MORSE ESTIMATES FOR TRANSLATED POINTS ON UNIT TANGENT BUNDLES

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ABSTRACT. In this article, we study conjectures of Sandon on the minimal number of translated points in the special case of the unit tangent bundle of a Riemannian manifold. We restrict ourselves to contactomorphisms of SM that lift diffeomorphisms of M homotopic to identity. We prove that there exist sequences (p_n, t_n) where p_n is a translated point of time-shift t_n with $t_n \to +\infty$ for a large class of manifolds. We also prove Morse estimates on the number of translated points in the case of Zoll Riemannian manifolds.

1. INTRODUCTION

In this article, we study conjectures essentially due to Sandon on the minimal number of translated points in the special case of the unit tangent bundle of a Riemannian manifold. Let us recall the definition of translated points. Let (V^{2n-1}, α) be an oriented contact manifold with a fixed contact form α (*i.e.* $\alpha \wedge (d\alpha)^{n-1}$ does not vanish). A contactomorphism $\varphi \in \mathrm{Cont}(V, \alpha)$ is a diffeomorphism of V such that $\varphi^* \alpha = e^g \alpha$ for some $g : V \to \mathbb{R}$. A point $p \in V$ is a discriminant point of φ if and only if it is fixed by φ and $g(p) = 0$ (this definition does not depend on the choice of the contact form associated with $\ker \alpha$). Let (ϕ_t^α) be the Reeb flow of α. A point $p \in V$ is a translated point of φ if and only if it is a discriminant point of $\phi_{-t}^\alpha \circ \varphi$ for some $t \in \mathbb{R}$ called a time-shift of p. Similarly to the Hamiltonian case, using a Weinstein neighborhood of the graph of the identity, one can prove that for every contactomorphism of a closed contact manifold $\varphi \in \mathrm{Cont}(V, \alpha)$ which is C^1-close to the identity,

$$\#\{p \in V \mid p \text{ is a translated point of } \varphi\} \geq \min_{f \in C^\infty(V, \mathbb{R})} \# \mathrm{Crit}(f), \qquad (1)$$

where $\mathrm{Crit}(f)$ denotes the set of critical points of f [24]. Moreover, if the Reeb flow is periodic, this inequality is sharp (see *e.g.* the introduction of [3]). In [24], Sandon proved that this inequality still holds without the "C^1-close" assumption, as long as φ is isotopic to the identity in the case of the real projective spaces $\mathbb{RP}^{2n-1} = \mathbb{S}^{2n-1}/(z \sim -z)$ endowed with the contact form induced by $\alpha_0 = \frac{1}{2}(x \mathrm{d}y - y \mathrm{d}x) \in \Omega^1(\mathbb{S}^{2n-1})$. Therefore, Sandon asked the following question, where $\mathrm{Cont}_0(V, \alpha)$ denotes the set of contactomorphisms isotopic to the identity.

Question 1. *Given a closed contact manifold endowed with a contact form (V, α), does every $\varphi \in \mathrm{Cont}_0(V, \alpha)$ satisfy (1)?*

Similarly to the Arnol'd conjecture, one can ask weaker estimates on the number of translated points, using the cup-length estimate or the category estimate of the Lusternik-Schnirelmann theory. Another variation of Question 1 can be asked for the non-degenerate contactomorphisms (see below), replacing the estimate on the minimal number of critical points of any $f : M \to \mathbb{R}$ by an estimate on the minimal number of critical points a Morse map can have. As mentioned earlier, Sandon proved that Question 1 is true for $(\mathbb{R}P^{2n-1}, \alpha_0)$ [24]. We proved that this is also the case for every quotient $L_k^{2n-1}(w)$ of $(\mathbb{S}^{2n-1}, \alpha_0)$ by free $\mathbb{Z}/k\mathbb{Z}$-actions of the form $(z_j) \mapsto (e^{2i\pi w_j/k} z_j)$ (with $k \geq 2$) in [3], improving an estimate of Granja-Karshon-Pabiniak-Sandon [12]. The question was also positively answered by the work of Albers-Fuchs-Merry [2] completed by Meiwes-Naef [19] in the case of hypertight contact manifolds (*i.e.* such that all Reeb orbits are non-contractible for some contact form supporting the contact structure) for generic contactomorphisms with a Morse estimate.

This first question seems better suited for contact manifolds all of whose Reeb orbits are closed: the so-called Besse contact manifolds. A result of Granja-Karshon-Pabiniak-Sandon suggests a second question that could cover a greater class of contact manifolds.

Question 2. *Given a closed contact manifold endowed with a contact form (V, α), does every $\varphi \in \mathrm{Cont}_0(V, \alpha)$ possess a sequence of couples $(p_n, t_n) \in V \times \mathbb{R}$ with $t_n \to +\infty$ such that p_n is a translated point of φ with time-shift t_n?*

Let us remark that the Weinstein conjecture is satisfied by the (V, α)'s satisfying the conclusion of Question 2 (by taking $\varphi = \phi_1^\alpha$). Let us further remark that a Besse contact manifold answering Question 1 positively also answers Question 2 positively. Granja-Karshon-Pabiniak-Sandon answered positively Question 2 for $(L_k^{2n-1}(w), \alpha)$, α being any contact form supporting the same contact structure as the quotient of α_0.

In this paper, we want to give motivations that Question 1 should be answered positively for unit tangent bundles of Riemannian manifolds, all of whose geodesics are closed and of same prime length whereas Question 2 should be answered positively for every unit tangent bundle. In order to do so, we will prove that this is indeed the case in a weak sense for a subclass of contactomorphisms of SM: the contactomorphic lift of the diffeomorphisms of M.

Let M be a Riemannian manifold, its unit tangent bundle SM is a contact manifold for the contact form α:

$$\alpha_{(x,v)} \cdot \xi = \langle v, \mathrm{d}\pi_x \cdot \xi \rangle, \quad \forall (x,v) \in SM, \forall \xi \in T_{(x,v)} SM,$$

where $\pi : SM \to M$ is the bundle map and $\langle \cdot, \cdot \rangle$ is the Riemannian metric. Given a diffeomorphism $f : M \to M$, we denote $\tilde{f} : SM \to SM$ the associated contactomorphism

$$\tilde{f}(x, v) := \left(f(x), \frac{\mathrm{d}f_x^{-T} \cdot v}{\|\mathrm{d}f_x^{-T} \cdot v\|} \right), \quad \forall (x, v) \in SM,$$

where $\mathrm{d}f^{-T}$ denotes the inverse of the adjoint $\mathrm{d}f^T$. We will study the minimal number of translated points of $\tilde{f}$ for f homotopic to the identity. The notion of translated points of $\tilde{f}$ with time-shift t can be naturally generalized to smooth maps $f : M \to M$ (and not only diffeomorphisms) in such a way that, in particular, a

translated point of a diffeomorphism $f : M \to M$ with time-shift t is exactly a translated point of the contactomorphism $\tilde{f}$ with time-shift t.

