MORREY GLOBAL BOUNDS AND QUASILINEAR RICCATI TYPE EQUATIONS BELOW THE NATURAL EXPONENT

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Abstract

We obtain global bounds in Lorentz-Morrey spaces for gradients of solutions to a class of quasilinear elliptic equations with low integrability data. The results are then applied to obtain sharp existence results in the framework of Morrey spaces for Riccati type equations with a gradient source term having growths below the natural exponent of the operator involved. A special feature of our results is that they hold under a very general assumption on the nonlinear structure, and under a mild natural restriction on the boundary of the ground domain.

Résumé

Nous dérivons des bornes globales dans les espaces de Lorentz-Morrey sur le gradient des solution d'une classe d'équations elliptiques quasi-linéaires pour des données faiblement intégrables. Ces résultats sont ensuite utilisés pour obtenir l'existence de solutions dans des espaces de Morrey à des équations de Riccati sous une hypothèse de croissance du gradient du terme source inférieure à celle de l'exposant naturel de l'opérateur. Une particularité de ces résultats est qu'ils s'appliquent sous des hypothèses très générales sur la structure de la non-linéarité, et la frontière du domaine.

Keywords: Quasilinear elliptic operator; Morrey space; Uniformly thick domain; Riccati type equation.

1. Introduction

There are two main goals that we wish to accomplish in this paper. The first goal is to obtain global regularity in Morrey and Lorentz-Morrey spaces for gradients of solutions to nonhomogeneous quasilinear equations of the form

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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The second goal is to address the solvability of the following quasilinear Riccati type equation

$$\left\{ \begin{array}{rcl} -\mathrm{div}\,\mathcal{A}(x,\nabla u) & = & |\nabla u|^q + f \quad \text{in } \Omega, \\ u & = & 0 \quad \text{on } \partial\Omega. \end{array} \right. \tag{1.2}$$

Here Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$, and for now the data f is a function in $L^1(\Omega)$ or a finite measures in Ω.

In (1.1) and (1.2) the nonlinearity $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector valued function, i.e., $\mathcal{A}(x,\xi)$ is measurable in x for every ξ and continuous in ξ for a.e. x. We assume that $\mathcal{A}$ satisfies the following growth and monotonicity conditions: for some $2 - 1/n < p \leq n$ there holds

$$|\mathcal{A}(x,\xi)| \leq \beta\, |\xi|^{p-1}\,, \tag{1.3}$$

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\eta), \xi - \eta\rangle \geq \alpha(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2 \tag{1.4}$$

for every $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$ and a.e. $x \in \mathbb{R}^n$. Here α and β are positive constants.

A typical example of such $\mathcal{A}$ is given by $\mathcal{A}(x,\xi) = |\xi|^{p-2}\xi$ which gives rise to the p-Laplacian $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. However, in general no smoothness is assumed in the x-variable of the nonlinearity $\mathcal{A}$ throughout the paper.

Most of the results in this paper are obtained under a very mild condition on the domain Ω. That is the the p-capacity uniform thickness condition (with constants $r_0, c_0 > 0$) imposed on $\mathbb{R}^n \setminus \Omega$. In this case we also say that $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick with constants $r_0, c_0 > 0$. By definition this means that there exist constants $c_0, r_0 > 0$ such that for all $0 < t \leq r_0$ and all $x \in \mathbb{R}^n \setminus \Omega$ there holds

$$\mathrm{cap}_p(\overline{B_t(x)} \cap (\mathbb{R}^n \setminus \Omega), B_{2t}(x)) \geq c_0\, \mathrm{cap}_p(\overline{B_t(x)}, B_{2t}(x)). \tag{1.5}$$

Here for a compact set $K \subset B_{2t}(x)$ we define its p-capacity by

$$\mathrm{cap}_p(K, B_{2t}(x)) = \inf\left\{ \int_{B_{2t}(x)} |\nabla\varphi|^p dy : \varphi \in C_0^\infty(B_{2t}(x)), \varphi \geq \chi_K \right\}.$$

It is easy to see that domains satisfying (1.5) include those with Lipschitz boundaries or even those that satisfy a uniform exterior corkscrew condition, where the latter means that there exist constants $c_0, r_0 > 0$ such that for all $0 < t \leq r_0$ and all $x \in \mathbb{R}^n \setminus \Omega$, there is $y \in B_t(x)$ such that $B_{t/c_0}(y) \subset \mathbb{R}^n \setminus \Omega$.

In this paper solutions u to the boundary value problems (1.1) and (1.2) are understood in the *renormalized sense*. It is well known that when the datum is not regular enough, a solution to nonlinear equations of Leray-Lions type does not necessarily belong to the natural Sobolev space $\mathrm{W}_0^{1,p}(\Omega)$. This in particular brings up a major unsettling issue regarding the uniqueness of solutions. Therefore, we find it is most convenient to work with the notion of renormalized solutions (see [18, 10, 7]). However, for global estimates involving equation (1.1) it is enough to use a milder notion of solutions (see

Remark 1.2 below). The notion of renormalized solutions will be recalled in the next section.

We now recall the definitions of Lorentz and Lorentz-Morrey spaces. For $0 < s < \infty$ and $0 < t \leq \infty$, the Lorentz space $L^{s,\,t}(\Omega)$ is the set of measurable functions g on Ω such that

$$\|g\|_{L^{s,\,t}(\Omega)} := \left[s \int_0^\infty (\alpha^s |\{x \in \Omega : |g(x)| > \alpha\}|)^{\frac{t}{s}} \frac{d\alpha}{\alpha} \right]^{\frac{1}{t}} < +\infty$$

when $t \neq \infty$; for $t = \infty$ the space $L^{s,\,\infty}(\Omega)$ is set to be the usual weak L^s or Marcinkiewicz space with quasinorm

$$\|g\|_{L^{s,\,\infty}(\Omega)} := \sup_{\alpha > 0} \alpha |\{x \in \Omega : |g(x)| > \alpha\}|^{\frac{1}{s}}.$$

It is easy to see that when $t = s$ the Lorentz space $L^{s,\,s}(\Omega)$ is nothing but the Lebesgue space $L^s(\Omega)$. On the other hand, the Lorentz-Morrey function space $\mathcal{L}^{q,\,t;\,\theta}(\Omega)$, where $0 < \theta \leq n$, $0 < q < \infty$, $0 < t \leq \infty$, is the set of measurable functions g on Ω such that

$$\|g\|_{\mathcal{L}^{q,\,t;\,\theta}(\Omega)} := \sup_{\substack{0 < r \leq \mathrm{diam}(\Omega) \\ z \in \Omega}} r^{\frac{\theta - n}{q}} \|g\|_{L^{q,\,t}(B_r(z) \cap \Omega)} < +\infty.$$

Clearly, $\mathcal{L}^{q,\,t;\,n}(\Omega) = L^{q,\,t}(\Omega)$. Moreover, when $q = t$ the space $\mathcal{L}^{q,\,t;\,\theta}(\Omega)$ becomes the usual Morrey space which will be denoted by $\mathcal{L}^{q;\,\theta}(\Omega)$.

We are now ready to state the first main result of the paper.

Theorem 1.1. *Let $2 - \frac{1}{n} < p < \theta \leq n$ and $0 < t \leq \infty$, and suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain whose complement satisfies a p-capacity uniform thickness condition with constants $c_0, r_0 > 0$. Then, under (1.3)-(1.4), there exists $\epsilon_0 = \epsilon_0(n, p, \alpha, \beta, c_0) > 0$ such that for $1 < \gamma < \frac{\theta(p+\epsilon_0)}{\theta(p-1)+p+\epsilon_0}$, and for any renormalized solution u to (1.1) with $f \in \mathcal{L}^{\gamma,\,t;\,\theta}(\Omega)$ there holds*

$$\left\| |\nabla u|^{p-1} \right\|_{\mathcal{L}^{\frac{\theta\gamma}{\theta-\gamma},\,\frac{\theta t}{\theta-\gamma};\,\theta}(\Omega)} \leq C \, \|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(\Omega)}. \tag{1.6}$$

Here the constants C depends only on $n, p, \gamma, \theta, t, c_0$ and $\mathrm{diam}(\Omega)/r_0$.

We note that the upper bound of γ and (1.6) imply that the integrability of ∇u is below $p + \epsilon_0$. This is natural as we assume no smoothness assumption on $\mathcal{A}$. The constant ϵ_0 in the above theorem is related to the celebrated higher integrability results of N. G. Meyers [50] and F. W. Gehring [32] (see Lemmas 3.1 and 3.5 below).

The restriction $p > 2 - 1/n$ is linked to the fact that, in general, solutions to $-\Delta_p u = \mu$ for a measure μ may not belong to $W^{1,1}_{\mathrm{loc}}$ when $1 < p \leq 2 - 1/n$. For this reason an important comparison estimate (see Lemmas 3.2 and 3.6 below) needed in the proof of Theorem 1.1 is not available for $1 < p \leq 2 - 1/n$. We notice that estimates in Morrey spaces are in nature different from those in Marcinkiewicz spaces as obtained, e.g., in [18], and the former could not be obtained via interpolation from the latter even in a linear situation.

We observe that inequality (1.6) can be viewed as a nonlinear version of a classical result due to D. R. Adams [2] regarding the optimal bound for Riesz potentials on Morrey spaces.

Some remarks are now in order.

Remark 1.2. From its proof one finds that Theorem 1.1 holds under the following milder notion of solutions. For each integer $k > 0$ the truncation

$$T_k(u) := \max\{-k, \min\{k, u\}\}$$

belongs to $W_0^{1,p}(\Omega)$ and satisfies

$$-\operatorname{div} \mathcal{A}(x, \nabla T_k(u)) = f_k$$

in the sense of distributions in Ω for a finite measure f_k in Ω. Moreover, if we extend both f and f_k by zero to $\mathbb{R}^n \setminus \Omega$ then f_k (resp. $|f_k|$) converges to f (resp. $|f|$) weakly as measures in $\mathbb{R}^n$. It is known that renormalized solutions satisfy these conditions (see Remark 2.4). Alternatively, one can also adopt the notion of *Solutions Obtained by Limit of Approximations* (SOLA) (see [8, 9, 19]) as having been employed, e.g., in [24, 54].

Remark 1.3. In this paper we confine ourselves to zero boundary condition which, due to the possible low regularity of u, is understood in a very weak sense, i.e., $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$. A reason for such a restriction is that we are not aware of any reasonable existence theory for p-Laplace type equations with general measure data and non-zero boundary conditions. Moreover, we observe that related gradient estimates below the natural exponent p for p-Laplace type equations with non-zero right-hand sides and boundary data having low integrability remain largely open (see [42]).

Remark 1.4. We notice that, at least in the case $2 \le p \le n$, a local version of inequality (1.6) has already been obtained by G. Mingione for the first time in [53] and the possibility of extending such local results to global ones was also mentioned in the same paper. We borrow some of the key ideas in [53], but technically our presentation instead resembles that of [56].

Remark 1.5. Under our conditions on $\mathcal{A}$ and $\partial\Omega$, the range of γ in Theorem 1.1 is sharp.

Next we address the solvability of the Riccati type equation (1.2). Equation (1.2) is a typical model for a class of quasilinear equations with an arbitrary power law growth $q > 0$ in the gradient that has been widely studied in the literature. It is now known that this equation exhibits different behaviors in the case $0 < q \le p-1$ and in the case $p-1 < q < +\infty$. As was shown in [7] (see also [14, 21, 22]), for $0 < q \le p-1$ equation (1.2) admits at least a solution as long as f is a finite measure in Ω. On the other hand, in order for (1.2) to have a solution when $q > p-1$ it is necessary to have both smallness and regularity assumptions on the datum f. It was shown in [55] (see also [40] for the case $p = 2$) that such necessary conditions on f can be quantified by the following trace inequality

$$\int_{\Omega} \varphi^{\frac{q}{q-p+1}} f dx \leq C \int_{\Omega} |\nabla \varphi|^{\frac{q}{q-p+1}} dx \tag{1.7}$$

for all $\varphi \in C_0^\infty(\Omega)$ and $\varphi \geq 0$.

