Morphological Scale-Space Representation with Levelings

Fernand Meyer[1] and Petros Maragos[2]

[1] Centre de Morphologie Mathématique, Ecole des Mines de Paris,
35, Rue Saint Honoré, 77305 Fontainebleau, France. Email: meyer@cmm.ensmp.fr
[2] National Technical University of Athens, Dept. of Electrical & Computer Engineering, Zografou 15773, Athens, Greece. Email: maragos@cs.ntua.gr

Abstract. A morphological scale-space representation is presented based on a morphological strong filter, the levelings. The scale-properties are analysed and illustrated. From one scale to the next, details vanish, but the contours of the remaining objects are preserved sharp and perfectly localised. This paper is followed by a companion paper on pde formulations of levelings.

1 Introduction

In many circumstances, the objects of interest which have to be measured, segmented or recognised in an image belong to a scale, and all remaining objects, to be discarded, to another scale. In some cases, however, such a threshold in the scales is not possible, and the information of interest is present at several scales: it has to be extracted from various scales. For such situations, multiscale approaches have been developed, where a series of coarser and coarser representations of the same image are derived. The recognition of the objects or segmentation will use the complete set of representations at various scales and not only the initial image.

A multiscale representation will be completely specified, if one has defined the transformations from a finer scale to a coarser scale. In order to reduce the freedom of choice, some properties of these transformations may be specified. Invariance properties are the most general:

- spatial invariance = invariance by translation
- isotropy = invariance by rotation
- invariance under a change of illumination: the transformation should commute with an increasing anamorphosis of the luminance

One may add some requirements on the effect of the transformation itself:

- The transformation should really be a simplification of the image. As such it will not be reversible: some information has to be lost from one scale to the next.

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– A particular form of simplification is expressed by the maximum principle: at any scale change, the maximal luminance at the coarser scale is always lower than the maximum intensity at the finer scale, the minimum always larger. [1]
– Causality: coarser scales can only be caused by what happened at finer scales [2]
– It should not create new structures at coarser scales ; the most frequent requirement is that it should not create new regional extrema.[3][4]

Furthermore, if the goal is image segmentation, one may require that the contours remain sharp and not displaced. Finally, one has to care for the relations between the various scales. Many scale-space representations in the literature verify a semi-group property: if g_λ is the representation at scale λ of image g, then the representation at scale μ of g_λ should be the same as the representation at scale $\lambda+\mu$ of $g : g_{\lambda+\mu} = (g_\lambda)_\mu$. We will present another structure by introducing an order relation among scales.

Since one rarely adds images, there is no particular reason, except mathematical tractability, to ask for linear transforms. If one however choses linearity, then various groups of the constraints listed above lead to the same solution: linear scale space theory. The evolution of images with the scale follows the physics of luminance diffusion: the decrease of luminance with scale is equal to the divergence of the luminance gradient [2]. The discrete operator for changing scale is a convolution by a Gaussian kernel. Its major utility is to regularize the images, permitting to compute derivatives: the spatial derivatives of the Gaussian are solutions of the diffusion equation too, and together with the zeroth order Gaussian, they form a complete family of differential operators. Besides this advantage, linear scale space cumulates the disadvantages. After convolution with a Gaussian kernerl, the images are uniformly blurred, also the regions of particular interest like the edges. Furthermore, the localisation of the structures of interest becomes extremely imprecise ; if an object is found at one scale, one has to refine its contours along all finer scales. At very large scales, the objects are not recognisable at all, for excess of blurring, but also due to the apparition of spurious extrema in 2 dimensins. Various solutions have been proposed to reduce this problem. Perona and Malik were the first to propose a diffusion inhibited by high gradient values[5]. Weickert introduced a tensor dependent diffusion [6]. Such approaches reduce the problems but do not eliminate them completely: spurious extrema may still appear.