Definition 1.1. A point $(x, v) \in SM$ is a translated point of $f : M \to M$ with time-shift $t \neq 0$ if their exists a geodesic $\gamma : [0, 1] \to \mathbb{R}$ of length $|t|$ such that

$$\begin{cases} \dot{\gamma}(0) = v\|\dot{\gamma}(0)\| \text{ and } \mathrm{d}f_x^T \cdot \dot{\gamma}(1) = \dot{\gamma}(0) & \text{when } t > 0, \\ \dot{\gamma}(1) = v\|\dot{\gamma}(1)\| \text{ and } \mathrm{d}f_x^T \cdot \dot{\gamma}(0) = \dot{\gamma}(1) & \text{when } t < 0; \end{cases}$$

it is a translated point with time-shift $t = 0$ if $\mathrm{d}f_x^T \cdot v = v$ (in particular, $f(x) = x$).

The correspondence between translated points of $\tilde{f}$ and translated points of f is due to the fact that the Reeb flow of SM is the geodesic flow (see *e.g.* [22, §1.3.3]). We give the following partial answer to Question 2.

Theorem 1.2. *Let M be a closed Riemannian manifold that has a finite cover $\widetilde{M}$, the singular homology group $H_*(\Lambda\widetilde{M})$ of the free loop space of which is not finitely generated. Every smooth map $f : M \to M$ homotopic to the identity admits a sequence of couples $(p_n, t_n) \in SM \times (0, +\infty)$ with $t_n \to +\infty$ such that p_n is a translated point of f with time-shift t_n.*

The assumption on M that $H_*(\Lambda\widetilde{M})$ is not finitely generated is satisfied by a large class of closed Riemannian manifolds (in fact, the author does not know if there are counter-examples). On the one hand, when the number of conjugacy classes of $\pi_1(M)$ is infinite, the group $H_0(\Lambda M)$ is already not finitely generated. On the other hand, when $\pi_1(M)$ is finite, Vigué-Poirrier and Sullivan proved that $H_*(\Lambda\widetilde{M})$ is not finitely generated for the universal cover $\widetilde{M}$ [25]. Therefore, the only possible counter-examples are among the closed manifolds that have an infinite fundamental group with a finite number of conjugacy classes.

Our most satisfying answer to Question 1 concerns the non-degenerate case. Let us first discuss the notion of non-degenerate translated points. Let (G_t) be the geodesic flow of TM and, for a diffeomorphism $f : M \to M$, let $\hat{f} : TM \to TM$ be its symplectic lift $\hat{f}(x, v) = (f(x), \mathrm{d}f^{-T} \cdot v)$. In our particular case, a translated point $p \in SM$ of a contactomorphic lift $\tilde{f}$ of a diffeomorphism $f : M \to M$ is non-degenerate for its time-shift $t \in \mathbb{R}$ if and only if $\mathrm{d}(G_{-t} \circ \hat{f})(p)$ does not have 1 as an eigenvalue. One can check that this definition coincides with the definition given by Sandon in the general setting [24]. Similarly to the definition of translated points, one can extend this notion to any smooth map $M \to M$ (for simplicity, we only give the definition for positive time-shifts).

Definition 1.3. A translated point (x, v) of $f : M \to M$ is non-degenerate for the time-shift $t > 0$ if the subspace of the Jacobi fields J along the associated geodesic γ satisfying

$$\begin{cases} J(1) = \mathrm{d}f \cdot J(0), \\ (\mathrm{d}^2 f \cdot J(0))^T \cdot \dot{\gamma}(1) = \dot{J}(0) - \mathrm{d}f_x^T \cdot \dot{J}(1), \end{cases}$$

is reduced to 0 (the linear morphism $\mathrm{d}^2 f \cdot J(0)$ denotes $u \mapsto \mathrm{d}^2 f[J(0), u]$, see Section 2.4).

The equivalence of both definitions in the case of diffeomorphisms is proven in Proposition 2.8. We give the following partial answer to Question 1 in the non-degenerate case. Let us recall that a Riemannian manifold M is called Zoll (of length ℓ) if all its geodesics are closed and of the same prime length (equal to ℓ).

Theorem 1.4. *Let M be a closed Zoll Riemannian manifold and let $R := \mathbb{Z}$ if M is orientable and $R := \mathbb{Z}/2\mathbb{Z}$ otherwise. For every smooth map $f : M \to M$ homotopic to the identity with finitely many translated points in SM all of which are non-degenerate, the number of translated points is not less than $\sum_j \beta_j(SM; R)$, where $\beta_j(SM; R) = \operatorname{rank} H_j(SM; R)$ denotes the j-th Betti number of SM.*

In the degenerate case, our result is less satisfying as it requires a C^0-closeness assumption. Let us recall that the cup-length $CL(X; R) \in \mathbb{N}$ of a space X is the maximal k such that $u_1 \smile \cdots \smile u_k \neq 0$ for some non-zero $u_j \in H^*(X; R)$ of positive degree. By a H^1-homotopy (f_s) of maps $M \to M$, we will mean that $t \mapsto f_t(x)$ is in the Sobolev space $H^1([0, 1], M)$ for all $x \in M$. The set of time-shifts of $f : M \to M$ will denote the set of $t \in \mathbb{R}$ that are time-shifts of a translated point; when M is Zoll of length ℓ, this set is $\ell\mathbb{Z}$-invariant so it can be seen as a subset of $\mathbb{R}/\ell\mathbb{Z}$.

Theorem 1.5. *Let M be a closed Zoll Riemannian manifold of length ℓ and let $R := \mathbb{Z}$ if M is orientable and $R := \mathbb{Z}/2\mathbb{Z}$ otherwise. Let (f_s) be a H^1-homotopy of maps $M \to M$ such that $f_0 = \mathrm{id}$ and*

$$\int_0^1 \|\partial_t f_t(x)\|^2 \mathrm{d}t < \left(\frac{\ell}{2}\right)^2, \quad \forall x \in M.$$

If the number of time-shifts of f_1 seen in $\mathbb{R}/\ell\mathbb{Z}$ is less than $1 + CL(SM; R)$, then f_1 has infinitely many translated points. In particular, the number of translated points of f_1 is not less than $1 + CL(SM; R)$.

The only closed Zoll Riemannian manifolds known by the author are diffeomorphic to the compact rank-one symmetric spaces: $\mathbb{S}^n$, $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ and $\mathbb{C}a\mathrm{P}^2$ (see however [5] for examples of manifolds with exotic structures all of whose geodesics starting from a special point go back to this point at the same length). According to a result of Bott and Samelson, every Zoll Riemannian manifold has a cohomology ring isomorphic to the cohomology ring of one of these spaces [7, 23]. Let us refer to [6] for a comprehensive introduction to the theory of Zoll and Besse Riemannian manifolds. Let us also point out that the study of Besse Riemannian manifolds and more generally of Besse contact forms from the variational viewpoint has recently known significant advances [1, 10, 18].