In this paper we confine ourselves to the solvability of (1.2) in the subnatural range $q \in (p-1, p]$. For $q > p$ an existence result in the frame work of Morrey spaces has been obtained in [55], where it was shown that there exists a constant $C_0 > 0$ such that if $f \in \mathcal{L}^{1+\delta; \frac{q(1+\delta)}{q-p+1}}(\Omega)$ for some $\delta > 0$ with

$$\|f\|_{\mathcal{L}^{1+\delta; \frac{q(1+\delta)}{q-p+1}}(\Omega)} \leq C_0, \tag{1.8}$$

then (1.2) has a solution provided Ω is a bounded $\mathcal{C}^1$ domain. Moreover, the condition (1.8) is sharp in the sense that it is not possible to take $\delta = 0$ there as the necessary (1.7) may fail then (see [48]).

There are numerous papers in the literature concerning the solvability of (1.2) in the natural growth case $q = p$, see for example [1, 4, 5, 12, 13, 16, 28, 29, 35, 36, 43, 44, 45, 49, 57]. See also [30] for the case $q = p = 2$ that is studied up to the boundary of the ground domain Ω.

For general $q \in (p-1, p]$, various sharp criteria of solvability were obtained in [40] but only in the semilinear case $p = 2$. This case was also studied in [37] for datum $f \in L^{n(q-1)/q}(\Omega)$. For general $p \in (1, n]$, existence results for this subnatural range of q have been obtained recently in [27, 20] under the assumption that the datum f is at least in the Lebesgue space $L^{n/p}(\Omega)$ (with $p - 1 < q \leq p$), a sufficient condition that is far from being necessary.

In the present paper we are concerned with the solvability of (1.2) for $2 - 1/n < p \leq n$ and when the growth q is in the subnatural range $p - 1 < q \leq p$. In particular, we are mainly interested in the so-called supercritical case $n(p-1)/(n-1) \leq q \leq p$. In this case, we present a sharp existence result in the framework of Morrey spaces under a very general structural assumption on $\mathcal{A}$ and a mild natural restriction on the boundary of Ω.

Theorem 1.6. *Let $2 - \frac{1}{n} < p \leq n$, $\frac{n(p-1)}{n-1} \leq q \leq p$, and suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain whose complement satisfies a p-capacity uniform thickness condition with constants $c_0, r_0 > 0$. Assume (1.3)-(1.4) and that $f \in \mathcal{L}^{1+\delta; \frac{q(1+\delta)}{q-p+1}}(\Omega)$ for some $\delta > 0$. There exists a constant $c > 0$ depending on $n, p, q, \alpha, \beta, \delta, c_0$, and $\mathrm{diam}(\Omega)/r_0$ such that if*

$$\|f\|_{\mathcal{L}^{1+\delta; \frac{q(1+\delta)}{q-p+1}}(\Omega)} \leq c,$$

then there is a renormalized solution $u \in W_0^{1, q(1+\delta)}(\Omega)$ to the equation (1.2) such that $\nabla u \in \mathcal{L}^{q(1+\delta), \frac{q(1+\delta)}{q-p+1}}(\Omega)$ with

$$\|\nabla u\|^q_{\mathcal{L}^{q(1+\delta), \frac{q(1+\delta)}{q-p+1}}(\Omega)} \leq \frac{qc}{q-p+1} - \|f\|_{\mathcal{L}^{1+\delta, \frac{q(1+\delta)}{q-p+1}}(\Omega)}.$$

Remark 1.7. The condition $f \in \mathcal{L}^{1+\delta;\, \frac{q(1+\delta)}{q-p+1}}(\Omega)$ for some $\delta > 0$ is satisfied when $f \in L^{n(q-p+1)/q,\,\infty}(\Omega)$ provided $n(q-p+1)/q > 1$, i.e., $q > \frac{n(p-1)}{n-1}$. Thus this substantially improves earlier existence results obtained in [27, 20] for datum f being at least in $L^{n/p}(\Omega)$ since $n(q-p+1)/q < n/p$ holds whenever $q < p$. Moreover, in view of the necessary condition (1.7), the condition on f in Theorem 1.6 is sharp. In particular, it is not possible to take $\delta = 0$ in the above theorem (see [48]).

It is worth mentioning recent results of [38] and [3], which prove existence of solutions and a priori estimates for more general problems of this type by using different nonlinear techniques, either truncation arguments or rearrangement methods. Those results deal with data in optimal Lebesgue or Lorentz spaces. We notice that those methods, naturally applied to rearrangement invariant spaces, are not likely to apply to Morrey spaces, a fact which gives extra motivation for the result of this paper.

Finally, we discuss the subcritical case $p-1 < q < n(p-1)/(n-1)$. In this case, to obtain existence results it is enough to require the datum f to be a finite measure (plus a smallness condition). This is possible since for this range of q the necessary condition (1.7) holds for any finite measure f. Moreover, when $p \neq n$ we do not need to impose any regularity condition on $\partial\Omega$.

Theorem 1.8. *Let $2 - \frac{1}{n} < p \leq n$, $p-1 < q < \frac{n(p-1)}{n-1}$, and let Ω be a bounded domain in $\mathbb{R}^n$. In the case $p = n$ assume in addition that $\mathbb{R}^n \setminus \Omega$ satisfies an n-capacity uniform thickness condition with constants $r_0, c_0 > 0$. Let f be a finite measure in Ω. Under (1.3)-(1.4), there exists a constant $c > 0$ such that if*

$$|\Omega|^{\frac{q}{n(q-p+1)}-1}|f|(\Omega) \leq c, \tag{1.9}$$

then there is a renormalized solution u to equation (1.2) with

$$\|\nabla u\|^q_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \leq |\Omega|^{\frac{q(n-1)}{n(p-1)}-\frac{q}{n(q-p+1)}}\left[\frac{qc}{q-p+1} - |\Omega|^{\frac{q}{n(q-p+1)}-1}|f|(\Omega)\right].$$

Here c depends only on n, p, q, α, β for $p \neq n$, and also on c_0 and $\mathrm{diam}(\Omega)/r_0$ for $p = n$.

We notice that existence results in this subcritical case have been obtained recently in [38] even for $1 < p \leq 2 - 1/n$. Since our proof of Theorem 1.8 is not long we choose to present it here for the sake of completeness.

2. The notion of renormalized solutions

In this section we recall the notion of renormalized solutions. Let $\mathcal{M}_B(\Omega)$ be the set of all signed measures in Ω with bounded total variations. We denote by $\mathcal{M}_0(\Omega)$ (respectively $\mathcal{M}_s(\Omega)$) the set of all measures in $\mathcal{M}_B(\Omega)$ which are absolutely continuous (respectively singular) with respect to the capacity $\mathrm{cap}_p(\cdot,\Omega)$. Here $\mathrm{cap}_p(\cdot,\Omega)$ is the capacity relative to the domain Ω defined by

$$\mathrm{cap}_p(K,\Omega) = \inf\Big\{ \int_\Omega |\nabla\phi|^p \, dx : \phi \in C_0^\infty(\Omega), \phi \geq 1 \text{ on } K \Big\}$$

for any compact set $K \subset \Omega$. Thus every measure in $\mathcal{M}_s(\Omega)$ is supported on a Borel set E with $\mathrm{cap}_p(E,\Omega) = 0$. Recall from [31, Lemma 2.1] that, for every measure μ in $\mathcal{M}_B(\Omega)$, there exists a unique pair of measures (μ_0, μ_s) with $\mu_0 \in \mathcal{M}_0(\Omega)$ and $\mu_s \in \mathcal{M}_s(\Omega)$, such that $\mu = \mu_0 + \mu_s$.

For a measure μ in $\mathcal{M}_B(\Omega)$, its positive and negative parts are denoted by μ^+ and μ^-, respectively. We say that a sequence of measures $\{\mu_k\}$ in $\mathcal{M}_B(\Omega)$ converges in the narrow topology to $\mu \in \mathcal{M}_B(\Omega)$ if

$$\lim_{k\to\infty} \int_\Omega \varphi \, d\mu_k = \int_\Omega \varphi \, d\mu$$

for every bounded and continuous function φ on Ω.

The notion of renormalized solutions is a generalization of that of entropy solutions introduced in [6] and [10], where the measure data are assumed to be in $L^1(\Omega)$ or in $\mathcal{M}_0(\Omega)$. Several equivalent definitions of renormalized solutions were given in [18], two of which are the following ones.

Definition 2.1. Let $\mu \in \mathcal{M}_B(\Omega)$. Then u is said to be a renormalized solution of

$$\begin{cases} -\mathrm{div}\, \mathcal{A}(x,\nabla u) &= \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

if the following conditions hold:

(a) The function u is measurable and finite almost everywhere, and $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$.
(b) The gradient ∇u of u satisfies $|\nabla u|^{p-1} \in L^q(\Omega)$ for all $q < \frac{n}{n-1}$.
(c) If w belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and if there exist $w^{+\infty}$ and $w^{-\infty}$ in $W^{1,r}(\Omega) \cap L^\infty(\Omega)$, with $r > n$, such that

$$\begin{cases} w = w^{+\infty} & \text{a.e. on the set } \{u > k\}, \\ w = w^{-\infty} & \text{a.e. on the set } \{u < -k\} \end{cases}$$

for some $k > 0$ then

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla w dx = \int_\Omega w d\mu_0 + \int_\Omega w^{+\infty} d\mu_s^+ - \int_\Omega w^{-\infty} d\mu_s^-.$$

Definition 2.2. Let $\mu \in \mathcal{M}_B(\Omega)$. Then u is a renormalized solution of (2.1) if u satisfies (a) and (b) in Definition 2.1, and if the following conditions hold:

(c) For every $k > 0$ there exist two nonnegative measures in $\mathcal{M}_0(\Omega)$, λ_k^+ and λ_k^-, concentrated on the sets $\{u = k\}$ and $\{u = -k\}$, respectively, such that $\lambda_k^+ \to \mu_s^+$ and $\lambda_k^- \to \mu_s^-$ in the narrow topology of measures.

(d) For every $k > 0$

$$\int_{\{|u|<k\}} \mathcal{A}(x, Du) \cdot \nabla\varphi dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_\Omega \varphi d\lambda_k^+ - \int_\Omega \varphi d\lambda_k^- \tag{2.2}$$

for every φ in $\mathrm{W}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 2.3. By [18, Remark 2.18], if u is a renormalized solution of (2.1) then (the cap_p-quasi continuous representative of) u is finite quasieverywhere with respect to $\mathrm{cap}_p(\cdot, \Omega)$. Therefore, u is finite μ_0-almost everywhere.

Remark 2.4. By (2.2), if u is a renormalized solution of (2.1) then

$$-\mathrm{div}\mathcal{A}(x, \nabla T_k(u)) = \mu_k \quad \text{in } \Omega,$$

with

$$\mu_k = \chi_{\{|u|<k\}}\mu_0 + \lambda_k^+ - \lambda_k^-.$$

Since $T_k(u) \in W_0^{1,p}(\Omega)$, by (1.4) we see that μ_k belongs to the dual space of $W_0^{1,p}(\Omega)$. Moreover, by Remark 2.3, $|u| < \infty$ μ_0-almost everywhere and hence $\chi_{\{|u|<k\}} \to \chi_\Omega$ μ_0-almost everywhere as $k \to \infty$. Therefore, μ_k (resp. $|\mu_k|$) converges to μ (resp. $|\mu|$) in the narrow topology of measures as well.

Remark 2.5. If u is a renormalized solution to (2.1) then for $1 < p < n$ the following global gradient estimate

$$\|\nabla u\|^{p-1}_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \leq C|\mu|(\Omega)$$

holds with $C = C(n, p, \alpha, \beta)$ for any bounded domain Ω (see [18, Theorem 4.1]). For $p = n$ this estimate holds as well provided the complement of Ω satisfies an n-capacity uniform thickness condition (see [56]).