Other non linear scale-spaces consider the evolution of curves and surfaces as a function of their geometry. Among them we find the morphological approaches producing dilations of increasing size for representing the successive scales [7]. These approaches have also the disadvantage to displace the boundaries. The first morphological scale-space approaches have been the granulometries associated to a family of openings or of closings ; openings operate only on the peaks and the closings only on the valleys [8],[9]. They obey a semi-group relation: $g_{\max(\lambda,\mu)} = (g_\lambda)_\mu$. Using morphological openings also displaces the contours, however openings and closings do not create spurious extrema. If one desires to preserve the contours, one uses openings and closings by reconstruction. If one desires a symmetric treatment of peaks and valleys, one uses alternate sequential filters, which are extremely costly in terms of computation, specially if one uses openings and closings by reconstruction [10][11].

In this paper we present a new and extremely general non linear scale-space representation with many extremely interesting features. The most interesting of them is the preservation of contours. Furthermore, no spurious extrema appear. As a matter of fact, the transformation from one scale to the next, called leveling, respects all the criteria listed abovve, except that it is not linear. From one scale to the next, the structures of the image progressively vanish, becoming flat or "quasi-flat" zone ; however, as long they are visible, they keep exactly the same localisation as in the initial image. Levelings have been introduced by F.Meyer. They have been studied by G.Matheron [12], F.Meyer [13], [14], and J.Serra [15].

In the first section, we present a characterisation and the scale-space properties of the simplest levelings. In a second section we show how to transform any function g into a leveling of a function f. We also present extensions of levelings. The analysis of the algorithm for constructing levelings leads to a PDE formulation, presented in a second paper.In a last section we illustrate the result.

2 Multiscale representation of images through levelings

2.1 Flat and quasi-flat zones.

We are working here on grey-tone functions defined on a digital grid. We call $N_G(p)$ the set of neighbors of a pixel p. The maximal (resp. minimal) value of a function g within $N_G(p)$ represents the elementary dilation δg (resp; erosion εg) of the function f at pixel p.

A path P of cardinal n between two pixels p and q on the grid G is an n-tuple of pixels $(p_1, p_2, ..., p_n)$ such that $p_1 = p$ and $p_n = q$, and for all i, (p_i, p_{i+1}) are neighbors.

We will see that simple levelings are a subclass of connected operators [16], that means they extend flat zones and do not create new contours. More general levelings will extend quasi-flat zones, defined as follows.

Definition 1. *Two pixels x, y belong to the same R-flat-zone of a function f if and only if there exists a n-tuple of pixels $(p_1, p_2, ..., p_n)$ such that $p_1 = x$ and $p_n = y$, and for all i, (p_i, p_{i+1}) are neighbours and verify the symmetrical relation: $f_{p_i}\ R\ f_{p_i+1}$.*

The simplest symmetrical relation R is equality: $f_{p_i} = f_{p_i+1}$ for which the quasi-flat zones are flat. As an example of a more complex relation R, let us define for two neighbouring pixels p and q, $f_p \approx f_q$ by $|f_p - f_q| \leq \lambda$. This relation is symmetrical and defines quasi-flat-zones with a maximal slope equal to λ.

2.2 Characterisation of levelings

We will define a non linear scale-space representation of images based on levelings. An image g will be a representation of an image f at a coarser scale, if g is a leveling of f, characterised by the following definition.

Definition 2. *An image g is a a leveling of the image f iff* $\forall (p,q)$ *neighbors:*
$g_p > g_q \quad \Rightarrow \quad f_p \geq g_p$ *and* $g_q \geq f_q$

Remark 1. If the function g is constant, no couple of neighboring pixels (p,q) may be found for which $g_p > g_q$. Hence the implication $\{g_p > g_q \Rightarrow f_p \geq g_p\}$ is always true, showing that a flat function is a leveling of any other function.

