5 THEOREM. Let ${}_A P_{\Gamma}$ be a bicomodule.

a) The following are equivalent.

(i) The functor $\mathbf{Com}_{-A} \rightarrow \mathbf{Com}_{-\Gamma}, Z_A \mapsto Z \square_A P$ is an equivalence.

(ii) The functor $\mathbf{Com}_{\Gamma} \rightarrow \mathbf{Com}_A, {}_{\Gamma} Y \mapsto P \square_{\Gamma} Y$ is an equivalence.

(iii) The comodule P_{Γ} is a quasi-finite injective cogenerator and $e_{-\Gamma}(P) \simeq A$ as coalgebras.

(iv) The comodule ${}_A P$ is a quasi-finite injective cogenerator and $e_A(P) \simeq \Gamma$ as coalgebras.

(v) There is a set of equivalence data $(\Lambda, \Gamma, P, Q, f, g)$.

(vi) There is a set of equivalence data $(\Gamma, \Lambda, Q', P, f', g')$.

b) When the above equivalent conditions hold, there is a canonical bicomodule isomorphism $h_{-\Gamma}(P, \Gamma) \simeq h_{\Lambda}(P, \Lambda)$. Let ${}_{\Gamma} Q_{\Lambda}$ denote this bicomodule. Then ${}_{\Gamma} \square_{\Lambda} Q$ (resp. $Q \square_{\Lambda} ?$) is a quasi-inverse of the functor of (i) (resp. (ii)).

PROOF. This follows immediately from (2.1), (2.2), (2.5), and (3.4).

3.6 COROLLARY. If P_{Γ} is a quasi-finite injective cogenerator comodule, there are a Γ - $e_{-\Gamma}(P)$ -bicomodule isomorphism $h_{-\Gamma}(P, \Gamma) \simeq h_{e_{-\Gamma}(P), -}(P, e_{-\Gamma}(P))$ and a coalgebra isomorphism $e_{e_{-\Gamma}(P), -}(P) \simeq \Gamma$. They are canonical.

§4. Locally finite abelian categories

4.1 DEFINITION [2, p. 356]. An abelian category \mathbf{A} is *locally finite*, if i) \mathbf{A} has direct sums, ii) for each *directed* family $\{P_{\alpha}\}$ of subobjects of an object $P \in \mathbf{A}$ the canonical map: $\lim_{\rightarrow} P_{\alpha} \rightarrow P$ induces an isomorphism: $\lim_{\rightarrow} P_{\alpha} \simeq \cup_{\alpha} P_{\alpha}$, and iii) there is a set of generators $\{M_i\}$ of \mathbf{A} where each M_i is of *finite length*.

The conditions i) and ii) mean that \mathbf{A} has *exact directed colimits* [2, p. 337, Prop. 6]. The subobjects of an object of \mathbf{A} form a 'set' by iii).

The category $\mathbf{Com}_{-\Gamma}$ is locally finite, since it is generated by finite dimensional comodules. (Note that the isomorphism classes of finite dimensional Γ -comodules clearly form a set).

4.2 Let \mathbf{A} be a locally finite abelian category. \mathbf{A} has *injective hulls* [2, p. 362, Th. 2]. The direct sum of a set of injective objects is injective [2, p. 387, Prop. 6]. Each object $M \in \mathbf{A}$ is clearly an *essential extension* [2, p. 358] of its *socle* $s(M)$ (=the sum of all *simple* subobjects of M). Hence an injective object I of \mathbf{A} is *indecomposable* if and only if the socle $s(I)$ is *simple* by [2, p. 361, Prop. 11].

Let $\{S_{\omega}\}_{\omega \in \Omega}$ be a complete set of representatives of isomorphism classes of simple objects of \mathbf{A} . (The set Ω exists by condition iii) of (4.1)). Let I_{ω} be the injective hull of S_{ω} . Then the I_{ω} , $\omega \in \Omega$, are injective indecomposable non isomorphic with each other, since $s(I_{\omega}) = S_{\omega}$. If I is an indecomposable injective object of \mathbf{A} , then $s(I) \simeq S_{\omega}$ for some $\omega \in \Omega$. Since I is the injective hull of $s(I)$, $I \simeq I_{\omega}$. Thus $\{I_{\omega}\}_{\omega \in \Omega}$ is a complete set of representatives of isomorphism classes of indecomposable injective objects of \mathbf{A} .