3. Comparison and decay estimates

In this section, we obtain some local interior and boundary comparison and decay estimates that are essential to our development later. First let us consider the interior ones. With $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, for each ball $B_{2R} = B_{2R}(x_0) \Subset \Omega$ we defined $w \in u + W_0^{1,p}(B_{2R})$ as the unique solution to the Dirichlet problem

$$\begin{cases} \mathrm{div}\ \mathcal{A}(x, \nabla w) = 0 & \text{in } B_{2R}, \\ w = u & \text{on } \partial B_{2R}. \end{cases} \tag{3.1}$$

Then a well-known version of Gehring's lemma applied to the function w defined above yields the following result (see [34, Theorem 6.7] and [34, Remark 6.12]).

Lemma 3.1. *With $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, let w be as in (3.1). Then there exists a constant $\theta_0 = \theta_0(n, p, \alpha, \beta) > 1$ such that for any $t \in (0, p]$ the reverse Hölder type inequality*

$$\left(\fint_{B_{\rho/2}(z)} |\nabla w|^{\theta_0 p} dx\right)^{\frac{1}{\theta_0 p}} \leq C \left(\fint_{B_\rho(z)} |\nabla w|^t dx\right)^{\frac{1}{t}}$$

holds for all balls $B_\rho(z) \subset B_{2R}(x_0)$ for a constant C depending only on n, p, α, β, t.

The following important comparison lemma involving an estimate "below the natural growth exponent" was established in [52] (see also [24, Lemma 3.3]) for the degenerate case $p \geq 2$. This lemma was later obtained in [25, Lemma 4.2] for the singular case $2 - 1/n < p < 2$.

Lemma 3.2. *With $p > 2 - 1/n$, let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution of (2.1) and let w be as in (3.1). Then there is a constant $C = C(n, p, \alpha, \beta)$ such that*

$$\begin{aligned} \fint_{B_{2R}} |\nabla u - \nabla w| dx \quad &\leq \quad C\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p-1}} \\ &\qquad + C\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]\left(\fint_{B_{2R}} |\nabla u| dx\right)^{2-p}. \end{aligned}$$

Moreover, when $p \geq 2$ the second term on the right-hand side can be dropped.

The next lemma follows from the standard interior Hölder continuity of solutions, which can be found in [34, Theorem 7.7].

Lemma 3.3. *With $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, let w be as in (3.1). Then there exists a constant $\beta_0 = \beta_0(n, p, \alpha, \beta) \in (0, 1/2]$ such that*

$$\left(\fint_{B_\rho(z)} |w - \overline{w}_{B_\rho(z)}|^p dx\right)^{\frac{1}{p}} \leq C(\rho/r)^{\beta_0} \left(\fint_{B_r(z)} |w - \overline{w}_{B_r(z)}|^p dx\right)^{\frac{1}{p}}$$

for any $z \in B_{2R}(x_0)$ with $B_\rho(z) \subset B_r(z) \subset B_{2R}(x_0)$. Moreover, there holds

$$\left(\fint_{B_\rho(z)} |\nabla w|^p dx\right)^{\frac{1}{p}} \leq C(\rho/r)^{\beta_0 - 1} \left(\fint_{B_r(z)} |\nabla w|^p dx\right)^{\frac{1}{p}} \tag{3.2}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \subset B_{2R}(x_0)$. Here $C = C(n, p, \alpha, \beta)$.

Using Lemma 3.1, inequality (3.2) can be further improved as in the following lemma. This lemma appears for the first time in [52] and has been used, e.g., in [53, 54].

Lemma 3.4. *With $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, let w be as in (3.1). Then there exists a constant $\beta_0 = \beta_0(n, p, \alpha, \beta) \in (0, 1/2]$ such that for any $t \in (0, p]$ there holds*

$$\left(\fint_{B_\rho(z)} |\nabla w|^t dx\right)^{\frac{1}{t}} \leq C(\rho/r)^{\beta_0 - 1} \left(\fint_{B_r(z)} |\nabla w|^t dx\right)^{\frac{1}{t}}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \subset B_{2R}(x_0)$. Here $C = C(n, p, t, \alpha, \beta)$.

Next we consider the counterparts of the above lemmas up to the boundary. As $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick with constants $c_0, r_0 > 0$, there exists $1 < p_0 = p_0(n,p,c_0) < p$ such that $\mathbb{R}^n \setminus \Omega$ is uniformly p_0-thick with constants $c_* = c(n,p,c_0)$ and r_0. This is by now a classical result due to J. Lewis [47] (see also [51]). Moreover, p_0 can be chosen near p so that $p_0 \in (\frac{np}{n+p}, p)$. Thus, since $p_0 < n$, we have

$$\begin{aligned} \text{cap}_{p_0}(\overline{B_t(x)} \cap (\mathbb{R}^n \setminus \Omega), B_{2t}(x)) &\geq c_* \, \text{cap}_{p_0}(\overline{B_t(x)}, B_{2t}(x)) \\ &\geq C(n,p,c_0)\, t^{n-p_0} \end{aligned} \tag{3.3}$$

for all $0 < t \leq r_0$ and all $x \in \mathbb{R}^n \setminus \Omega$.

Now let $x_0 \in \partial\Omega$ be a boundary point and for $0 < 2R \leq r_0$ we set $\Omega_{2R} = \Omega_{2R}(x_0) = B_{2R}(x_0) \cap \Omega$. For $u \in W_0^{1,p}(\Omega)$ we consider the unique solution $w \in u + W_0^{1,p}(\Omega_{2R})$ to the equation

$$\begin{cases} \text{div } \mathcal{A}(x, \nabla w) = 0 & \text{in } \Omega_{2R}, \\ w = u & \text{on } \partial\Omega_{2R}. \end{cases} \tag{3.4}$$

In what follows we extend μ and u by zero to $\mathbb{R}^n \setminus \Omega$ and then extend w by u to $\mathbb{R}^n \setminus \Omega_{2R}$.

The next two lemmas are the boundary counterparts of Lemmas 3.1 and 3.2, that have been obtained in [56]. Note that the proof Lemma 3.5 uses (3.3), whereas Lemma 3.6 holds for general domains and thus the p-capacity uniform thickness condition is not needed there.

Lemma 3.5. *With $u \in W_0^{1,p}(\Omega)$, let w be as in (3.4). Then there exists a constant $\theta_0 = \theta_0(n,p,\alpha,\beta,c_0) > 1$ such that for every $t \in (0,p]$ the reverse Hölder type inequality*

$$\left(\fint_{B_{\rho/2}(z)} |\nabla w|^{\theta_0 p} dx \right)^{\frac{1}{\theta_0 p}} \leq C \left(\fint_{B_{3\rho}(z)} |\nabla w|^t dx \right)^{\frac{1}{t}}$$

holds for all balls $B_{3\rho}(z) \subset B_{2R}(x_0)$ for a constant $C = C(n,p,t,\alpha,\beta,c_0)$.

Lemma 3.6. *With $p > 2 - 1/n$, let $u \in W_0^{1,p}(\Omega)$ be a solution of (2.1) and let w be as in (3.4). Then there is a constant $C = C(n,p,\alpha,\beta)$ such that*

$$\begin{aligned} \fint_{B_{2R}} |\nabla u - \nabla w| dx \;&\leq\; C \left[\frac{|\mu|(B_{2R})}{R^{n-1}} \right]^{\frac{1}{p-1}} \\ &\quad + C \left[\frac{|\mu|(B_{2R})}{R^{n-1}} \right] \left(\fint_{B_{2R}} |\nabla u| dx \right)^{2-p}. \end{aligned}$$

Moreover, when $p \geq 2$ the second term on the right-hand side can be dropped.

We now consider the boundary version of Lemma 3.3.

Lemma 3.7. *With $u \in W_0^{1,p}(\Omega)$, let w be as in (3.4). Then there exists a constant $\beta_0 = \beta_0(n,p,\alpha,\beta,c_0) \in (0,1/2]$ such that*

$$\left(\fint_{B_\rho(z)} |w|^p dx\right)^{\frac{1}{p}} \le C(\rho/r)^{\beta_0} \left(\fint_{B_r(z)} |w|^p dx\right)^{\frac{1}{p}} \tag{3.5}$$

for any $z \in \partial\Omega$ with $B_\rho(z) \subset B_r(z) \subset B_{2R}(x_0)$. Moreover, there holds

$$\left(\fint_{B_\rho(z)} |\nabla w|^p dx\right)^{\frac{1}{p}} \le C(\rho/r)^{\beta_0 - 1} \left(\fint_{B_r(z)} |\nabla w|^p dx\right)^{\frac{1}{p}} \tag{3.6}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \subset B_{2R}(x_0)$. Here $C = C(n,p,\alpha,\beta,c_0)$.

Proof. It is enough to consider the case $\rho \le r/20$. For a set U we set $\mathrm{osc}(w,U) = \sup_U w - \inf_U w$. Then by [41, Corollary 6.36] we can find a constant $\beta_0 = \beta_0(n,p,\alpha,\beta,c_0) \in (0,1/2]$ such that

$$\mathrm{osc}(w,\Omega_\rho(z)) \le C(\rho/r)^{\beta_0}\, \mathrm{osc}(w,\Omega_{r/4}(z)),$$

and since $w = 0$ on $\mathbb{R}^n \setminus \Omega$ this yields

$$\left(\fint_{B_\rho(z)} |w|^p dx\right)^{\frac{1}{p}} \le C(\rho/r)^{\beta_0}\, \mathrm{osc}(w,\Omega_{r/4}(z)).$$

Thus to prove (3.5) it is enough to show that

$$\mathrm{osc}(w,\Omega_{r/4}(z)) \le C \left(\fint_{B_{r/2}(z)} |w|^p dx\right)^{\frac{1}{p}}. \tag{3.7}$$

For any $y \in \partial\Omega \cap B_{r/2}(z)$, consider the balls $B_t(y) \subset B_T(y)$ with $0 < t < T \le r/2$. Let $\varphi \in C_0^\infty(B_T(y))$ be such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_t(y)$ and $|\nabla\varphi| \le c(T-t)^{-1}$. Using $\phi = \varphi^p(w-k)^+$, $k \ge 0$, as a test function for (3.4) we find

$$\int_{B_t(y)} |\nabla[(w-k)^+]|^p dx \le \frac{C}{(T-t)^p} \int_{B_T(y)} |(w-k)^+|^p dx. \tag{3.8}$$

Likewise, using $\phi = \varphi^p(w-k)^-$, $k \le 0$, as a test function for (3.4) we get

$$\int_{B_t(y)} |\nabla[(w-k)^-]|^p dx \le \frac{C}{(T-t)^p} \int_{B_T(y)} |(w-k)^-|^p dx. \tag{3.9}$$

In the language of [23, Chap. **10**, Sec. 7], these inequalities say that the function w belongs to the (homogeneous) boundary De Giorgi classes DG^+ and DG^- with zero Dirichlet data on $\partial\Omega \cap B_{r/2}(z)$. Thus following the proof of Theorem 2.1 in [23, Chap. **10**, Sec. 2] (see also [34, Remark 7.6]) we have for any $y \in \partial\Omega \cap B_{r/2}(z)$,

$$\sup_{\Omega_{r/4}(y)} w^+ \le C \left(\fint_{B_{r/2}(y)} (w^+)^p dx\right)^{\frac{1}{p}}$$

and

$$\sup_{\Omega_{r/4}(y)} w^- \leq C \left(\fint_{B_{r/2}(y)} (w^-)^p dx \right)^{\frac{1}{p}}.$$

Adding the last two inequalities with $y = z$ we arrive at (3.7).

Next, we prove (3.6) for the case $z = z_0 \in \partial\Omega$. By (3.8) and (3.9) with $k = 0$ we have

$$\int_{B_\rho(z_0)} |\nabla w|^p dx \leq \frac{C}{\rho^p} \int_{B_{2\rho}(z_0)} |w|^p dx.$$

On the other hand, by a Sobolev inequality (see, e.g., Lemma 8.11 and Remark 8.14 in [51]) there holds

$$\left(\fint_{B_r(z_0)} |w|^p dx \right)^{\frac{1}{p}} \leq C \left(\frac{1}{\mathrm{cap}_p(K, B_{2r}(z_0))} \int_{B_r(z_0)} |\nabla w|^p dx \right)^{1/p},$$

where $K = \overline{B_{r/2}(z_0)} \cap \{w = 0\}$. Thus by our condition on $\partial\Omega$ we get

$$\left(\fint_{B_r(z_0)} |w|^p dx \right)^{\frac{1}{p}} \leq C \left(r^p \fint_{B_r(z_0)} |\nabla w|^p dx \right)^{1/p}. \tag{3.10}$$

These inequalities and the relation (3.5) gives (3.6) for $z = z_0 \in \partial\Omega$.