The proofs of this article are based on a rather classical variational principle that goes back to Grove [13]. Indeed, we remark in Section 2.1 that geodesics corresponding to translated points of positive time-shift are exactly the critical points of the restriction of the energy functional to paths $\gamma : [0, 1] \to M$ such that $f(\gamma(0)) = \gamma(1)$. This variational principle was initially applied in the specific case where f is an isometry in order to study isometry-invariant geodesics [13, 14, 16, 17, 4, 15].

Theorem 1.2 is a direct application of Morse theory. Theorems 1.4 and 1.5 are more subtle consequences of Morse and Lusternik-Schnirelmann theories. In order to translate the symmetry $(p, t) \mapsto (p, t + \ell)$ of the set of couples (translated point, associated time-shift), we make use of the Chas-Sullivan product [9]. More precisely, we apply the results of Goresky-Hingston concerning the product-structure of the homology groups of the free loop space of Zoll Riemannian manifolds [11].

Organization of the paper. In Section 2, we study the variational principle satisfied by translated points of positive time-shifts and prove Theorem 1.2. In Section 3, we study the special case of Zoll Riemannian manifolds and prove Theorems 1.4 and 1.5.

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2. The variational principle

2.1. **The variational principle.** Let M be a closed Riemannian manifold. We will denote $PM := H^1([0,1], M)$ the path space of M endowed with its usual structure of Hilbert manifold (here H^1 denotes the Sobolev space also denoted $W^{1,2}$). Given $f : M \to M$, let

$$\Lambda(f) := \{\gamma \in PM | \; \gamma(1) = f(\gamma(0))\} ;$$

in particular $\Lambda(\mathrm{id}) = \Lambda M$ is the free loop space of M. We will also use the notation Λ_f for $\Lambda(f)$ and Λ for ΛM. This space was already introduced in [13] as $\Lambda_{G(f)} M$, we refer to this article for details in the properties recalled here. The space $\Lambda(f)$ is a Hilbert-submanifold as $\gamma \mapsto (\gamma(0), \gamma(1))$ is a submersion. Given $\gamma \in \Lambda(f)$, a vector $U \in T_\gamma \Lambda(f)$ is a H^1-vector field along γ satisfying $\mathrm{d}f \cdot U(0) = U(1)$. Let us denote $E : PM \to \mathbb{R}$ the energy functional,

$$E(\gamma) := \int_0^1 \|\dot\gamma\|^2 \mathrm{d}t,$$

and $E_f : \Lambda(f) \to \mathbb{R}$ the restriction to $\Lambda(f)$ for $f : M \to M$. According to [13, Theorem 2.4], the functional E_f satisfies the Palais-Smale condition.

Let us denote ∇ the Levi-Civita covariant derivative of M. Taken along a curve $t \mapsto c(t)$, the covariant derivative of the vector field X will be either denoted $\nabla_{\dot c} X$, $\frac{DX}{\mathrm{d}t}$ or $\dot X$ (this last notation is reserved to the time variable t). By convention, a geodesic will always mean a geodesic with constant speed: $\gamma \in PM$ is a geodesic if and only if

$$\nabla_{\dot\gamma} \dot\gamma = \ddot\gamma = 0.$$

Proposition 2.1. *Given $f : M \to M$, $\gamma \in \Lambda(f)$ is a critical point of E_f if and only if γ is a geodesic and $\mathrm{d}f_{\gamma(0)}^\perp \cdot \dot\gamma(1) = \dot\gamma(0)$.*

Proof. Let $\gamma \in \Lambda(f)$, let $U \in T_\gamma \Lambda(f)$ and let (γ_u) be a smooth family in $\Lambda(f)$ such that $U = \partial_u \gamma_u$ (the derivative being taken at $u = 0$). Then

$$\frac{1}{2} \mathrm{d}E(\gamma) \cdot U = \int_0^1 \left\langle \frac{D}{\partial u} \dot\gamma_u, \dot\gamma \right\rangle \mathrm{d}t = \int_0^1 \left\langle \dot U, \dot\gamma \right\rangle \mathrm{d}t = \left[\langle U, \dot\gamma \rangle \right]_0^1 - \int_0^1 \langle U, \ddot\gamma \rangle \, \mathrm{d}t.$$

As $\mathrm{d}f \cdot U(0) = U(1)$, one has

$$\left[\langle U, \dot\gamma \rangle \right]_0^1 = \langle U(0), \mathrm{d}f^T \cdot \dot\gamma(1) - \dot\gamma(0) \rangle.$$

The identity $\mathrm{d}E_f(\gamma) \cdot U = 0$ for the vector fields along γ such that $U(0) = 0$ implies that $\ddot\gamma = 0$, *i.e.* γ is a geodesic. The identity $\mathrm{d}E_f(\gamma) \cdot U = 0$ for every $U \in T_\gamma \Lambda(f)$ then implies that $\mathrm{d}f^T \cdot \dot\gamma(1) = \dot\gamma(0)$ (although we will often write $\mathrm{d}f^T$ without mention of the base point of M at which we take the adjoint, we point out that this notation can be ambiguous as $\mathrm{d}f^T$ is define on f^*TM rather than TM). $\qquad\square$

Corollary 2.2. *For every diffeomorphism $f : M \to M$, critical points of E_f with energy value $e > 0$ are in bijection with translated points (x, v) of $\tilde{f}$ with time-shift $\sqrt{e}$.*

As the correspondence suggests, we will rather be interested in the values of the functional $P_f := \sqrt{E_f}$ (which is C^1 away from $\{E_f = 0\}$) as they correspond to time-shifts at critical points with positive critical values. For every $\lambda \geq 0$, let $\Lambda_f^{<\lambda}$ and $\Lambda_{\overline{f}}^{\leq\lambda}$ be the respective subsets $\{P_f < \lambda\}$ and $\{P_f \leq \lambda\}$ of $\Lambda(f)$; when λ^2 is not a critical value of E_f, these subsets are submanifolds (with boundary in the second case).

2.2. **The topology of $\Lambda(f)$.** Let us first extend the join of loops defined in [11, §2.3] to the spaces $\Lambda(f)$, for $f : M \to M$ smooth, in the obvious way. Let us define the submanifold

$$PM \times_M PM := \{(\alpha, \beta) \in PM \times PM \mid \alpha(1) = \beta(0)\}$$

as well as $\Lambda(f) \times_M \Lambda(g) := (\Lambda(f) \times \Lambda(g)) \cap (PM \times_M PM)$. The concatenation $\phi : PM \times_M PM \to PM$ will denote the following continuous map:

$$\phi(\alpha, \beta)(t) := \begin{cases} \alpha(\frac{t}{s}), & t \in [0, s] \\ \beta(\frac{t-s}{1-s}), & t \in [s, 1] \end{cases}, \quad \text{where} \quad s = \frac{\sqrt{E(\alpha)}}{\sqrt{E(\alpha)} + \sqrt{E(\beta)}},$$

in the case $E(\alpha) = E(\beta) = 0$, both paths α and β are constant $\equiv p$ and one sets $\phi(\alpha, \beta) \equiv p$. This map satisfies

$$\sqrt{E(\phi(\alpha, \beta))} = \sqrt{E(\alpha)} + \sqrt{E(\beta)}. \tag{2}$$

The concatenation ϕ is associative: for every $\alpha, \beta, \gamma \in PM$ such that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$,

$$\phi(\phi(\alpha, \beta), \gamma) = \phi(\alpha, \phi(\beta, \gamma)). \tag{3}$$

By restriction, one gets a continuous map $\phi : \Lambda(f) \times_M \Lambda(g) \to \Lambda(g \circ f)$ satisfying $P_{g \circ f}(\phi(\alpha, \beta)) = P_f(\alpha) + P_g(\beta)$, for every $f, g : M \to M$.

