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$\sqrt{\text{Morita}}$ theory

— Formal ring laws and monoidal equivalences of categories of bimodules —

To the memory of Professor Akira Hattori

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Introduction.

By an *equivalence data* between two categories $\mathcal{A}$, $\mathcal{B}$ we mean a 4-tuple $(\boldsymbol{\Gamma}, \boldsymbol{\Delta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$, where $\boldsymbol{\Gamma}: \mathcal{A} \to \mathcal{B}$ and $\boldsymbol{\Delta}: \mathcal{B} \to \mathcal{A}$ are functors and $\boldsymbol{\gamma}: \boldsymbol{\Gamma\Delta} \simeq \boldsymbol{I}$, $\boldsymbol{\delta}: \boldsymbol{\Delta\Gamma} \simeq \boldsymbol{I}$ are isomorphisms of functors such that

$$\boldsymbol{\Delta\gamma} = \boldsymbol{\delta\Delta}, \qquad \boldsymbol{\gamma\Gamma} = \boldsymbol{\Gamma\delta}.$$

The Morita theory deals with equivalence data between left module categories ${}_R\mathcal{M}$, ${}_S\mathcal{M}$ for rings R, S. It is known that every equivalence data up to isomorphism is described in terms of some *Morita equivalence data* $({}_SP_R, {}_RQ_S, \alpha, \beta)$ with bimodule isomorphisms

$$\alpha : P \otimes_R Q \simeq S, \qquad \beta : Q \otimes_S P \simeq R$$

as follows: $\boldsymbol{\Gamma}$ takes $M \in {}_R\mathcal{M}$ to $P \otimes_R M \in {}_S\mathcal{M}$ and $\boldsymbol{\Delta}$ takes $N \in {}_S\mathcal{M}$ to $Q \otimes_S N \in {}_R\mathcal{M}$. The isomorphisms $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ come from α, β respectively.

When $\mathcal{A}$, $\mathcal{B}$ are *monoidal categories*, the 4-tuple $(\boldsymbol{\Gamma}, \boldsymbol{\Delta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ is called a *monoidal equivalence data* if in addition $\boldsymbol{\Gamma}$, $\boldsymbol{\Delta}$ are *monoidal functors* and $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ are isomorphisms of monoidal functors. A basic example of a monoidal category is provided by ${}_R\mathcal{M}_R$ the category of all R-bimodules. For R-bimodules M, N, the tensor product $M \otimes_R N$ (of M_R with ${}_RN$) has an R-bimodule structure (coming from ${}_RM$ and N_R). Together with unit R, this tensor product makes ${}_R\mathcal{M}_R$ into a monoidal category.

A natural question arises: *What happens if we consider monoidal equivalence data between bimodule monoidal categories* ${}_R\mathcal{M}_R$ *and* ${}_S\mathcal{M}_S$?

We begin with two simple examples of monoidal equivalence data. Let $({}_SP_R, {}_RQ_S, \alpha, \beta)$ be a Morita equivalence data as before. There is an associated

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monoidal equivalence data $(\Gamma, \Delta, \gamma, \delta)$ between ${}_R\mathcal{M}_R$, ${}_S\mathcal{M}_S$: Γ takes $M \in {}_R\mathcal{M}_R$ to $P \otimes_R M \otimes_R Q \in {}_S\mathcal{M}_S$ and Δ takes $N \in {}_S\mathcal{M}_S$ to $Q \otimes_S N \otimes_S P \in {}_R\mathcal{M}_R$. By means of α, β, we can define a natural monoidal structure on Γ, Δ as well as natural isomorphisms γ, δ. (See (2.2) for details).

To give the second example, we note that we can relate all the consideration to a fixed base ring k. This means R, S are k-algebras, and all bimodules such as ${}_RM_R$, ${}_SN_S$, ${}_SP_R$ and so on have *one* underlying k-module structure. (We omit the symbol k but the monoidal categories ${}_R\mathcal{M}_R$, ${}_S\mathcal{M}_S$ are related with k). Further this means we consider only *k-linear* (monoidal) equivalence data in the sense that Γ, Δ are k-linear functors.

Let A be an Azumaya, i.e., a central separable k-algebra. There is a natural (k-linear) equivalence data (Γ, Δ) between $\mathcal{M}_k$ and ${}_A\mathcal{M}_A$: Γ takes $V \in \mathcal{M}_k$ to $A \otimes_k V \in {}_A\mathcal{M}_A$ and Δ takes $M \in {}_A\mathcal{M}_A$ to $M^A = \{x \in M \mid ax = xa, \ \forall a \in A\}$. One sees (2.4) Γ, Δ have a natural monoidal structure together with natural isomorphisms γ, δ.

Two k-algebras R, S are called *$\sqrt{\text{Morita}}$ equivalent* (over k), written $R \sim_{\sqrt{\mathrm{M}}} S$, if there is a k-linear monoidal equivalence data between ${}_R\mathcal{M}_R$, ${}_S\mathcal{M}_S$. Similarly, we define the Morita equivalence relation $R \sim_{\mathrm{M}} S$. Obviously, $R \sim_{\sqrt{\mathrm{M}}} S$ implies $R \otimes_k R^{\mathrm{op}} \sim_{\mathrm{M}} S \otimes_k S^{\mathrm{op}}$. (The name $\sqrt{\text{Morita}}$ stems from this property, suggested by Moss Sweedler). The above examples show that we have

$$R \sim_{\mathrm{M}} S \implies R \sim_{\sqrt{\mathrm{M}}} S,$$
$$A \text{ is Azumaya} \implies A \sim_{\sqrt{\mathrm{M}}} k.$$

One of the main results (2.5d) tells that the converse of the last implication holds true. Thus, 'being $\sqrt{\text{Morita}}$ equivalent with the base ring' characterizes Azumaya algebras.

Just as usual Morita theory, the main problem of $\sqrt{\text{Morita}}$ theory is *how to describe all $\sqrt{\text{Morita}}$ equivalences.* We have two kinds of description, one coalgebraic (§ 3) and the other algebraic (§ 4). The coalgebraic description is universal in the sense that every monoidal equivalence data between two bimodule monoidal categories comes from some coalgebraic $\sqrt{\text{Morita}}$ equivalence data, but the appearance is quite technical, so we do not reproduce it here. The coalgebraic description is applied to deduce

$$R_i \sim_{\sqrt{\mathrm{M}}} S_i \ (i=1, 2) \implies R_1 \otimes R_2 \sim_{\sqrt{\mathrm{M}}} S_1 \otimes S_2.$$

The algebraic description has an appearance quite similar to the usual Morita equivalence data, while it applies only between those algebras which are *finite projective k-modules.* Let R, S be finite projective k-algebras. An *$S|R$-ring* means a pair (A, i) of an algebra A and an algebra map $i: S \otimes R^{\mathrm{op}} \to A$. One of the simplest example is the $R|R$-ring $\mathrm{End} R$. If A is an $S|R$-ring, and B is an

$R|U$-ring (let U be a finite projective algebra), then there is an $S|U$-ring

$$A\times_R B.$$

(This corresponds to the usual procedure to get a bimodule ${}_S(P\otimes_R Q)_U$ from bimodules ${}_SP_R$, ${}_RQ_U$). The product $\times_R$ was first introduced by Sweedler [**5**] in case R is commutative, and generalized by the author [**6**]=[**GA**] to the noncommutative case. The $R|R$-ring $\operatorname{End} R$ is the unit with respect to $\times_R$, thus we have

$$A\times_R \operatorname{End} R \simeq A, \qquad \operatorname{End} R\times_R B \simeq B.$$

An *algebraic* $\sqrt{Morita}$ *equivalence data* between R, S means a 4-tuple $(A_{S|R}, B_{R|S}, \lambda, \mu)$ where A and B are an $S|R$-ring and an $R|S$-ring respectively, and

$$\lambda : A\times_R B \simeq \operatorname{End} S \quad \text{an } S|S\text{-ring isomorphism,}$$
$$\mu : B\times_S A \simeq \operatorname{End} R \quad \text{an } R|R\text{-ring isomorphism}$$

satisfying some coherence condition. Just as the usual Morita equivalence data, this induces a monoidal equivalence data $(\boldsymbol{\Gamma}, \boldsymbol{\Delta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ between ${}_R\mathcal{M}_R$ and ${}_S\mathcal{M}_S$: $\boldsymbol{\Gamma}$ takes $M\in {}_R\mathcal{M}_R$ to $A\times_R M\in {}_S\mathcal{M}_S$ and $\boldsymbol{\Delta}$ takes $N\in {}_S\mathcal{M}_S$ to $B\times_S N\in {}_R\mathcal{M}_R$. The isomorphisms $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ are induced from λ, μ. The fact that $\boldsymbol{\Gamma}$, $\boldsymbol{\Delta}$ have a natural monoidal structure, together with the fact that $A\times_R B$ is an algebra for an algebra A over $\bar{R}$ $(=R^{\mathrm{op}})$ and an algebra B over R, follows from the fact that the bifunctor [**GA**, p. 465, (1.1)]

$$(-)\times_R(-) : {}_{\bar{R}}\mathcal{M}_{\bar{R}}\times {}_R\mathcal{M}_R \longrightarrow \mathcal{M}_k, \qquad (M, N)\longmapsto M\times_R N$$

has a natural monoidal structure. We prove (5.10) that *every monoidal equivalence data between* ${}_R\mathcal{M}_R$ *and* ${}_S\mathcal{M}_S$ *comes from some algebraic* $\sqrt{Morita}$ *equivalence data up to isomorphism if the algebras* R, S *are finite projective.*

In § 1, we discuss formal ring laws. The notion of an Amitsur 2-cocycle has been generalized to noncommutative algebras by Sweedler [**4**]=[**MA**]. When $\sigma=\sum a_i\otimes b_i\otimes c_i\in R\otimes R\otimes R$ is a 2-cocycle in his sense [**MA**, (2.1)] we say $\sigma(X, Y)=\sum a_iXb_iYc_i$ with noncommuting indeterminates X, Y, is a *formal ring law* over R because the defining formulas (ibid., (2.2) and (2.3)) can be read as

$$\sigma(X, \sigma(Y, Z)) = \sigma(\sigma(X, Y), Z),$$
$$\exists e_\sigma\in R \quad \text{such that} \quad \sigma(e_\sigma, X) = X = \sigma(X, e_\sigma).$$

For a formal ring law $\sigma(X, Y)$ over R, there is some associated monoidal functor

$$(-)^\sigma : {}_R\mathcal{M}_R \longrightarrow {}_{R^\sigma}\mathcal{M}_{R^\sigma}, \qquad M\longmapsto M^\sigma.$$

We give a criterion (1.17) for this to be a monoidal equivalence. The formal ring law $\sigma(X, Y)$ is called *invertible* if this is the case. Our criterion tells that if σ is invertible, then it has an inverse, i.e., a formal ring law $\tau(X, Y)$ over R^σ such that $(-)^\tau$ is the strict inverse of the monoidal functor $(-)^\sigma$. Therefore, the monoidal functor $(-)^\sigma$ is an *isomorphism* if σ is invertible.

If R is commutative, σ is invertible if and only if it is invertible in the usual sense. If R is Azumaya, σ is invertible if and only if the algebra R^σ is Azumaya.

The brief summary [8] offers another approach to the $\sqrt{\text{Morita}}$ theory. The idea of generalizing Morita equivalences to monoidal categories appears in Pareigis [2]. This paper gives some influence to our theory. Most main results of the paper were obtained while the author was staying at the University of Munich in 1983/84. He thanks the Alexander von Humboldt Foundation for its support, Professor B. Pareigis for his friendship and hospitality, and Moss Sweedler for conversation with the author.

§ 0. Conventions.

Throughout we work over a fixed commutative ring k with 1. All algebras and modules are unitary k-algebras and k-modules. All algebra maps are unitary. Unadorned $\otimes$, Hom, and End mean $\otimes_k$, Hom_k, and End_k respectively. $\mathcal{M}_k$ denotes the category of all modules. For algebras $R_1, \cdots, R_n$, ${}_{R_1,\cdots,R_t}\mathcal{M}_{R_{t+1},\cdots,R_n}$ denotes the category of all *left* $R_1, \cdots, R_t$ *right* $R_{t+1}, \cdots, R_n$ *multimodules*. Such a multimodule is defined to be a module M with $\theta_i : R_i \to \mathrm{End}\,M$ an algebra map for $1 \leqq i \leqq t$ and an opposite algebra map for $t < i \leqq n$ such that $\theta_i(r_i)\theta_j(r_j) = \theta_j(r_j)\theta_i(r_i)$ for all $i \neq j$, $r_i \in R_i$, $r_j \in R_j$.

For an algebra R and R-bimodules M, N, the tensor product $M \otimes_R N$ (taken for M_R and ${}_RN$) is again an R-bimodule (the structure coming from ${}_RM$ and N_R). The *monoidal category* $({}_R\mathcal{M}_R, \otimes_R, R)$ plays an essential role. A *monoid object* in ${}_R\mathcal{M}_R$ is called an *R-ring* (*algebra over* R [5, 6], R/k-*algebra* [7, 8]). It is identified with a pair of an algebra E and an algebra map $R \to E$. The category of all R-rings is denoted by $\mathcal{R}_R$.

We deal with *monoidal functors*

$$\boldsymbol{\Gamma} : {}_R\mathcal{M}_R \longrightarrow {}_S\mathcal{M}_S$$

for algebras R, S. Refer to [1] (and [9], too) for a general theory of monoidal categories and functors. We always assume $\boldsymbol{\Gamma}$ is k-*linear* in the sense that $\boldsymbol{\Gamma} : {}_R\mathcal{M}_R(M, N) \to {}_S\mathcal{M}_S(\boldsymbol{\Gamma}M, \boldsymbol{\Gamma}N)$ is a k-module map for all $M, N \in {}_R\mathcal{M}_R$. The monoidal structure on $\boldsymbol{\Gamma}$ is given by defining a *product* $xy \in \boldsymbol{\Gamma}(M \otimes_R N)$ for $x \in \boldsymbol{\Gamma}M$ and $y \in \boldsymbol{\Gamma}N$ in such a way that the product map $(x, y) \mapsto xy$ gives an

S-bimodule map

$$\boldsymbol{\Gamma}M\otimes_S\boldsymbol{\Gamma}N \longrightarrow \boldsymbol{\Gamma}(M\otimes_R N)$$

which is natural in M, N, associative in the sense that

$$(xy)z = x(yz) \qquad \text{in } \boldsymbol{\Gamma}(M\otimes_R N\otimes_R P)$$

for $z\in\boldsymbol{\Gamma}P$ with $P\in {}_R\mathcal{M}_R$, and having a *unit* $1\in\boldsymbol{\Gamma}R$. The monoidal functor $\boldsymbol{\Gamma}$ is called a *monoidal equivalence* if it is an equivalence as a functor and the structure maps (product and unit) are isomorphisms.

§ 1. Formal ring laws.

We give a new interpretation to Sweedler's 2-cocycles [**MA**, p. 308, Def. 2.1]. We work over an algebra R.

1.1. DEFINITION. For a finite set of indeterminates $X_1, \cdots, X_n$, we denote by $R\langle X_1, \cdots, X_n\rangle$ *the free* R*-ring on* $X_1, \cdots, X_n$. If U is an R-ring, then for any elements $u_1, \cdots, u_n$ in U, there is a unique R-ring map $R\langle X_1, \cdots, X_n\rangle\to U$ which takes X_i to u_i. The image of $f(X_1, \cdots, X_n)\in R\langle X_1, \cdots, X_n\rangle$ is denoted by $f(u_1, \cdots, u_n)$. The induced map

$$U^n \longrightarrow U, \qquad (u_1, \cdots, u_n)\longmapsto f(u_1, \cdots, u_n)$$

plays a role later.

1.2. DEFINITION. The sub-R-bimodule $RX_1RX_2\cdots X_nR$ of $R\langle X_1, \cdots, X_n\rangle$ is denoted by $R(X_1, \cdots, X_n)$. This is isomorphic to $R\otimes R\otimes\cdots\otimes R$ ($n+1$ copies). Its elements are called *n-forms* (in $X_1, \cdots, X_n$ over R). Any n-form is a finite sum of *monomial forms* $a_0X_1a_1X_2\cdots X_na_n$ with $a_i\in R$. The 0-forms are identified with elements in R.

1.3. SUBSTITUTION LEMMA. *Let* $[1, N]=I_1\cup\cdots\cup I_n$ *be a division into* n *disjoint intervals. We assume* $x<y$ *for* $x\in I_i$ *and* $y\in I_j$ *if* $i<j$. *Let* $f(X_1, \cdots, X_n)$ *and* $g_i(Y_j, \cdots, Y_k)$, *where* $\{j, \cdots, k\}=I_i$, *for* $1\leq i\leq n$, *be forms over* R. *(Here* $Y_1, \cdots, Y_N$ *are non-commuting indeterminates). Then* $f(g_1, \cdots, g_n)$, *which is a well-defined element in* $R\langle Y_1, \cdots, Y_N\rangle$, *is an* N*-form in* $Y_1, \cdots, Y_N$ *over* R.

This is easily verified. In particular, for $f(X), g(X)\in R(X)$, we have $f(g(X))\in R(X)$. We put $(f\circ g)(X)=f(g(X))$. The 1-forms $R(X)$ form an algebra with product $\circ$ and unit X, which is isomorphic to $R\otimes R^{\mathrm{op}}$.

1.4. DEFINITION. A 2-form $\sigma(X, Y)$ over R is called a *formal ring law over* R if we have

(1.4a) $\sigma(X, \sigma(Y, Z)) = \sigma(\sigma(X, Y), Z)$ as 3-forms in X, Y, Z,

(1.4b) $\exists e_\sigma \in R$ such that $\sigma(X, e_\sigma) = X = \sigma(e_\sigma, X)$.