In order to prove (3.6) for general $z \in B_{2R}(x_0)$ we reduce it to the case $z = z_0 \in \partial\Omega$ by considering the following two cases.

Case 1: $B_{r/4}(z) \subset \Omega$. Then inequality (3.6) follows from the standard interior Hölder continuity of solutions; see, e.g, [34, Theorem 7.7].

Case 2: $B_{r/4}(z) \cap \partial\Omega \neq \emptyset$. In this case we let $z_0 \in \partial\Omega \cap B_{r/4}(z)$ such that $|z - z_0| = \mathrm{dist}(z, \partial\Omega)$. Note then that $|z - z_0| \leq r/4$, and thus

$$B_{r/4}(z_0) \subset B_{r/2}(z), \qquad \text{and} \quad B_{r/2}(z_0) \subset B_{3r/4}(z). \tag{3.11}$$

Now if $\rho \geq |z - z_0|/4$ then since $B_\rho(z) \subset B_{5\rho}(z_0)$ we find

$$\begin{aligned} \left(\fint_{B_\rho(z)} |\nabla w|^p dx \right)^{\frac{1}{p}} &\leq C \left(\fint_{B_{5\rho}(z_0)} |\nabla w|^p dx \right)^{\frac{1}{p}} \\ &\leq C(\rho/r)^{\beta_0 - 1} \left(\fint_{B_{r/4}(z_0)} |\nabla w|^p dx \right)^{\frac{1}{p}} \\ &\leq C(\rho/r)^{\beta_0 - 1} \left(\fint_{B_{r/2}(z)} |\nabla w|^p dx \right)^{\frac{1}{p}}, \end{aligned}$$

where we use (3.11) and the fact that $z_0 \in \partial\Omega$. This gives (3.6) when $\rho \geq |z - z_0|/4$.

On the other hand, if $\rho < |z - z_0|/4$ then by interior Caccioppoli inequality we have

$$\int_{B_\rho(z)} |\nabla w|^p dx \leq \frac{C}{\rho^p} \int_{B_{2\rho}(z)} |w - \overline{w}_{B_{2\rho}(z)}|^p dx, \tag{3.12}$$

and also by [34, Theorem 7.7]

$$\begin{aligned}
&\fint_{B_{2\rho}(z)} |w - \overline{w}_{B_{2\rho}(z)}|^p dx \\
&\leq C \left(\frac{\rho}{|z - z_0|}\right)^{\beta_0 p} \fint_{B_{|z-z_0|/2}(z)} |w - \overline{w}_{B_{|z-z_0|/2}(z)}|^p dx \\
&\leq C \left(\frac{\rho}{|z - z_0|}\right)^{\beta_0 p} \fint_{B_{|z-z_0|/2}(z)} |w|^p dx \\
&\leq C \left(\frac{\rho}{|z - z_0|}\right)^{\beta_0 p} \fint_{B_{3|z-z_0|/2}(z_0)} |w|^p dx.
\end{aligned}$$

Thus using (3.10) and the case $z = z_0 \in \partial\Omega$ we get

$$\begin{aligned}
&\fint_{B_{2\rho}(z)} |w - \overline{w}_{B_{2\rho}(z)}|^p dx \\
&\leq C \left(\frac{\rho}{|z - z_0|}\right)^{\beta_0 p} |z - z_0|^p \fint_{B_{3|z-z_0|/2}(z_0)} |\nabla w|^p dx \\
&\leq C \left(\frac{\rho}{|z - z_0|}\right)^{\beta_0 p} |z - z_0|^p \left(\frac{|z - z_0|}{r}\right)^{(\beta_0 - 1)p} \fint_{B_{r/2}(z_0)} |\nabla w|^p dx \\
&\leq C \left(\frac{\rho}{r}\right)^{\beta_0 p} r^p \fint_{B_r(z)} |\nabla w|^p dx,
\end{aligned}$$

where the last inequality follows from (3.11). Therefore, in view of (3.12) we see that

$$\fint_{B_\rho(z)} |\nabla w|^p dx \leq C \left(\frac{\rho}{r}\right)^{(\beta_0 - 1)p} \fint_{B_r(z)} |\nabla w|^p dx.$$

This completes the proof of the lemma. □

Lemmas 3.5 and 3.7 now yield the following boundary version of Lemma 3.4.

Lemma 3.8. *With $u \in W_0^{1,p}(\Omega)$, let w be as in (3.4). Then there exists a constant $\beta_0 = \beta_0(n, p, \alpha, \beta, c_0) \in (0, 1/2]$ such that for any $t \in (0, p]$ there holds*

$$\left(\fint_{B_\rho(z)} |\nabla w|^t dx\right)^{\frac{1}{t}} \leq C(\rho/r)^{\beta_0 - 1} \left(\fint_{B_r(z)} |\nabla w|^t dx\right)^{\frac{1}{t}}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \subset B_{2R}(x_0)$. Here $C = C(n, p, t, \alpha, \beta, c_0)$.

4. Applications of comparison estimates

Our approach to Theorem 1.1 is based on following technical lemma which allows ones to work with balls instead of cubes. A version of this lemma appeared for the first time in [58]. It can be viewed as a version of the Calderón-Zygmund-Krylov-Safonov decomposition that has been used in [17, 53]. A proof of this lemma, which uses Lebesgue Differentiation Theorem and the standard Vitali covering lemma, can be found in [15] with obvious modifications to fit the setting here.

Lemma 4.1. *Assume that $A \subset \mathbb{R}^n$ is a measurable set for which there exist $c_1, r_1 > 0$ such that*

$$|B_t(x) \cap A| \geq c_1 \, |B_t(x)| \tag{4.1}$$

holds for all $x \in A$ and $0 < t \leq r_1$. Fix $0 < r \leq r_1$ and let $C \subset D \subset A$ be measurable sets for which there exists $0 < \epsilon < 1$ such that

1. *$|C| < \epsilon \, r^n |B_1|$ and*
2. *for all $x \in A$ and $\rho \in (0, r]$, if $|C \cap B_\rho(x)| \geq \epsilon \, |B_\rho(x)|$, then $B_\rho(x) \cap A \subset D$.*

Then we have the estimate

$$|C| \leq (c_1)^{-1} \epsilon \, |D|.$$

We now recall that for a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$ the Hardy-Littlewood maximal function of f is defined by

$$\mathbf{M} f(x) = \sup_{r>0} \fint_{B_r(x)} |f(y)| dy.$$

It is well known that $\mathbf{M}$ is of weak type $(1,1)$, i.e., there exists a constant $C(n) > 0$ such that

$$t |\{x \in \mathbb{R}^n : \mathbf{M} f(x) > t\}| \leq C(n) \, \|f\|_{L^1(\mathbb{R}^n)} \tag{4.2}$$

for every $t > 0$.

We will also use the first order fractional maximal function $\mathbf{M}_1$ defined for each nonnegative locally finite measure ν by

$$\mathbf{M}_1(\nu)(x) = \sup_{r>0} \frac{r \, \nu(B_r(x))}{|B_r(x)|}, \qquad x \in \mathbb{R}^n.$$

In order to apply Lemma 4.1 we need the following proposition, whose proof relies essentially on the comparison estimates obtained in the previous section.

Proposition 4.2. *There exist constants $A, \theta_0 > 1$, depending only on n, p, α, β, c_0, so that the following holds for any $T > 1$ and any $\lambda > 0$. Suppose that u is a solution of (2.1) with $\mathcal{A}$ satisfying (1.3)-(1.4). Fix a ball $B_0 = B_{R_0}$ and let $4B_0 = B_{4R_0}$. Assume that for some ball $B_\rho(y)$ with $\rho \leq \min\{r_0, 2R_0\}/16$ we have*

$$B_\rho(y) \cap B_0 \cap \{\mathbf{M}(\chi_{4B_0} |\nabla u|) \leq \lambda\} \cap \{[\mathbf{M}_1(\chi_{4B_0} |\mu|)]^{\frac{1}{p-1}} \leq \epsilon(T) \lambda\} \neq \emptyset,$$

where $\epsilon(T)$ *is defined by*

$$\epsilon(T) = \begin{cases} T^{-p\theta_0+1} & if\ 2 \leq p \leq n, \\ T^{(-p\theta_0+1)/(p-1)} & if\ 2-\frac{1}{n} < p < 2. \end{cases} \tag{4.3}$$

Then there holds

$$|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{4B_0}|\nabla u|) > AT\lambda\} \cap B_\rho(y)| < T^{-p\theta_0}\,|B_\rho(y)|. \tag{4.4}$$

Proof. By hypothesis, there exists $x_0 \in B_\rho(y) \cap B_0$ such that for any $r > 0$

$$\fint_{B_r(x_0)} \chi_{4B_0}|\nabla u|dz \leq \lambda \quad \text{and } r\fint_{B_r(x_0)} \chi_{4B_0}d|\mu| \leq [\epsilon(T)\lambda]^{p-1}. \tag{4.5}$$

Moreover, since $8\rho \leq R_0$ we have

$$B_{23\rho}(y) \subset B_{24\rho}(x_0) \subset 4B_0.$$

We first claim that for $x \in B_\rho(y)$ there holds

$$\mathbf{M}(\chi_{4B_0}|\nabla u|)(x) \leq \max\left\{\mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u|)(x), 3^n\lambda\right\}. \tag{4.6}$$

Indeed, for $r \leq \rho$ we have $B_r(x) \cap 4B_0 \subset B_{2\rho}(y) \cap 4B_0 = B_{2\rho}(y)$ and thus

$$\fint_{B_r(x)} \chi_{4B_0}|\nabla u|dz = \fint_{B_r(x)} \chi_{B_{2\rho}(y)}|\nabla u|dz,$$

whereas for $r > \rho$ we have $B_r(x) \subset B_{3r}(x_0)$ from which by (4.5) yields

$$\fint_{B_r(x)} \chi_{4B_0}|\nabla u|dz \leq 3^n \fint_{B_{3r}(x_0)} \chi_{4B_0}|\nabla u|dz \leq 3^n\lambda.$$

We now restrict A to the range $A \geq 3^n$. Then in view of (4.6) we see that to obtain (4.4) it is enough to show that

$$|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)| < T^{-p\theta_0}\,|B_\rho(y)|. \tag{4.7}$$

Moreover, since $|\nabla u| = 0$ outside Ω the later inequality trivially holds provided $B_{4\rho}(y) \subset \mathbb{R}^n \setminus \Omega$. Thus it is enough to consider (4.7) for the case $B_{4\rho}(y) \subset \Omega$ and the case $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$.