The relation $\{g$ is a leveling of $f\}$will be written $g \prec f$. The characterisation using neighboring points, defining the levelings is illustrated by fig.1b. In [14] we have shown that adopting a different order relation, giving a new meaning to $g_p > g_q$ leads to larger classes of levelings.

2.3 Properties of levelings

Algebraic properties If two functions g_1 and g_2 both are levelings of the same function f then $g_1 \vee g_2$ and $g_1 \wedge g_2$ are both levelings of f. This property permits to associate new levelings to family of levelings. In particular if (g_i) is a family of levelings of f, the morphological centre $(f \vee \bigwedge g_i) \wedge \bigvee g_i$ of this family also is a leveling of f.

Invariance properties In the introduction, we have listed a number of desirable properties of transformations on which to build a scale-space. They are obviously satisfied by levelings:

- Invariance by spatial translation
- isotropy: invariance by rotation
- invariance to a change of illumination: g being a leveling of f, if g and f are submitted to a same increasing anamorphosis, then the transformed function g' will still be a leveling of the transformed function f'.

Relation between 2 scales Levelings really will construct a scale-space, when a true simplification of the image occurs between two scales. Let us now characterize the type of simplifications implied by levelings.

In this section we always suppose that g is a leveling of f. As shown by the definition, if there is a transition for the function g between two neighboring pixels $g_p > g_q$, then there exists an even greater transition between f_p and f_q, as $f_p \geq g_p > g_q \geq f_q$. In other words to any contour of the function g corresponds a stronger contour of the function f at the very same location, and the localisation of this contour is exactly the same. This bracketing of each transition of the function g by a transition of the function f also shows that the "causality principle" is verified: coarser scales can only be caused by what happened at finer scale.

Furthermore, if we exclude the case where g is a completely flat function, then the "maximum principle" also is satisfied: at any scale change, the maximal luminance at the coarser scale is always lower than the maximum intensity at the finer scale, the minimum is always larger.

Let us now analyse what happens on the zones where the leveling g departs from the function f. Let us consider two neighboring points (p, q) for which $f_p > g_p$ and $f_q > g_q$. For such a couple of pixels, the second half of the definition: $f_p \geq g_p$ and $g_q \geq f_q$ is wrong, showing that the first half must also be wrong: $g_p \leq g_q$. By reason of symmetry we also have $g_p \geq g_q$, and hence $g_p = g_q$. This means that if g is a leveling of f, the connected components of the anti-extensivity zones $\{f > g\}$ are necessarily flat. By duality, the same holds for the extensivity zones $\{f < g\}$.

The last criterion "no new extrema at larger scales" also is satisfied as shown by the following section.

Life and death of the regional minima and maxima Levelings are a particular case of monotone planings:

Definition 3. *An image g is a a monotone planing of the image f iff $\forall\,(p, q)$ neighbors:*
$g_p > g_q \qquad \Rightarrow \qquad f_p > f_q$

Theorem 1. *A monotone planing does not create regional minima or maxima. In other words, if g is a monotone planing of f, and if g has a regional minimum (resp. maximum) X, then f possesses a regional minimum (resp. maximum) $Z \subset X$.*

Hint of the proof: If X is a regional minimum of g all its neighbors have a higher altitude. To these increasing transitions correspond increasing transitions of f. It is then easy to show that the lowest pixel for f within X belongs to a regional minimum Z for f included in X.

Relations between multiple scales: preorder relation We have now to consider the relations between multiple scales. Until now, we have presented how levelings simplify images. For speaking about scales, we need some structure among scales. This structure is a lattice structure. To be a leveling is in fact an order relation as shown by the following two lemmas.
Lemma: The relation $\{g$ is a leveling of $f\}$is symmetric and transitive: it is a preorder relations.
Lemma: The family of levelings, from which we exclude the trivial constant functions, verify the anti-symmetry relation: if f is a non constant function and a leveling of g, and simultaneously g is a leveling of f, then $f = g$.