Conditions (1.4) are simply restatements of [**MA**, (2.2) and (2.3)]. Hence $\sigma(X, Y)=\sum a_i X b_i X c_i$ is a formal ring law if and only if $\sum a_i \otimes b_i \otimes c_i \in R\otimes R\otimes R$ is a *2-cocycle*. The element e_σ, called *the unit*, is unique. The *trivial* formal ring law XY is denoted by $\omega(X, Y)$.

Sometimes we use the *n-times iterated product* $\sigma_n(X_0, X_1, \cdots, X_n)$ of the formal ring law $\sigma(X, Y)$, defined inductively by

(1.5) $\sigma_k(X_0, \cdots, X_k) = \sigma(X_0, \sigma_{k-1}(X_1, \cdots, X_k))$

starting with $\sigma_1(X_0, X_1)=\sigma(X_0, X_1)$, or $\sigma_0(X_0)=X_0$.

For an R-ring U, the formal ring law $\sigma(X, Y)$ over R induces an associative product

(1.6) $\sigma : U\times U \longrightarrow U, \quad (u, v) \longmapsto \sigma(u, v)$

to get a new algebra written U^σ with unit e_σ [**MA**, p. 308]. In particular we have an algebra R^σ, and the algebra map $R\to U$ induces an algebra map $R^\sigma \to U^\sigma$. Thus we have a functor

(1.7) $(-)^\sigma : \mathcal{R}_R \longrightarrow \mathcal{R}_{R^\sigma}, \quad U \longmapsto U^\sigma$ (cf. [**3**, p. 144]).

The trivial formal ring law induces the identity.

1.8. DEFINITION. Let $\sigma(X, Y)$ and $\tau(X, Y)$ be formal ring laws over R. A 1-form $f(Z)$ is a *map* $\sigma\to\tau$ if we have

(1.8a) $f(\sigma(X, Y)) = \tau(f(X), f(Y))$ in $R(X, Y)$,

(1.8b) $f(e_\sigma) = e_\tau$.

If $f:\sigma\to\tau$ and $g:\tau\to\rho$ are maps of formal ring laws over R, then $g\circ f$, the product in the algebra $R(Z)$, is a map $\sigma\to\rho$. The (small) category of all formal ring laws over R is denoted by $\mathcal{F}_R$.

$f(Z)$ is a map $\sigma\to\tau$ if and only if *σ is cohomologous to τ via f* [**MA**, p. 309, Def. 2.7]. It follows directly from (1.8) that if this is the case, the induced map

(1.9) $f_U : U^\sigma \longrightarrow U^\tau, \quad u \longmapsto f(u)$

is an algebra map for all R-ring U. The algebra map (1.9) is natural in U.

We show the functor (1.7) comes from some *monoidal functor* ${}_R\mathcal{M}_R \to {}_{R^\sigma}\mathcal{M}_{R^\sigma}$. For $M\in {}_R\mathcal{M}_R$, we have $\sigma(R, M)$, $\sigma(M, R)\subset M$ in any R-ring containing M. This σ-product makes M into a R^σ-bimodule, written M^σ, and we get a functor

(1.10) $(-)^\sigma : {}_R\mathcal{M}_R \longrightarrow {}_{R^\sigma}\mathcal{M}_{R^\sigma}$ (cf. [**3**, p. 149]).

For $m \in M$, $n \in N$ with $M, N \in {}_R\mathcal{M}_R$, consider the product

$$(1.10a) \qquad \sigma(m, n) \underset{\text{def}}{=} \sum a_i m b_i \otimes n c_i = \sum a_i m \otimes b_i n c_i \in M \otimes_R N$$

where $\sigma(X, Y) = \sum a_i X b_i Y c_i$. The σ-product induces a R^σ-bimodule map

$$(1.10b) \qquad M^\sigma \otimes_{R^\sigma} N^\sigma \longrightarrow (M \otimes_R N)^\sigma, \qquad m \otimes n \longmapsto \sigma(m, n).$$

Together with the identity $R^\sigma = R^\sigma$, this makes (1.10) into a monoidal functor, and the functor (1.7) is induced from this.

For a map $f(Z) = \sum p_j Z q_j : \sigma \to \tau$ in $\mathcal{F}_R$ and $M \in {}_R\mathcal{M}_R$,

$$(1.11a) \qquad f_M : M^\sigma \longrightarrow M^\tau, \qquad m \longmapsto f(m) = \sum p_j m q_j$$

is semilinear with respect to f_R (1.9) taking σ-product to τ-product. Hence this gives rise to a *map of monoidal functors*

$$(1.11b) \qquad f_- : (-)^\sigma \longrightarrow \mathbf{for} \circ (-)^\tau$$

with the forgetful monoidal functor $\mathbf{for} : {}_{R^\tau}\mathcal{M}_{R^\tau} \to {}_{R^\sigma}\mathcal{M}_{R^\sigma}$ with respect to $f_R : R^\sigma \to R^\tau$.

We show the composite of monoidal functors of type (1.10) is of the same type. To establish this, we require some technical map which is contained implicitly in the proof of [**3**, Prop. 4.3].

Let $\sigma(X, Y)$ be a formal ring law over R. Let

$$(1.12a) \qquad \phi_\sigma^{(n)} : R^\sigma\langle X_1, \cdots, X_n\rangle \longrightarrow R\langle X_1, \cdots, X_n\rangle^\sigma$$

be the R^σ-ring map taking X_i to X_i. This induces a R^σ-bimodule map

$$(1.12b) \qquad \phi_\sigma^{(n)} : R^\sigma(X_1, \cdots, X_n) \longrightarrow R(X_1, \cdots, X_n)^\sigma.$$

$\phi_\sigma^{(0)}$ is the identity of R^σ. In terms of iterated product, we have

$$(1.12c) \qquad \phi_\sigma^{(n)}(a_0 X_1 a_1 X_2 \cdots X_n a_n) = \sigma_{2n}(a_0, X_1, a_1, X_2, \cdots, X_n, a_n)$$

for $a_i \in R$. The ϕ_σ-map commutes with substitution. For forms $f, g_1, \cdots, g_n$ over R^σ as in (1.3), let $f' = \phi_\sigma(f)$ and $g_i' = \phi_\sigma(g_i)$. Then we have

$$\phi_\sigma^{(N)}(f(g_1, \cdots, g_n)) = f'(g_1', \cdots, g_n').$$

From this follow the following facts easily:

(1.13a) $\phi_\sigma^{(1)} : R^\sigma(X) \to R(X)^\sigma = R(X)$ *is an algebra map* (*cf.* [**3**, Lemma 5.1]). (*The unit* $e_\sigma X e_\sigma$ *goes to* X).

(1.13b) *If* $\tau(X, Y)$ *is a formal ring law over* R^σ *with unit* e_τ, *then* $\phi_\sigma^{(2)}(\tau)(X, Y)$ *is a formal ring law over* R *with the same unit.*

(1.13c) *If* $f(Z):\tau_1\to\tau_2$ *is a map in* $\mathcal{F}_{R^\sigma}$*, then* $\phi_\sigma^{(1)}(f)(Z):\phi_\sigma^{(2)}(\tau_1)\to\phi_\sigma^{(2)}(\tau_2)$ *is a map in* $\mathcal{F}_R$.

Summarizing these facts, we get a functor

(1.13) $$\boldsymbol{\phi}_\sigma : \mathcal{F}_{R^\sigma} \longrightarrow \mathcal{F}_R$$

taking the trivial formal ring law ω^σ to σ.

1.14. DEFINITION. For $\sigma\in\mathcal{F}_R$ and $\tau\in\mathcal{F}_{R^\sigma}$, we put

$$\sigma*\tau = \boldsymbol{\phi}_\sigma(\tau) \in \mathcal{F}_R.$$

1.15. LEMMA. *Let* $\sigma(X, Y)$ *be a formal ring law over* R, *and let* $f(X, Y)$ *be a 2-form over* R^σ. *Put* $f'(X, Y)=\phi_\sigma^{(2)}(f)(X, Y)$. *For any* R*-ring* U, *the* f*-product on* U^σ *is the same as the* f'*-product on* U.

PROOF. We may assume $f(X, Y)=aXbYc$ $(a, b, c\in R^\sigma)$. We have $f'(X, Y)=\sigma_4(a, X, b, Y, c)$ taken in $R\langle X, Y\rangle$. For $u, v\in U^\sigma$, the value $f(u, v)$ is the product $aubvc$ in U^σ, namely, $\sigma_4(a, u, b, v, c)$ taken in U, which is precisely $f'(u, v)$. Q. E. D.

It follows immediately that we have

$$U^{\sigma*\tau} = (U^\sigma)^\tau$$

as algebra for any R-ring U and $\tau\in\mathcal{F}_{R^\sigma}$. We write $U^{\sigma\tau}=U^{\sigma*\tau}$.

1.16. THEOREM. *Let* $\sigma\in\mathcal{F}_R$, $\tau\in\mathcal{F}_{R^\sigma}$ *and* $\rho\in\mathcal{F}_{R^{\sigma\tau}}$. *We have*

a) $$(\sigma*\tau)*\rho = \sigma*(\tau*\rho),$$

b) $$\omega*\sigma = \sigma = \sigma*\omega^\sigma,$$

c) $$\phi_{\sigma*\tau}^{(n)} : R^{\sigma\tau}(X_1, \cdots, X_n) \xrightarrow{\phi_\tau^{(n)}} R^\sigma(X_1, \cdots, X_n)^\tau \xrightarrow{\phi_\sigma^{(n)}} R(X_1, \cdots, X_n)^{\sigma\tau},$$

d) $$\boldsymbol{\phi}_{\sigma*\tau} : \mathcal{F}_{R^{\sigma\tau}} \xrightarrow{\boldsymbol{\phi}_\tau} \mathcal{F}_{R^\sigma} \xrightarrow{\boldsymbol{\phi}_\sigma} \mathcal{F}_R,$$

e) $$(-)^{\sigma*\tau} : {}_R\mathcal{M}_R \xrightarrow{(-)^\sigma} {}_{R^\sigma}\mathcal{M}_{R^\sigma} \xrightarrow{(-)^\tau} {}_{R^{\sigma\tau}}\mathcal{M}_{R^{\sigma\tau}}$$ *as monoidal functors.*

PROOF. For any R-ring U, we have $U^{(\sigma\tau)\rho}=U^{\sigma(\tau\rho)}$. Taking $U=R\langle X, Y\rangle$ and considering the product of X and Y, we conclude a). b) follows similarly. We have $\phi_{\sigma*\tau}^{(n)}=\phi_\sigma^{(n)}\circ\phi_\tau^{(n)}$ as $R^{\sigma\tau}$-ring map by definition. This yields c). d) We have $\boldsymbol{\phi}_{\sigma*\tau}=\boldsymbol{\phi}_\sigma\circ\boldsymbol{\phi}_\tau$ on objects by a), and on morphisms by c) for $n=1$. e) Take $M\in{}_R\mathcal{M}_R$. Embed M into an R-ring U. Since $U^{\sigma*\tau}=(U^\sigma)^\tau$, it follows that $M^{\sigma*\tau}=(M^\sigma)^\tau$ as $R^{\sigma\tau}$-bimodules. We may write it as $M^{\sigma\tau}$ in the sequel. Take another $N\in{}_R\mathcal{M}_R$, and embed M and N into an R-ring V so that the product of V induces an injection $M\otimes_R N\to V$. Since $V^{\sigma*\tau}=(V^\sigma)^\tau$ as algebras, it follows that the $\sigma*\tau$-product map for M, N (1.10b) factors as

$$M^{\sigma\tau}\otimes_{R^{\sigma\tau}}N^{\sigma\tau} \longrightarrow (M^{\sigma}\otimes_{R^{\sigma}}N^{\sigma})^{\tau} \longrightarrow (M\otimes_{R}N)^{\sigma\tau}$$

with the τ-product for M^{σ}, N^{σ} and the σ-product for M, N. This means we have $(-)^{\sigma*\tau}=(-)^{\tau}\circ(-)^{\sigma}$ as monoidal functors. Q. E. D.

1.17. INVERTIBILITY THEOREM. *For $\sigma\in\mathcal{F}_R$, the following conditions are equivalent with each other.*

1) *There is $\tau\in\mathcal{F}_{R^{\sigma}}$ such that $\sigma*\tau=\omega$ and $\tau*\sigma=\omega^{\sigma}$.*
2) *There are $\tau\in\mathcal{F}_{R^{\sigma}}$ and $\rho\in\mathcal{F}_{R^{\sigma\tau}}$ such that $\sigma*\tau\simeq\omega$ in $\mathcal{F}_R$ and $\tau*\rho\simeq\omega^{\sigma}$ in $\mathcal{F}_{R^{\sigma}}$.*
3) *The functor $\boldsymbol{\phi}_{\sigma}$* (1.13) *is an equivalence.*
4) *The functor $\boldsymbol{\phi}_{\sigma}$* (1.13) *is an isomorphism.*
5) *The functor $(-)^{\sigma}$* (1.7) *is an equivalence.*
6) *The functor $(-)^{\sigma}$* (1.7) *is an isomorphism.*
7) *The monoidal functor $(-)^{\sigma}$* (1.10) *is a monoidal equivalence.*
8) *The monoidal functor $(-)^{\sigma}$* (1.10) *is an isomorphism.*
9) *The R^{σ}-ring map $\psi_{\sigma}^{(n)}$* (1.12a) *is an isomorphism for all n.*
10) *The map $\phi_{\sigma}^{(n)}$* (1.12b) *is an isomorphism for all n.*
11) *The R^{σ}-ring map $\psi_{\sigma}^{(1)}$* (1.12a) *is an isomorphism.*
12) *The map $\phi_{\sigma}^{(1)}$* (1.12b) *is an isomorphism and the structure map* (1.10b) *is an isomorphism for all $M, N\in{}_R\mathcal{M}_R$.*

The formal ring law $\sigma(X, Y)$ is called *invertible* (with inverse $\tau(X, Y)$ in 1)) if these equivalent conditions hold. It follows from 1) that invertible formal ring laws are closed under the $*$-product.

In § 3, we refine conditions 10) and 12) as follows (3.20):

10)′ *The map $\phi_{\sigma}^{(n)}$* (1.12b) *is an isomorphism for $n=1, 2$.*

12)′ *The map $\phi_{\sigma}^{(1)}$* (1.12b) *is an isomorphism, and the structure map* (1.10b) *is an isomorphism for $M=N=R\otimes R$.*

The algebra R is *categorically rigid* [**3**, Def. 6.1] if the functor (1.7) is an equivalence for all $\sigma\in\mathcal{F}_R$. In our terminology, this is the case if all $\sigma\in\mathcal{F}_R$ is invertible.

PROOF. We prove the theorem as follows:

$$\begin{tikzcd}
6 \arrow[r] & 5 \arrow[r] & 9 \arrow[r] & 11 \arrow[d] & 1 \arrow[r] & 2 \arrow[dlllll, bend right=40] \\
8 \arrow[u] \arrow[r] & 7 \arrow[u] \arrow[r] & 12 \arrow[r] & 10 \arrow[r] & 4 \arrow[u] \arrow[r] & 3 \arrow[u]
\end{tikzcd}$$

Trivial implications: 6)$\Rightarrow$5), 8)$\Rightarrow$7), 9)$\Rightarrow$11), 1)$\Rightarrow$2), and 4)$\Rightarrow$3).

5)$\Rightarrow$9). There is a R^{σ}-ring isomorphism $R^{\sigma}\langle X_1, \cdots, X_n\rangle\simeq U^{\sigma}$ for some

R-ring U. The composite

$$U^\sigma \simeq R^\sigma\langle X_1, \cdots, X_n\rangle \xrightarrow{\phi_\sigma^{(n)}} R\langle X_1, \cdots, X_n\rangle^\sigma$$

is of the form χ^σ for some R-ring map $\chi: U \to R\langle X_1, \cdots, X_n\rangle$. Assume X_i in $R^\sigma\langle X_1, \cdots, X_n\rangle$ corresponds with u_i in U^σ $(=U)$. Then we have $\chi(u_i)=X_i$. Define an R-ring map $\beta: R\langle X_1, \cdots, X_n\rangle \to U$ by $\beta(X_i)=u_i$. Then $\chi\circ\beta=\mathrm{Id}$. On the other hand, $\beta^\sigma\circ\chi^\sigma=\mathrm{Id}$ since $U^\sigma=R^\sigma\langle u_1, \cdots, u_n\rangle$. We have $\beta\circ\chi=\mathrm{Id}$ since $(-)^\sigma$ (1.7) is an equivalence. Thus χ, hence χ^σ too, is an isomorphism. This means the map $\phi_\sigma^{(n)}$ is an isomorphism for all n.

11)$\Rightarrow$10). The substitution $f(X_1, \cdots, X_n) \mapsto f(X, \cdots, X)$, $R(X_1, \cdots, X_n) \to R\langle X\rangle$ gives rise to an isomorphism of R-bimodules

$$\bigoplus_{n=0}^{\infty} R(X_1, \cdots, X_n) \simeq R\langle X\rangle .$$

Through this isomorphism, the R^σ-ring map $\phi_\sigma^{(1)}: R^\sigma\langle X\rangle \to R\langle X\rangle^\sigma$ is identified with the direct sum of $\phi_\sigma^{(n)}$. It follows that 11) is equivalent to 10).

7)$\Rightarrow$(5) and 12)). Since the monoidal functor (1.10) induces the functor (1.7), we have 7)$\Rightarrow$5). The latter half of condition 12) is contained in 7). The first half is a special case of 10), and we know 7)$\Rightarrow$5)$\Rightarrow$9)$\Rightarrow$11)$\Rightarrow$10) already.