Suppose for now that $u \in W_0^{1,p}(\Omega)$. First we consider the case that $B_{4\rho}(y) \subset \Omega$. Let $w \in u + W_0^{1,p}(B_{4\rho}(y))$ be the unique solution to the Dirichlet problem

$$\begin{cases} \text{div } \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } B_{4\rho}(y), \\ w &= u \quad \text{on } \partial B_{4\rho}(y). \end{cases} \tag{4.8}$$

By the weak type $(1,1)$ estimate for the maximal function, see (4.2), we have

$$\begin{aligned} &\left|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \\ &\leq \left|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla w|) > AT\lambda/2\} \cap B_\rho(y)\right| \\ &\quad + \left|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u - \nabla w|) > AT\lambda/2\} \cap B_\rho(y)\right| \\ &\leq C(AT\lambda)^{-p\theta_0}\int_{B_{2\rho}(y)} |\nabla w|^{p\theta_0}dx + C(AT\lambda)^{-1}\int_{B_{2\rho}(y)} |\nabla u - \nabla w|dx. \end{aligned}$$

Note that by Lemma 3.1 we have

$$\begin{aligned}\left(\fint_{B_{2\rho}(y)} |\nabla w|^{p\theta_0} dx\right)^{\frac{1}{p\theta_0}} &\leq C\fint_{B_{4\rho}(y)} |\nabla w| dx \\ &\leq C\fint_{B_{4\rho}(y)} |\nabla u| dx + C\fint_{B_{4\rho}(y)} |\nabla u - \nabla w| dx\end{aligned}$$

and thus

$$\begin{aligned}(4.9)\qquad & \left|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \\ &\leq C(AT\lambda)^{-p\theta_0}|B_\rho(y)|\left(\fint_{B_{4\rho}(y)} |\nabla u| dx\right)^{p\theta_0} \\ &\quad + C(AT\lambda)^{-p\theta_0}|B_\rho(y)|\left(\fint_{B_{4\rho}(y)} |\nabla u - \nabla w| dx\right)^{p\theta_0} \\ &\quad + C(AT\lambda)^{-1}|B_\rho(y)|\fint_{B_{4\rho}(y)} |\nabla u - \nabla w| dx.\end{aligned}$$

On the other hand, by Lemma 3.2 we have

$$\begin{aligned}(4.10)\qquad \fint_{B_{4\rho}(y)} |\nabla u - \nabla w| dx &\leq C\left[\frac{|\mu|(B_{5\rho}(x_0))}{\rho^{n-1}}\right]^{\frac{1}{p-1}} \\ &\quad + C\left[\frac{|\mu|(B_{5\rho}(x_0))}{\rho^{n-1}}\right]\left(\fint_{B_{5\rho}(x_0)} |\nabla u| dx\right)^{2-p},\end{aligned}$$

where the last term should be dropped when $p \geq 2$. Thus by (4.5) and the definition of $\epsilon(T)$ we get

$$\fint_{B_{4\rho}(y)} |\nabla u - \nabla w| dx \leq CT^{-p\theta_0+1}\lambda$$

if $p \geq 2$ and

$$\fint_{B_{4\rho}(y)} |\nabla u - \nabla w| dx \leq CT^{(-p\theta_0+1)/(p-1)}\lambda + CT^{-p\theta_0+1}\lambda$$

if $2 - \frac{1}{n} < p < 2$. In any case, since $T > 1$, we have

$$(4.11)\qquad \fint_{B_{4\rho}(y)} |\nabla u - \nabla w| dx \leq CT^{-p\theta_0+1}\lambda.$$

At this point combining (4.9),(4.11) and using $T > 1$ we find

$$\begin{aligned}&\left|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \\ &\leq (CA^{-p\theta_0} + CA^{-1})T^{-p\theta_0}|B_\rho(y)|.\end{aligned}$$

We now choose A so that $A \geq 3^n$ and $2CA^{-1} \leq 1/2$, i.e., $A \geq \max\{3^n, 4C\}$. Then we have

$$\left|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \leq (1/2)T^{-p\theta_0}|B_\rho(y)|,$$

which in view of (4.6) yields (4.4).

Next, also with $u \in W_0^{1,p}(\Omega)$, we consider the case that $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$. Let $y_0 \in \partial\Omega$ be a boundary point such that $|y - y_0| = \mathrm{dist}(y, \partial\Omega)$. Define $w \in u + W_0^{1,p}(\Omega_{16\rho}(y_0))$ as the unique solution to the Dirichlet problem

$$\begin{cases} \mathrm{div}\, \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } \Omega_{16\rho}(y_0), \\ w &= u \quad \text{on } \partial\Omega_{16\rho}(y_0). \end{cases}$$

Here we also extend u by zero to $\mathbb{R}^n \setminus \Omega$ and then extend w by u to $\mathbb{R}^n \setminus \Omega_{16\rho}(y_0)$. As in (4.9) in this case we have

$$\begin{aligned} &\left|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{\Omega_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \\ &\leq C(AT\lambda)^{-p\theta_0}|B_\rho(y)| \left(\fint_{B_{12\rho}(y)} |\nabla u| dx\right)^{p\theta_0} \\ &\quad + C(AT\lambda)^{-p\theta_0}|B_\rho(y)| \left(\fint_{B_{12\rho}(y)} |\nabla u - \nabla w| dx\right)^{p\theta_0} \\ &\quad + C(AT\lambda)^{-1}|B_\rho(y)| \fint_{B_{12\rho}(y)} |\nabla u - \nabla w| dx, \end{aligned} \tag{4.12}$$

where Lemma 3.5 is used in stead of Lemma 3.1. Since

$$B_{12\rho}(y) \subset B_{16\rho}(y_0) \subset B_{20\rho}(y) \subset B_{21\rho}(x_0) \subset 4B_0$$

by Lemma 3.6, as in (4.11), we find

$$\fint_{B_{12\rho}(y)} |\nabla u - \nabla w| dx \leq CT^{-p\theta_0+1}\lambda. \tag{4.13}$$

Inequalities (4.12)-(4.13) and the fact that $T > 1$ now yield

$$\begin{aligned} &\left|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{\Omega_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \\ &\leq (CA^{-p\theta_0} + CA^{-1})T^{-p\theta_0}|B_\rho(y)|, \end{aligned}$$

and thus we arrive at

$$\left|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{\Omega_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \leq (1/2)T^{-p\theta_0}|B_\rho(y)|.$$

provided $A \geq \max\{3^n, 4C\}$. The last bound and (4.6) yield (4.4) as desired.

Finally, to remove that assumption $u \in W_0^{1,p}(\Omega)$ we argue via approximation as follows. Let $u_k = T_k(u)$ for each integer $k > 0$. Since u is a renormalized solution we see that $u_k \in W_0^{1,p}(\Omega)$ solves

$$-\,\mathrm{div}\, \mathcal{A}(x, \nabla u_k) = \mu_k \tag{4.14}$$

for a finite measure μ_k in Ω. Moreover, if we extend both μ and μ_k by zero to $\mathbb{R}^n \setminus \Omega$ then μ_k (resp. $|\mu_k|$) converges to μ (resp. $|\mu|$) weakly as measures in $\mathbb{R}^n$ (see Remark 2.4). This implies in particular that

$$\limsup_{k\to\infty} |\mu_k|(B_r(z)) \leq |\mu|(\overline{B_r(z)}) \tag{4.15}$$

for any ball $B_r(z) \subset \mathbb{R}^n$. To show (4.7) it is enough to consider the case $B_{4\rho}(y) \subset \Omega$ as the case $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$ is just similar. Now by working with (4.14) then, instead of (4.10), we have

$$\begin{aligned} \fint_{B_{4\rho}(y)} |\nabla u_k - \nabla w_k| dx &\leq C \left[\frac{|\mu_k|(B_{5\rho}(x_0))}{\rho^{n-1}}\right]^{\frac{1}{p-1}} \\ &\quad + C \left[\frac{|\mu_k|(B_{5\rho}(x_0))}{\rho^{n-1}}\right] \left(\fint_{B_{5\rho}(x_0)} |\nabla u_k| dx\right)^{2-p} \end{aligned}$$

and the last term should be dropped when $p \geq 2$. Here w_k is the solution of (4.8) with u_k in place of u. Thus using (4.5) and (4.15) we have the following analogue of (4.11)

$$\limsup_{k\to\infty} \fint_{B_{4\rho}(y)} |\nabla u_k - \nabla w_k| dx \leq C T^{-p\theta_0+1}\lambda,$$

from which we obtain, for large enough A,

$$\begin{aligned} \limsup_{k\to\infty} |\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u_k|) > AT\lambda\} \cap B_\rho(y)| \\ \leq (1/2) T^{-p\theta_0} |B_\rho(y)|. \end{aligned} \tag{4.16}$$

Then inequality (4.7) (with $2A$ in place of A) follows from (4.16) by first observing that

$$\begin{aligned} &|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u|) > 2AT\lambda\} \cap B_\rho(y)| \\ &\quad\leq |\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u_k|) > AT\lambda\} \cap B_\rho(y)| \\ &\qquad + |\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u - \nabla u_k|) > AT\lambda\} \cap B_\rho(y)| \\ &\quad\leq |\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{B_{2\rho}(y)}|\nabla u_k|) > AT\lambda\} \cap B_\rho(y)| \\ &\qquad + \frac{C(n)}{AT\lambda} \int_{\Omega\cap B_{2\rho}(y)} |\nabla u - \nabla u_k| dx, \end{aligned}$$

and then taking $\limsup_{k\to\infty}$. □

Proposition 4.2 can be restated as follows.

Proposition 4.3. *There exist constants $A, \theta_0 > 1$, depending only on n, p, α, β, c_0, so that the following holds for any $T > 1$ and any $\lambda > 0$. Let u be a solution of (2.1) with $\mathcal{A}$ satisfying (1.3)-(1.4). Fix a ball $B_0 = B_{R_0}$ and let $4B_0 = B_{4R_0}$. Suppose that for some ball $B_\rho(y)$ with $\rho \leq \min\{r_0, 2R_0\}/16$ we have*

$$|\{x \in \mathbb{R}^n : \mathbf{M}(\chi_{4B_0}|\nabla u|) > AT\lambda\} \cap B_\rho(y)| \geq T^{-p\theta_0} |B_\rho(y)|.$$

Then there holds

$$B_\rho(y)\cap B_0 \subset \{\mathbf{M}(\chi_{4B_0}|\nabla u|)>\lambda\}\cup\{[\mathbf{M}_1(\chi_{4B_0}|\mu|)]^{\frac{1}{p-1}}>\epsilon(T)\lambda\},$$

where $\epsilon(T)$ *is as defined in* (4.3).

We can now apply Lemma 4.1 and the last proposition to get the following result.

Lemma 4.4. *There exist constants* $A,\theta_0>1$*, depending only on* n*,* p*,* α*,* β*,* c_0*, so that the following holds for any* $T>1$*. Let* u *be a solution of (2.1) with* $\mathcal{A}$ *satisfying* (1.3)-(1.4)*. Let* B_0 *be a ball of radius* R_0*. Fix a real number* $0<r\le\min\{r_0,2R_0\}/16$ *and suppose that there exists* $N>0$ *such that*

$$|\{x\in\mathbb{R}^n:\mathbf{M}(\chi_{4B_0}|\nabla u|)>N\}|<T^{-p\theta_0}\,r^n|B_1|. \tag{4.17}$$

Then for any integer $k\ge 0$ *there holds*

$$\begin{aligned}&|\{x\in B_0:\mathbf{M}(\chi_{4B_0}|\nabla u|)>N(AT)^{k+1}\}|\\ &\quad\le\ c(n)T^{-p\theta_0}|\{x\in B_0:\mathbf{M}(\chi_{4B_0}|\nabla u|)>N(AT)^k\}|\\ &\qquad+c(n)|\{x\in B_0:[\mathbf{M}_1(\chi_{4B_0}|\mu|)]^{\frac{1}{p-1}}>\epsilon(T)N(AT)^k\}|,\end{aligned}$$

where $\epsilon(T)$ *is as defined in* (4.3).

Proof. Let A and $\theta_0>1$ be as in Proposition 4.3 and set

$$C=\{\mathbf{M}(\chi_{4B_0}|\nabla u|)>N(AT)^{k+1}\}\cap B_0\quad\text{and}\quad D=D_1\cap B_0,$$

where

$$D_1=\{\mathbf{M}(\chi_{4B_0}|\nabla u|)>N(AT)^k\}\cup\{[\mathbf{M}_1(\chi_{4B_0}|\mu|)]^{\frac{1}{p-1}}>\epsilon(T)N(AT)^k\}.$$

with $\epsilon(T)$ being as defined in (4.3).