For each object $M \in \mathbf{A}$ and a cardinal number a let $M^{(a)}$ denote the direct sum of a isomorphic copies of M .

Then by [2, p. 388, Th. 2], each injective object I of \mathbf{A} is isomorphic to the direct sum $\bigoplus_{\omega \in \Omega} I_{\omega}^{(a_{\omega})}$ with a uniquely determined set of cardinal numbers $\{a_{\omega}\}_{\omega \in \Omega}$.

The direct sum $I = \bigoplus_{\omega \in \Omega} I_{\omega}^{(a_{\omega})}$ is an *injective cogenerator* of \mathbf{A} if and only if $a_{\omega} > 0$ for all $\omega \in \Omega$. Indeed I is an injective cogenerator if and only if $\mathbf{A}(S_{\omega}, I) \neq 0$ for all $\omega \in \Omega$. Since $s(I) = \bigoplus_{\omega \in \Omega} S_{\omega}^{(a_{\omega})}$, the assertion follows.

4.3 PROPOSITION. *Let \mathbf{A} be a locally finite k -abelian category. The following are equivalent.*

a) *If M and N are objects of finite length of \mathbf{A} , then the vector space $\mathbf{A}(M, N)$ is finite dimensional over k .*

b) *For each simple object S of \mathbf{A} , the endomorphism algebra $\mathbf{A}(S, S)$ is finite dimensional over k .*

PROOF. Let M and N be objects of finite length of \mathbf{A} . Let S be a simple subobject of M . Then the sequence $0 \rightarrow \mathbf{A}(M/S, N) \rightarrow \mathbf{A}(M, N) \rightarrow \mathbf{A}(S, N)$ is exact. Since condition b) means that $\mathbf{A}(S, N)$ is finite dimensional over k , it follows by induction on length of M that $\mathbf{A}(M, N)$ is finite dimensional.

4.4 DEFINITION. A k -abelian category \mathbf{A} is of *finite type* if \mathbf{A} is locally finite (as an abelian category) and the equivalent conditions of (4.3) are satisfied.

The category $\mathbf{Com}\text{-}_F$ is of finite type, since $\mathbf{Com}\text{-}_F(M, N)$ is finite by dimensional, if M and N are finite dimensional comodules.

4.5 PROPOSITION. *Let \mathbf{A} be a finite type k -abelian category and F an object of \mathbf{A} . The following are equivalent.*

a) *For each object M of finite length of \mathbf{A} , the space $\mathbf{A}(M, F)$ is finite dimensional over k .*

b) *For each simple object S of \mathbf{A} , the space $\mathbf{A}(S, F)$ is finite dimensional over k .*

c) *The socle $s(F)$ is isomorphic to $\bigoplus_{\omega \in \Omega} S_{\omega}^{n_{\omega}}$ where $\{S_{\omega}\}_{\omega \in \Omega}$ is a complete set of representatives of isomorphism classes of simple objects and n_{ω} are finite cardinal numbers.*

PROOF. The equivalence a) \Leftrightarrow b) is proved by induction on length of M . The equivalence b) \Leftrightarrow c) is obvious.

4.6 DEFINITION. An object F of a finite type k -abelian category \mathbf{A} is *quasi-finite* if the equivalent conditions of (4.5) are satisfied.

With the same notations as in (4.2), the injective object $I = \bigoplus_{\omega \in \Omega} I_\omega^{(a_\omega)}$ is quasi-finite if and only if each cardinal number a_ω is finite, since $s(I) = \bigoplus_{\omega \in \Omega} S_\omega^{(a_\omega)}$. In particular \mathbf{A} always has a quasi-finite injective cogenerator. (Take $a_\omega = 1$ for all $\omega \in \Omega$).

§ 5. Characterization of categories of comodules.

5.1 THEOREM. Let \mathbf{A} be a k -abelian category. \mathbf{A} is k -linearly equivalent to \mathbf{Com}_Γ for some coalgebra Γ if and only if \mathbf{A} is of finite type.