Let $(f_s)_{s \in [0,1]}$ be a H^1-homotopy of smooth maps $M \to M$, *i.e.* such that $s \mapsto f_s(x)$ is in PM for all $x \in M$. It induces the following map $\tau(f_s) : \Lambda(f_0) \to \Lambda(f_1)$,

$$\tau(f_s)(\alpha) := \phi\big(\alpha, t \mapsto f_t(\alpha(0))\big).$$

Let

$$\delta(f_s) := \sup_{x \in M} \sqrt{E(t \mapsto f_t(x))},$$

then $\tau(f_s) : \Lambda_{f_0}^{\leq\lambda} \to \Lambda_{f_1}^{\leq\lambda + \delta(f_s)}$ according to (2). Following the same lines as [13, Lemma 3.6] for the more usual concatenation of continuous paths, one proves the following counterpart for ϕ.

Lemma 2.3 ([13, Lemma 3.6]). *Let $(f_s)_{s \in [0,1]}$ be a homotopy of smooth maps $M \to M$ such that $s \mapsto f_s(x)$ is in PM for all $x \in M$. Then $\tau(f_s)$ and $\tau(f_{1-s})$ are homotopy inverses.*

In particular, when $f : M \to M$ is homotopic to identity, $\Lambda(f)$ is homotopy equivalent to the free loop space ΛM whereas when $f : M \to M$ is homotopic to a constant, $\Lambda(f)$ is homotopy equivalent to a point.

Corollary 2.4. *Under the hypothesis of Lemma 2.3, the maps $\tau(f_s)$ and $\tau(f_{1-s})$ induce a $\delta(f_s)$-interleaving between the persistence modules $t \mapsto H_*(\Lambda_{f_j}^{<t})$, $j \in \{0,1\}$, i.e., for all $t \geq 0$, the induced morphisms*

$$\begin{cases} \tau(f_s)_* : H_*(\Lambda_{f_0}^{<t}) \to H_*(\Lambda_{f_1}^{<t+\delta(f_s)}), \\ \tau(f_{1-s})_* : H_*(\Lambda_{f_1}^{<t}) \to H_*(\Lambda_{f_0}^{<t+\delta(f_s)}), \end{cases}$$

commute with the inclusion morphisms (the same is true for $t \mapsto H_*(\Lambda_{\overline{f}_j}^{\leq t})$).

2.3. **Proof of Theorem 1.2.** The following fact is well-known to the experts when $f = \mathrm{id}$ (and this case will be enough for us).

Lemma 2.5. *Given $f : M \to M$, for every $\lambda \geq 0$, the image of the inclusion morphism $H_*(\Lambda_{\overline{f}}^{\leq \lambda}) \to H_*(\Lambda_f)$ is finitely generated.*

Proof. Let $f : M \to M$. It is enough to prove that for every $\lambda \geq 0$, the inclusion morphism $H_*(\Lambda_{\overline{f}}^{\leq \lambda}) \to H_*(\Lambda_{\overline{f}}^{\leq K})$ is finitely generated for some $K \geq \lambda$. Let $K > 0$ be a regular value of E_f. Using a subspace of broken geodesics, one can retract $\Lambda_{\overline{f}}^{\leq K}$ on a finite-dimensional submanifold N with a retraction $r : \Lambda_{\overline{f}}^{\leq K} \to N$ by deformation such that $E_f \circ r \leq E_f$ and the critical points of E_f are exactly the critical points of $g := E_f \circ r$ (see *e.g.* [20, §16]). Therefore, the desired result boils down to proving that the image of the homology morphism induced by $\{g \leq \lambda\} \hookrightarrow N$ has a finitely generated image for $g : N \to \mathbb{R}$ a map on a finite-dimensional manifold satisfying the Palais-Smale condition. As this property is C^0-open, one can assume that g is a Morse map satisfying the Palais-Smale condition (one can adapt indeed the proof of [21, Theorem 2.7] using the fact that the critical set of g in $\{g \leq \lambda\}$ is compact and the norm of $\mathrm{d}g$ is uniformly bounded from below outside any neighborhood of this set). The Palais-Smale condition then implies that $\{g \leq \lambda\}$ contains only a finite number $m \in \mathbb{N}$ of critical points (which are non-degenerate), so $H_*(\{g \leq \lambda\})$ can be generated by m elements. $\square$

Proof of Theorem 1.2. Let us first prove that one can assume $M = \widetilde{M}$. Let us assume that the theorem is true for a finite Riemannian cover $q : \widetilde{M} \to M$ satisfying the homological hypothesis. Let $f = f_1 : M \to M$ be homotopic to the id $= f_0$ through (f_t). By applying the homotopy lifting property to $(f_t \circ q)$ one gets a homotopy $(\tilde{f}_t)$ in $\widetilde{M} \to \widetilde{M}$ from the identity to a map $\tilde{f}_1$ commuting with f. If $(\tilde{p}_n, t_n)$ is a sequence of $\widetilde{M} \times \mathbb{R}$ satisfying the conclusion of the theorem for $\tilde{f}_1$, then $(q(\tilde{p}_n), t_n)$ is the desired sequence for f.

Let us now assume that $M = \widetilde{M}$. Let $f : M \to M$ be a smooth map homotopic to the identity. According to Lemma 2.3, $H_*(\Lambda(f))$ is isomorphic to $H_*(\Lambda M)$ which is not finitely generated. Therefore, Lemma 2.5 implies that $H_*(\Lambda_{\overline{f}}^{\leq \lambda}) \to H_*(\Lambda_f)$ is never onto for $\lambda \in [0, +\infty)$. According to the Morse deformation lemma, there exists a sequence (t_n) of positive critical values with $t_n \to +\infty$. The conclusion follows from Corollary 2.2. $\square$

2.4. **Non-degeneracy of a translated point.** In this section, we show the equivalence between the non-degeneracy of a translated point in the sense of contact geometry and the non-degeneracy of the associated critical point of E_f.