8)$\Rightarrow$6). The inverse of the monoidal functor (1.10) induces the inverse of the functor (1.7).

12)$\Rightarrow$10). The product induces an isomorphism of R-bimodules $R(X_1, \cdots, X_n) \otimes_R R(X_{n+1}) \simeq R(X_1, \cdots, X_n, X_{n+1})$. Through this isomorphism, the map $\phi_\sigma^{(n+1)}$ is identified with the composite

$$R^\sigma(X_1, \cdots, X_n) \otimes_{R^\sigma} R^\sigma(X_{n+1}) \xrightarrow{\phi_\sigma^{(n)} \otimes \phi_\sigma^{(1)}} R(X_1, \cdots, X_n)^\sigma \otimes_{R^\sigma} R(X_{n+1})^\sigma$$
$$\xrightarrow[(1.10\mathrm{b})]{} [R(X_1, \cdots, X_n) \otimes_R R(X_{n+1})]^\sigma .$$

Hence by induction, the map $\phi_\sigma^{(n)}$ is an isomorphism for all n if it is for $n=1$.

10)$\Rightarrow$4). We have $\phi_\sigma^{(2)}: R^\sigma(X, Y) \simeq R(X, Y)$ and $\phi_\sigma^{(1)}: R^\sigma(Z) \simeq R(Z)$ by assumption. Since the ϕ_σ-map commutes with substitution, it follows directly that $\tau(X, Y)$ in $R^\sigma(X, Y)$ is a formal ring law over R^σ if and only if $\phi_\sigma^{(2)}(\tau)(X, Y)$ is a formal ring law over R. (We are using condition 10) for $n=1, 2, 3$). It also follows that for τ, ρ in $\mathcal{F}_{R^\sigma}$ and $f(Z)$ in $R^\sigma(Z)$, we have $f: \tau \to \rho$ if and only if $\phi_\sigma^{(1)}(f): \phi_\sigma^{(2)}(\tau) \to \phi_\sigma^{(2)}(\rho)$. Hence $\boldsymbol{\phi}_\sigma: \mathcal{F}_{R^\sigma} \to \mathcal{F}_R$ (1.13) is an isomorphism of categories.

The following lemma is required to prove the rest implications.

1.18. LEMMA. *8) follows if* $\sigma \simeq \omega$ *in* $\mathcal{F}_R$.

PROOF. The monoidal functor $(-)^\sigma$ (1.10) is isomorphic with the isomorphism

${}_R\mathcal{M}_R \to {}_{R^\sigma}\mathcal{M}_{R^\sigma}$ induced from some isomorphism $R^\sigma \to R$ (1.11). This means $(-)^\sigma$ is a monoidal equivalence, i. e., 7) holds. We know 7)$\Rightarrow$12)$\Rightarrow$10)$\Rightarrow$4). Hence $\boldsymbol{\phi}_\sigma : \mathcal{F}_{R^\sigma} \simeq \mathcal{F}_R$. Assume the isomorphism $\sigma \simeq \omega$ comes from an isomorphism $\omega^\sigma \simeq \tau$ in $\mathcal{F}_{R^\sigma}$. Since this yields $\sigma * \tau = \omega$, it follows from (1.16) that the monoidal functor $(-)^\sigma$ has a left inverse $(-)^\tau$. Since $\tau \simeq \omega^\sigma$, the monoidal functor $(-)^\tau$ has a left inverse, too. Hence $(-)^\tau$ is an isomorphism, and so is $(-)^\sigma$. This means 8). Q. E. D.

2)$\Rightarrow$8). The above lemma together with (1.16) implies $(-)^\tau \circ (-)^\sigma$ and $(-)^\rho \circ (-)^\tau$ are isomorphisms. Hence $(-)^\tau$ is an isomorphism, and so is $(-)^\sigma$.

3)$\Rightarrow$2). There is τ in $\mathcal{F}_{R^\sigma}$ such that $\boldsymbol{\phi}_\sigma(\tau) = \sigma * \tau \simeq \omega$ in $\mathcal{F}_R$. By the above lemma, 8), hence 4) too, holds for $\sigma * \tau$. By (1.16d), the composite functor $\boldsymbol{\phi}_\sigma \boldsymbol{\phi}_\tau : \mathcal{F}_{R^{\sigma\tau}} \to \mathcal{F}_R$ is an isomorphism. Since $\boldsymbol{\phi}_\sigma$ is an equivalence, so is $\boldsymbol{\phi}_\tau$. The argument applied to τ shows there is ρ in $\mathcal{F}_{R^{\sigma\tau}}$ with $\tau * \rho \simeq \omega^\sigma$.

4)$\Rightarrow$1). In the above proof 3)$\Rightarrow$2), we can replace '$\simeq$' (two places) by '$=$'. We have $\sigma = \rho$ by group theory. Q. E. D.

1.19. EXAMPLE. Let R be an *Azumaya* (i. e., central separable) algebra. There is a canonical isomorphism [**MA**, (1.3) and (1.6)]

$$\begin{aligned} R(X_1, \cdots, X_n) &\simeq \operatorname{Hom}(\otimes^n R, R) \\ f(X_1, \cdots, X_n) &\longleftrightarrow \bar{f} : u_1 \otimes \cdots \otimes u_n \mapsto f(u_1, \cdots, u_n) \end{aligned}$$

for all n. In particular $R(X) \simeq \operatorname{End} R$ as algebras (definition of Azumaya!). Since the isomorphism commutes with substitution, the following facts follow:

(1.19a) *A 2-form $\sigma(X, Y)$ over R is a formal ring law if and only if $(R, \bar{\sigma})$ is an (associative unitary) algebra.*

(1.19b) *For σ, τ in $\mathcal{F}_R$, the isomorphism $R(X) \simeq \operatorname{End} R$ induces $\mathcal{F}_R(\sigma, \tau) \simeq$* $\operatorname{Alg}(R^\sigma, R^\tau)$. *In particular $\sigma \simeq \tau$ in $\mathcal{F}_R$ if and only if $R^\sigma \simeq R^\tau$ as algebras.*

1.20. THEOREM. *Let $\sigma(X, Y)$ be a formal ring law over the Azumaya algebra R. σ is invertible if and only if R^σ is an Azumaya algebra.*

PROOF. The 'only if' part follows from (2.5). Assume R^σ is Azumaya. There is a unique τ in $\mathcal{F}_{R^\sigma}$ such that $R = (R^\sigma)^\tau$. This means $\omega = \sigma * \tau$. Replacing (R, σ) with (R^σ, τ) we see there is ρ in $\mathcal{F}_R$ with $\omega^\sigma = \tau * \rho$. Obviously, $\sigma = \rho$. Hence σ is invertible. Q. E. D.

1.21. EXAMPLE. Let R be a *commutative* algebra. If $\sigma(X, Y)$ is a formal ring law over R, then e_σ is a unit since $1 = \sigma(1, e_\sigma) \in e_\sigma R$, and we have an algebra isomorphism $R \simeq R^\sigma$, $a \leftrightarrow a e_\sigma$. For a in R, m in M with an R-bimodule M, we have $\sigma(a e_\sigma, m) = am$ and $\sigma(m, a e_\sigma) = ma$, hence the R^σ-bimodule M^σ is simply the transport along $R \simeq R^{\sigma}$. For M, N in ${}_R\mathcal{M}_R$, we view $M \otimes_R N$ as a $R \otimes R \otimes R$-module as usual. If $\sigma(X, Y) = \sum a_i X b_i Y c_i$, then we have 7)=8) if and only if $\sum a_i \otimes b_i \otimes c_i$ is a unit in $R \otimes R \otimes R$. Next, consider $\phi_\sigma^{(1)} : R^{\sigma}(X) \to R(X)$. For a, b in R, $e_\sigma a X e_\sigma b$ goes to $\sigma_2(e_\sigma a, X, e_\sigma b) = aXb$. Hence this is always an isomorphism. Summarizing we have:

1.22. THEOREM. *Let $\sigma(X, Y) = \sum a_i X b_i Y c_i$ be a formal ring law over the commutative algebra R. σ is invertible if and only if $\sum a_i \otimes b_i \otimes c_i$ is a unit in $R \otimes R \otimes R$. Hence invertible formal ring laws (or Sweedler 2-cocycles) are identified with Amitsur 2-cocycles. The first half of condition 12) is always true.*

Simple criteria such as (1.20) and (1.22) for σ to be invertible cannot be expected to exist in general. Riffelmacher's criterion [3, Lemma 6.3] seems to contain some error (ibid., p. 154, $\downarrow 12 \sim \downarrow 16$).

§ 2. $\sqrt{\text{Morita}}$ equivalences.

We work over two algebras R and S.

2.1. DEFINITION. A *$\sqrt{\text{Morita}}$ equivalence data between R and S* means a 4-tuple $(\boldsymbol{\Gamma}, \boldsymbol{\Delta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ where $\boldsymbol{\Gamma} : {}_R\mathcal{M}_R \to {}_S\mathcal{M}_S$ and $\boldsymbol{\Delta} : {}_S\mathcal{M}_S \to {}_R\mathcal{M}_R$ are monoidal functors, and

$$(2.1a) \qquad \boldsymbol{\gamma} : \boldsymbol{\Gamma}\boldsymbol{\Delta} \simeq \boldsymbol{I}, \qquad \boldsymbol{\delta} : \boldsymbol{\Delta}\boldsymbol{\Gamma} \simeq \boldsymbol{I}$$

are isomorphisms of monoidal functors such that

$$(2.1b) \qquad \boldsymbol{\Delta}\boldsymbol{\gamma} = \boldsymbol{\delta}\boldsymbol{\Delta}, \qquad \boldsymbol{\gamma}\boldsymbol{\Gamma} = \boldsymbol{\Gamma}\boldsymbol{\delta}.$$

If we have a monoidal equivalence $\boldsymbol{\Gamma} : {}_R\mathcal{M}_R \to {}_S\mathcal{M}_S$, then it has a quasi-inverse since ${}_S\mathcal{M}_S$ has a generator, and the quasi-inverse has a natural monoidal structure. It follows that $\boldsymbol{\Gamma}$ embeds into a $\sqrt{\text{Morita}}$ equivalence data between R and S.

We write $R \sim_{\sqrt{\mathrm{M}}} S$ (resp. $R \sim_{\mathrm{M}} S$) if there is a $\sqrt{\text{Morita}}$ (resp. Morita) equivalence data between R and S. (We note that we are considering only *k-linear* equivalences).

$R \sim_{\sqrt{\mathrm{M}}} S$ implies $R \otimes R^{\mathrm{op}} \sim_{\mathrm{M}} S \otimes S^{\mathrm{op}}$ obviously. The name $\sqrt{\text{Morita}}$ stems from this (suggested by M. Sweedler).

2.2. EXAMPLE. Let $({}_SP_R, {}_RQ_S, \alpha, \beta)$ be a Morita equivalence data between R and S with bimodule isomorphisms $\alpha : P \otimes_R Q \simeq S$ and $\beta : Q \otimes_S P \simeq R$. This induces a $\sqrt{\text{Morita}}$ equivalence data $(\boldsymbol{\Gamma}, \boldsymbol{\Delta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ as follows: $\boldsymbol{\Gamma}$ takes $M \in {}_R\mathcal{M}_R$ to $\boldsymbol{\Gamma}(M) = P \otimes_R M \otimes_R Q \in {}_S\mathcal{M}_S$, and $\boldsymbol{\Delta}$ takes $N \in {}_S\mathcal{M}_S$ to $\boldsymbol{\Delta}(N) = Q \otimes_S N \otimes_S P \in {}_R\mathcal{M}_R$. We put

$$\gamma_N : \Gamma\Delta(N)=P\otimes_R Q\otimes_S N\otimes_S P\otimes_R Q \xrightarrow{\alpha\otimes I_N\otimes\alpha} S\otimes_S N\otimes_S S=N$$

and $\delta_M=\beta\otimes I_M\otimes\beta$ similarly. We have (2.1b). For $M_i\in{}_R\mathcal{M}_R$ $(i=1, 2)$, there is a product S-bimodule map

$$\Gamma M_1\otimes_S\Gamma M_2=P\otimes_R M_1\otimes_R Q\otimes_S P\otimes_R M_2\otimes_R Q$$
$$\xrightarrow{I\otimes\beta\otimes I} P\otimes_R M_1\otimes_R R\otimes_R M_2\otimes_R Q=\Gamma(M_1\otimes_R M_2).$$

Together with $\alpha^{-1}: S\to P\otimes_R Q=\Gamma(R)$, this makes Γ into a monoidal functor. Similarly Δ is a monoidal functor. We see γ and δ are isomorphisms of monoidal functors. Hence the 4-tuple $(\Gamma, \Delta, \gamma, \delta)$ is a $\sqrt{\text{Morita}}$ equivalence data between R and S. This means $R\sim_{\mathbf{M}}S$ implies $R\sim_{\sqrt{\mathbf{M}}}S$.

2.3. EXAMPLE. Let $\sigma(X, Y)$ be a formal ring law over R. We have a monoidal functor $(-)^\sigma: {}_R\mathcal{M}_R\to{}_{R^\sigma}\mathcal{M}_{R^\sigma}$ (1.10). By (1.17), this is a monoidal equivalence if and only if σ is invertible. If $\tau\in\mathcal{F}_{R^\sigma}$ is its inverse, we see the pair $((-)^\sigma, (-)^\tau)$ is a *strict* $\sqrt{\text{Morita}}$ equivalence data between R and R^σ in the sense that we can take the identities as γ and δ (2.1). Thus we have $R\sim_{\sqrt{\mathbf{M}}}R^\sigma$ for $\sigma\in\mathcal{F}_R$ invertible.

2.4. EXAMPLE. Let A be an Azumaya algebra. We have an equivalence

(2.4a) $\qquad A\otimes- : \mathcal{M}_k \longrightarrow {}_A\mathcal{M}_A, \qquad V\longmapsto A\otimes V.$

For $V_i\in\mathcal{M}_k$ $(i=1, 2)$, the canonical A-bimodule isomorphism

$$(A\otimes V_1)\otimes_A(A\otimes V_2)\simeq A\otimes(V_1\otimes V_2)$$

and the identity $A=A\otimes k$ make the functor (2.4a) into a monoidal equivalence. The quasi-inverse is given by

(2.4b) $\qquad (-)^A : {}_A\mathcal{M}_A \longrightarrow \mathcal{M}_k, \qquad M\longmapsto M^A=\{x\in M \mid ax=xa, \forall a\in A\}.$

Thus we have $A\sim_{\sqrt{\mathbf{M}}}k$.

2.5. THEOREM. *We have*

a) $R\sim_{\mathbf{M}}S\Rightarrow R\sim_{\sqrt{\mathbf{M}}}S\Rightarrow R\otimes R^{\text{op}}\sim_{\mathbf{M}}S\otimes S^{\text{op}}$,

b) $R\sim_{\sqrt{\mathbf{M}}}S\Rightarrow \text{cent}(R)\simeq\text{cent}(S)$,

c) $R\sim_{\sqrt{\mathbf{M}}}S$ *and* R *is central* (*resp. separable, resp. simple*) $\Rightarrow S$ *is central* (*resp. separable, resp. simple*),

d) *an algebra* A *is Azumaya if and only if* $A\sim_{\sqrt{\mathbf{M}}}k$.

PROOF. a) follows from (2.1) and (2.2). b) Any monoidal equivalence ${}_R\mathcal{M}_R\approx{}_S\mathcal{M}_S$ takes R to S inducing $\text{End}_{{}_R\mathcal{M}_R}(R)\simeq\text{End}_{{}_S\mathcal{M}_S}(S)$. We have only to identify $\text{cent}(R)=\text{End}_{{}_R\mathcal{M}_R}(R)$. c) Assume $R\sim_{\sqrt{\mathbf{M}}}S$. If R is central, so is S by b). Saying that R is separable (resp. simple) is equivalent with saying R is a

projective (resp. simple) object in ${}_R\mathcal{M}_R$. Since there is an equivalence ${}_R\mathcal{M}_R \approx {}_S\mathcal{M}_S$ such that $R \leftrightarrow S$, it follows that if R is separable (resp. simple), then so is S.
d) The 'only if' part follows from (2.4). The 'if' part from c). Q. E. D.

In the next section, we refine b) to claim that $R \sim_{\sqrt{\mathbf{M}}} S$ (over k) implies $R \sim_{\sqrt{\mathbf{M}}} S$ over the common center cent(R)=cent(S). We also prove that $R_i \sim_{\sqrt{\mathbf{M}}} S_i$ $(i=1, 2)$ imply $R_1 \otimes R_2 \sim_{\sqrt{\mathbf{M}}} S_1 \otimes S_2$.

§ 3. The coalgebraic description.

We work over algebras R, S, T, V.

3.1. DEFINITION. An *R-coring* means a *comonoid object* in the monoidal category ${}_R\mathcal{M}_R$, i.e., a triple $({}_RC_R, \Delta, \varepsilon)$ with R-bimodule maps $\Delta: C \to C \otimes_R C$ and $\varepsilon: C \to R$ satisfying usual coalgebra condition. It is called an R/k-coalgebra in [7].