Being an anti-symmetric preorder relation, the relation $\{g$ is a leveling of $f\}$is an order relation, except for functions which are constant everywhere. With the help of this order relation, we are now able to construct a multiscale representation of an image in the form of a series of levelings $(g_0 = f, g_1, ...g_n)$ where g_k is a leveling of g_{k-1} and as a consequence of the transitivity, g_k also is a leveling of each function g_l for $l < k$.

3 Construction of the levelings

3.1 A criterion characterizing levelings

It will be fruitful to consider the levelings as the intersection of two larger classes: the lower levelings and the upper levelings, defined as follows.

Definition 4. *A function g is a lower-leveling of a function f if and only if for any couple of neighbouring pixels (p,q): $g_p > g_q \Rightarrow g_q \geq f_q$*

Definition 5. *A function g is an upper-leveling of a function f if and only if for any couple of neighbouring pixels (p,q): $g_p > g_q \Rightarrow g_p \leq f_p$*

The name "upper-leveling" comes from the fact that all connected components where $g > f$ are flat: for any couple of neighbouring pixels (p,q): $\begin{vmatrix} g_q > f_q \\ g_p > f_p \end{vmatrix} \Rightarrow g_p = g_q.$
Similarly if g is a lower leveling of f, then all connected components where $g < f$ are flat.

Obviously, a function g is a leveling of a function f if and only if it is both an upper and a lower leveling of the function f. Let us now propose an equivalent formulation for the lower levelings:
Criterion: A function g is a lower-leveling of a function f if and only if for each pixel q with a neighbour p verifying $g_p > g_q$ the relation $g_q \geq f_q$ is satisfied.

But the pixels with this property are those for which the dilation δ will increase the value: $g_q < \delta_q g$. This leads to a new criterion
Criterion:A function g is a lower-leveling of a function f if and only if: $g_q < \delta_q g \Rightarrow g_q \geq f_q$

Recalling that the logical meaning of $[A \Rightarrow B]$ is $[notA \text{ or } B]$ we may interpret $[g_q < \delta_q g \Rightarrow g_q \geq f_q]$ as $[g_q \geq \delta_q g$ or $g_q \geq f_q]$ or in a equivalent manner $[g_q \geq f_q \wedge \delta_q g]$. This gives the following criterion
Criterion:A function g is a lower-leveling of a function f if and only if: $g \geq f \wedge \delta g$

In a similar way we derive a criterion for upper levelings:
Criterion Up:A function g is an upper-leveling of a function f if and only if: $g \leq f \vee \varepsilon g$

Putting everything together yields a criterion for levelings
Criterion A function g is a leveling of a function f if and only if: $f \wedge \delta g \leq g \leq f \vee \varepsilon g$ (see [12]).

3.2 Openings and closings by reconstruction

We recall that a function g is an opening (resp. closing) by reconstruction of a function f iff $g = f \wedge \delta g$ (resp. $g = f \vee \varepsilon g$). As it verifies the criterion Low (resp. Up), such a function g is then a lower (resp. upper) leveling of f. The reciprocal is also true. Hence:

Proposition 1. *g is an opening (resp. closing) by reconstruction of a function f if and only if g is a lower (resp. upper) leveling of f verifying $g \leq f$ (resp. $g \geq f$).*

Using this characterisation, we may particularize the initial definition of lower levelings in the case where $f \geq g$:

Proposition 2. *g is an opening by reconstruction of a function f if and only if $g \leq f$ and for any couple of neighbouring pixels (p, q): $g_p > g_q \Rightarrow g_q = f_q$.*

Proposition 3. *g is a closing by reconstruction of a function f if and only if $g \geq f$ and for any couple of neighbouring pixels (p, q): $g_p > g_q \Rightarrow g_p = f_p$.*

Remark 2. If g is a (lower) leveling of f then $g \wedge f$ is a lower leveling of f verifying $g \wedge f \leq f$, i.e. an opening by reconstruction. Similarly if g is an upper leveling of f then $g \vee f$ is a closing by reconstruction.