Let us denote R the Riemann tensor defined by

$$R(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

so that a vector field J along γ is a Jacobi field if and only if

$$\ddot{J} = R(\dot{\gamma}, J)\dot{\gamma}.$$

Let us also recall that the second derivative $\mathrm{d}^2 f$ of a map $f : M \to N$ is well-defined as $\nabla(\mathrm{d}f)$, *i.e.* by the tensorial expression: for all vector fields X, Y,

$$\mathrm{d}^2 f[X, Y] := X \cdot (Y \cdot f) - \mathrm{d}f \cdot \nabla_X Y.$$

Let us recall that the Hessian $\mathrm{d}^2 E_f(\gamma)$ of the energy functional is well-defined at a critical point γ (see *e.g.* [20, §13]).

Proposition 2.6. *Let $\gamma \in \Lambda(f)$ be a critical point of E_f, for every $U, V \in T_\gamma \Lambda(f)$,*

$$\frac{1}{2}\mathrm{d}^2 E_f(\gamma)[U, V] = \int_0^1 \left[R(V, \dot{\gamma})U, \dot{\gamma}) + \langle \dot{U}, \dot{V} \rangle \right] \mathrm{d}t + \langle \mathrm{d}^2 f_{\gamma(0)}[U(0), V(0)], \dot{\gamma}(1) \rangle.$$

Proof. Let $U, V \in T_\gamma \Lambda(f)$ and let $(\gamma_{u,v})$ be a smooth family of $\Lambda(f)$ such that $\gamma_{0,0} = \gamma$, $U = \partial_u \gamma_{u,0}$, $V = \partial_v \gamma_{0,v}$ (the partial derivatives being taken at 0). Then

$$\frac{1}{2}\mathrm{d}^2 E_f(\gamma)[U, V] = \int_0^1 \frac{\partial}{\partial v} \left\langle \frac{D}{\partial u}\dot{\gamma}_{u,v}, \gamma_{0,v} \right\rangle \mathrm{d}t$$

$$= \int_0^1 \left[\left\langle \frac{D^2}{\partial v \partial u}\dot{\gamma}_{u,v}, \dot{\gamma} \right\rangle + \left\langle \frac{D}{\partial u}\dot{\gamma}_{u,0}, \frac{D}{\partial v}\dot{\gamma}_{0,v} \right\rangle \right] \mathrm{d}t$$

$$= \int_0^1 \left[\langle R(V, \dot{\gamma})U, \dot{\gamma}) + \langle \dot{U}, \dot{V} \rangle \right] \mathrm{d}t + \left[\langle \nabla_V U, \dot{\gamma} \rangle \right]_0^1.$$

By derivating the identity $f(\gamma_{u,v}(0)) = \gamma_{u,v}(1)$, one gets

$$\mathrm{d}^2 f_{\gamma(0)}[V(0), U(0)] + \mathrm{d}f_{\gamma(0)} \cdot \nabla_V U(0) = \nabla_V U(1),$$

so that, using $\mathrm{d}f^T \cdot \dot{\gamma}(1) = \dot{\gamma}(0)$,

$$\left[\langle \nabla_V U, \dot{\gamma} \rangle \right]_0^1 = \langle \nabla_V U(1) - \mathrm{d}f \cdot \nabla_V U(0), \dot{\gamma}(1) \rangle = \langle \mathrm{d}^2 f[U(0), V(0)], \dot{\gamma}(1) \rangle.$$

$\square$

Corollary 2.7. *Let $\gamma \in \Lambda(f)$ be a critical point of E_f, a vector field $J \in T_\gamma \Lambda(f)$ belongs to the kernel of the quadratic form $\mathrm{d}^2 E_f(\gamma)$ if and only if J is a Jacobi field satisfying*

$$(\mathrm{d}^2 f \cdot J(0))^T \cdot \dot{\gamma}(1) = \dot{J}(0) - \mathrm{d}f^T \cdot \dot{J}(1),$$

where $\mathrm{d}^2 f \cdot J(0)$ denotes the linear morphism $T_{\gamma(0)}M \to T_{\gamma(1)}M$, $u \mapsto \mathrm{d}^2 f[J(0), u]$.

In particular, the kernel of $\mathrm{d}^2 E_f(\gamma)$ has a dimension bounded by $2 \dim M - 1$. Indeed, the vector space of Jacobi fields of γ has dimension $2 \dim M$, and $J \in T_\gamma \Lambda(f)$ implies that $J(1) = 0$ if $J(0) = 0$, so, for instance, the Jacobi field $J(t) = t\dot{\gamma}(t)$ does not belong to $T_\gamma \Lambda(f)$.

Proof. Let $J \in \ker \mathrm{d}^2 E_f(\gamma)$. Applying Proposition 2.6 with $U := J$ and every vector field V with $V(0) = 0$, one proves that J is a Jacobi field in a classical manner. The conclusion now follows from the identity $\forall V \in T_\gamma \Lambda(f)$,

$$\int_0^1 \langle \dot{J}, \dot{V} \rangle \mathrm{d}t = [\langle \dot{J}, V \rangle]_0^1 - \int_0^1 \langle \ddot{J}, V \rangle \mathrm{d}t = \langle \mathrm{d}f^T \cdot \dot{J}(1) - \dot{J}(0), V(0) \rangle - \int_0^1 \langle \ddot{J}, V \rangle \mathrm{d}t.$$

$\square$

Proposition 2.8. *Let $f : M \to M$ be a diffeomorphism, a translated point $(x, v) \in SM$ of $\tilde{f}$ is non-degenerate for the time-shift $t > 0$ if and only if the associated geodesic $\gamma \in \Lambda(f)$ of length t is a non-degenerate critical point of E_f.*

Proof. We will need the following differential identity: for every diffeomorphism $f : M \to M$ and every vector field U of M,

$$\nabla_U(\mathrm{d}f^{-T}) = -\mathrm{d}f^{-T} \cdot (\nabla_U \mathrm{d}f)^T \cdot \mathrm{d}f^{-T}. \tag{4}$$

This identity can be derived by taking the covariant derivative of the identity $\mathrm{d}f^{-T} \cdot \mathrm{d}f^T = \mathrm{id}$ (id meaning the section $x \mapsto \mathrm{id}_{T_x M}$) and using $\nabla_U(\mathrm{d}f^T) = (\nabla_U \mathrm{d}f)^T$ (which can be obtained by derivating the definition of the adjoint operator).