Since $AT>1$ the assumption (4.17) implies that $|C|<T^{-p\theta_0}\,r^n|B_1|$. Moreover, if $x\in B_0$ and $\rho\in(0,r]$ such that $|C\cap B_\rho(x)|\ge T^{-p\theta_0}\,|B_\rho(x)|$, then using Proposition 4.3 with $\lambda=N(AT)^k$ we have

$$B_\rho(x)\cap B_0\subset D.$$

Thus the hypotheses of Lemma 4.1 are satisfied with $A=B_0$ and $\epsilon=T^{-p\theta_0}$ (note that condition (4.1) holds for all $0<t\le 2R_0$). Since $T>1$, this yields

$$\begin{aligned}|C|\ &\le\ c(n)\,T^{-p\theta_0}\,|D|\\ &\le\ c(n)\,T^{-p\theta_0}\,|\{x\in B_0:\mathbf{M}(\chi_{4B_0}|\nabla u|)>N(AT)^k\}|+\\ &\qquad+c(n)\,|\{x\in B_0:[\mathbf{M}_1(\chi_{4B_0}|\mu|)]^{\frac{1}{p-1}}>\epsilon(T)N(AT)^k\}|.\end{aligned}$$

□

5. Lorentz and Lorentz-Morrey estimates

Theorem 5.1. *Let $2-\frac{1}{n}<p\leq n$ and suppose that $\Omega\subset\mathbb{R}^n$ is a bounded domain whose complement satisfies a p-capacity uniform thickness condition with constants $c_0, r_0>0$. Then there exists $\epsilon=\epsilon(n,p,\alpha,\beta,c_0)>0$ such that for any $0<q<p+\epsilon$, and $0<t\leq\infty$ and for any solution u to (2.1) with a finite measure μ there holds*

$$\begin{aligned}(5.1)\qquad \|\nabla u\|_{L^{q,\,t}(B_0)} \;\leq\;& C\,R_0^{\frac{n}{q}}\min\{R_0, r_0/2\}^{-n}\|\nabla u\|_{L^1(4B_0)}\\ &+C\left\|[\mathbf{M}_1(\chi_{4B_0}|\mu|)]^{\frac{1}{p-1}}\right\|_{L^{q,\,t}(B_0)}.\end{aligned}$$

Here $B_0=B_{R_0}(z_0)$ is any ball with $z_0\in\Omega$ and $R_0>0$, and the constant C depends only on n,p,q,t,c_0.

Proof. Let B_0 be a ball of radius $R_0>0$ and set $r=\min\{r_0, 2R_0\}/16$. As usual we set u and μ to be zero in $\mathbb{R}^n\setminus\Omega$. In what follows we consider only the case $t\neq\infty$ as for $t=\infty$ the proof is similar. Moreover, to prove (5.1) we may assume that

$$\|\nabla u\|_{L^1(B_0)}\neq 0.$$

For $T>1$ to be determined, we claim that there exists $N>0$ such that

$$|\{x\in\mathbb{R}^n:\mathbf{M}(\chi_{4B_0}|\nabla u|)(x)>N\}|<T^{-p\theta_0}\,r^n|B_1|.$$

To see this, we first use the weak type $(1,1)$ estimate for the maximal function, see (4.2), to get

$$|\{x\in\mathbb{R}^n:\mathbf{M}(\chi_{4B_0}|\nabla u|)(x)>N\}|<\frac{C(n)}{N}\int_{4B_0}|\nabla u|dx.$$

Then we choose $N>0$ so that

$$(5.2)\qquad \frac{C(n)}{N}\int_{4B_0}|\nabla u|dx=T^{-p\theta_0}\,[\min\{r_0,2R_0\}/16]^n|B_1|.$$

Let $A,\theta_0>1$ be as in Lemma 4.4 and let $\epsilon(T)$ be as in (4.3). For $0<t<\infty$ we now consider the sum

$$S=\sum_{k=1}^{\infty}\left[(AT)^{qk}|\{x\in B_0:\mathbf{M}(\chi_{4B_0}|\nabla u|)(x)>N(AT)^k\}|\right]^{\frac{t}{q}}.$$

Note that we have

$$(5.3)\qquad C^{-1}\,S\leq\|\mathbf{M}(\chi_{4B_0}|\nabla u/N|)\|^t_{L^{q,\,t}(B_0)}\leq C\,(|B_0|^{\frac{t}{q}}+S).$$

By Lemma 4.4 we find

$$\begin{aligned} S &\leq C\sum_{k=1}^{\infty}\Big[(AT)^{qk}T^{-p\theta_0}|\{x\in B_0:\mathbf{M}(\chi_{4B_0}|\nabla u|)(x)>N(AT)^{k-1}\}|\Big]^{\frac{t}{q}}\\ &\quad+C\sum_{k=1}^{\infty}\Big[(AT)^{qk}|\{x\in B_0:[\mathbf{M}_1(\chi_{4B_0}|\mu|)]^{\frac{1}{p-1}}>\epsilon(T)N(AT)^{k-1}\}|\Big]^{\frac{t}{q}}\\ &\leq C\,[(AT)^qT^{-p\theta_0}]^{\frac{t}{q}}(S+|B_0|^{\frac{t}{q}})\\ &\quad+C_1\left\|[\mathbf{M}_1(\chi_{4B_0}|\mu|/N^{p-1})]^{\frac{1}{p-1}}\right\|^t_{L^{q,\,t}(B_0)}.\end{aligned}$$

Thus for $q<p\theta_0$, i.e., $q<p+\epsilon$ with $\epsilon=p(\theta_0-1)$, and T sufficiently large we have

$$S\leq C\left(|B_0|^{\frac{t}{q}}+\left\|[\mathbf{M}_1(\chi_{4B_0}|\mu|/N^{p-1})]^{\frac{1}{p-1}}\right\|^t_{L^{q,t}(B_0)}\right).$$

By (5.3) this yields

$$\|\nabla u/N\|_{L^{q,\,t}(B_0)}\leq C\left(|B_0|^{\frac{1}{q}}+\left\|[\mathbf{M}_1(\chi_{4B_0}|\mu|/N^{p-1})]^{\frac{1}{p-1}}\right\|_{L^{q,\,t}(B_0)}\right),$$

and thus by (5.2)

$$\begin{aligned}\|\nabla u\|_{L^{q,\,t}(B_0)} &\leq C\left(|B_0|^{\frac{1}{q}}N+\left\|[\mathbf{M}_1(\chi_{4B_0}|\mu|)]^{\frac{1}{p-1}}\right\|_{L^{q,\,t}(B_0)}\right)\\ &\leq C\,|B_0|^{\frac{1}{q}}[\min\{r_0,2R_0\}]^{-n}\|\nabla u\|_{L^1(4B_0)}\\ &\quad+C\left\|[\mathbf{M}_1(\chi_{4B_0}|\mu|)]^{\frac{1}{p-1}}\right\|_{L^{q,\,t}(B_0)}.\end{aligned}$$

This gives (5.1) as desired and completes the proof of the theorem. □

The rest of this section is devoted to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\epsilon_0=\min\{\epsilon,\frac{\theta}{1-\beta_0}-p\}$ where

$$\epsilon=\epsilon(n,p,\alpha,\beta,c_0)>0$$

is as in Theorem 5.1 and β_0 is as in Lemmas 3.4 and 3.8. Note that, as usual, $|\nabla u|$ and f are zero outside Ω.

Fix a ball $B_0=B_{R_0}(z_0)$ with $z_0\in\Omega$ and $0<R_0\leq \mathrm{diam}(\Omega)$. Then by Theorem 5.1 we have

$$\begin{aligned}\|\nabla u\|^{p-1}_{L^{\frac{\theta\gamma(p-1)}{\theta-\gamma},\,\frac{\theta t(p-1)}{\theta-\gamma}}(B_0)} &\leq C\,R_0^{\frac{n(\theta-\gamma)}{\theta\gamma}}\min\{r_0/2,R_0\}^{-n(p-1)}\|\nabla u\|^{p-1}_{L^1(4B_0)}\\ &\quad+C\left\|[\mathbf{M}_1(\chi_{4B_0}|f|)]^{\frac{1}{p-1}}\right\|^{p-1}_{L^{\frac{\theta\gamma(p-1)}{\theta-\gamma},\,\frac{\theta t(p-1)}{\theta-\gamma}}(B_0)}.\end{aligned}\tag{5.4}$$

Thus further restricting $R_0 \le \min\{r_0/8, \mathrm{diam}(\Omega)\}$ we find

$$\|\nabla u\|^{p-1}_{L^{\frac{\theta\gamma(p-1)}{\theta-\gamma},\, \frac{\theta t(p-1)}{\theta-\gamma}}(B_0)} \le C\, R_0^{\frac{n(\theta-\gamma)}{\theta\gamma}-n(p-1)} \|\nabla u\|^{p-1}_{L^1(4B_0)} + C\left\|[\mathbf{M}_1(\chi_{4B_0}|f|)]^{\frac{1}{p-1}}\right\|^{p-1}_{L^{\frac{\theta\gamma(p-1)}{\theta-\gamma},\, \frac{\theta t(p-1)}{\theta-\gamma}}(B_0)}. \tag{5.5}$$

With $\tilde{f} = \chi_{4B_0}|f|$, for any $t > 0$ and $x \in B_0$ we have

$$\begin{aligned} t^{1-n}\int_{B_t(x)} \tilde{f} dy &= \left(t^{-n}\int_{B_t(x)} \tilde{f} dy\right)^{1-\frac{\gamma}{\theta}} \left(t^{-n+\frac{\theta}{\gamma}}\int_{B_t(x)} \tilde{f} dy\right)^{\frac{\gamma}{\theta}} \\ &\le C[\mathbf{M}(\tilde{f})(x)]^{1-\frac{\gamma}{\theta}} \left(t^{\frac{\theta-n}{\gamma}} \left\|\tilde{f}\right\|_{L^{\gamma,\,\infty}(B_t(x))}\right)^{\frac{\gamma}{\theta}} \\ &\le C[\mathbf{M}(\tilde{f})(x)]^{1-\frac{\gamma}{\theta}} \left(t^{\frac{\theta-n}{\gamma}} \left\|\tilde{f}\right\|_{L^{\gamma,\,t}(B_t(x))}\right)^{\frac{\gamma}{\theta}} \\ &\le C[\mathbf{M}(\tilde{f})(x)]^{1-\frac{\gamma}{\theta}} \left(\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(4B_0)}\right)^{\frac{\gamma}{\theta}}. \end{aligned}$$

This gives

$$\mathbf{M}_1(\chi_{4B_0}|f|)(x) \le C[\mathbf{M}(\chi_{4B_0}|f|)(x)]^{1-\frac{\gamma}{\theta}} \left(\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(4B_0)}\right)^{\frac{\gamma}{\theta}},$$

and thus it can be used to estimate the second term on the right-hand side of (5.5) yielding

$$\begin{aligned} &\left\|[\mathbf{M}_1(\chi_{4B_0}|f|)]^{\frac{1}{p-1}}\right\|^{p-1}_{L^{\frac{\theta\gamma(p-1)}{\theta-\gamma},\, \frac{\theta t(p-1)}{\theta-\gamma}}(B_0)} \\ &= \|\mathbf{M}_1(\chi_{4B_0}|f|)\|_{L^{\frac{\theta\gamma}{\theta-\gamma},\, \frac{\theta t}{\theta-\gamma}}(B_0)} \\ &\le C\left\|[\mathbf{M}(\chi_{4B_0}|f|)]^{1-\frac{\gamma}{\theta}}\right\|_{L^{\frac{\theta\gamma}{\theta-\gamma},\, \frac{\theta t}{\theta-\gamma}}(B_0)} \left(\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(4B_0)}\right)^{\frac{\gamma}{\theta}} \\ &\le C\,\|[\mathbf{M}(\chi_{4B_0}|f|)]\|^{1-\frac{\gamma}{\theta}}_{L^{\gamma,\,t}(B_0)} \left(\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(4B_0)}\right)^{\frac{\gamma}{\theta}}. \end{aligned}$$

Therefore, by the boundedness property of the maximal function we find

$$\begin{aligned} &\left\|[\mathbf{M}_1(\chi_{4B_0}|f|)]^{\frac{1}{p-1}}\right\|^{p-1}_{L^{\frac{\theta\gamma(p-1)}{\theta-\gamma},\, \frac{\theta t(p-1)}{\theta-\gamma}}(B_0)} \\ &\le C\,\|f\|^{1-\frac{\gamma}{\theta}}_{L^{\gamma,\,t}(4B_0)} \left(\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(4B_0)}\right)^{\frac{\gamma}{\theta}} \\ &\le CR_0^{(n-\theta)\frac{\theta-\gamma}{\theta\gamma}} \|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(4B_0)} \\ &\le CR_0^{(n-\theta)\frac{\theta-\gamma}{\theta\gamma}} \|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(\Omega)}. \end{aligned} \tag{5.6}$$

We next aim to estimate the first term on the right-hand side of (5.5). To that end, we assume for the moment that $u \in W^{1,\,p}_0(\Omega)$. If $B_{r_0/4}(z_0) \subset \Omega$

we let $w \in u + W_0^{1,p}(B_{r_0/5}(z_0))$ be the unique solution to

$$\begin{cases} \text{div } \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } B_{r_0/5}(z_0), \\ w &= u \quad \text{on } \partial B_{r_0/5}(z_0). \end{cases}$$

Otherwise, i.e., $B_{r_0/4}(z_0) \cap \partial\Omega \neq \emptyset$, we let $w \in u + W_0^{1,p}(\Omega_{r_0/2}(x_0))$ be the unique solution to

$$\begin{cases} \text{div } \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } \Omega_{r_0/2}(x_0), \\ w &= u \quad \text{on } \partial\Omega_{r_0/2}(x_0). \end{cases}$$

Here $x_0 \in \partial\Omega \cap B_{r_0/4}(z_0)$ is chosen so that $|z_0 - x_0| = \text{dist}(z_0, \partial\Omega)$, and thus it follows that $B_{r_0/5}(z_0) \subset B_{r_0/2}(x_0)$.