3.3 An algorithm for constructing levelings

We finally adopt the following general criterion of levelings
Criterion: A function g is a leveling of a function f if and only if: $f \wedge \alpha g \leq g \leq f \vee \beta g$, where α is an extensive operator $\alpha g \geq g$ and β an anti-extensive operator $\beta g \leq g$

With the help of this criterion, we may turn each function g into the leveling of a function f. We will call the function f reference function and the function g marker function. Given two functions g and f, we want to transform g into a leveling of f. If g is not a leveling of f, then the criterion $[f \wedge \alpha g \leq g \leq f \vee \beta g]$ is false for at least a pixel p. The criterion is not verified in two cases:

- $g_p < f_p \wedge \alpha_p g$. Hence the smallest modification of g_p for which the criterion becomes true is $g'_p = f_p \wedge \alpha_p g$. We remark that $g_p \leq g'_p \leq f_p$
- $g_p > f_p \vee \beta_p g$. Hence the smallest modification of g_p for which the criterion becomes true is $g'_p = f_p \vee \beta_p g$. We remark that $g_p \geq g'_p \geq f_p$

We remark that for $\{g_p = f_p\}$ the criterion is always satisfied. Hence another formulation of the algorithm:

- *lev^-*: On $\{g < f\}$ do $g = f \wedge \alpha g$.
- *lev^+*: On $\{g > f\}$ do $g = f \vee \beta g$

It is easy to check that this algorithm amounts to replace everywhere g by the new value $g = (f \wedge \alpha g) \vee \beta g = (f \vee \beta g) \wedge \alpha g$

We repeat the algorithm until the criterion is satisfied everywhere. We are sure that the algorithm will converge, since the modifications of g are pointwise monotonous: the successive values of g get closer and closer to f until convergence.

In order to optimize the speed of the algorithm, we use a unique parallel step of the algorithm $g = (f \wedge \alpha g) \vee \beta g$ After this first step both algorithms $[lev^-]$ and $[lev^+]$ have no effect on each other and may be used in any order. In particular one may use them as sequential algorithms in which the new value of any pixel is used for computing the values of their neighboring pixels. This may be done during alternating raster scans, a direct scan from top to bottom and left to right being followed by an inverse scan from bottom to top and right to left. Or hierarchical queues may be used, allowing to process the pixels in decreasing order on $\{g < f\}$ and on increasing order on $\{g > f\}$.

Let us illustrate in fig.1a how a a marker function h is transformed until it becomes a function g which is a leveling of f. This leveling uses for α the dilation δ and for β the erosion ε. On $\{h < f\}$, the leveling increases h as little as possible until a flat zone is created or the function g hits the function f: hence on $\{g < f\}$, the function g is flat. On $\{h > f\}$, the leveling decreases h as little as possible until a flat zone is created or the function g hits the function f: hence on $\{g > f\}$, the function g also is flat. For more general levelings, quasi-flat zones are created.

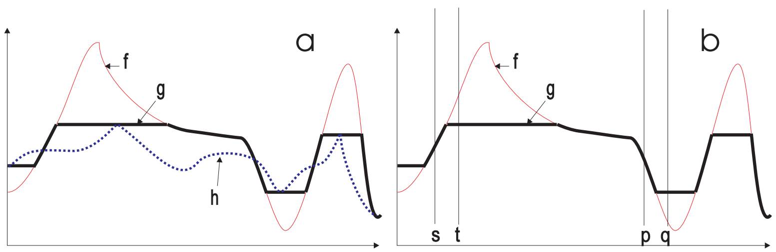

Fig. 1. a) f = reference function ; h = marker function ; g = associated leveling ; b) characterisation of levelings on the transition zones.