Let C be an R-coring. For $M \in {}_R\mathcal{M}_R$, we put

$$\Phi_C(M) = {}_R\mathcal{M}_R(C, M)$$

the R-bimodule maps $C \to M$. If $f \in \Phi_C(M)$ and $g \in \Phi_C(N)$ with $M, N \in {}_R\mathcal{M}_R$, then the composite

$$f * g : C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{f \otimes g} M \otimes_R N$$

is in $\Phi_C(M \otimes_R N)$, and we get a linear map

$$\Phi_C(M) \otimes \Phi_C(N) \longrightarrow \Phi_C(M \otimes_R N), \qquad f \otimes g \longmapsto f * g. \tag{3.2}$$

We see the functor Φ_C becomes a monoidal functor with product (3.2) and unit $k \to \Phi_C(R)$, $1 \mapsto \varepsilon$. This means for any R-ring U, $\Phi_C(U)$ becomes an algebra. In particular

$$C^* \underset{\text{def}}{=} \Phi_C(R) \tag{3.3a}$$

has an algebra structure, and $\Phi_C(U)$ becomes naturally a C^*-ring. In fact, for $M \in {}_R\mathcal{M}_R$, the map

$$C^* \otimes \Phi_C(M) \otimes C^* = \Phi_C(R) \otimes \Phi_C(M) \otimes \Phi_C(R)$$
$$\xrightarrow{*\text{-product}} \Phi_C(R \otimes_R M \otimes_R R) = \Phi_C(M)$$

makes $\Phi_C(M)$ into a C^*-bimodule, and the map (3.2) induces a C^*-bimodule map

$$\Phi_C(M) \otimes_{C^*} \Phi_C(N) \longrightarrow \Phi_C(M \otimes_R N). \tag{3.3b}$$

Together with the identity (3.3a), this makes the functor

(3.3) $\qquad \Phi_C : {}_R\mathcal{M}_R \longrightarrow {}_{C*}\mathcal{M}_{C*}$

into a monoidal functor, and the previous C^*-ring $\Phi_C(U)$ for an R-ring U is induced from this.

3.4. LEMMA. *Let C be an R-coring. There is a* 1-1 *correspondence between algebra maps $\eta : S \to C^*$ and multi-module structures ${}_{R,S}C_{R,S}$ such that for $a, b \in S$ and $c \in C$ with $\Delta(c) = \sum c_1 \otimes c_2$,*

(3.4a) $\qquad \Delta(acb) = \sum c_1 b \otimes ac_2\,,$

(3.4b) $\qquad \varepsilon(ac) = \varepsilon(ca)\,.$

For an algebra map η, the action ${}_SC_S$ is defined by

(3.4c) $\qquad ac = \sum c_1 \cdot \eta(a)(c_2)\,,$

(3.4d) $\qquad ca = \sum \eta(a)(c_1) \cdot c_2\,.$

For a multi-module ${}_{R,S}C_{R,S}$ satisfying a), b), *the algebra map η is defined by*

(3.4e) $\qquad \eta(a)(c) = \varepsilon(ac) = \varepsilon(ca)\,.$

PROOF. Begin with a multi-module. The map $\eta(a)$ of e) is an R-bimodule map, hence is in C^*. a) implies $\varepsilon(acb) = \sum \varepsilon(c_1 b)\varepsilon(ac_2) = (\eta(b) * \eta(a))(c)$, and b) implies $\varepsilon(acb) = \varepsilon(cba) = \eta(ba)(c)$. Hence η is an algebra map. Letting $b=1$ in a), we have $ac = \sum c_1 \cdot \varepsilon(ac_2)$ which means c). Similarly d) follows. Conversely, assume we are given an algebra map η. The operations c) and d) are R-bimodule endomorphisms, hence we get a multi-module ${}_{R,S}C_{R,S}$. If we put $\Delta_2(c) = \sum c_1 \otimes c_2 \otimes c_3 \in C \otimes_R C \otimes_R C$, it follows from c) that we have $\Delta(ac) = \sum c_1 \otimes c_2 \cdot \eta(a)(c_3) = \sum c_1 \otimes ac_2$, and $\Delta(ca) = \sum c_1 a \otimes c_2$ similarly from d). This yields a). Applying ε to c) and d), we get b) and e). This establishes the 1-1 correspondence. Q. E. D.

3.5. DEFINITION. An $S|R$-*coring* means a pair (C, η) with an R-coring C and an algebra map $\eta : S \to C^*$, or the R-coring with the corresponding multi-module structure ${}_{R,S}C_{R,S}$ (3.4a, b).

We give several comments on an $S|R$-coring C. It follows from (3.4c, d) that we have

(3.5a) $\qquad \sum ac_1 \otimes c_2 = \sum c_1 \otimes c_2 a\,.$

This means the image $\Delta(C)$ is contained in the centralizer $[C \otimes_R C]^S$. Secondly, the algebra map η induces a forgetful monoidal functor ${}_{C*}\mathcal{M}_{C*} \to {}_S\mathcal{M}_S$. By abuse of notation, we let Φ_C mean the following composite monoidal functor

(3.5b) $\qquad \Phi_C : {}_R\mathcal{M}_R \xrightarrow{\Phi_C\ (3.3)} {}_{C*}\mathcal{M}_{C*} \xrightarrow{\textbf{forg.}\,(\eta)} {}_S\mathcal{M}_S\,.$

Thus the *$S|R$-coring C induces a monoidal functor* ${}_R\mathcal{M}_R \to {}_S\mathcal{M}_S$. For $M \in {}_R\mathcal{M}_R$, $\Phi_C(M) = {}_R\mathcal{M}_R({}_{R,S}C_{R,S}, {}_RM_R)$ inherits an S-bimodule structure from ${}_SC_S$ (in the opposite way). The last remark is that the S-bimodule structure of $\Phi_C(M)$ in (3.5b) is precisely this.

More generally, a multi-module ${}_{R,S}\Lambda_{R,S}$ represents a functor

$$\text{(3.5c)} \qquad \Phi_\Lambda : {}_R\mathcal{M}_R \longrightarrow {}_S\mathcal{M}_S, \qquad M \longmapsto \Phi_\Lambda(M) = {}_R\mathcal{M}_R(\Lambda, M).$$

The last remark means an $S|R$-coring structure on Λ yields a monoidal structure on Φ_Λ. We have the converse:

3.6. THEOREM. *Let ${}_{R,S}\Lambda_{R,S}$ be a multi-module. There is a 1-1 correspondence between R-coring structures (Δ, ε) on Λ making it into an $S|R$-coring, and monoidal structures on Φ_Λ (3.5c). If Φ_Λ is a monoidal functor, then the corresponding coring structure (Δ, ε) is defined as follows: $\Delta : \Lambda \to \Lambda \otimes_R \Lambda$ is the image of $I \otimes_S I \in \Phi_\Lambda(\Lambda) \otimes_S \Phi_\Lambda(\Lambda)$ under the product $\Phi_\Lambda(\Lambda) \otimes_S \Phi_\Lambda(\Lambda) \to \Phi_\Lambda(\Lambda \otimes_R \Lambda)$, and $\varepsilon : \Lambda \to R$ is the image of $1 \in S$ under the unit $S \to \Phi_\Lambda(\Lambda)$.*

We leave it to the reader to establish this standard result.

As the easiest example, the multi-module ${}_{\cdot}R_{\times} \otimes_{\times} R_{\cdot}$ ($\cdot$, $\times$ meaning R) represents the identity ${}_{\cdot}\mathcal{M}_{\cdot} \to {}_{\times}\mathcal{M}_{\times}$. The corresponding $\times|\cdot$-coring structure on ${}_{\cdot}R_{\times} \otimes_{\times} R_{\cdot}$ is given by

$$\text{(3.7a)} \qquad \Delta(a \otimes b) = (a \otimes 1) \otimes_R (1 \otimes b),$$

$$\text{(3.7b)} \qquad \varepsilon(a \otimes b) = ab$$

for $a, b \in R$.

3.8. COROLLARY. *Let C, D be $S|R$-corings. There is a 1-1 correspondence between $S|R$-coring maps $C \to D$ and maps of monoidal functors $\Phi_D \to \Phi_C$.*

PROOF. We have a 1-1 correspondence between R, S-bimodule maps $C \to D$ and maps of representable functors $\Phi_D \to \Phi_C$ (3.5c). It follows from (3.6) that the multi-module map $C \to D$ commutes with the coring structure if and only if the natural transformation $\Phi_D \to \Phi_C$ commutes with the monoidal structure. Q. E. D.

It follows that if $\Gamma : {}_R\mathcal{M}_R \to {}_S\mathcal{M}_S$ is a monoidal functor *representable as a functor*, then it is represented by an $S|R$-coring which is *uniquely determined up to isomorphism*. We say Γ is a *representable monoidal functor*.

3.9. PROPOSITION. *The composite of two representable monoidal functors*

$${}_R\mathcal{M}_R \xrightarrow{\Gamma} {}_S\mathcal{M}_S \xrightarrow{\Delta} {}_T\mathcal{M}_T$$

is still representable.

PROOF. We can ignore the monoidal structure. Assume functors $\boldsymbol{\Gamma}$, $\boldsymbol{\Delta}$ are represented by multi-modules ${}_{R,S}P_{R,S}$, ${}_{S,T}Q_{S,T}$ respectively. Let $Q\circ P$ denote the quotient of $Q\otimes P$ by the submodule

$$\{ayb\otimes x-y\otimes bxa \mid y\in Q,\ x\in P,\ a,\ b\in S\}\,.$$

The multi-module ${}_{R,T}[Q\circ P]_{R,T}$ represents the composite $\boldsymbol{\Delta}\circ\boldsymbol{\Gamma}$. Q. E. D.

The above proposition leads to the concept of the *composite coring.*

3.10. THEOREM-DEFINITION. *Let C be an $S|R$-coring and let D be a $T|S$-coring. Define a multi-module ${}_{R,T}[D\circ C]_{R,T}$ as in the proof of* (3.9). *Let $d\circ c\in D\circ C$ denote the image of $d\otimes c\in D\otimes C$. If $\Delta(c)=\sum c_1\otimes c_2$ and $\Delta(d)=\sum d_1\otimes d_2$, then we can well-define an R-coring structure (Δ, ε) on $D\circ C$ by*

(3.10a) $\quad \Delta(d\circ c)=\sum (d_1\circ c_1)\otimes(d_2\circ c_2)\in(D\circ C)\otimes_R(D\circ C)$,

(3.10b) $\quad \varepsilon(d\circ c)=\varepsilon(\varepsilon(d)c)$

to get a $T|R$-coring $D\circ C$. There is a canonical isomorphism of monoidal functors (3.5b)

(3.10c) $\quad \boldsymbol{\Phi}_{D\circ C}\simeq\boldsymbol{\Phi}_D\circ\boldsymbol{\Phi}_C\,.$

PROOF. It follows from (3.9) that there is a natural isomorphism of functors (3.10c). Let us transport the monoidal structure of $\boldsymbol{\Phi}_D\circ\boldsymbol{\Phi}_C$ onto $\boldsymbol{\Phi}_{D\circ C}$ through this isomorphism. By (3.6), there is a corresponding $T|R$-coring structure on $D\circ C$. One sees easily the structure (Δ, ε) is described by (3.10 a, b). Q. E. D.

Thus we have defined the composite $T|R$-coring $D\circ C$ for a $T|S$-coring D and an $S|R$-coring C.

3.11. PROPOSITION. *For a $V|T$-coring E, a $T|S$-coring D, and an $S|R$-coring C, we have canonical isomorphisms:*

(3.11a) $\quad (E\circ D)\circ C\simeq E\circ(D\circ C)\quad$ *as $V|R$-corings,*

(3.11b) $\quad (S\otimes S)\circ C\simeq C\simeq C\circ(R\otimes R)\quad$ *as $S|R$-corings.*

This follows immediately from (3.10c). The isomorphism (3.11a) is induced from the canonical isomorphism $(E\otimes D)\otimes C\simeq E\otimes(D\otimes C)$. Under (3.11b) we have $(1\otimes 1)\circ c\leftrightarrow c\leftrightarrow c\circ(1\otimes 1)$ for $c\in C$.

Let T_i, S_i, R_i $(i=1, 2)$ be algebras. For a $T_i|S_i$-coring D_i and an $S_i|R_i$-coring C_i $(i=1, 2)$, we see $D_1\otimes D_2$ and $C_1\otimes C_2$ are a $(T_1\otimes T_2)|(S_1\otimes S_2)$-coring and a $(S_1\otimes S_2)|(R_1\otimes R_2)$-coring respectively. One sees there is an isomorphism of $(T_1\otimes T_2)|(R_1\otimes R_2)$-corings

$$(3.12) \qquad (D_1 \circ C_1) \otimes (D_2 \circ C_2) \simeq (D_1 \otimes D_2) \circ (C_1 \otimes C_2),$$
$$d_1 \circ c_1 \otimes d_2 \circ c_2 \longleftrightarrow (d_1 \otimes d_2) \circ (c_1 \otimes c_2).$$

3.13. DEFINITION. A *coalgebraic* $\sqrt{\mathit{Morita}}$ *equivalence data* between R and S means a 4-tuple $(C_{S|R}, D_{R|S}, \gamma, \delta)$ where C and D are an $S|R$-coring and an $R|S$-coring respectively, and

$$(3.13a) \qquad \gamma : S \otimes S \simeq C \circ D, \qquad \delta : R \otimes R \simeq D \circ C$$

are an $S|S$-coring isomorphism and an $R|R$-coring isomorphism respectively such that the following diagrams commute:

$$(3.13b) \qquad \begin{array}{ccc} (S\otimes S)\circ C & \simeq & C\circ(R\otimes R) \\ \downarrow{\scriptstyle \gamma\circ C} & & \downarrow{\scriptstyle C\circ\delta} \\ (C\circ D)\circ C & \simeq & C\circ(D\circ C) \end{array} \qquad \begin{array}{ccc} (R\otimes R)\circ D & \simeq & D\circ(S\otimes S) \\ \downarrow{\scriptstyle \delta\circ D} & & \downarrow{\scriptstyle D\circ\gamma} \\ (D\circ C)\circ D & \simeq & D\circ(C\circ D) \end{array}$$

where we use the canonical isomorphisms (3.11).

A coalgebraic $\sqrt{\text{Morita}}$ equivalence data (C, D, γ, δ) induces a $\sqrt{\text{Morita}}$ equivalence data $(\boldsymbol{\Phi}_C, \boldsymbol{\Phi}_D, \boldsymbol{\gamma}, \boldsymbol{\delta})$ (2.1) where

$$(3.14a) \qquad \boldsymbol{\gamma} : \boldsymbol{\Phi}_C \boldsymbol{\Phi}_D \simeq \boldsymbol{\Phi}_{C \circ D} \xrightarrow{\boldsymbol{\Phi}_\gamma} \boldsymbol{\Phi}_{S \otimes S} \simeq \mathbf{Id},$$

$$(3.14b) \qquad \boldsymbol{\delta} : \boldsymbol{\Phi}_D \boldsymbol{\Phi}_C \simeq \boldsymbol{\Phi}_{D \circ C} \xrightarrow{\boldsymbol{\Phi}_\delta} \boldsymbol{\Phi}_{R \otimes R} \simeq \mathbf{Id}$$

with isomorphism (3.10c). ((3.13b) implies (2.1b)).

Any equivalence of module categories is representable, by Morita theory. It follows that any monoidal equivalence of bimodule monoidal categories is representable. In view of (3.8)-(3.10), this means any $\sqrt{\text{Morita}}$ equivalence data between R and S is isomorphic with the data determined by some coalgebraic data as above. In addition the coalgebraic data is determined up to isomorphism. Thus we get the following coalgebraic description theorem.

3.14. COALGEBRAIC DESCRIPTION THEOREM. *If* (C, D, γ, δ) *is a coalgebraic* $\sqrt{\mathit{Morita}}$ *equivalence data between* R *and* S, *then we have a* $\sqrt{\mathit{Morita}}$ *equivalence data* $(\boldsymbol{\Phi}_C, \boldsymbol{\Phi}_D, \boldsymbol{\gamma}, \boldsymbol{\delta})$ *with* (3.14 a, b). *Any* $\sqrt{\mathit{Morita}}$ *equivalence data between* R *and* S *is of this form up to isomorphism with a uniquely determined coalgebraic data* (C, D, γ, δ) *up to isomorphism.*

We describe briefly the coalgebraic $\sqrt{\text{Morita}}$ equivalence data corresponding to Examples 2.2-2.4.

3.15. EXAMPLE. Let $({}_SP_R, {}_RQ_S, \alpha, \beta)$ be a Morita equivalence data (2.2). There is an induced coalgebraic $\sqrt{\text{Morita}}$ equivalence data $(C_{S|R}, D_{R|S}, \gamma, \delta)$

defined as follows: We put

$$_{R,S}C_{R,S} = (_RQ_S)\otimes(_SP_R), \qquad _{S,R}D_{S,R} = (_SP_R)\otimes(_RQ_S).$$

We define an R-coring structure (Δ, ε) on C as follows:

(3.15a) $$\Delta : Q\otimes P \longrightarrow Q\otimes S\otimes P \xrightarrow{I\otimes\alpha^{-1}\otimes I} (Q\otimes P)\otimes_R(Q\otimes P),$$
$$y\otimes x \longmapsto y\otimes 1\otimes x$$

(3.15b) $$\varepsilon : Q\otimes P \xrightarrow{\text{cano.}} Q\otimes_S P \xrightarrow{\beta} R.$$

Then C becomes an $S|R$-coring. Similarly, D is an $R|S$-coring. The $S|S$-coring isomorphism γ is defined by

$$\gamma : S\otimes S \xrightarrow{\alpha^{-1}\otimes\alpha^{-1}} (P\otimes_R Q)\otimes(P\otimes_R Q) \simeq (Q\otimes P)\circ(P\otimes Q)$$
$$(x\otimes y)\otimes(x'\otimes y') \longleftrightarrow (y\otimes x')\circ(x\otimes y').$$

The $R|R$-coring isomorphism δ is defined similarly.