PROOF. We have only to prove the 'if' part. Let \mathbf{A} be a finite type k -abelian category. \mathbf{A} has a quasi-finite injective cogenerator U (4.6).

5.2 Since \mathbf{A} has direct sums, for each $W \in \mathbf{Mod}$ and $X \in \mathbf{A}$, there is an object $W \otimes X \in \mathbf{A}$ such that

$$\mathbf{A}(W \otimes X, Y) \simeq \text{Hom}(W, \mathbf{A}(X, Y))$$

naturally for all $Y \in \mathbf{A}$.

If Z is an object of finite length of \mathbf{A} , then $\mathbf{A}(Z, W \otimes X) \simeq W \otimes \mathbf{A}(Z, X)$, since the image $f(Z)$, where $f \in \mathbf{A}(Z, W \otimes X)$, must be contained in $W' \otimes X$ for some finite dimensional subspace W' of W .

In particular, since $\mathbf{A}(Z, U)$ is finite dimensional, $\mathbf{A}(Z, W \otimes U) \simeq \text{Hom}(\mathbf{A}(Z, U)^*, W)$.

5.3 LEMMA. For each object $X \in \mathbf{A}$, there is a vector space $h(X)$ such that $\text{Hom}(h(X), W) \simeq \mathbf{A}(X, W \otimes U)$ naturally for all $W \in \mathbf{Mod}$.

PROOF. When X is of finite length, we have only to put $h(X) = \mathbf{A}(X, U)^*$. In general let $\{X_\lambda\}$ be the subobjects of finite length of X . Since $X = \lim_{\rightarrow \lambda} X_\lambda$, it is enough to put $h(X) = \lim_{\rightarrow \lambda} h(X_\lambda)$.

5.4 Let $\alpha_X: X \rightarrow h(X) \otimes U$ denote the \mathbf{A} -morphism corresponding to the identity of $h(X)$. For each \mathbf{A} -morphism $f: X \rightarrow W \otimes U$, where $W \in \mathbf{Mod}$, there is a unique linear map $q: h(X) \rightarrow W$ with $f = (q \otimes I) \circ \alpha_X$. If $u: X \rightarrow X'$ is an \mathbf{A} -map, there is a unique linear map $h(u): h(X) \rightarrow h(X')$ such that $(h(u) \otimes I) \circ \alpha_X = \alpha_{X'} \circ u: X \rightarrow h(X') \otimes U$. The functor $h: \mathbf{A} \rightarrow \mathbf{Mod}$, $X \mapsto h(X)$ is the left adjoint of $W \mapsto W \otimes U$, $\mathbf{Mod} \rightarrow \mathbf{A}$. Hence h is linear and preserves direct sums.

In particular for each $X \in \mathbf{A}$ and $W \in \mathbf{Mod}$, the map $I \otimes \alpha_X: W \otimes X \rightarrow W \otimes h(X) \otimes U$ induces an isomorphism

$$h(W \otimes X) \simeq W \otimes h(X).$$

5.5 LEMMA. *The functor h is exact.*

PROOF. Indeed $X \mapsto h(X)^* \simeq \mathbf{A}(X, U)$ is exact, since U is injective.

5.6 Put $\Gamma = h(U)$. Let $\Delta: \Gamma \rightarrow \Gamma \otimes \Gamma$ and $\eta: \Gamma \rightarrow k$ be the unique linear maps such that $(I \otimes \alpha_U) \circ \alpha_U = (\Delta \otimes I) \circ \alpha_U: U \rightarrow \Gamma \otimes \Gamma \otimes U$ and $I = (\eta \otimes I) \circ \alpha_U: U \rightarrow k \otimes U = U$.

Then (Γ, Δ, η) is a coalgebra.

5.7 For each object $X \in \mathbf{A}$, let $\rho: h(X) \rightarrow \Gamma \otimes h(X)$ be the unique linear map such that $(I \otimes \alpha_U) \circ \alpha_X = (\rho \otimes I) \circ \alpha_X: X \rightarrow h(X) \otimes \Gamma \otimes U$.

Then $(h(X), \rho)$ is a right Γ -comodule. If $u: X \rightarrow X'$ is an \mathbf{A} -morphism, then $h(u): (h(X) \rightarrow h(X'))$ is Γ -colinear.

