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MORSE THEORY FOR FIXED POINTS OF SYMPLECTIC DIFFEOMORPHISMS

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ABSTRACT. We prove the following special case of the Arnold conjecture on the fixed points of an exact deformation φ of a compact closed symplectic manifold P: If $\pi_2(P) = 0$ and all fixed points of φ are nondegenerate, then their number is greater than or equal to the sum of the Betti numbers of P with respect to Z_2 coefficients.

Let P be a symplectic manifold, i.e. P is a smooth manifold equipped with a closed and nondegenerate 2-form ω. Then we can assign to each smooth function

$$H: P \times \mathbf{R} \to \mathbf{R}; \quad H(x,t) = H_t(x) \tag{1}$$

a family X_t of vector fields on P defined by $\omega(\cdot, X_t) = dH_t$. This vector field is called the (exact) Hamiltonian vector field associated with the (time-dependent) Hamiltonian H. If P is compact, then the differential equation

$$\frac{d}{dt}\varphi_{H,t}(x) = X_t(\varphi_{H,t}(x)) \tag{2}$$

with initial condition $\varphi_{H,0}(x) = x$ defines a family of smooth diffeomorphisms of P, which also preserve the symplectic structure, i.e. for each $t \in \mathbf{R}$ we have $\varphi_t^* \omega = \omega$. In fact, the set

$$\mathcal{D} = \{\varphi_{H,t} \mid t \in \mathbf{R} \text{ and } H \in C^\infty(P \times \mathbf{R})\} \tag{3}$$

of exact diffeomorphisms turns out to be a subgroup of the group of symplectic diffeomorphisms on P.

Since each $\varphi \in \mathcal{D}$ is homotopic to the identity, the Lefschetz fixed point theorem implies that if all fixed points x of φ are nondegenerate in the sense that

$$\det(D\varphi(x) - \mathrm{id}) \neq 0, \tag{4}$$

then the sum of the signs of (4) over all fixed points of φ is equal to the Euler characteristic $\chi(P)$. In particular, if all fixed points are nondegenerate, their number must be equal to or greater than the absolute value of $\chi(P)$. It has been conjectured by V. Arnold that a stronger result holds for exact diffeomorphisms: the number of fixed points of each $\varphi \in \mathcal{D}$ should satisfy estimates similar to those obtained by Morse theory for the number of critical points of a smooth function on P.

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Results in this direction have been proved in [**2**] for the standard symplectic structure on the even-dimensional torus, in [**3** and **7**] for surfaces and other hyperbolic manifolds and in [**4**] for the complex projective space. Moreover, there has been a perturbation result for general symplectic manifolds, see [**8**]. Recently, Gromov [**5**] proved the existence of at least one fixed point for any exact deformation on P provided that $\pi_2(P) = 0$. In this note, we announce the extension of this existence result to a Morse theory of nondegenerate fixed points.

THEOREM 1. *Let P be a compact closed symplectic manifold with $\pi_2(P) = 0$. Let $\varphi: P \to P$ be an exact diffeomorphism all of whose fixed points are nondegenerate. Then the number of fixed points is greater than or equal to the sum of the Betti numbers of P with respect to Z_2-coefficients.*

It is conceivable that the ideas underlying the proof of Theorem 1 also work in the case of a general symplectic manifold. The estimate is a consequence of the following more precise relation between the fixed point set and the cohomology of P.

THEOREM 2. *With P and φ as in Theorem 1, let C^* denote the Z_2-vector space over the set of fixed points of φ. Then there exists a homomorphism*

$$\delta: C^* \to C^* \tag{5}$$

of Z_2-vector spaces so that $\delta\delta = 0$ and so that

$$\ker \delta / \operatorname{im} \delta = H^*(P, Z_2). \tag{6}$$

The coboundary operator δ is constructed as follows. For $\varphi \in \mathcal{D}$, define

$$\Omega(\varphi) = \{z \in C^\infty([0,1], P) |\ z(1) = \varphi(z(0))\}. \tag{7}$$

Moreover, choose an almost complex structure J so that the bilinear form $g = \omega(J\cdot, \cdot)$ is a metric, i.e. is symmetric and positive. Then we consider formally the flow generated by the "vector field" $V(z) = J\dot{z}$ on $\Omega(\varphi)$. To be more precise, we consider 1-parameter families $u: \mathbf{R} \times [0,1] \to P$ in Ω satisfying

$$\frac{\partial u(\tau, t)}{\partial \tau} + J(u(\tau, t))\frac{\partial u(\tau, t)}{\partial t} = 0. \tag{8}$$

Clearly, the fixed points of this "flow" are in 1-1 correspondence with fixed points of φ. We are particularly interested in the sets $\mathcal{M}(x, y)$ of solutions of (8) converging to fixed points x and y for $\tau \to \pm\infty$. Applying a small perturbation to the almost complex structure J if necessary, we find that these sets are smooth finite-dimensional manifolds. Moreover, the group $\mathbf{R}$ acts freely on $\mathcal{M}(x, y)$ by translation in the first variable. We now define

$$\langle x, \delta y \rangle = \begin{cases} \#(\mathcal{M}(x, y)/\mathbf{R} \mod 2 & \text{if } \dim \mathcal{M}(x, y) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

For an analogous construction in finite-dimensional Morse theory, see Milnor [**6**]. In order for (9) to be well defined, we need a compactness property of holomorphic curves, see also [**5**].

It turns out that (9) defines the matrix elements of an operator δ satisfying $\delta\delta = 0$. In order to show that it also satisfies (6), we show that the quotient $\ker\delta/\mathrm{im}\,\delta$ is invariant under deformation of φ within $\mathcal{D}$. By the definition of $\mathcal{D}$, we can now deform φ into the identity. The relation (6) is then proved by a perturbation argument.

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