With the notation of (2.2), for $M\in {}_R\mathcal{M}_R$, an element $x\otimes m\otimes y\in P\otimes_R M\otimes_R Q = \boldsymbol{\Gamma}(M)$ induces an R-bimodule map

$$C = Q\otimes P \longrightarrow M, \qquad y'\otimes x' \longmapsto \beta(y'\otimes x)m\beta(y\otimes x').$$

This gives rise to an S-bimodule isomorphism

$$\boldsymbol{\Gamma}(M) \simeq {}_R\mathcal{M}_R(C, M) = \boldsymbol{\Phi}_C(M).$$

One sees the $S|R$-coring structure on C corresponds to the monoidal structure on $\boldsymbol{\Gamma}$. Similarly, the monoidal functor $\boldsymbol{\Delta}$ is represented by the $R|S$-coring D. Thus the $\sqrt{\text{Morita}}$ equivalence data $(\boldsymbol{\Gamma}, \boldsymbol{\Delta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ of (2.2) is isomorphic to the data coming from the coalgebraic data (C, D, γ, δ) defined above.

3.16. Example. Let $\sigma(X, Y)=\sum a_iXb_iYc_i$ $(a_i, b_i, c_i\in R)$ be a formal ring law over R. The R-bimodule $R\otimes R$ has the following R-coring structure $(\Delta_\sigma, \varepsilon_\sigma)$ ('*comultiplication alteration*'):

(3.16a) $$\Delta_\sigma : R\otimes R \longrightarrow R\otimes R\otimes R=(R\otimes R)\otimes_R(R\otimes R)$$
$$\Delta_\sigma(u\otimes v) = \sum ua_i\otimes b_i\otimes c_iv,$$

(3.16b) $$\varepsilon_\sigma : R\otimes R \longrightarrow R, \qquad \varepsilon_\sigma(u\otimes v) = ue_\sigma v.$$

This R-coring is denoted by $R\otimes_\sigma R$. For $a\in R$, we have an R-bimodule map

$$\tilde{a} : R\otimes_\sigma R \longrightarrow R, \qquad u\otimes_\sigma v \longmapsto uav.$$

The map $a\mapsto\tilde{a}$ is seen to give an algebra isomorphism

(3.16c) $\quad R^{\sigma} \simeq (R\otimes_{\sigma}R)^{*}.$

This makes $R\otimes_{\sigma}R$ into a $R^{\sigma}|R$-coring.

For $m\in M$ with $M\in {}_R\mathcal{M}_R$, we have an R-bimodule map

$$\tilde{m} : R\otimes_{\sigma}R \longrightarrow M, \qquad u\otimes_{\sigma}v \longmapsto umv.$$

As before, we see the map $m\mapsto\tilde{m}$ gives a R^{σ}-bimodule isomorphism

(3.16d) $\quad M^{\sigma} \simeq \Phi_{R\otimes_{\sigma}R}(M).$

For $n\in N$ with $N\in {}_R\mathcal{M}_R$, the product $\tilde{m}*\tilde{n}\in\Phi_{R\otimes_{\sigma}R}(M\otimes_R N)$ (3.2) comes from $\sigma(m, n)\in M\otimes_R N$ (1.10a). This means (3.16d) gives an isomorphism of monoidal functors

(3.16e) $\quad (-)^{\sigma} \simeq \Phi_{R\otimes_{\sigma}R}.$

If $\tau(X, Y)$ is a formal ring law over R^{σ}, then there is a natural isomorphism of $R^{\sigma\tau}|R$-corings

(3.16f) $\quad R\otimes_{\sigma*\tau}R \simeq (R^{\sigma}\otimes_{\tau}R^{\sigma})\circ(R\otimes_{\sigma}R), \qquad 1\otimes_{\sigma*\tau}1 \longleftrightarrow (e_{\sigma}\otimes_{\tau}e_{\sigma})\circ(1\otimes_{\sigma}1).$

If σ is invertible with inverse τ, then (3.16f) induces isomorphisms

$$\delta : R\otimes R \simeq (R^{\sigma}\otimes_{\tau}R^{\sigma})\circ(R\otimes_{\sigma}R) \qquad \text{as } R|R\text{-corings},$$
$$\gamma : R^{\sigma}\otimes R^{\sigma} \simeq (R\otimes_{\sigma}R)\circ(R^{\sigma}\otimes_{\tau}R^{\sigma}) \qquad \text{as } R^{\sigma}|R^{\sigma}\text{-corings}.$$

We see the coalgebraic $\sqrt{\text{Morita}}$ equivalence data $(R\otimes_{\sigma}R, R^{\sigma}\otimes_{\tau}R^{\sigma}, \gamma, \delta)$ represents the strict $\sqrt{\text{Morita}}$ equivalence data $((-)^{\sigma}, (-)^{\tau})$ of (2.3).

3.17. EXAMPLE. Let A be an Azumaya algebra. The dual k-coalgebra $A^*=\mathrm{Hom}(A, k)$ is a k-coring with $A=A^{**}$, hence an $A|k$-coring. In general R has a trivial R-coring structure. Hence A is an A-coring with $k=\mathrm{cent}(A)$ $={}_R\mathcal{M}_R(A, A)$, thus a $k|A$-coring. We show the pair $(A^*_{k|A}, A_{A|k})$ has a coalgebraic $\sqrt{\text{Morita}}$ structure (γ, δ) representing the $\sqrt{\text{Morita}}$ equivalence data of (2.4). We see from definition the canonical isomorphisms

$$A\otimes V \simeq \mathrm{Hom}(A^*, V) = \Phi_{A^*}(V), \qquad V \in \mathcal{M}_k,$$
$$M^{A} \simeq {}_A\mathcal{M}_A(A, M) = \Phi_A(M), \qquad M \in {}_A\mathcal{M}_A$$

give isomorphisms of monoidal functors

$$A\otimes(-) \simeq \Phi_{A^*} \quad \text{and} \quad (-)^{A} \simeq \Phi_A.$$

Let (γ, δ) be the coalgebraic $\sqrt{\text{Morita}}$ structure corresponding to the $\sqrt{\text{Morita}}$ equivalence data $(A\otimes(-), (-)^{A})$ (2.4). We see

$$\delta : k \simeq A\circ A^*$$

is the unique coalgebra map, i.e., the inverse of the counit. If we identify $A^*\circ A=A^*\otimes A=\operatorname{End} A$, we see

$$\gamma : A\otimes A \simeq A^*\circ A$$

is given by $\gamma(1\otimes 1)=\mathrm{id}$.

The next theorem is an easy application of the coalgebraic description.

3.18. THEOREM. *For algebras R_i, S_i $(i=1, 2)$, $R_i\sim_{\sqrt{\mathrm{M}}}S_i$ $(i=1, 2)$ imply $R_1\otimes R_2\sim_{\sqrt{\mathrm{M}}}S_1\otimes S_2$.*

PROOF. Let $(C_i, D_i, \gamma_i, \delta_i)$ be a coalgebraic $\sqrt{\text{Morita}}$ equivalence data between R_i and S_i. Then we have a coalgebraic $\sqrt{\text{Morita}}$ equivalence data $(C_1\otimes C_2, D_1\otimes D_2, \gamma, \delta)$ between $R_1\otimes R_2$ and $S_1\otimes S_2$, where γ is the composite

$$\begin{aligned}\gamma : (S_1\otimes S_2)\otimes(S_1\otimes S_2) &\simeq (S_1\otimes S_1)\otimes(S_2\otimes S_2)\\ &\xrightarrow{\gamma_1\otimes\gamma_2} (C_1\circ D_1)\otimes(C_2\circ D_2) \underset{(3.12)}{\simeq} (C_1\otimes C_2)\circ(D_1\otimes D_2)\end{aligned}$$

and δ is defined similarly. Therefore $R_1\otimes R_2$ and $S_1\otimes S_2$ are $\sqrt{\text{Morita}}$ equivalent. Q.E.D.

An $S|R$-coring C is *invertible* if the monoidal functor $\Phi_C : {}_R\mathcal{M}_R\to{}_S\mathcal{M}_S$ (3.5b) is a monoidal equivalence. This is equivalent with saying that C embeds into a coalgebraic $\sqrt{\text{Morita}}$ equivalence data between R and S.

3.19. THEOREM. *The $S|R$-coring C is invertible if and only if the following conditions hold:*

a) *${}_RC_R$ is a progenerator in ${}_R\mathcal{M}_R$.*

b) $S\otimes S^{\mathrm{op}} \xrightarrow{\simeq} \operatorname{End}_{{}_R\mathcal{M}_R}(C)$.

c) $\Phi_C(R\otimes R)\otimes_S\Phi_C(R\otimes R) \xrightarrow[\text{(prod. str.)}]{\simeq} \Phi_C(R\otimes R\otimes R)$.

PROOF. By Morita theory, a) plus b) is equivalent with Φ_C being an equivalence. If this is the case, the functor Φ_C preserves colimits, hence the product $\Phi_C(M)\otimes_S\Phi_C(N)\to\Phi_C(M\otimes_R N)$ becomes an isomorphism for all $M, N\in{}_R\mathcal{M}_R$ if condition c) holds. Thus, a)-c) imply that Φ_C is a *nonunitary monoidal equivalence*. It follows from [**9**, Lemma 5.12] that we have $S\xrightarrow{\simeq}C^*$. Q.E.D.

Conditions a)-c) are independent of each other. We assume $S=C^*$. If $C=R\otimes_\sigma R$ (3.16) with commutative R, then a) and b) are always true, and c) is enough for σ to be invertible. If $R=k$ (and $S=C^*$), then c) is always true, and a) plus b) means C^* is an Azumaya algebra.

Applying (3.19) to $C=R\otimes_\sigma R$ (3.16), we can refine conditions 10) and 12) of

(1.17) as follows:

3.20. COROLLARY. *With the notation of* (1.17), *conditions* 10) *and* 12) *are equivalent to* 10)′ *and* 12)′ *below:*

10)′ *The map* $\phi_\sigma^{(n)}$ (1.12b) *is an isomorphism for* $n=1, 2$.

12)′ *The map* $\phi_\sigma^{(1)}$ (1.12b) *is an isomorphism, and the structure map* (1.10b) *is an isomorphism for* $M=N=R\otimes R$.

PROOF. Put $S=R^\sigma$, $C=R\otimes_\sigma R$ in (3.19). a) is true. The algebra map of b) is identified with the map $\phi_\sigma^{(1)}$. (Remember $R(X)\simeq R\otimes R^{\mathrm{op}}$). The map $\phi_\sigma^{(2)}$ factors as

$$\phi_\sigma^{(2)} : R^\sigma(X_1, X_2) \xrightarrow{\phi_\sigma^{(1)}\otimes\phi_\sigma^{(1)}} R(X_1)^\sigma \otimes_{R^\sigma} R(X_2)^\sigma \xrightarrow{\text{(prod. str.)}} R(X_1, X_2)^\sigma .$$

Hence we have b) plus c)$\Leftrightarrow$10)′$\Leftrightarrow$12)′, and we know σ is invertible if and only if the $R^\sigma|R$-coring $R\otimes_\sigma R$ is invertible. Q. E. D.

Finally we refine (2.5b) as an application of the coalgebraic description.

3.21. THEOREM. *If* $R\sim_{\sqrt{\mathrm{M}}}S$ *(over* k*), then there is a commutative* k*-algebra* K *such that* R, S *have a central* K*-algebra structure in such a way that* $R\sim_{\sqrt{\mathrm{M}}}S$ *over* K.

PROOF. Take a coalgebraic $\sqrt{\text{Morita}}$ equivalence data $(C_{S|R}, D_{R|S}, \gamma, \delta)$. The equivalence Φ_C, together with the structure isomorphism $S\simeq C^*=\Phi_C(R)$, induces an algebra isomorphism $\mathrm{cent}(R)=\mathrm{End}_{{}_R\mathcal{M}_R}(R)\simeq\mathrm{End}_{{}_S\mathcal{M}_S}(\Phi_C(R))\simeq\mathrm{End}_{{}_S\mathcal{M}_S}(S)$ $=\mathrm{cent}(S)$. We see the inverse is induced from Φ_D in a similar way. We can take a commutative k-algebra K and make R, S into central K-algebras in such a way that this algebra isomorphism $\mathrm{cent}(R)\simeq\mathrm{cent}(S)$ becomes a K-algebra isomorphism. For $a\in K$ with images $a_R\in R$ and $a_S\in S$, we conclude easily from (3.4c, d) that we have

$$\text{(3.21a)} \qquad a_R x = x a_S \quad \text{and} \quad a_S x = x a_R$$

for all x in C or D. Let $\bar{C}$ be the quotient module of C by $\{a_R x - x a_R\,(=x a_S - a_S x) \mid x\in C,\ a\in K\}$. Define $\bar{D}$ similarly. $\bar{C}$ and $\bar{D}$ have one K-module structure, and we see they inherit the structure of an $S|R$-coring and an $R|S$-coring respectively. It follows from (3.21a) that γ and δ induce K-module isomorphisms

$$\bar{\gamma} : S\otimes_K S \simeq \bar{C}\circ\bar{D} \quad \text{and} \quad \bar{\delta} : R\otimes_K R \simeq \bar{D}\circ\bar{C}$$

respectively. They are obviously an $S|S$-coring isomorphism and an $R|R$-coring isomorphism respectively. We have a coalgebraic $\sqrt{\text{Morita}}$ equivalence data $(\bar{C}, \bar{D}, \bar{\gamma}, \bar{\delta})$ between R and S over K. Hence $R\sim_{\sqrt{\mathrm{M}}}S$ over K. Q. E. D.

3.22. COROLLARY. *If R is $\sqrt{Morita}$ equivalent with a commutative algebra, then R is separable over the center.*

This follows from (2.5d) and (3.21).

§ 4. The algebraic description.

The concept of $\times$-product introduced by Sweedler [**5**] and generalized by the author [**GA**] plays an essential role in this section. We work over algebras R, S, and T. As in [**GA**], $\bar{R}$ denotes the opposite algebra to R with anti-isomorphism $a \in R \leftrightarrow \bar{a} \in \bar{R}$. For left modules ${}_{\bar{R}}M$ and ${}_RN$, the quotient k-module of $M \otimes N$ obtained by identifying $\bar{a}x \otimes y = x \otimes ay$ for $a \in R$, $x \in M$, $y \in N$ is denoted by

$$ {}_{\bar{a}}M \otimes_{a \in R} {}_aN. $$

This is the same as $M \otimes_R N$ if we view ${}_{\bar{R}}M$ as M_R.

Recall [**GA**, (1.1)] that for bimodules ${}_{\bar{R}}M_{\bar{R}}$ and ${}_RN_R$, the k-submodule of all $\sum x_i \otimes y_i \in {}_{\bar{a}}M \otimes_{a \in R} {}_aN$ such that $\sum x_i \bar{b} \otimes y_i = \sum x_i \otimes y_i b$ for all $b \in R$ is denoted by

$$ M \times_R N. $$

We get a bifunctor

$$ (4.1) \qquad (-)\times_R(-) : {}_{\bar{R}}\mathcal{M}_{\bar{R}} \times {}_R\mathcal{M}_R \longrightarrow \mathcal{M}_k, \qquad (M, N) \longmapsto M \times_R N. $$

If we have some additional operations on M or N commuting with the $\bar{R}$- or R-biactions, then we get the corresponding operations on $M \times_R N$. Thus, for example, the bifunctor (4.1) induces

$$ (4.1a) \qquad {}_{S,\bar{R}}\mathcal{M}_{S,\bar{R}} \times {}_R\mathcal{M}_R \longrightarrow {}_S\mathcal{M}_S, $$

$$ (4.1b) \qquad {}_{\bar{R}}\mathcal{M}_{\bar{R}} \times {}_{R,T}\mathcal{M}_{R,T} \longrightarrow {}_T\mathcal{M}_T, $$

$$ (4.1c) \qquad {}_{S,\bar{R}}\mathcal{M}_{S,\bar{R}} \times {}_{R,T}\mathcal{M}_{R,T} \longrightarrow {}_{S,T}\mathcal{M}_{S,T} \qquad \text{and so on.} $$

The direct product of two monoidal categories has a natural structure of a monoidal category. We show *the bifunctor* (4.1) *has a natural structure of a monoidal functor*. This is done by modifying [**GA**, Propositions 1.11, 1.12] as follows: For $M_i \in {}_{\bar{R}}\mathcal{M}_{\bar{R}}$, $N_i \in {}_R\mathcal{M}_R$ $(i=1, 2)$, the twist isomorphism $M_1 \otimes N_1 \otimes M_2 \otimes N_2 \simeq M_1 \otimes M_2 \otimes N_1 \otimes N_2$ induces a k-linear map

$$ (M_1 \times_R N_1) \otimes ({}_{\bar{a}}M_2 \otimes_{a \in R} {}_aN_2) \longrightarrow ({}_{\bar{b}}M_1 \otimes_{\bar{R}} M_2) \otimes_{b \in R} ({}_bN_1 \otimes_R N_2) $$

similar to the ϕ-map [**GA**, (1.11)]. Further, this map induces

$$(4.2)\qquad (M_1\times_R N_1)\otimes(M_2\times_R N_2) \longrightarrow (M_1\otimes_{\bar{R}} M_2)\times_R(N_1\otimes_R N_2)$$
$$(\sum x_i\otimes y_i)\otimes(\sum x'_j\otimes y'_j) \longmapsto \sum (x_i\otimes x'_j)\otimes(y_i\otimes y'_j)$$

similar to the ξ-map (ibid., (1.12)). Together with the trivial map $k\to\bar{R}\times_R R$, $1\mapsto 1\otimes 1$, (4.2) makes (4.1) into a monoidal bifunctor. Similarly, bifunctors (4.1a, b, c) have a structure of a monoidal functor.