Let $f : M \to M$ be a diffeomorphism and let $\hat{f} : TM \to TM$ be its symplectic lift,

$$f(x, v) = (f(x), \mathrm{d}f_x^{-T} \cdot v), \quad \forall (x, v) \in TM.$$

The differential of $\hat{f}$ at $(x, v) \in TM$ is

$$\mathrm{d}\hat{f}_{(x,v)} \cdot (\xi_1, \xi_2) = \left(\mathrm{d}f_x \cdot \xi_1, \mathrm{d}f_x^{-T} \cdot \xi_2 + \nabla_{\xi_1}(\mathrm{d}f^{-T}) \cdot v \right), \quad \forall \xi_1, \xi_2 \in T_x M,$$

where the identification $T_{(x,v)}TM \simeq T_x M \times T_x M$ is given by the Levi-Civita connection (see *e.g.* [22, §1.3.1]). Let $(x, v) \in SM$ be a translated point of $\tilde{f}$ for the time-shift $t > 0$ and $\gamma \in \Lambda(f)$ be the associated geodesic of length t. It is non-degenerate if and only if $\mathrm{d}(\hat{f} \circ G_{-t})_{G_t(x,v)}$ does not have 1 as an eigenvalue. We recall that, for $t > 0$,

$$(\mathrm{d}G_{-t})_{G_t(x,v)} \cdot \left(J(1), \frac{1}{t}\dot{J}(1) \right) = \left(J(0), \frac{1}{t}\dot{J}(0) \right),$$

for every Jacobi field J along γ (the t^{-1} factor being due to the reparametrization, see *e.g.* [22, §1.5]). Since $J \mapsto (J(1), \dot{J}(1))$ is an isomorphism and $tv = \dot{\gamma}(0)$, (x, v) is degenerate if and only if there exists a Jacobi field J along γ such that

$$\begin{cases} \mathrm{d}f \cdot J(0) = J(1), \\ \mathrm{d}f^{-T} \cdot \dot{J}(0) + \nabla_{J(0)}(\mathrm{d}f^{-T}) \cdot \dot{\gamma}(0) = \dot{J}(1). \end{cases} \tag{5}$$

The first equation of (5) means that $J \in T_\gamma \Lambda(f)$. Using identity (4) and $\mathrm{d}f^{-T} \cdot \dot{\gamma}(0) = \dot{\gamma}(1)$, the second equation of (5) becomes

$$\mathrm{d}f^{-T} \cdot \dot{J}(0) - \mathrm{d}f^{-T} \cdot (\nabla_{J(0)} \mathrm{d}f)^T \cdot \dot{\gamma}(1) = \dot{J}(1).$$

Finally, applying $\mathrm{d}f^T$ to this last equation, we find the equation of Corollary 2.7. $\square$

3. The Zoll case

3.1. **The Chas-Sullivan product.** Let us recall and extend in an obvious manner the filtered Chas-Sullivan product defined in [11]. We refer to [11] for technical details. For now, we do not need to assume that the closed Riemannian manifold M is Zoll. Let $f, g : M \to M$ be smooth maps, as the space $\Lambda(f) \times_M \Lambda(g)$ is a submanifold of $\Lambda(f) \times \Lambda(g)$ of codimension $n := \dim M$, there is a well-defined Gysin morphism

$$H_*(\Lambda(f) \times \Lambda(g)) \to H_{*-n}(\Lambda(f) \times_M \Lambda(g))$$

(see *e.g.* [11, Proposition B.2]). The Chas-Sullivan product is defined by composing this morphism with the concatenation ϕ:

$$* : H_*(\Lambda(f)) \otimes H_*(\Lambda(g)) \to H_{*-n}(\Lambda(g \circ f)).$$

Let $a, b > 0$ be regular values of P_f and P_g, then $\Lambda_{\overline{f}}^{\leq a}$, $\Lambda_{\overline{g}}^{\leq b}$ are submanifolds with boundary and one can thus define the Gysin morphism

$$H_*(\Lambda_{\overline{f}}^{\leq a} \times \Lambda_{\overline{g}}^{\leq b}) \to H_{*-n}(\Lambda_{\overline{f}}^{\leq a} \times_M \Lambda_{\overline{g}}^{\leq b}).$$

According to (2), one can thus define a filtered Chas-Sullivan product

$$* : H_*(\Lambda_{\overline{f}}^{\leq a}) \otimes H_*(\Lambda_{\overline{g}}^{\leq b}) \to H_{*-n}(\Lambda_{\overline{g \circ f}}^{\leq a+b})$$

commuting with the inclusion morphisms. A relative version of this product is also available

$$H_*(\Lambda_{\overline{f}}^{\leq a}, \Lambda_{\overline{f}}^{\leq a'}) \otimes H_*(\Lambda_{\overline{g}}^{\leq b}, \Lambda_{\overline{g}}^{\leq b'}) \to H_{*-n}(\Lambda_{\overline{g \circ f}}^{\leq a+b}, \Lambda_{\overline{g \circ f}}^{\leq \max(a+b', a'+b)}),$$
$$H_*(\Lambda_{\overline{f}}^{\leq a}, \Lambda_f^{<a}) \otimes H_*(\Lambda_{\overline{g}}^{\leq b}, \Lambda_g^{<b}) \to H_{*-n}(\Lambda_{\overline{g \circ f}}^{\leq a+b}, \Lambda_{\overline{g \circ f}}^{<a+b}),$$

that commutes with each other through inclusion morphisms.

Lemma 3.1. *Let $g : M \to M$ be a smooth map, let (f_s) be an H^1-homotopy of smooth maps $M \to M$, Let $\delta := \delta(f_s)$ (which is $\geq \delta(f_s \circ g)$). For every $\alpha \in H_*(\Lambda_{\overline{g}}^{\leq a})$, $\beta \in H_*(\Lambda_{\overline{f_0}}^{\leq b})$,*

$$\alpha * \tau(f_s)_* \beta = \tau(f_s \circ g)_* (\alpha * \beta) \in H_*(\Lambda_{\overline{f_1 \circ g}}^{\leq a+b+\delta}),$$

as long as the product $*$ is well-defined.

Proof. Let us consider the following diagram:

$$
\begin{array}{ccccc}
\Lambda_{\overline{g}}^{\leq a} \times \Lambda_{\overline{f_0}}^{\leq b} & \longleftarrow & \Lambda_{\overline{g}}^{\leq a} \times_M \Lambda_{\overline{f_0}}^{\leq b} & \overset{\phi}{\longrightarrow} & \Lambda_{\overline{f_0 \circ g}}^{\leq a+b} \\
{\scriptstyle \mathrm{id} \times \tau(f_s)} \downarrow & & {\scriptstyle \mathrm{id} \times \tau(f_s)} \downarrow & & \downarrow {\scriptstyle \tau(f_s \circ g)} \\
\Lambda_{\overline{g}}^{\leq a} \times \Lambda_{\overline{f_1}}^{\leq b+\delta} & \longleftarrow & \Lambda_{\overline{g}}^{\leq a} \times_M \Lambda_{\overline{f_1}}^{\leq b+\delta} & \overset{\phi}{\longrightarrow} & \Lambda_{\overline{f_1 \circ g}}^{\leq a+b+\delta}
\end{array}
\quad,
$$

where the unlabeled arrows are inclusion maps. By definition of $\tau(f_s)$ and $\tau(f_s \circ g)$ and by associativity of ϕ (3), this diagram commutes. By naturality of the Gysin morphisms and by definition of the product $*$, the conclusion follows. $\square$

In the same way, one proves that $*$ is associative.