By Lemmas 3.4 and 3.8 we have in any case and for any $0 < \rho \leq r_0/5$ there holds

$$\int_{B_\rho(z_0)} |\nabla w| dx \leq C(\rho/r_0)^{n+\beta_0-1} \int_{B_{r_0/5}(z_0)} |\nabla w| dx.$$

Thus it follows that

$$\begin{aligned} \int_{B_\rho(z_0)} |\nabla u| dx &\leq \int_{B_\rho(z_0)} |\nabla w| dx + \int_{B_\rho(z_0)} |\nabla u - \nabla w| dx \\ &\leq C(\rho/r_0)^{n+\beta_0-1} \int_{B_{r_0/5}(z_0)} |\nabla w| dx + C \int_{B_{r_0/5}(z_0)} |\nabla u - \nabla w| dx \\ &\leq C(\rho/r_0)^{n+\beta_0-1} \int_{B_{r_0/5}(z_0)} |\nabla u| dx + C \int_{B_{r_0/5}(z_0)} |\nabla u - \nabla w| dx. \end{aligned} \tag{5.7}$$

On the other hand, by Lemmas 3.2 and 3.6, and using Young's inequality in the case $2 - \frac{1}{n} < p < 2$ we infer that

$$\int_{B_{r_0/5}(z_0)} |\nabla u - \nabla w| dx \leq C_1\, r_0^n \left[\frac{\int_{B_{r_0/2}(z_0)} |f| dx}{r_0^{n-1}} \right]^{\frac{1}{p-1}} + \epsilon \int_{B_{r_0/2}(z_0)} |\nabla u| dx,$$

which holds for all $\epsilon > 0$, with $C_1 = C_1(n, p, \alpha, \beta, c_0, \epsilon)$.

Therefore, for some $C_1 = C_(n, p, \alpha, \beta, c_0, \epsilon)$ and $C = C(n, p, \alpha, \beta, c_0)$ we have

$$\begin{aligned} &\int_{B_{r_0/5}(z_0)} |\nabla u - \nabla w| dx \\ &\leq C_1 r_0^n \left[\frac{r_0^{n-n/\gamma} \|f\|_{L^{\gamma,\infty}(B_{r_0/2}(z_0))}}{r_0^{n-1}} \right]^{\frac{1}{p-1}} + C\epsilon \int_{B_{r_0/2}(z_0)} |\nabla u| dx \\ &\leq C_1 r_0^n \left[r_0^{1-n/\gamma} \|f\|_{L^{\gamma,t}(B_{r_0/2}(z_0))} \right]^{\frac{1}{p-1}} + C\epsilon \int_{B_{r_0/2}(z_0)} |\nabla u| dx \\ &\leq C_1 r_0^{n - \frac{\theta-\gamma}{\gamma(p-1)}} \left[\|f\|_{\mathcal{L}^{\gamma,t;\theta}(B_{r_0/2}(z_0))} \right]^{\frac{1}{p-1}} + C\epsilon \int_{B_{r_0/2}(z_0)} |\nabla u| dx, \end{aligned}$$

which in view of (5.7) yields

$$\int_{B_\rho(z_0)} |\nabla u| dx \leq C\left[(\rho/r_0)^{n+\beta_0-1}+\epsilon\right]\int_{B_{r_0/2}(z_0)}|\nabla u|dx + C_1 r_0^{n-\frac{\theta-\gamma}{\gamma(p-1)}}\left[\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(B_{r_0/2}(z_0))}\right]^{\frac{1}{p-1}}. \tag{5.8}$$

Inequality (5.8) holds with $u \in W_0^{1,p}(\Omega)$ for all $\epsilon > 0$ and $0 < \rho \leq r_0/5$. By means of approximation and using our notion of solutions, as in the proof of Proposition 4.2, it holds as well without assuming that $u \in W_0^{1,p}(\Omega)$. Moreover, inequality (5.8) holds also for $r_0/5 < \rho \leq r_0/2$ by enlarging the constant C if necessary. Thus letting

$$\phi(t) = \int_{B_t(z_0)} |\nabla u| dx, \qquad t \in (0, r_0/2],$$

we find that

$$\phi(\rho) \leq C\left[(\rho/r_0)^{n+\beta_0-1}+\epsilon\right]\phi(r_0/2) + C_1 A r_0^{n-\frac{\theta-\gamma}{\gamma(p-1)}} \tag{5.9}$$

for all $\rho \in (0, r_0/2]$, with

$$A = \left[\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(B_{r_0/2}(z_0))}\right]^{\frac{1}{p-1}}.$$

Next we observe that

$$\gamma < \frac{\theta(p+\epsilon_0)}{\theta(p-1)+p+\epsilon_0} \Longleftrightarrow \gamma\left[\frac{\theta(p-1)}{p+\epsilon_0}+1\right] < \theta,$$

and

$$\epsilon_0 \leq \frac{\theta}{1-\beta_0} - p \Longleftrightarrow (1-\beta_0)(p-1) \leq \frac{\theta(p-1)}{p+\epsilon_0}.$$

Thus we have

$$\gamma\left[(1-\beta_0)(p-1)+1\right] < \theta,$$

which is the same as

$$n - \frac{\theta-\gamma}{\gamma(p-1)} < n+\beta_0-1. \tag{5.10}$$

Also, since $p > 2 - \frac{1}{n}$, $\theta \leq n$, and $\gamma > 1$ we have

$$0 < n - \frac{\theta-\gamma}{\gamma(p-1)}. \tag{5.11}$$

With inequalities (5.9)-(5.11) in hands we can now apply Lemma 3.4 in [39] (as inequality (5.9) also holds with r in place of r_0 for any $0 < \rho \leq r/2 \leq r_0/2$) to get

$$\phi(\rho) \leq C(\rho/r_0)^{n-\frac{\theta-\gamma}{\gamma(p-1)}}\phi(r_0/2) + CA\rho^{n-\frac{\theta-\gamma}{\gamma(p-1)}}$$

for all $\rho \in (0, r_0/2]$. This gives

$$(5.12)\quad \begin{aligned} &\rho^{\frac{\theta-\gamma}{\gamma(p-1)}-n}\int_{B_\rho(z_0)}|\nabla u|dx \\ &\leq Cr_0^{\frac{\theta-\gamma}{\gamma(p-1)}-n}\int_{B_{r_0/2}(z_0)}|\nabla u|dx + C\left[\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(B_{r_0/2}(z_0))}\right]^{\frac{1}{p-1}} \\ &\leq C\,\mathrm{diam}(\Omega)^{\frac{\theta-\gamma}{\gamma(p-1)}-n}\|\nabla u\|_{L^1(\Omega)} + C\left[\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(\Omega)}\right]^{\frac{1}{p-1}}, \end{aligned}$$

where C now also depends on $\mathrm{diam}(\Omega)/r_0$.

On the other hand, from standard estimates for equations with measure data (see, e.g., [6, 18]) we have

$$(5.13)\quad \begin{aligned} \|\nabla u\|_{L^1(\Omega)} &\leq C\,|\Omega|^{1-\frac{n-1}{n(p-1)}}\,\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}} \\ &\leq C\,|\Omega|^{1-\frac{n-1}{n(p-1)}}\,|\Omega|^{(1-\frac{1}{\gamma})\frac{1}{p-1}}\,\|f\|_{L^{\gamma,\,t}(\Omega)}^{\frac{1}{p-1}} \\ &\leq C\,\mathrm{diam}(\Omega)^{n-\frac{\theta-\gamma}{\gamma(p-1)}}\,\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(\Omega)}^{\frac{1}{p-1}}, \end{aligned}$$

where we also use $p > 2-\frac{1}{n}$ in the last inequality. Therefore, taking $\rho = 4R_0 \leq r_0/2$ in (5.12) and using (5.13) we arrive at

$$R_0^{\frac{\theta-\gamma}{\gamma(p-1)}-n}\int_{B_{4R_0}(z_0)}|\nabla u|dx \leq C\left[\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(\Omega)}\right]^{\frac{1}{p-1}},$$

and thus

$$(5.14)\quad R_0^{\frac{n(\theta-\gamma)}{\theta\gamma}-n(p-1)}\|\nabla u\|_{L^1(4B_0)}^{p-1} \leq CR_0^{(n-\theta)\frac{\theta-\gamma}{\theta\gamma}}\,\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(\Omega)}.$$

At this point, using (5.6) and (5.14) in (5.5) we obtain

$$(5.15)\quad \|\nabla u\|^{p-1}_{L^{\frac{\theta\gamma(p-1)}{\theta-\gamma},\,\frac{\theta t(p-1)}{\theta-\gamma}}(B_0)} \leq CR_0^{(n-\theta)\frac{\theta-\gamma}{\theta\gamma}}\,\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(\Omega)},$$

which holds for all $R_0 \leq \min\{r_0/8, \mathrm{diam}(\Omega)\}$, with C depending on n, p, γ, θ, t, c_0, and $\mathrm{diam}(\Omega)/r_0$. Inequality (5.15) also holds for $r_0/8 < R_0 \leq \mathrm{diam}(\Omega)$. To see this, we first use (5.4) to obtain

$$\begin{aligned} &\|\nabla u\|^{p-1}_{L^{\frac{\theta\gamma(p-1)}{\theta-\gamma},\,\frac{\theta t(p-1)}{\theta-\gamma}}(B_0)} \\ &\leq C\,\mathrm{diam}(\Omega)^{\frac{n(\theta-\gamma)}{\theta\gamma}}\,r_0^{-n(p-1)}\|\nabla u\|_{L^1(4B_0)}^{p-1} \\ &\quad + C\,\|[\mathbf{M}_1(\chi_{4B_0}|f|)]\|^{p-1}_{L^{\frac{\theta\gamma(p-1)}{\theta-\gamma},\,\frac{\theta t(p-1)}{\theta-\gamma}}(B_0)} \\ &\leq C\,\mathrm{diam}(\Omega)^{\frac{n(\theta-\gamma)}{\theta\gamma}-n(p-1)}\|\nabla u\|_{L^1(4B_0)}^{p-1} \\ &\quad + C\,\|[\mathbf{M}_1(\chi_{4B_0}|f|)]\|^{p-1}_{L^{\frac{\theta\gamma(p-1)}{\theta-\gamma},\,\frac{\theta t(p-1)}{\theta-\gamma}}(B_0)}, \end{aligned}$$

where the second inequality follows since C is allowed to depend on $\mathrm{diam}(\Omega)/r_0$. Thus combining the last inequality with (5.13) and (5.6) we arrive at

$$\begin{aligned} \|\nabla u\|^{p-1}_{L^{\frac{\theta\gamma(p-1)}{\theta-\gamma},\, \frac{\theta t(p-1)}{\theta-\gamma}}(B_0)} &\leq C\,\mathrm{diam}(\Omega)^{(n-\theta)\frac{\theta-\gamma}{\theta\gamma}}\,\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(\Omega)} \\ &\quad + C R_0^{(n-\theta)\frac{\theta-\gamma}{\theta\gamma}}\,\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(\Omega)} \\ &\leq C R_0^{(n-\theta)\frac{\theta-\gamma}{\theta\gamma}}\,\|f\|_{\mathcal{L}^{\gamma,\,t;\,\theta}(\Omega)}, \end{aligned}$$

since $r_0/8 < R_0 \leq \mathrm{diam}(\Omega)$ and C allows to depend on the ratio $\mathrm{diam}(\Omega)/r_0$.