4.3. DEFINITION. An $S|R$-ring means an $S\otimes\bar{R}$-ring.

For example, $\operatorname{End} R$ is an $R|R$-ring [**GA**, p. 465].

Let A be an $S|R$-ring. By means of (4.1a), we have a functor

$$(4.4)\qquad A\times_R - \; : {}_R\mathcal{M}_R \longrightarrow {}_S\mathcal{M}_S\,,\qquad N\longmapsto A\times_R N.$$

The monoid structure of A and the monoidal structure of (4.1a) make (4.4) into a monoidal functor. The product is given by

$$(A\times_R N_1)\otimes_S(A\times_R N_2) \longrightarrow A\times_R(N_1\otimes_R N_2)$$
$$(\sum a_i\otimes y_i)\otimes(\sum a'_j\otimes y'_j) \longmapsto \sum a_i a'_j\otimes(y_i\otimes y'_j)$$

for $N_i\in{}_R\mathcal{M}_R$ $(i=1, 2)$. The unit is given by $S\to A\times_R R$, $s\mapsto s\otimes 1$. The monoidal functor (4.4) plays an essential role.

4.5. EXAMPLE. Consider the monoidal functor $\operatorname{End} R\times_R -$ of ${}_R\mathcal{M}_R$ into itself. For $M\in{}_R\mathcal{M}_R$, there is a canonical R-bimodule map [**GA**, (2.2)]

$$\boldsymbol{\theta}' : \operatorname{End} R\times_R M \longrightarrow M,\qquad \sum f_i\otimes x_i \longmapsto \sum f_i(1)x_i\,.$$

We have $\sum f_i(a)x_i=\sum f_i(1)x_i a$ for $a\in R$, $\sum f_i\otimes x_i\in\operatorname{End} R\times_R M$ (ibid.). It follows that for $\sum g_j\otimes y_j\in\operatorname{End} R\times_R N$ with $N\in{}_R\mathcal{M}_R$, the product

$$(\sum f_i\otimes x_i)(\sum g_j\otimes y_j) = \sum f_i\circ g_j\otimes(x_i\otimes y_j)$$

is mapped by $\boldsymbol{\theta}'$ to

$$\sum f_i(g_j(1))x_i\otimes y_j = \sum f_i(1)x_i\otimes g_j(1)y_j = \boldsymbol{\theta}'(\sum f_i\otimes x_i)\boldsymbol{\theta}'(\sum g_j\otimes y_j).$$

This means we have *a map of monoidal functors* $\boldsymbol{\theta}' : \operatorname{End} R\times_R - \to \mathbf{Id}$. *If R is a finite projective k-module, this is an isomorphism* [**GA**, (4.14)].

4.6. PROPOSITION [**GA**, (3.1)]. *If B is a $\bar{S}$-ring and A an S-ring, then $B\times_S A$ is an algebra with product*

$$(\sum b_i\otimes a_i)(\sum b'_j\otimes a'_j) = \sum b_i b'_j\otimes a_i a'_j$$

and with unit $1\otimes 1$. *If B is a $T|S$-ring and A an $S|R$-ring, then $B\times_S A$ becomes a $T|R$-ring.*

If μ_B and μ_A denote the multiplication of B and A, then the above product is simply the composite

$$(B\times_S A)\otimes(B\times_S A)\xrightarrow{(4.2)}(B\otimes_{\bar S}B)\times_S(A\otimes_S A)\xrightarrow{\mu_B\times_S\mu_A}B\times_S A.$$

The monoidal functor (4.1) for S (as R) takes the monoid object (B, A) into a monoid object. The algebra $B\times_S A$ is precisely this. Similarly, the last statement follows by using the monoidal functor (4.1c) after appropriate permutation of R, S, T.

For example, if A is an $S|R$-ring, then $\mathrm{End}\,S\times_S A$ is also an $S|R$-ring. It follows from (4.5) that

(4.7a) $\quad \boldsymbol{\theta}' : \mathrm{End}\,S\times_S A\longrightarrow A$

is an $S|R$-ring map. Similarly, we have an $S|R$-ring map [**GA**, (2.2)]

(4.7b) $\quad \boldsymbol{\theta} : A\times_R\mathrm{End}\,R\longrightarrow A,\qquad \sum a_i\otimes f_i\longmapsto\sum\overline{f_i(1)}a_i.$

These are isomorphisms if S and R are finite projective k-modules.

For $L\in{}_{\bar S}\mathcal{M}_{\bar S}$, $M\in{}_{S,\bar R}\mathcal{M}_{S,\bar R}$, and $N\in{}_R\mathcal{M}_R$, we can define a k-module $L\times_S M\times_R N$ in the same way as [**GA**, (1.4)] (where the case $S=R$ is treated), and we get a trifunctor

$$(-)\times_S(-)\times_R(-) : {}_{\bar S}\mathcal{M}_{\bar S}\times{}_{S,\bar R}\mathcal{M}_{S,\bar R}\times{}_R\mathcal{M}_R\longrightarrow\mathcal{M}_k.$$

The previous arguments are easily generalized to see the trifunctor has a structure of a monoidal functor with product similar to (4.2), and further we have its variations — monoidal trifunctors similar to (4.1a-c). If B is a $T|S$-ring and A an $S|R$-ring, then we get a *monoidal functor* just as (4.4)

$$B\times_S A\times_R- : {}_R\mathcal{M}_R\longrightarrow{}_T\mathcal{M}_T,$$

and if E is an $R|U$-ring with an algebra U, then $B\times_S A\times_R E$ becomes a $T|U$*-ring* just as (4.6).

We have associativity maps [**GA**, (1.7)]

$$L\times_S(M\times_R N)\xrightarrow{\boldsymbol{\alpha}'}L\times_S M\times_R N\xleftarrow{\boldsymbol{\alpha}}(L\times_S M)\times_R N$$

for $L\in{}_{\bar S}\mathcal{M}_{\bar S}$, $M\in{}_{S,\bar R}\mathcal{M}_{S,\bar R}$, and $N\in{}_R\mathcal{M}_R$. One verifies directly $\boldsymbol{\alpha}$ *and* $\boldsymbol{\alpha}'$ *are maps of monoidal functors*. It follows that for a $T|S$-ring B and an $S|R$-ring A, we have *maps of monoidal functors*

(4.8a) $\quad B\times_S(A\times_R-)\xrightarrow{\boldsymbol{\alpha}'}B\times_S A\times_R-\xleftarrow{\boldsymbol{\alpha}}(B\times_S A)\times_R-$

and $T|U$-ring maps

(4.8b) $\quad B\times_S(A\times_R E)\xrightarrow{\boldsymbol{\alpha}'}B\times_S A\times_R E\xleftarrow{\boldsymbol{\alpha}}(B\times_S A)\times_R E$

for an $R|U$-ring E.

4.9. LEMMA. *Let* $B \in {}_{\bar{S}}\mathcal{M}_{\bar{S}}$ *and* $A \in {}_{S,\bar{R}}\mathcal{M}_{S,\bar{R}}$. *If* S *is a finite projective* k*-module and* B *is a direct summand of some finite direct sum of* End S *as a* $\bar{S}$*-bimodule, then the natural transformations* $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$ *of* (4.8a) *are isomorphisms.*

PROOF. We may assume $B = \operatorname{End} S$. For $N \in {}_R\mathcal{M}_R$, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{End} S \times_S (A \times_R N) \arrow[r, "\boldsymbol{\alpha}'"] \arrow[dr, "\boldsymbol{\theta}_2'"'] & \operatorname{End} S \times_S A \times_R N \arrow[d, "\boldsymbol{\theta}^{(\prime)}"] & (\operatorname{End} S \times_S A) \times_R N \arrow[l, "\boldsymbol{\alpha}"'] \arrow[dl, "\boldsymbol{\theta}_1' \times_R N"] \\
 & A \times_R N &
\end{tikzcd}$$

where $\boldsymbol{\theta}_1' : \operatorname{End} S \times_S A \to A$ and $\boldsymbol{\theta}_2' : \operatorname{End} S \times_S (A \times_R N) \to A \times_R N$ are the $\boldsymbol{\theta}'$-map (4.5), and the map $\boldsymbol{\theta}^{(\prime)}$ is defined by

$$\boldsymbol{\theta}^{(\prime)}(\sum f_i \otimes a_i \otimes y_i) = \sum f_i(1) a_i \otimes y_i .$$

The commutativity is obvious. It follows from [**GA**, Cor. 4.14] that $\boldsymbol{\theta}_1'$ and $\boldsymbol{\theta}_2'$ are isomorphisms. Since End S is left $\bar{S}$-projective, it follows from [**GA**, Prop. 1.7, ii)] that $\boldsymbol{\alpha}'$ is an isomorphism. It follows that $\boldsymbol{\theta}^{(\prime)}$ and $\boldsymbol{\alpha}$ are isomorphisms.

Q. E. D.

Throughout the rest of this section, *we assume all algebras* R, S, T, U, $\cdots$ *are finite projective* k*-modules.*

4.10. DEFINITION. An $S|R$-ring A is *admissible* if A is a direct summand of a finite direct power of End R as $\bar{R}$-bimodule.

4.11. THEOREM. *Let* B *be a* $T|S$*-ring, let* A *be an* $S|R$*-ring, and let* E *be an* $R|U$*-ring.*

i) *If the* $T|S$*-ring* B *is admissible, then the maps* (4.8) *are isomorphisms. Thus we have an isomorphism of monoidal functors* ${}_R\mathcal{M}_R \to {}_T\mathcal{M}_T$

(4.11a) $\qquad B \times_S (A \times_R -) \simeq (B \times_S A) \times_R -$

and a $T|U$*-ring isomorphism*

(4.11b) $\qquad B \times_S (A \times_R E) \simeq (B \times_S A) \times_R E .$

ii) *The* $S|R$*-ring maps* (4.7) *are isomorphisms, thus we have*

(4.11c) $\qquad \operatorname{End} S \times_S A \simeq A \simeq A \times_R \operatorname{End} R .$

iii) *If both* B *and* A *are admissible, then the* $T|R$*-ring* $B \times_S A$ *is admissible, too.*

PROOF. i) follows from Lemma 4.9. ii) is obvious. iii). Assume

$B \lessdot\!\!\oplus (\mathrm{End}\, S)^m$ as $\bar{S}$-bimodule and $A \lessdot\!\!\oplus (\mathrm{End}\, R)^n$ as $\bar{R}$-bimodule. We have $B \times_S A \lessdot\!\!\oplus (\mathrm{End}\, S)^m \times_S A = (\mathrm{End}\, S \times_S A)^m \simeq A^m \lessdot\!\!\oplus (\mathrm{End}\, R)^{mn}$ as $\bar{R}$-bimodule.

Q. E. D.

One sees the isomorphisms (4.11b, c) satisfy the *coherence* condition.

4.12. DEFINITION. An *algebraic* $\sqrt{\text{Morita}}$ *equivalence data* between R and S means a 4-tuple $(A_{S|R}, B_{R|S}, \lambda, \mu)$ where A and B are an admissible $S|R$-ring and an admissible $R|S$-ring respectively and

$$\begin{aligned} &\lambda : A \times_R B \simeq \mathrm{End}\, S \quad \text{an } S|S\text{-ring isomorphism,} \\ &\mu : B \times_S A \simeq \mathrm{End}\, R \quad \text{an } R|R\text{-ring isomorphism} \end{aligned} \tag{4.12a}$$

such that the following diagrams commute (cf. (3.13)):

$$\begin{tikzcd} (A \times_R B) \times_S A \arrow[r, phantom, "\simeq"] \arrow[d, "\lambda \times_S A"] & A \times_R (B \times_S A) \arrow[d, "A \times_R \mu"] \\ \mathrm{End}\, S \times_S A \arrow[r, phantom, "\simeq"] & A \times_R \mathrm{End}\, R \end{tikzcd} \qquad \begin{tikzcd} (B \times_S A) \times_R B \arrow[r, phantom, "\simeq"] \arrow[d, "\mu \times_R B"] & B \times_S (A \times_R B) \arrow[d, "B \times_S \lambda"] \\ \mathrm{End}\, R \times_R B \arrow[r, phantom, "\simeq"] & B \times_S \mathrm{End}\, S \end{tikzcd} \tag{4.12b}$$

where we use the canonical isomorphisms of (4.11).

Just as the coalgebraic case, an algebraic $\sqrt{\text{Morita}}$ equivalence data (A, B, λ, μ) induces a $\sqrt{\text{Morita}}$ equivalence data $(A \times_R -, B \times_S -, \boldsymbol{\lambda}, \boldsymbol{\mu})$ with

$$\boldsymbol{\lambda} : A \times_R (B \times_S -) \simeq (A \times_R B) \times_S - \xrightarrow{\lambda \times_S -} \mathrm{End}\, S \times_S - \simeq \mathbf{Id}, \tag{4.13a}$$

$$\boldsymbol{\mu} : B \times_S (A \times_R -) \simeq (B \times_S A) \times_R - \xrightarrow{\mu \times_R -} \mathrm{End}\, R \times_R - \simeq \mathbf{Id}, \tag{4.13b}$$

where we use (4.11a) and (4.5).

4.14. LEMMA. *Let* ${}_S P_R$ *and* ${}_R Q_U$ *be bimodules. Viewing* P *and* Q *as a left* $S \otimes \bar{R}$*-module and a left* $R \otimes \bar{U}$*-module respectively, we consider* $\mathrm{End}\, P$ *and* $\mathrm{End}\, Q$ *as an* $S|R$*-ring and an* $R|U$*-ring respectively. (Similarly,* $\mathrm{End}(P \otimes_R Q)$ *is an* $S|U$*-ring). If* P *and* Q *are finite projective* k*-modules, then there exists an* $S|U$*-ring isomorphism*

$$\begin{aligned} \mathrm{End}\, P \times_R \mathrm{End}\, Q \ &\simeq \ \mathrm{End}(P \otimes_R Q) \\ \textstyle\sum f_i \otimes g_i &\longmapsto (x \otimes y \mapsto \textstyle\sum f_i(x) \otimes g_i(y)). \end{aligned}$$

PROOF. From the assumption, we have an isomorphism

$$\begin{aligned} {}_{\bar{a}}(\mathrm{End}\, P) \otimes_{a \in R} {}_a(\mathrm{End}\, Q) \ &\simeq \ \mathrm{Hom}(P \otimes Q, P \otimes_R Q) \\ f \otimes g &\longmapsto (x \otimes y \mapsto f(x) \otimes g(y)). \end{aligned}$$

Since we have $(f\bar{a} \otimes g)(x \otimes y) = (f \otimes g)(xa \otimes y)$ and $(f \otimes ga)(x \otimes y) = (f \otimes g)(x \otimes ay)$

for $a \in R$, this yields the isomorphism in the statement. It is obviously a S, $\bar{U}$-bimodule isomorphism. For $\sum f_i \otimes g_i$, $\sum f'_j \otimes g'_j \in \operatorname{End} P \times_R \operatorname{End} Q$, the product $\sum f_i f'_j \otimes g_i g'_j$ acts on $P \otimes_R Q$ as the composite of the corresponding endomorphisms. Hence the isomorphism is an $R|U$-ring map. Q. E. D.

We briefly describe the algebraic $\sqrt{\text{Morita}}$ equivalence data corresponding to Examples 2.2–2.4.

4.15. EXAMPLE. Let $({}_S P_R, {}_R Q_S, \alpha, \beta)$ be a Morita equivalence data between R and S (2.2). We put $A = \operatorname{End} P$ and $B = \operatorname{End} Q$. A and B are an $S|R$-ring and an $R|S$-ring respectively by (4.14). Since $P_R \langle\oplus (R_R)^n$, we have $A \langle\oplus (\operatorname{End} R)^{n^2}$ as $\bar{R}$-bimodule, hence A is admissible. Similarly, B is admissible. In view of Lemma 4.14, there is a canonical $S|S$-ring isomorphism $\operatorname{End} P \times_R \operatorname{End} Q \simeq \operatorname{End}(P \otimes_R Q)$, and α induces an $S|S$-ring isomorphism $\operatorname{End}(P \otimes_R Q) \simeq \operatorname{End} S$. Let

$$\lambda : A \times_R B \simeq \operatorname{End} S$$

be the composite. Define an $R|R$-ring isomorphism $\mu : B \times_S A \simeq \operatorname{End} R$ in a similar way. One verifies the 4-tuple (A, B, λ, μ) satisfies condition (4.12b), hence it is an algebraic $\sqrt{\text{Morita}}$ equivalence data. We show the corresponding $\sqrt{\text{Morita}}$ equivalence data $(A \times_R -, B \times_S -)$ is canonically isomorphic with the $\sqrt{\text{Morita}}$ equivalence data $(\boldsymbol{\Gamma}, \boldsymbol{\Delta})$ of (2.2). For $M \in {}_R\mathcal{M}_R$, the canonical isomorphism

$${}_{\bar{a}}A \otimes_{a \in R} {}_a M \simeq \operatorname{Hom}(P, P \otimes_R M), \qquad a \otimes m \longmapsto (x \mapsto a(x) \otimes m)$$

induces an S-bimodule isomorphism

$$A \times_R M \simeq \operatorname{Hom}_{-R}(P, P \otimes_R M)$$

which becomes an isomorphism of *monoidal* functors if we define the monoidal structure on $\operatorname{Hom}_{-R}(P, P \otimes_R -)$ as follows: For $f \in \operatorname{Hom}_{-R}(P, P \otimes_R M)$ and $g \in \operatorname{Hom}_{-R}(P, P \otimes_R N)$ with $N \in {}_R\mathcal{M}_R$, the product fg is the composite

$$fg : P \xrightarrow{g} P \otimes_R N \xrightarrow{f \otimes I} P \otimes_R M \otimes_R N.$$

The unit is the identity in $\operatorname{Hom}_{-R}(P, P \otimes_R R)$. On the other hand, there is an isomorphism of monoidal functors

$$\boldsymbol{\Gamma}(M) = P \otimes_R M \otimes_R Q \simeq \operatorname{Hom}_{-R}(P, P \otimes_R M),$$
$$x \otimes m \otimes y \longmapsto (x' \mapsto x \otimes m\beta(y, x')).$$

Composing the above two, we have an isomorphism of monoidal functors

$$A \times_R - \simeq \boldsymbol{\Gamma}$$

and $B\times_S-\simeq\boldsymbol{\Delta}$ similarly. Through the induced isomorphisms $A\times_R(B\times_S-)\simeq\boldsymbol{\Gamma\Delta}$ and $B\times_S(A\times_R-)\simeq\boldsymbol{\Delta\Gamma}$, one sees the structure isomorphisms $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ correspond with $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ respectively. Hence $\sqrt{\text{Morita}}$ equivalence data $(A\times_R-, B\times_S-)$ and $(\boldsymbol{\Gamma}, \boldsymbol{\Delta})$ are isomorphic with each other.