The product structure of the homology group $H_*(\Lambda)$ was studied by Goresky-Hingston, especially in the case where all geodesics are closed and of same prime length ℓ [11, §13-15]. In the sequel, M will satisfy this assumption and the coefficient ring of the singular homology groups will be $\mathbb{Z}$ if M is orientable and $\mathbb{Z}/2\mathbb{Z}$ otherwise. In this case, the energy functional E_{id} is a perfect Morse-Bott functional: it implies in particular that the following sequences of inclusion morphisms are exact

$$
\begin{aligned}
0 \to H_*(\Lambda^{\leq a}) \to H_*(\Lambda^{\leq b}) \to H_*(\Lambda^{\leq b}, \Lambda^{\leq a}) \to 0, \\
0 \to H^*(\Lambda^{\leq b}, \Lambda^{\leq a}) \to H^*(\Lambda^{\leq b}) \to H^*(\Lambda^{\leq a}) \to 0,
\end{aligned}
\tag{6}
$$

for every $0 \leq a \leq b \leq c \leq +\infty$, with $\Lambda^{\leq +\infty} := \Lambda$, the same being true replacing $\Lambda^{\leq \lambda}$ with $\Lambda^{<\lambda}$. Moreover, the filtered product $*$ can be defined for all values. The critical values of P_f correspond to the positive multiples of ℓ with associated critical submanifolds diffeomorphic to SM *via* $\gamma \mapsto \dot{\gamma}(0)/\|\dot{\gamma}(0)\|$. Let $\lambda_1 \geq 0$ be the Morse index of the critical submanifold Σ_1 associated with the critical value ℓ. The manifold Σ_1 can be oriented and one defines

$$\Theta \in H_{2n-1+\lambda_1}(\Lambda^{\leq \ell})$$

the image of the fundamental class of Σ_1 under the canonical isomorphism

$$H_*(\Lambda^{\leq \ell}) \simeq H_*(\Lambda^{\leq 0}) \oplus H_{*-\lambda_1}(\Sigma_1) \tag{7}$$

resulting from the splitting of the exact sequence (6) for the pair $(\Lambda^{\leq \ell}, \Lambda^{\leq 0})$ induced by $\mathrm{ev} : \Lambda \to \Lambda^{\leq 0}$, $\mathrm{ev}(\gamma) \equiv \gamma(0)$, and the Morse-Bott isomorphism $H_*(\Lambda^{\leq \ell}, \Lambda^{\leq 0}) \simeq H_{*-\lambda_1}(\Sigma_1)$. Let us set $b := \lambda_1 + n - 1$.

Theorem 3.2 ([11, Theorem 13.4]). *Let M be an n-dimensional closed Zoll Riemannian manifold of length ℓ. The product*

$$\Theta * \cdot : H_*(\Lambda, \Lambda^{\leq 0}) \to H_{*+b}(\Lambda, \Lambda^{\leq 0})$$

with the class Θ is injective and induces an isomorphism

$$H_*(\Lambda^{\leq r\ell}, \Lambda^{<r\ell}) \to H_{*+b}(\Lambda^{\leq (r+1)\ell}, \Lambda^{<(r+1)\ell})$$

for every positive integer r.

3.2. **Min-max critical values.** Given $f : M \to M$, let us define min-max critical values of P_f associated with homology classes of $\Lambda(f)$. Given $\alpha \in H_*(\Lambda(f))$, let us define

$$c(\alpha, f) := \inf \left\{ \lambda \geq 0 \mid \alpha \in \mathrm{im} \left(H_*(\Lambda_f^{\leq \lambda}) \to H_*(\Lambda_f) \right) \right\}.$$

By the Morse deformation lemma, if $c(\alpha, f) > 0$ then it is a critical value of P_f. A consequence of Theorem 3.2 and the perfectness of E_{id} in the Zoll case is the following corollary.

Corollary 3.3. *Let M be a Zoll Riemannian manifold with prime length ℓ. For every $\alpha \in H_*(\Lambda)$, one has $c(\alpha, \mathrm{id}) \in \mathbb{N}\ell$ and if $c(\alpha, \mathrm{id}) > 0$, then*

$$c(\Theta^k * \alpha, \mathrm{id}) = c(\alpha, \mathrm{id}) + k\ell, \quad \forall k \in \mathbb{N}.$$

Proof. Since the $c(\alpha, \mathrm{id})$ are either 0 or critical values of P_{id}, one has $c(\alpha, \mathrm{id}) \in \mathbb{N}\ell$ for all $\alpha \in H_*(\Lambda)$. Let $\alpha \in H_*(\Lambda)$ be such that $c(\alpha, \mathrm{id}) > 0$, so $c(\alpha, \mathrm{id}) = r\ell$ with $r \in \mathbb{N}^*$ and there exists $\beta \in H_*(\Lambda^{\leq r\ell})$ the image of which is α under the inclusion morphism. Since $\Theta^k * \cdot$ commutes with inclusion morphisms, the image of $\Theta^k * \beta \in H_{*+kb}(\Lambda^{\leq (r+k)\ell})$ in $H_*(\Lambda)$ is $\Theta^k * \alpha$, so $c(\Theta^k * \alpha, \mathrm{id}) \leq (r+k)\ell$.

The image γ of β under the inclusion morphism $H_*(\Lambda^{\leq r\ell}) \to H_*(\Lambda^{\leq r\ell}, \Lambda^{<r\ell})$ is non-zero since $c(\alpha, \mathrm{id}) \geq r\ell$ (by looking at the long exact sequence of the couple $(\Lambda^{\leq r\ell}, \Lambda^{<r\ell})$). Let us consider the commuting diagram:

$$
\begin{array}{ccc}
H_*(\Lambda^{\leq r\ell}) & \xrightarrow{\ \Theta^k *\cdot\ } & H_{*+kb}(\Lambda^{\leq (r+k)\ell}) \\
\downarrow & & \downarrow \\
H_*(\Lambda^{\leq r\ell}, \Lambda^{<r\ell}) & \xrightarrow[\simeq]{\Theta^k *\cdot} & H_{*+kb}(\Lambda^{\leq (r+k)\ell}, \Lambda^{<(r+k)\ell})
\end{array}
\ ,
$$

where the vertical arrows are inclusion morphisms. Since the bottom arrow is an isomorphism according to Theorem 3.2, the image of γ is non-zero and so is the image of $\Theta^k * \beta$ under the right vertical arrow. This implies that $\Theta^k * \beta$ is not in the image of $H_{*+kb}(\Lambda^{<(r+k)\ell})$ under the inclusion morphism so $c(\Theta^k * \alpha, \mathrm{id}) \geq (r+k)\ell$. $\square$

Proposition 3.4. *Let M be a Zoll Riemannian manifold with prime length ℓ and let (f_s) be a H^1-homotopy from id to f and let us set $\tau := \tau(f_s)$ and $\delta := \delta(f_s)$.*

(1) For every $\alpha \in H_*(\Lambda)$, $|c(\alpha, \mathrm{id}) - c(\tau_* \alpha, f)| \leq \delta$.