If g is not modified, while applying this complete algorithm to a couple of functions (f, g), then g is a leveling of f. If on the other hand g is modified, one repeats the same algorithm until convergence as explained above.

3.4 Robustness of levelings

In this section, we will see that levelings are particularly robust: they are strong morphological filters. We recall that an operator ϕ is called morphological filter if it is:

- increasing: $g > h \Rightarrow \phi g > \phi h$. This implies that $\phi(h \wedge k) < \phi h \wedge \phi k$ and $\phi(h \vee k) > \phi h \vee \phi k$
- idempotent: $\phi\phi = \phi$. This means that the operator is stable: it is sufficient to apply it once in order to get the result (for instance, the median filter, which is not a morphological filter is not stable, it may oscillate when iterated)

It is strong, if furthermore $\phi\left(Id \vee \phi\right) = \phi\left(Id \wedge \phi\right) = \phi$, where Id represents the identity operator. This property defines that functions within a given range will yield the same result, for any function h verifying $f \wedge \phi f < h < f \vee \phi f$, we have $\phi f = \phi h$.

In our case, we define an operator $\nu_g\left(f\right)$ which constructs the leveling of the marker g with reference function f. For a fixed function g and varying f, this operator is a strong morphological filter. If we call $\nu_g^-\left(f\right)$ the opening by reconstruction and $\nu_g^+\left(f\right))$ the closing by reconstruction of f based on the marker g it can be shown that : $\nu_g\left(f\right) = \nu_g^-\left(\nu_g^+\left(f\right)\right) = \nu_g^+\left(\nu_g^-\left(f\right)\right)$, an opening followed by a closing and simultaneously a closing followed by an opening, a sufficient condition for a leveling to be a strong morphological filter. We use this property for showing that yet another scale space dimension exists, based on levelings. We use here a family of leveling operators, based on a family (α_i) of extensive dilations and the family of adjunct erosions (β_i), verifying for $i > j : \alpha_i < \alpha_j$ and $\beta_i > \beta_j$. We call Λ_i the leveling built with α_i and β_i. Then using the same marker g and the same reference function f, we obtain a family of increasing levelings: for $i > j$ the leveling $\Lambda_i(f; g)$ is a leveling of $\Lambda_j(f; g)$.

4 Illustration

Levelings depend upon several parameters. First of all the type of leveling has to be chosen, this depends upon the choice of the operators α and β. Fig.2 presents three different levelings, applied to the same reference and marker image. The operators α and β used for producing them are, from the left to the right, the following:1) $\alpha = \delta$; $\beta = \varepsilon$; 2) $\alpha = Id \vee (\delta - 1)$; $\beta = Id \wedge (\varepsilon + 1)$; 3) $\alpha = Id \vee \gamma\delta$; $\beta = Id \wedge \varphi\varepsilon$, where γ and φ are respectively an opening and a closing. In Fig.3 a flat leveling based on δ and ε is applied to the same reference image (in the centre of the figure), using different markers produced by an alternate sequential filter applied to the reference image : "marker 1" using disks as structuring elements, and "marker 2" using line segments.

The last series of illustrations presents how levelings may be used in order to derive a multiscale representation of an image. We use as markers alternate sequential filters with disks: m_0 = original image ; $m_i = \varphi_i\gamma_i m_{i-1}$. The levelings are produced in the following manner: l_0 is the original image and l_i is the leveling obtained if one takes as reference the image l_{i-1} and as marker the image m_i. The resulting levelings inherit in this case the semi-group property of the markers [17]. The illustrations are disposed as follows:

m_1 *original* l_1
m_3 *original* l_3
m_5 *original* l_5

5 Conclusion

A morphological scale space representation has been presented, with all desirable features of a scale space. It has been applied with success in order to reduce the bitstream of an MPEG-4 encoder, when the simplified sequence replaces