4.16. EXAMPLE. Let $\sigma(X, Y)=\sum a_iXb_iYc_i$ be a formal ring law over R. If Λ is an $R|S$-ring, then the R^σ-ring Λ^σ made from the R-ring Λ becomes a $R^\sigma|S$-ring with algebra map $\bar{S}\to\Lambda^\sigma$, $\bar{a}\mapsto\bar{a}e_\sigma$. We note the underlying $\bar{S}$-bimodule of Λ^σ is Λ itself. From definition we see $(\Lambda\times_SN)^\sigma=\Lambda^\sigma\times_SN$ *as* R^σ*-bimodules* for $N\in{}_S\mathcal{M}_S$. If $x=\sum u_j\otimes m_j\in\Lambda\times_SM$ and $y=\sum v_k\otimes n_k\in\Lambda\times_SN$ with $M, N\in{}_S\mathcal{M}_S$, then the product $\sigma(x, y)=\sum a_ixb_iyc_i$ in $(\Lambda\times_S(M\otimes_SN))^\sigma$ (1.10a and 4.4) is

$$\sigma(x, y)=\sum_i\Big(\sum_j a_iu_jb_i\otimes m_j\Big)\Big(\sum_k v_kc_i\otimes n_k\Big)$$

$$=\sum_{j,k}\Big(\sum_i a_iu_jb_iv_kc_i\Big)\otimes(m_j\otimes n_k)=\sum_{j,k}\sigma(u_j, v_k)\otimes(m_j\otimes n_k)$$

which is just the product in $\Lambda^\sigma\times_S(M\otimes_SN)$. This means we have

$$(\Lambda\times_S-)^\sigma=\Lambda^\sigma\times_S-$$

as monoidal functors ${}_S\mathcal{M}_S\to{}_{R^\sigma}\mathcal{M}_{R^\sigma}$. In particular, the monoidal functor $(-)^\sigma$ of (1.10) is isomorphic to $(\operatorname{End}R)^\sigma\times_R-$ where $(\operatorname{End}R)^\sigma$ is a $R^\sigma|R$-ring. If σ has an inverse τ, it follows there is a canonical algebraic $\sqrt{\text{Morita}}$ equivalence data $((\operatorname{End}R)^\sigma, (\operatorname{End}R^\sigma)^\tau)$ between R and R^σ.

4.17. EXAMPLE. Let A be an Azumaya algebra. A is an $A|k$-ring, and $\bar{A}$ is a $k|A$-ring. The map

$$\lambda : A\times_k\bar{A}=A\otimes\bar{A}\simeq\operatorname{End}A,\qquad \lambda(a\otimes\bar{b})(x)=axb$$

is an $A|A$-ring isomorphism. The map

$$\mu : \bar{A}\times_AA=\operatorname{cent}A=\operatorname{cent}\bar{A}=k$$

is a $k|k$-ring isomorphism. The 4-tuple $(A, \bar{A}, \lambda, \mu)$ is an algebraic $\sqrt{\text{Morita}}$ equivalence data corresponding to the $\sqrt{\text{Morita}}$ equivalence data of (2.4).

§ 5. Duality.

Throughout the section, we assume all algebras $R, S, T, \cdots$ are *finite projective k-modules.* We compare coalgebraic and algebraic $\sqrt{\text{Morita}}$ equivalence data by means of the duality introduced in the section in order to prove that every $\sqrt{\text{Morita}}$ equivalence data between finite projective algebras comes from an algebraic $\sqrt{\text{Morita}}$ equivalence data.

5.1. DEFINITION. We define a *contravariant monoidal functor*

(5.1a) $\quad \boldsymbol{D} : {}_R\mathcal{M}_R \longrightarrow {}_{\bar{R}}\mathcal{M}_{\bar{R}}$.

For $M \in {}_R\mathcal{M}_R$, $\boldsymbol{D}(M) = \mathrm{Hom}_{R\text{-}}(M, R)$ the left R-homomorphisms. For $x \in \boldsymbol{D}(M)$ and $m \in M$, we put $\langle x, m \rangle = x(m)$. $\boldsymbol{D}(M)$ is made into a $\bar{R}$-bimodule by setting

$$\langle \bar{a}x, m \rangle = \langle x, m \rangle a, \qquad \langle x\bar{a}, m \rangle = \langle x, ma \rangle$$

for $\bar{a} \in \bar{R}$. For $x \in \boldsymbol{D}(M)$ and $y \in \boldsymbol{D}(N)$ with $N \in {}_R\mathcal{M}_R$, the product $xy \in \boldsymbol{D}(M \otimes_R N)$ is defined by

$$\langle xy, m \otimes n \rangle = \langle x, m \langle y, n \rangle \rangle$$

for $m \in M$, $n \in N$. The product is associative giving a $\bar{R}$-bimodule map

(5.1b) $\quad \boldsymbol{D}(M) \otimes_{\bar{R}} \boldsymbol{D}(N) \longrightarrow \boldsymbol{D}(M \otimes_R N), \qquad x \otimes y \longmapsto xy$.

The contravariant functor (5.1a) becomes a *monoidal functor* with the product and unit $1 \in \boldsymbol{D}(R) = \mathrm{End}_{R\text{-}}(R)$. If it is necessary to specify R, we write $\boldsymbol{D} = \boldsymbol{D}_R$.

The contravariant monoidal functor $\boldsymbol{D}$ *takes a comonoid object to a monoid object.* This means if C is an R-coring, then $\boldsymbol{D}(C)$ is a $\bar{R}$-ring with product $x * y$ ($x, y \in \boldsymbol{D}(C)$) defined by

$$\langle x * y, c \rangle = \sum \langle x, c_1 \langle y, c_2 \rangle \rangle \qquad (c \in C)$$

where $\Delta(c) = \sum c_1 \otimes c_2$ as usual. The unit is ε.

Let ${}_R\mathcal{M}_R{}^{\mathrm{f}}$ be the subcategory of all R-bimodules which are *left R-finite projective*. This is a *monoidal subcategory* of ${}_R\mathcal{M}_R$, i.e., $M \otimes_R N \in {}_R\mathcal{M}_R{}^{\mathrm{f}}$ for $M, N \in {}_R\mathcal{M}_R{}^{\mathrm{f}}$. The functor $\boldsymbol{D}$ induces an *anti-equivalence* ${}_R\mathcal{M}_R{}^{\mathrm{f}} \to {}_{\bar{R}}\mathcal{M}_{\bar{R}}{}^{\mathrm{f}}$, and the map (5.1b) is an *isomorphism* for $M, N \in {}_R\mathcal{M}_R{}^{\mathrm{f}}$. This means we have a *monoidal anti-equivalence*

$$\boldsymbol{D} : {}_R\mathcal{M}_R{}^{\mathrm{f}} \longrightarrow {}_{\bar{R}}\mathcal{M}_{\bar{R}}{}^{\mathrm{f}}.$$

Put $\bar{\boldsymbol{D}} = \boldsymbol{D}_{\bar{R}} : {}_{\bar{R}}\mathcal{M}_{\bar{R}} \to {}_R\mathcal{M}_R$. For $m \in M$, we have a left $\bar{R}$-homomorphism $x \mapsto \langle x, m \rangle$, $\boldsymbol{D}(M) \to \bar{R}$, i.e., an element in $\bar{\boldsymbol{D}}(\boldsymbol{D}(M))$. This gives an R-bimodule map $M \to \bar{\boldsymbol{D}}(\boldsymbol{D}(M))$ which is an *isomorphism* if $M \in {}_R\mathcal{M}_R{}^{\mathrm{f}}$. Thus we have natural transformations

(5.2) $\quad \boldsymbol{I} \longrightarrow \bar{\boldsymbol{D}} \circ \boldsymbol{D} \quad$ and $\quad \boldsymbol{I} \longrightarrow \boldsymbol{D} \circ \bar{\boldsymbol{D}}$.

If we restrict $\boldsymbol{D}$ and $\bar{\boldsymbol{D}}$ on ${}_R\mathcal{M}_R{}^{\mathrm{f}}$ and ${}_{\bar{R}}\mathcal{M}_{\bar{R}}{}^{\mathrm{f}}$, then we see (5.2) gives *isomorphisms of monoidal equivalences.* (Note the composite of two monoidal anti-equivalences is a monoidal equivalence). Thus we have a *monoidal duality* $(\boldsymbol{D}, \bar{\boldsymbol{D}})$ between ${}_R\mathcal{M}_R{}^{\mathrm{f}}$ and ${}_{\bar{R}}\mathcal{M}_{\bar{R}}{}^{\mathrm{f}}$ with adjunction (5.2). This monoidal anti-equivalence takes a monoid object to a comonoid object. This yields:

5.3. PROPOSITION. a) *If C is an R-coring, then $\boldsymbol{D}(C)$ is a $\bar{R}$-ring.*

b) *If A is a $\bar{R}$-ring which is left $\bar{R}$-finite projective, then there is a unique R-coring structure on $\bar{\boldsymbol{D}}(A)$ such that $A \simeq \boldsymbol{D}(\bar{\boldsymbol{D}}(A))$* (5.2) *as $\bar{R}$-rings.*

c) *The monoidal duality $(\boldsymbol{D}, \bar{\boldsymbol{D}})$ between ${}_R\mathcal{M}_R{}^{\mathrm{f}}$ and ${}_{\bar{R}}\mathcal{M}_{\bar{R}}{}^{\mathrm{f}}$ induces a duality $C \mapsto \boldsymbol{D}(C)$ and $\bar{\boldsymbol{D}}(A) \leftarrow A$ between left R-finite projective R-corings C and left $\bar{R}$-finite projective $\bar{R}$-rings A.*

The comultiplication of $\bar{\boldsymbol{D}}(A)$ in b) is the composite of $\bar{\boldsymbol{D}}(\mu): \bar{\boldsymbol{D}}(A) \to \bar{\boldsymbol{D}}(A \otimes_{\bar{R}} A)$ (μ denoting the multiplication of A) with the *inverse* of the product $\bar{\boldsymbol{D}}(A) \otimes_R \bar{\boldsymbol{D}}(A) \simeq \bar{\boldsymbol{D}}(A \otimes_{\bar{R}} A)$. Under the duality of c), the isomorphisms (5.2) give an R-coring isomorphism $C \simeq \bar{\boldsymbol{D}}(\boldsymbol{D}(C))$ and a $\bar{R}$-ring isomorphism $A \simeq \boldsymbol{D}(\bar{\boldsymbol{D}}(A))$.

If R is commutative, (5.3) gives the duality of [**9**, §4].

For an R-coring C, C^* is the submodule of all R-bimodule maps $C \to R$ in $\boldsymbol{D}(C)$, namely we have

$$C^* = \mathrm{cent}_{\boldsymbol{D}(C)}(\bar{R})$$

the centralizer of $\bar{R}$ in $\boldsymbol{D}(C)$. For $x, y \in \boldsymbol{D}(C)$ and $c \in C$, we have

$$\langle x*y, c\rangle = \sum \langle x, c_1\langle y, c_2\rangle\rangle = \sum \langle x, c_1\rangle\langle y, c_2\rangle.$$

This means C^* is a subalgebra of $\boldsymbol{D}(C)$. This gives immediately a) of the following:

5.4. PROPOSITION. a) *If C is an $S|R$-coring, then $\boldsymbol{D}(C)$ is an $S|R$-ring.*

b) *If A is an $S|R$-ring which is left $\bar{R}$-finite projective, then there is a unique $S|R$-coring structure on $\bar{\boldsymbol{D}}(A)$ such that $A \simeq \boldsymbol{D}(\bar{\boldsymbol{D}}(A))$* (5.2) *as $S|R$-rings.*

c) *There is a duality $C \mapsto \boldsymbol{D}(C)$ and $\bar{\boldsymbol{D}}(A) \leftarrow A$ between left R-finite projective $S|R$-corings C and left $\bar{R}$-finite projective $S|R$-rings A.*

d) *Under the duality of* c), *the $S|R$-ring A is admissible if and only if ${}_RC_R$ is a finite projective object in ${}_R\mathcal{M}_R$.*

We note if the $S|R$-ring A is admissible, then it is left $\bar{R}$-finite projective since R is k-finite projective.

PROOF. b) The isomorphism $A \simeq \boldsymbol{D}(\bar{\boldsymbol{D}}(A))$ (5.2) induces algebra isomorphism $\mathrm{cent}_A(\bar{R}) \simeq \bar{\boldsymbol{D}}(A)^*$. The structure algebra map $S \to \mathrm{cent}_A(\bar{R})$ makes $\bar{\boldsymbol{D}}(A)$ into an $S|R$-coring. c) follows from a) and b). d) We have $\boldsymbol{D}(R \otimes R) = \mathrm{End}\, R$. Hence $C \langle\!\oplus (R \otimes R)^n$ in ${}_R\mathcal{M}_R$ if and only if $A \langle\!\oplus (\mathrm{End}\, R)^n$ in ${}_{\bar{R}}\mathcal{M}_{\bar{R}}$. Q.E.D.

We note under the duality of c), the bimodules ${}_SC_S$ and ${}_SA_S$ are combined with each other by the relation

(5.5) $\langle axb, c\rangle = \langle x, bca\rangle \quad (a, b \in S,\ x \in A,\ c \in C).$

In fact, $bca=\sum\langle a, c_1\rangle c_2\langle b, c_3\rangle$ by (3.4c, d), hence $\langle x, bca\rangle=\sum\langle a, c_1\rangle\langle x, c_2\langle b, c_3\rangle\rangle$ the left-hand side.

In the following technical lemma, we use the following functors:

$$\text{(5.6a)}\qquad \begin{aligned} &\boldsymbol{D}_R : {}_{S,R}\mathcal{M}_{S,R} \longrightarrow {}_{S,\bar{R}}\mathcal{M}_{S,\bar{R}},\\ &\boldsymbol{D}_S : {}_S\mathcal{M}_S \longrightarrow {}_{\bar{S}}\mathcal{M}_{\bar{S}}\end{aligned}$$

where in the above the S-bimodule structure of $\boldsymbol{D}_R(M)$ for $M\in{}_{S,R}\mathcal{M}_{S,R}$ is defined by (5.5). We use $\{ , \}$ to denote the pairing for $\boldsymbol{D}_S$.

5.6. LEMMA. *Let $M\in{}_S\mathcal{M}_S$ and $N\in{}_{S,R}\mathcal{M}_{S,R}$. Let $M\circ N$ be the quotient of $M\otimes N$ obtained by identifying $amb\otimes n=m\otimes bna$ for $a, b\in S$, $m\in M$, $n\in N$. This has a natural R-bimodule structure coming from ${}_RN_R$. There is a $\bar{R}$-bimodule map*

$$\xi : \boldsymbol{D}_S(M)\times_S\boldsymbol{D}_R(N) \longrightarrow \boldsymbol{D}_R(M\circ N)$$

defined by $\langle\xi(\sum x_i\otimes y_i), m\otimes n\rangle=\sum\langle y_i, n\{x_i, m\}\rangle$ for $\sum x_i\otimes y_i\in\boldsymbol{D}_S(M)\times_S\boldsymbol{D}_R(N)$ and $m\in M$, $n\in N$.

PROOF. For $x\in\boldsymbol{D}_S(M)$ and $y\in\boldsymbol{D}_R(N)$, we define $\xi(x, y)\in\boldsymbol{D}_R(M\otimes N)$ (the R-bimodule structure of $M\otimes N$ coming from ${}_RN_R$) by setting

$$\langle\xi(x, y), m\otimes n\rangle = \langle y, n\{x, m\}\rangle \qquad (m\in M,\ n\in N).$$

The functional $\xi(x, y)$ takes the same value on $am\otimes n$ and $m\otimes na$ for $a\in S$, and we have $\xi(\bar{a}x, y)=\xi(x, ay)$. Hence ξ induces a map

$$\text{(5.6b)}\qquad \begin{aligned}\xi : {}_{\bar{b}}[\boldsymbol{D}_S(M)]\otimes_{b\in S}{}_b[\boldsymbol{D}_R(N)] &\longrightarrow \boldsymbol{D}_R({}_aM\otimes_{a\in S}N_a),\\ x\otimes y &\longmapsto \xi(x, y).\end{aligned}$$

Since we have

$$\langle\xi(x\bar{a}, y), m\otimes n\rangle = \langle\xi(x, y), ma\otimes n\rangle, \qquad \langle\xi(x, ya), m\otimes n\rangle = \langle\xi(x, y), m\otimes an\rangle$$

for $a\in S$, it follows that (5.6b) induces the required map. One sees easily the ξ-map is a $\bar{R}$-bimodule map. Q. E. D.