5.8 The functor $h: \mathbf{A} \rightarrow \mathbf{Com}_{-\Gamma}$ is linear exact, and commutes with colimits.

5.9 LEMMA. *For each $X \in \mathbf{A}$, the map $\alpha_X: X \rightarrow h(X) \otimes U$ is a monomorphism.*

PROOF. Let X' be a subobject of finite length of X contained in $\text{Ker}(\alpha_X)$. Then $\alpha_{X'} = 0$, since $h(X') \subset h(X)$. This means that $h(X') = \mathbf{A}(X', U)^* = 0$. Hence $X' = 0$.

5.10 LEMMA. *The functor $h: \mathbf{A} \rightarrow \mathbf{Com}_{-\Gamma}$ is fully faithful.*

PROOF. Let X and $Y \in \mathbf{A}$. Consider the natural map

$$\mathbf{A}(X, Y) \longrightarrow \mathbf{Com}_{-\Gamma}(h(X), h(Y))$$

induced by h . Both hand sides are left exact as functors of Y . Since there is an exact sequence of the form $0 \rightarrow Y \rightarrow W_1 \otimes U \rightarrow W_2 \otimes U$, where $W_i \in \mathbf{Mod}$, by (5.9), it is enough to consider the case $Y = W \otimes U$ in order to say that the above map is an isomorphism.

But then

$$\begin{aligned} \mathbf{A}(X, W \otimes U) &\simeq \text{Hom}(h(X), W) \simeq \mathbf{Com}_{-\Gamma}(h(X), W \otimes \Gamma) \\ &\simeq \mathbf{Com}_{-\Gamma}(h(X), h(W \otimes U)), \end{aligned}$$

where the composite coincides with the above map. Hence h is fully faithful.

5.11 We claim that $h: \mathbf{A} \rightarrow \mathbf{Com}_{-\Gamma}$ is an equivalence. This completes the proof of (5.1). Let Z_Γ be an arbitrary comodule. There is an exact sequence $0 \rightarrow Z \rightarrow W_1 \otimes \Gamma \xrightarrow{u} W_2 \otimes \Gamma$ of Γ -comodules, where $W_i \in \mathbf{Mod}$. Since $W_i \otimes \Gamma \simeq h(W_i \otimes U)$, there is a unique \mathbf{A} -morphism $\tilde{u}: W_1 \otimes U \rightarrow W_2 \otimes U$ such that $u = h(\tilde{u})$, since h is fully faithful. If $X = \text{Ker}(\tilde{u})$, then $h(X) \simeq Z$, since h is exact. Therefore h is an equivalence.

Note: An object X of a category \mathbf{A} which has direct products is a *cogenerator* if each $Y \in \mathbf{A}$ is embeddable into the direct product of a set of isomorphic copies of X .

If Γ is a coalgebra, Γ^* is an algebra, and each right Γ -comodule is a left Γ^* -module. Each left Γ^* -module M contains a unique maximal right Γ -comodule M' . If $(M_\alpha)_{\alpha \in I}$ is a family of right Γ -comodules, $(\prod_{\alpha \in I} M_\alpha)'$ gives the direct product in \mathbf{Com}_Γ . This contains $\bigoplus_{\alpha \in I} M_\alpha$ as a subcomodule. Hence a comodule P_Γ is a cogenerator of \mathbf{Com}_Γ if $\Gamma_\Gamma \subset W \otimes P_\Gamma$ for some $W \in \mathbf{Mod}$. The converse is true if P_Γ is quasi-finite (3.3).

After this paper was completed, it came to the author's attention that similar subjects were treated by Bertrand I-peng Lin, Morita's Theorem for Coalgebras, Communications in Algebra, vol. 1 (1974), 311-344. The results of the present paper are not contained in his work. He considers only *strong equivalences* between categories of comodules. It seems that he does not use the co-tensor product nor the co-hom functor. The characterization of the categories of comodules is not given in his paper. The author thanks Professor E. Taft for informing him of the Lin's paper.

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Department of Mathematics
University of Tsukuba
Sakura-Mura
Ibaraki-Ken
300-31 Japan