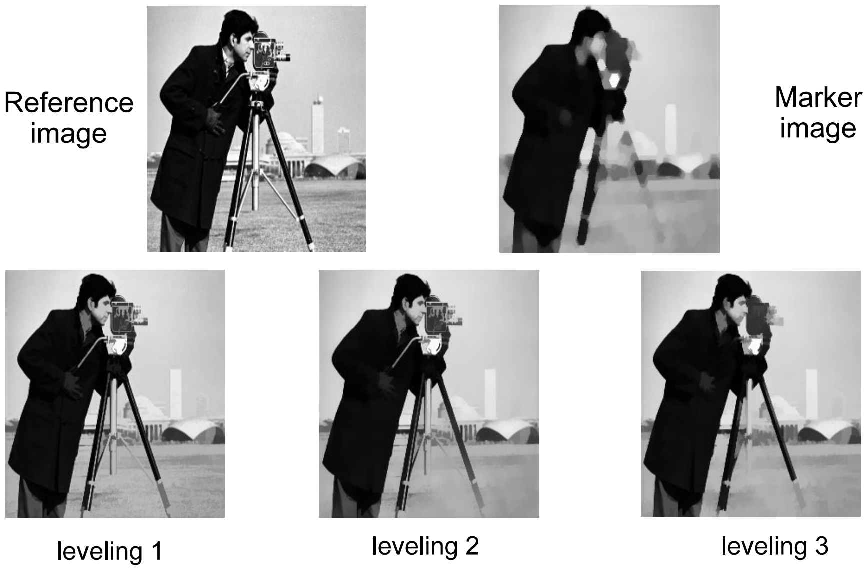

Fig. 2. Three different levelings, applied to the same reference and marker image.

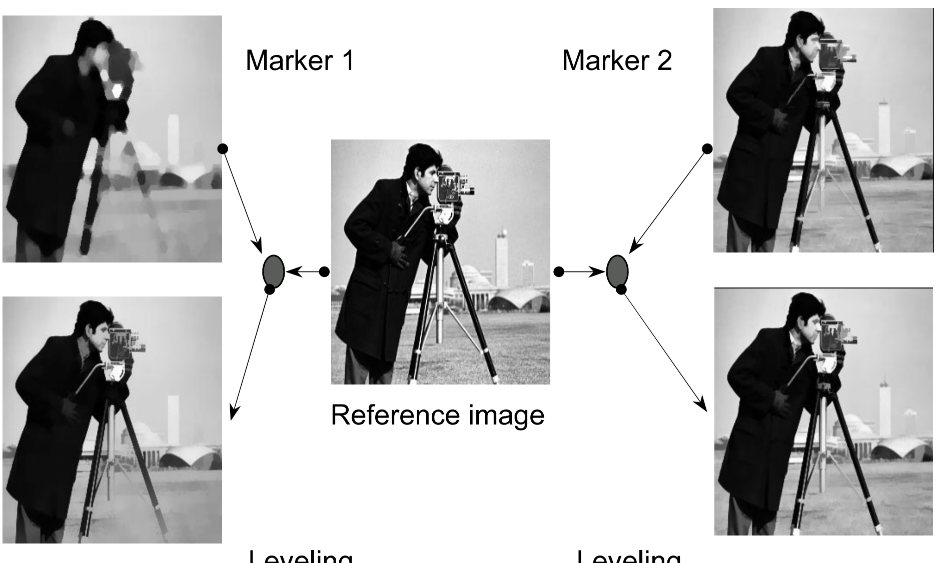

Fig. 3. A same leveling applied to the same reference image with distinct marker images

Fig. 4. Illustration of a multiscale representation

the original sequence. In this case, a sliding temporal window is processed and treated as a 3D volume, with 2 spatial dimensions and one temporal dimension: 3D markers and 3D levelings are then used. Another important application is the simplification of the images prior to segmentation. Since the levelings enlarge flat zones, these flat zones may be used as seeds for a segmentation algorithm.

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