The map (5.6b) is an isomorphism if M is left S-finite projective. Hence in this case, the map ξ of (5.6) is an isomorphism, too.

5.7. PROPOSITION. *Let Λ be an S-coring, and let C be an $S|R$-coring. Then $\boldsymbol{D}_S(\Lambda)$ is a $\bar{S}$-ring and $\boldsymbol{D}_R(C)$ an $S|R$-ring. We have an R-coring $\Lambda\circ C$ (take $T=k$ in* (3.10)), *hence a $\bar{R}$-ring $\boldsymbol{D}_R(\Lambda\circ C)$. The $\bar{R}$-bimodule map ξ* (5.6)

$$\xi : \boldsymbol{D}_S(\Lambda)\times_S\boldsymbol{D}_R(C) \longrightarrow \boldsymbol{D}_R(\Lambda\circ C)$$

is a $\bar{R}$-ring map. If Λ is left S-finite projective, this is an isomorphism. If Λ is a $T|S$-coring in addition, then this is a $T|R$-ring map.

PROOF. With the assumption in the last statement, $\Lambda\circ C$ becomes a $T|R$-coring, hence $\boldsymbol{D}_R(\Lambda\circ C)$ a $T|R$-ring. It is easy to see $\boldsymbol{\xi}$ is a T-bimodule map, too. Hence all we have to do is show $\boldsymbol{\xi}$ preserves the product and unit. Take $z=\sum x_i\otimes y_i$, $w=\sum u_j\otimes v_j\in \boldsymbol{D}_S(\Lambda)\times_S\boldsymbol{D}_R(C)$ and $b\in\Lambda$, $c\in C$. We have $zw=\sum_{i,j}x_iu_j\otimes y_iv_j$ (4.6) and $\Delta(b\circ c)=\sum(b_1\circ c_1)\otimes(b_2\circ c_2)$ (3.10a), hence we have

$$
\begin{aligned}
&\langle \boldsymbol{\xi}(zw),\, b\circ c\rangle = \sum_{i,j}\langle y_iv_j,\, c\{x_iu_j,\, b\}\rangle\\
&=\sum_{i,j}\langle y_i,\, c_1\{x_i,\, b_1\{u_j,\, b_2\}\}\langle v_j,\, c_2\rangle\rangle \qquad \text{(use (3.4a))}\\
&=\sum_{i,j}\langle y_i,\, c_1\langle v_j,\, c_2\rangle\{x_i,\, b_1\{u_j,\, b_2\}\}\rangle = \langle \boldsymbol{\xi}(z),\, \sum_j b_1\{u_j,\, b_2\}\circ c_1\langle v_j,\, c_2\rangle\rangle.
\end{aligned}
$$

Since $\Delta(C)\subset[C\otimes_R C]^S$ (3.5a), we have

$$
\begin{aligned}
&\sum_j b_1\{u_j,\, b_2\}\circ c_1\langle v_j,\, c_2\rangle = \sum_j b_1\circ\{u_j,\, b_2\}c_1\langle v_j,\, c_2\rangle\\
&=\sum_j b_1\circ c_1\langle v_j,\, c_2\{u_j,\, b_2\}\rangle = \sum b_1\circ c_1\langle \boldsymbol{\xi}(w),\, b_2\circ c_2\rangle.
\end{aligned}
$$

Hence $\langle \boldsymbol{\xi}(zw),\, b\circ c\rangle=\sum\langle \boldsymbol{\xi}(z),\, b_1\circ c_1\langle \boldsymbol{\xi}(w),\, b_2\circ c_2\rangle\rangle=\langle \boldsymbol{\xi}(z)\boldsymbol{\xi}(w),\, b\circ c\rangle$. This means $\boldsymbol{\xi}(zw)=\boldsymbol{\xi}(z)\boldsymbol{\xi}(w)$. On the other hand, we have $\boldsymbol{\xi}(1)=1$ easily. Q. E. D.

The ring map $\boldsymbol{\xi}$ (5.7) can be generalized to a triple, or more generally to an n-tuple, of composable corings. To be precise, let Λ be a $T|S$-coring, let C be an $S|R$-coring, and let Γ be an $R|U$-coring. There is a natural $T|U$-ring map

$$\boldsymbol{\xi}_2 : \boldsymbol{D}_S(\Lambda)\times_S\boldsymbol{D}_R(C)\times_R\boldsymbol{D}_U(\Gamma)\longrightarrow \boldsymbol{D}_U(\Lambda\circ C\circ\Gamma)$$

defined by

$$\langle \boldsymbol{\xi}(\sum x_i\otimes y_i\otimes z_i),\, b\circ c\circ d\rangle = \sum\langle z_i,\, d\langle y_i,\, c\langle x_i,\, b\rangle\rangle\rangle$$

for $b\in\Lambda$, $c\in C$, $d\in\Gamma$, $\sum x_i\otimes y_i\otimes z_i\in\boldsymbol{D}_S(\Lambda)\times_S\boldsymbol{D}_R(C)\times_R\boldsymbol{D}_U(\Gamma)$. We have a commutative diagram

$$
\begin{array}{ccc}
[\boldsymbol{D}_S(\Lambda)\times_S\boldsymbol{D}_R(C)]\times_R\boldsymbol{D}_U(\Gamma) & \xrightarrow{\boldsymbol{\xi}\times_R I} & \boldsymbol{D}_R(\Lambda\circ C)\times_R\boldsymbol{D}_U(\Gamma)\\
\Big\downarrow{\scriptstyle \boldsymbol{\alpha}\ (4.8)} & & \Big\downarrow{\scriptstyle \boldsymbol{\xi}}\\
\boldsymbol{D}_S(\Lambda)\times_S\boldsymbol{D}_R(C)\times_R\boldsymbol{D}_U(\Gamma) & \xrightarrow{\boldsymbol{\xi}_2} & \boldsymbol{D}_U(\Lambda\circ C\circ\Gamma)
\end{array}
$$

and a similar diagram for $\boldsymbol{\alpha}'$.

Take $\Lambda=S\otimes S$ in (5.7). Then $\boldsymbol{D}_S(\Lambda)=\operatorname{End}S$. For $x=\sum f_i\otimes y_i\in\operatorname{End}S\times_S\boldsymbol{D}_R(C)$ and $c\in C$, we have

$$\langle \boldsymbol{\xi}(x),\, (1\otimes 1)\circ c\rangle = \sum\langle y_i,\, cf_i(1)\rangle = \langle\sum f_i(1)y_i,\, c\rangle.$$

This means the composite

$$\operatorname{End}S\times_S\boldsymbol{D}_R(C) = \boldsymbol{D}_S(S\otimes S)\times_S\boldsymbol{D}_R(C)\xrightarrow{\boldsymbol{\xi}}\boldsymbol{D}_R((S\otimes S)\circ C)\simeq\boldsymbol{D}_R(C)$$

is the $\boldsymbol{\theta}'$-map (4.7a). Similarly, we reobtain the $\boldsymbol{\theta}$-map from $\boldsymbol{\xi}$ by taking $C=S\otimes S$ and $S=R$ in (5.7).

The $S|R$-coring C is called *admissible* if ${}_R C_R$ is a finite projective object in ${}_R\mathcal{M}_R$. By (5.4d), we have a duality between admissible $S|R$-corings and admissible $S|R$-rings.

5.8. THEOREM. *Let Λ be a $T|S$-coring, let C be an $S|R$-coring, and let Γ be an $R|U$-coring, all admissible.*

a) *There is a canonical $T|U$-ring isomorphism*

$$\boldsymbol{\xi} : \boldsymbol{D}_S(\Lambda)\times_S \boldsymbol{D}_R(C) \simeq \boldsymbol{D}_R(\Lambda\circ C).$$

b) *We have a commutative diagram*

$$\begin{tikzcd}
(\boldsymbol{D}_S(\Lambda)\times_S \boldsymbol{D}_R(C))\times_R \boldsymbol{D}_U(\Gamma) \arrow[r, phantom, "\simeq"] \arrow[d, no head, "\wr"] & \boldsymbol{D}_R(\Lambda\circ C)\times_R \boldsymbol{D}_U(\Gamma) \arrow[r, phantom, "\simeq"] & \boldsymbol{D}_U((\Lambda\circ C)\circ\Gamma) \arrow[d, no head, "\wr"] \\
\boldsymbol{D}_S(\Lambda)\times_S(\boldsymbol{D}_R(C)\times_R \boldsymbol{D}_U(\Gamma)) \arrow[r, phantom, "\simeq"] & \boldsymbol{D}_S(\Lambda)\times_S \boldsymbol{D}_U(C\circ\Gamma) \arrow[r, phantom, "\simeq"] & \boldsymbol{D}_U(\Lambda\circ(C\circ\Gamma))
\end{tikzcd}$$

where we use a), (4.11b), *and* (3.11a).

c) *We have a commutative diagram*

$$\begin{tikzcd}
\boldsymbol{D}_S(S\otimes S)\times_S \boldsymbol{D}_R(C) \arrow[r, phantom, "\simeq"] \arrow[d, no head, "\wr"] & \boldsymbol{D}_R((S\otimes S)\circ C) \arrow[rr, phantom, "\simeq"] \arrow[dr, no head, "\wr"] & & \boldsymbol{D}_R(C\circ(R\otimes R)) \arrow[r, phantom, "\simeq"] \arrow[dl, no head, "\wr"'] & \boldsymbol{D}_R(C)\times_R \boldsymbol{D}_R(R\otimes R) \arrow[d, no head, "\wr"] \\
\operatorname{End} S\times_S \boldsymbol{D}_R(C) \arrow[rr, phantom, "\simeq"] & & \boldsymbol{D}_R(C) \arrow[rr, phantom, "\simeq"] & & \boldsymbol{D}_R(C)\times_R \operatorname{End} R
\end{tikzcd}$$

where we use a), (4.11c), (3.11b).

This follows from the previous arguments.

5.9. THEOREM. a) *If $(C_{S|R}, D_{R|S}, \gamma, \delta)$ is a coalgebraic $\sqrt{\text{Morita}}$ equivalence data between R and S, then we have an algebraic $\sqrt{\text{Morita}}$ equivalence data $(\boldsymbol{D}_R(C), \boldsymbol{D}_S(D), \bar{\gamma}, \bar{\delta})$, where*

$$\bar{\gamma} : \boldsymbol{D}_R(C)\times_R \boldsymbol{D}_S(D) \simeq \boldsymbol{D}_S(C\circ D) \xrightarrow{\boldsymbol{D}_S(\gamma)} \boldsymbol{D}_S(S\otimes S) \simeq \operatorname{End} S,$$

$$\bar{\delta} : \boldsymbol{D}_S(D)\times_S \boldsymbol{D}_R(C) \simeq \boldsymbol{D}_R(D\circ C) \xrightarrow{\boldsymbol{D}_R(\delta)} \boldsymbol{D}_R(R\otimes R) \simeq \operatorname{End} R.$$

b) *If $(A_{S|R}, B_{R|S}, \lambda, \mu)$ is an algebraic $\sqrt{\text{Morita}}$ equivalence data between R and S, then we have a coalgebraic $\sqrt{\text{Morita}}$ equivalence data $(\boldsymbol{D}_{\bar{R}}(A), \boldsymbol{D}_{\bar{S}}(B), \bar{\lambda}, \bar{\mu})$ where $\gamma=\bar{\lambda}$ and $\delta=\bar{\mu}$ are uniquely determined coring isomorphisms such that*

$$\lambda : A\times_R B \simeq \boldsymbol{D}_R(\boldsymbol{D}_{\bar{R}}(A))\times_R \boldsymbol{D}_S(\boldsymbol{D}_{\bar{S}}(B)) \xrightarrow{\bar{\gamma}} \operatorname{End} S,$$

$$\mu : B\times_S A \simeq \boldsymbol{D}_S(\boldsymbol{D}_{\bar{S}}(B))\times_S \boldsymbol{D}_R(\boldsymbol{D}_{\bar{R}}(A)) \xrightarrow{\bar{\delta}} \operatorname{End} R.$$

c) *The correspondence* $(C, D, \gamma, \delta)\leftrightarrow(A, B, \lambda, \mu)$ *given in* a) *and* b) *establishes a duality between coalgebraic and algebraic* $\sqrt{\text{Morita}}$ *equivalence data between* R *and* S. *The corresponding data determine isomorphic* $\sqrt{\text{Morita}}$ *equivalence data, namely we have*

$$(\boldsymbol{\Phi}_C, \boldsymbol{\Phi}_D, \boldsymbol{\gamma}, \boldsymbol{\delta}) \simeq (A\times_R -, B\times_S -, \boldsymbol{\lambda}, \boldsymbol{\mu}).$$

PROOF. a), b), and the first part of c) are easy consequences of (5.8) in view of the duality (5.4). We establish the last statement. Put $(A, B, \lambda, \mu)=(\boldsymbol{D}_R(C), \boldsymbol{D}_S(D), \bar{\gamma}, \bar{\delta})$. For $M\in {}_R\mathcal{M}_R$, the isomorphism

$${}_{\bar{a}}A\otimes_{a\in R}\, {}_aM \simeq \mathrm{Hom}_{R\text{-}}(C, M), \qquad x\otimes m \longmapsto (c \mapsto \langle x, c\rangle m)$$

induces an S-bimodule isomorphism

$$\boldsymbol{\zeta}_M : A\times_R M \simeq \boldsymbol{\Phi}_C(M) \qquad \text{(cf. (4.15))}.$$

Take $u=\sum x_i\otimes m_i\in A\times_R M$, $v=\sum y_j\otimes n_j\in A\times_R N$ with $M, N\in {}_R\mathcal{M}_R$. We have $uv=\sum x_i y_j\otimes(m_i\otimes n_j)$ in $A\times_R(M\otimes_R N)$ (4.4), and for $c\in C$

$$\begin{aligned}\boldsymbol{\zeta}(uv)(c) &= \textstyle\sum \langle x_i y_j, c\rangle m_i\otimes n_j = \sum \langle x_i, c_1\langle y_j, c_2\rangle\rangle m_i\otimes n_j \\ &= \sum_j \boldsymbol{\zeta}(u)(c_1\langle y_j, c_2\rangle)\otimes n_j = \sum_j \boldsymbol{\zeta}(u)(c_1)\otimes\langle y_j, c_2\rangle n_j = \textstyle\sum \boldsymbol{\zeta}(u)(c_1)\otimes\boldsymbol{\zeta}(v)(c_2).\end{aligned}$$

This means we have an *isomorphism of monoidal functors* $\boldsymbol{\zeta} : A\times_R - \simeq \boldsymbol{\Phi}_C$. Similarly, we have $\boldsymbol{\zeta}' : B\times_S - \simeq \boldsymbol{\Phi}_D$. For $z=\sum x_i\otimes y_i\otimes z_i\in A\times_R B\times_S N$ with $N\in {}_S\mathcal{M}_S$ and $c\circ d\in C\circ D$, we have

$$\begin{aligned}\boldsymbol{\zeta}'(\boldsymbol{\zeta}(z)(c))(d) &= \textstyle\sum \{\langle x_i, c\rangle y_i, d\} n_i = \sum \{y_i, d\langle x_i, c\rangle\} n_i \\ &= \textstyle\sum \langle \boldsymbol{\xi}(x_i, y_i), c\circ d\rangle n_i.\end{aligned}$$

This means we have a commutative diagram

$$\begin{tikzcd} A\times_R B\times_S N \arrow[d, "\boldsymbol{\xi}\times_S I"'] \arrow[r, phantom, "\simeq"] & \boldsymbol{\Phi}_C(\boldsymbol{\Phi}_D(N)) \arrow[d, dash, "\wr"] \\ \boldsymbol{D}_R(C\circ D)\times_S N \arrow[r, phantom, "\simeq"] & \boldsymbol{\Phi}_{C\circ D}(N). \end{tikzcd}$$

It follows that $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ correspond with each other through the induced isomorphism $A\times_R B\times_S - \simeq \boldsymbol{\Phi}_C\circ\boldsymbol{\Phi}_D \simeq \boldsymbol{\Phi}_{C\circ D}$. Similarly, $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ correspond with each other. Q. E. D.

As a direct consequence of (3.14) and (5.9) we have:

5.10. ALGEBRAIC DESCRIPTION THEOREM. *Let* R *and* S *be finite projective* k*-algebras. Every* $\sqrt{\text{Morita}}$ *equivalence data between them is isomorphic to the data* $(A\times_R -, B\times_S -, \boldsymbol{\lambda}, \boldsymbol{\mu})$ *coming from an algebraic* $\sqrt{\text{Morita}}$ *equivalence data* (A, B, λ, μ) *uniquely determined within isomorphism.*

The algebraic data of (4.15-17) correspond to the coalgebraic data of (3.15-17) under the duality (5.9c). As a final remark,

5.11. PROPOSITION. *Assume* $R\sim_{\sqrt{\mathrm{M}}}S$ *over* k. *If* R *is a finite projective* k-*module, then so is* S.

PROOF. First assume $R\sim_{\mathrm{M}}S$. There is a Morita data $({}_SP_R, \cdots)$. Since P_R and R_k are finite projective, P is k-finite projective. Since ${}_SS \langle\oplus {}_SP^n$ for some n, S is k-finite projective, too. Now we have $R\otimes\bar{R}\sim_{\mathrm{M}}S\otimes\bar{S}$. It follows that $S\otimes\bar{S}$ is k-finite projective. Hence its direct summand S is, too. Q.E.D.

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