(2) *For every* $\alpha \in H_*(\Lambda)$,

$$c(\Theta^{k+1} * \tau_* \alpha, f) = c(\Theta^k * \tau_* \alpha, f) + \ell - \varepsilon_k(\alpha), \quad \forall k \in \mathbb{N},$$

where $(\varepsilon_k(\alpha))$ *is a sequence of non-negative reals that converges to* 0 *when* $c(\alpha, \mathrm{id}) > 0$.

Proof. The first statement is a direct consequence of the fact that $\tau(f_s)$ and $\tau(f_{1-s})$ induce a δ-interleaving in the sense of Corollary 2.4. Let us consider the commuting diagram

$$\begin{array}{ccc} H_*(\Lambda_f^{\leq \lambda}) & \xrightarrow{\Theta*\cdot} & H_{*+b}(\Lambda_f^{\leq \lambda+\ell}) \\ \downarrow & & \downarrow \\ H_*(\Lambda_f) & \xrightarrow{\Theta*\cdot} & H_{*+b}(\Lambda_f) \end{array},$$

where the vertical arrows are inclusion morphisms and $\lambda \geq 0$. It implies that if $c(\alpha, f) < \lambda$ then $c(\Theta * \alpha, f) < \lambda + \ell$. In particular, for every $\alpha \in H_*(\Lambda)$,

$$c(\Theta^{k+1} * \tau_* \alpha, f) = c(\Theta^k * \tau_* \alpha, f) + \ell - \varepsilon_k(\alpha), \quad \forall k \in \mathbb{N},$$

for some non-negative sequence $(\varepsilon_k(\alpha))$. According to the first statement of this proposition, for every $k \in \mathbb{N}$,

$$\sum_{i=0}^{k-1} \varepsilon_i(\alpha) = c(\tau_* \alpha, f) - c(\Theta^k * \tau_* \alpha, f) + k\ell \leq 2\delta + c(\alpha, \mathrm{id}) - c(\Theta^k * \alpha, \mathrm{id}) + k\ell,$$

where we have used that $\Theta^k * \tau_* \alpha = \tau_*(\Theta^k * \alpha)$ (Lemma 3.1). Therefore, Corollary 3.3 implies that the series $\sum_i \varepsilon_i(\alpha)$ is bounded by 2δ when $c(\alpha, \mathrm{id}) > 0$, bringing the conclusion. $\square$

3.3. **Proof of Theorems 1.4 and 1.5.**

Proof of Theorem 1.4. Let $N := \sum_i \beta_i(SM)$ and let $\alpha_1, \ldots, \alpha_N \in H_*(\Lambda)$ be an independent family satisfying $c(\alpha_i, \mathrm{id}) = \ell$ for all i. Such a family can be obtained as follows: one takes the image in $H_*(\Lambda^{\leq \ell})$ of an independent family of $0 \oplus H_*(\Sigma_1) \simeq H_*(SM)$ under the isomorphism (7), then takes the image of this family under the inclusion morphism $H_*(\Lambda^{\leq \ell}) \to H_*(\Lambda)$ (this last morphism is injective by exactness of (6)). The exactness of (6) together with Theorem 3.2 implies that the family $(\Theta^k * \alpha_i)_{k,i}$, $k \in \mathbb{N}$, $1 \leq i \leq N$, is independent in $H_*(\Lambda)$.

Let (f_s) be a H^1-homotopy from id to $f : M \to M$ and let $\tau := \tau(f_s)$. Applying the isomorphism $\tau_* : H_*(\Lambda) \to H_*(\Lambda(f))$, the family $(\Theta^k * \tau_* \alpha_i)_{k,i}$ is independent in $H_*(\Lambda(f))$ (we have used Lemma 3.1). Let us assume that f has finitely many translated points in SM. Since M is a Zoll Riemannian manifold, the set of positive time-shifts of a translated point of f is $t + \mathbb{N}\ell$ for some $t \in (0, \ell]$. Therefore, the set of (positive) critical values of P_f is invariant by $t \mapsto t + \ell$, discrete such that any interval $(a, a + \ell]$, $a \geq 0$, contains a fixed finite number of them. Let us study the sequences $(c_k^i) := (c(\Theta^k * \tau_* \alpha_i, f))$ for $1 \leq i \leq N$, the image of which is contained in this discrete set. Since $c_{k+1}^i = c_k^i + \ell - \varepsilon_k^i$ with $\varepsilon_k^i \to 0$ (Proposition 3.4), by discreetness of the $\mathbb{N}\ell$-invariant set of critical value, $\varepsilon_k^i = 0$ for large k's. We deduce that for $A > 0$ large, for each i, there exists a unique k_i such that $c_{k_i}^i \in (A, A + \ell]$. By non-degeneracy of the critical points of P_f, the Morse inequalities in the window of values $(A, A + \ell]$ imply that the number of critical points in $P_f^{-1}(A, A + \ell]$ is not

less than the cardinal of $(\Theta^{k_i} * \tau_* \alpha_i)_i$, which is N. For each translated point, only one time-shift belongs to $(A, A + \ell]$, which brings the conclusion. $\qquad\square$

Proof of Theorem 1.5. Let $N := CL(SM)$ and let $u_1, \ldots, u_N \in H^*(\Lambda)$ be non-zero classes of positive degree such that

$$c(\Theta \frown (u_1 \smile \cdots \smile u_N), \mathrm{id}) = \ell.$$

Such a family can be obtained as follows: let $v_1, \ldots, v_N \in H^*(SM)$ be non-zero classes of positive degree such that $v_1 \smile \cdots \smile v_N \neq 0$ (which exist by definition of $CL(SM)$), then $[SM] \frown (v_1 \smile \cdots \smile v_N) \neq 0$ and we apply (7) together with the exactness of (6) to send $[SM]$ to Θ (by definition of Θ) and the v_i's to the u_i's.

Let $\alpha_0, \ldots, \alpha_N \in H_*(\Lambda)$ be the subordinated classes $\alpha_i := \Theta \frown (u_1 \smile \cdots \smile u_i)$ such that $c(\alpha_i, \mathrm{id}) = \ell$ for all i (as by subordination, $c(\alpha_{i+1}, \mathrm{id}) \leq c(\alpha_i, \mathrm{id})$ and $c(\Theta, \mathrm{id}) = \ell$). Let (f_s) be a H^1-homotopy as in the statement of Theorem 1.5, so that $\delta(f_s) < \ell/2$, and let $\tau := \tau(f_s)$. By subordination, $c(\tau_* \alpha_i, f)$ is non-increasing with i and by the first point of Proposition 3.4,

$$\frac{\ell}{2} < c(\tau_* \alpha_N, f) \leq c(\tau_* \alpha_{N-1}, f) \leq \cdots \leq c(\tau_* \alpha_0, f) < \frac{3\ell}{2}.$$

According to the Lusternik-Schnirelmann theorem (see *e.g.* [8, §II.3.2]), if P_f has a finite number of critical points in the window $(\ell/2, 3\ell/2)$ then $c(\tau_* \alpha_i, f)$ is decreasing. Since the $c(\tau_* \alpha_i, f)$'s are critical values of P_f, the conclusion follows by Corollary 2.2. $\qquad\square$

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