Therefore, (5.15) holds for all $0 < R_0 \leq \mathrm{diam}(\Omega)$, and this completes the proof of the theorem. □

6. Quasilinear Riccati type equations

In this section, we provide the proofs of Theorems 1.6 and 1.8. We start with the proof of Theorem 1.6.

Proof of Theorem 1.6. We may assume that $f \not\equiv 0$ for otherwise $u \equiv 0$ is a valid solution. By Theorem 1.1 there is a constant $C_0 > 0$ such that

$$\left\||\nabla u|^{p-1}\right\|_{\mathcal{L}^{\frac{q(1+\delta)}{p-1};\,\frac{q(1+\delta)}{q-p+1}}(\Omega)} \leq C_0\,\|f\|_{\mathcal{L}^{1+\delta;\,\frac{q(1+\delta)}{q-p+1}}(\Omega)}. \tag{6.1}$$

With this C_0 we let

$$c = \frac{q-p+1}{C_0 q}\left(\frac{p-1}{C_0 q}\right)^{\frac{p-1}{q-p+1}},$$

and assume that

$$\|f\|_{\mathcal{L}^{1+\delta;\,\frac{q(1+\delta)}{q-p+1}}(\Omega)} \leq c,$$

For a number $a \in (0, c]$ let $g : [0, \infty) \to \mathbb{R}$ be a function defined by

$$g(t) = (C_0\,t + C_0\,a)^{\frac{q}{p-1}} - t.$$

Then we have $g(0) > 0$ and $\lim_{t\to\infty} g(t) = \infty$. Moreover, $g'(t) = 0$ if and only if

$$t = t_0 = \frac{1}{C_0}\left(\frac{p-1}{C_0 q}\right)^{\frac{p-1}{q-p+1}} - a = \frac{qc}{q-p+1} - a > 0.$$

Thus the minimum value of g on $[0, \infty)$ is

$$\begin{aligned} g(t_0) &= \left(\frac{p-1}{C_0 q}\right)^{\frac{q}{q-p+1}} + a - \frac{1}{C_0}\left(\frac{p-1}{C_0 q}\right)^{\frac{p-1}{q-p+1}} \\ &\leq \left(\frac{p-1}{C_0 q}\right)^{\frac{q}{q-p+1}} - \frac{p-1}{C_0 q}\left(\frac{p-1}{C_0 q}\right)^{\frac{p-1}{q-p+1}} = 0. \end{aligned}$$

This shows that g has exactly one root T in the interval $(0, t_0]$. We now choose

$$a = \|f\|_{\mathcal{L}^{1+\delta;\,\frac{q(1+\delta)}{q-p+1}}(\Omega)}$$

and let

$$E = \left\{ v \in W_0^{1,1}(\Omega) : v \in W_0^{1,q(1+\delta)} \text{ with } \|\nabla v\|_{\mathcal{L}^{q(1+\delta), \frac{q(1+\delta)}{q-p+1}}(\Omega)} \le T^{\frac{1}{q}} \right\}.$$

It is easy to see from Fatou Lemma that E is closed under the strong topology of $W_0^{1,1}(\Omega)$. Moreover, since $q(1+\delta) \geq 1$ we find that E is convex.

We next consider a map $S : E \to E$ defined for each $v \in E$ by $S(v) = u$, where $u \in W_0^{1,1}(\Omega)$ is the unique renormalized solution to

$$\begin{cases} -\mathrm{div}\, \mathcal{A}(x, \nabla u) &= |\nabla v|^q + f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Note that by (6.1) we have

$$\begin{aligned} \|\nabla u\|^{p-1}_{\mathcal{L}^{q(1+\delta); \frac{q(1+\delta)}{q-p+1}}(\Omega)} &= \left\| |\nabla u|^{p-1} \right\|_{\mathcal{L}^{\frac{q(1+\delta)}{p-1}; \frac{q(1+\delta)}{q-p+1}}(\Omega)} \\ &\le C_0 \left\| |\nabla v|^q + f \right\|_{\mathcal{L}^{1+\delta; \frac{q(1+\delta)}{q-p+1}}(\Omega)} \\ &\le C_0 (T + \|f\|_{\mathcal{L}^{1+\delta; \frac{q(1+\delta)}{q-p+1}}(\Omega)}) \\ &= T^{\frac{p-1}{q}}, \end{aligned}$$

where in the last inequality we used the fact that T is a root of g. Also, since $u \in W_0^{1,1}(\Omega)$ and $\nabla u \in L^{q(1+\delta)}(\Omega)$ we have $u \in W^{1,q(1+\delta)}(\Omega)$. Now if $1 \le q(1+\delta) \le p$ then it is easy to see that $u \in W_0^{1,q(1+\delta)}(\Omega)$ using the fact that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. On the other hand, if $q(1+\delta) > p$ then by the condition on Ω there holds the pointwise Hardy's inequality

$$|u(x)| \le c\, \mathrm{dist}(x, \partial\Omega) (\mathbf{M}(|Du|^p)(x))^{1/p} \qquad \text{a.e.}$$

(see [46]). Thus using the boundedness of $\mathbf{M}$ on $L^{\frac{q(1+\delta)}{p}}$ we see that

$$u(\cdot)/\mathrm{dist}(\cdot, \partial\Omega) \in L^{q(1+\delta)}(\Omega),$$

which yields that $u \in W_0^{1,q(1+\delta)}(\Omega)$ (see [26, p. 223] and [46]). We can now conclude that $u = S(v) \in E$.

Therefore, if we can show that the map $S : E \to E$ is continuous and $S(E)$ is precompact under the strong topology of $W_0^{1,1}(\Omega)$ then by Schauder Fixed Point Theorem (see, e.g., [33, Corollary 11.2]) S has a fixed point in E. This gives a solution u to problem (1.2) as desired. We will achieve the continuity and compactness of S in the next lemma, and thus completes the proof of the theorem. □

Lemma 6.1. *The map $S : E \to E$ is continuous and $S(E)$ is precompact under the strong topology of $W_0^{1,1}(\Omega)$.*

Proof. We first show the continuity of S. Let $\{v_j\}$ be a sequence in E such that v_j converges strongly in $W_0^{1,1}(\Omega)$ to a function $v \in E$. We need to show that $S(v_j) \to S(v)$ strongly in $W_0^{1,1}(\Omega)$. With $u_j = S(v_j)$, we have

$$\text{(6.2)} \qquad \begin{cases} -\mathrm{div}\,\mathcal{A}(x,\nabla u_j) &=& |\nabla v_j|^q + f \quad \text{in } \Omega, \\ u_j &=& 0 \quad \text{on } \partial\Omega, \end{cases}$$

with

$$\text{(6.3)} \qquad \|\nabla v_j\|_{L^{q(1+\delta)}(\Omega)} \leq C.$$

Let $\{v_{j'}\}$ be a subsequence of $\{v_j\}$ such that $\nabla v_{j'} \to \nabla v$ almost everywhere. Then by (6.3) and Vitali Convergence Theorem we have $\nabla v_{j'} \to \nabla v$ strongly in $L^q(\Omega)$. As the limit is independent of the subsequence, we see that $\nabla v_j \to \nabla v$ strongly in $L^q(\Omega)$.

Therefore, by the stability result of renormalized solutions [18, Theorem 3.4] there exists a subsequence $\{u_{j'}\}$ and a function $u \in W_0^{1,1}(\Omega)$ such that $u_{j'} \to u$ a.e. in Ω, where u is the unique renormalized solution to

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x,\nabla u) &=& |\nabla v|^q + f \quad \text{in } \Omega, \\ u &=& 0 \quad \text{on } \partial\Omega. \end{cases}$$

Note that by the proof of Theorem 3.4 in [18] (see also [9, 11]) we also have

$$\text{(6.4)} \qquad \nabla u_{j'} \to \nabla u \qquad \text{a.e. in } \Omega.$$

Thus $u = S(v)$ and since

$$\text{(6.5)} \qquad \|\nabla u_j\|_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \leq C,$$

with $\frac{n(p-1)}{n-1} > 1$, by Vitali Convergence Theorem $u_{j'}$ (and hence u_j) converges strongly to u in $W_0^{1,1}(\Omega)$, which gives the continuity of S.

Similarly, we can show that the set $S(E)$ is precompact under the strong topology of $W_0^{1,1}(\Omega)$. Indeed, if $\{u_j\} = \{S(v_j)\}$ is a sequence in $S(E)$ where $\{v_j\} \subset E$, then we have (6.2), (6.3), and (6.5). By (6.3) and again arguing as in the proof of [18, Theorem 3.4] one can find a subsequence $\{u_{j'}\}$ and a function $u \in W_0^{1,1}(\Omega)$ such that the pointwise a.e. convergence (6.4) holds. Thus (6.5) and Vitali Convergence Theorem yield that $u_{j'}$ strongly converges to u in $W_0^{1,1}(\Omega)$.

This completes the proof of Lemma 6.1. □

Finally, we prove Theorem 1.8.

Proof of Theorem 1.8. Recall that for each $g \in L^{s,\infty}(\Omega)$, $s > 1$, if we set

$$|||g|||_{L^{s,\infty}(\Omega)} := \sup\{|E|^{\frac{1-s}{s}} \int_E |g| dx : E \subset \Omega, |E| > 0\},$$

then $|||\cdot|||$ is a norm on $L^{s,\infty}(\Omega)$ with

$$\|g\|_{L^{s,\infty}(\Omega)} \leq |||g|||_{L^{s,\infty}(\Omega)} \leq \frac{s}{s-1} \|g\|_{L^{s,\infty}(\Omega)}.$$

Thus letting

$$E = \left\{ v \in W_0^{1,1}(\Omega) : |||\nabla v|||_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \leq T^{\frac{1}{q}} \right\},$$

with $T > 0$ to be chosen, we see that E is a closed and convex set of $W_0^{1,1}(\Omega)$.

Let f now be a finite measure in Ω. We observe that it is enough to prove the theorem for $f \in \mathcal{M}_0(\Omega)$. The general result follows by approximation and the stability result of [18]. Thus for each $v \in E$ there is a unique renormalized solution $u \in W_0^{1,1}(\Omega)$ to

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x, \nabla u) &= |\nabla v|^q + f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

We then let $S : E \to E$ be defined by $S(v) = u$. By Remark (2.5) we have

$$\begin{aligned} |||\nabla u|||^{p-1}_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} &\leq C(\||\nabla v|^q\|_{L^1(\Omega)} + |f|(\Omega)) \\ &\leq A\left[|\Omega|^{1-\frac{q(n-1)}{n(p-1)}} |||\nabla v|||^q_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} + |f|(\Omega)\right] \\ &\leq A\left[|\Omega|^{1-\frac{q(n-1)}{n(p-1)}} T + |f|(\Omega)\right]. \end{aligned}$$

Thus if T is the smallest positive root of the equation

$$\left[C_1\, t + C_1\, |\Omega|^{\frac{q(n-1)}{n(p-1)}-1} |f|(\Omega)\right]^{\frac{q}{p-1}} - t = 0,$$

where $C_1 = A|\Omega|^{1-\frac{q(n-1)}{n(p-1)}}$ then we find

$$|||\nabla u|||_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \leq T^{\frac{1}{q}}.$$

This justifies that $S(v) \in E$. It is easy to see that the existence of T is guaranteed if (1.9) holds for some appropriate constant c, and in that case one has

$$T \leq |\Omega|^{\frac{q(n-1)}{n(p-1)} - \frac{q}{n(q-p+1)}} \left[\frac{qc}{q-p+1} - |\Omega|^{\frac{q}{n(q-p+1)}-1} |f|(\Omega)\right].$$

Now arguing as in the proof of Lemma 6.1 we see that S is continuous and precompact under the strong topology of $W_0^{1,1}(\Omega)$. Thus it has a fixed point in E and the proof is complete. □

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