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MORSE THEORY FOR LAGRANGIAN INTERSECTIONS

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Abstract

Let P be a compact symplectic manifold and let $L \subset P$ be a Lagrangian submanifold with $\pi_2(P, L) = 0$. For any exact diffeomorphism ϕ of P with the property that $\phi(L)$ intersects L transverally, we prove a Morse inequality relating the set $\phi(L) \cap L$ to the cohomology of L. As a consequence, we prove a special case of the Arnold conjecture: If $\pi_2(P) = 0$ and ϕ is an exact diffeomorphism all of whose fixed points are nondegenerate, then the number of fixed points is greater than or equal to the sum over the $\mathbf{Z}_2$-Betti numbers of P.

1. Introduction

Let (P, ω) be a symplectic manifold, i.e., P is a smooth manifold with a closed and nondegenerate 2-form ω. We can then assign to each smooth function

$$H : P \times \mathbf{R} \to \mathbf{R} : H(x,t) = H_t(x)$$

a family of vector fields X_t on P defined by

$$\omega(\cdot, X_t) = dH_t. \tag{1.1}$$

This is called the Hamiltonian vector field associated with the time dependent Hamiltonian H. If P is compact, then the differential equation

$$\frac{d}{dt}\phi_{H,t}(x) = H_t(\phi_{H,t}(x)) \tag{1.2}$$

with initial condition $\phi_{H,0}(x) = x$ defines a family of smooth diffeomorphisms of P, which also preserve the symplectic structure, i.e. for each $t \in \mathbf{R}$, we have $\phi_t^*\omega = \omega$.

In fact, the set

$$\mathscr{D} = \{\phi_{H,t} \mid t \in \mathbf{R} \text{ and } H \in C^\infty(P \times \mathbf{R})\} \tag{1.3}$$

turns out to be a subgroup of the group of symplectic diffeomorphisms of P.

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As an immediate consequence of the existence of a nondegenerate 2-form ω, the manifold P must have even dimension $2n$. An n-dimensional submanifold L is called Lagrangian if

$$\omega_X(\xi, \varsigma) = 0 \quad \forall x \in L; \xi, \varsigma \in T_x L$$

(see, for example, [24]). It is well known that n is the maximal dimension for any manifold with this property. If two orientable n-dimensional submanifolds L and L' of P intersect transversally, then one can assign to each $x \in L \cap L'$ a sign $\sigma(x)$ by comparing the orientation of $T_x L \oplus T_x L'$ with the orientation of $T_x P$. The sum over these signs is a cohomological invariant of L and L' and is called their intersection number. As a consequence, the intersection number of L and L' gives a lower bound on the cardinality of $L \cap L'$ in the case of transverse intersection. For example, the intersection number of a Lagrangian submanifold L with itself equals the Euler characteristic of L, since the normal bundle of L in P is isomorphic to its tangent bundle (see [24]). It has been conjectured by V. I. Arnold [1], [2] that stronger estimates hold if L is Lagrangian and L' is obtained from L by an exact deformation $\phi = \{\phi_t\}$. Let us denote by $I(L, \phi)$ the set of all $x \in \phi_1(L) \cap L$ for which $\{\phi_t(x)\}$ defines the zero element in $\pi_1(P, L)$. It is the purpose of this paper to prove

Theorem 1. *Let P be a compact symplectic manifold and let $L \subset P$ be a compact Lagrangian submanifold of P with $\pi_2(P, L) = 0$. Moreover, let ϕ be an exact deformation so that $\phi_1(L)$ intersects L transversally. Then to each $x \in I(L, \phi)$, we can assign an integer $\mu(x)$ with the following property: Define the polynomials*

$$(1.4) \qquad \begin{aligned} \Pi_\phi(t) &= \sum_{x \in I(L,\phi)} t^{\mu(x)}, \\ \Pi_L(t) &= \sum_{k=0}^{\dim L} \dim_{\mathbf{Z}_2} H^k(L, \mathbf{Z}_2) t^k. \end{aligned}$$

Then there exists a polynomial $Q \in \mathbf{Z}[t]$ with nonnegative coefficients so that

$$(1.5) \qquad \Pi_\phi(T) = \Pi_L(t) + Q(t)(1+t).$$

Setting $t = 1$, we obtain an estimate on $|L \cap \phi(L)|$ by the sum of the $\mathbf{Z}_2$-Betti-numbers. In the orientable case, the restriction to $\mathbf{Z}_2$-coefficients in (1.4) does not seem to be essential, but simplifies the analysis considerably. It is conceivable that the methods of this paper can be refined to estimate $|L \cap \phi_1(L)|$ by the "cuplength" of L after dropping the transversality assumption. Note that if one extends the set of exact deformations of L to allow self-intersections of $\phi(L)$, then Weinstein proved in [25] that $L \cap \phi_1(L)$ may be empty.

By applying Theorem 1 to the symplectic manifold $P \times (-P)$ (where the sign of the symplectic form on the second factor has been reversed) and the Lagrangian submanifold $L = \{(x,x) \mid x \in P\}$ we obtain the following corollary:

Theorem 2. *If P is a compact symplectic manifold with $\pi_2(P) = 0$ and ϕ is an exact diffeomorphism of P all of whose fixed points are nondegenerate, then the number of fixed points of ϕ is greater than or equal to the sum over the $\mathbf{Z}_2$-Betti numbers of P.*

Again, extensions to arbitrary coefficients and to general fixed points are expected to hold. Such a result was obtained by Conley and Zehnder [5] for the torus T^{2n} with the standard symplectic structure. The main idea of [5] is to convert the fixed point problem into a variational problem on the loop space of P and to apply Conley's index theory. These methods were generalized by Sikorav [19] and the present author [6] to cover e.g. surfaces of higher genus. Theorem 2 is expected to remain true without any assumption on $\pi_2(P)$. In fact, the general Arnold conjecture has been proved for $P = \mathbf{C}P^n$ by Fortune [11]. We hope to extend our methods to prove Theorem 2 for general symplectic manifolds. Estimates of Lagrangian intersections have been proved for the diagonal in T^{2n} with the standard symplectic structure by Chaperon [3] and for the zero section of cotangent bundles by Hofer [13], and by Laudenbach and Sikorav [14]. For P and L as in Theorem 1, Gromov [12] proved the existence of at least one intersection and hence of one fixed point for any P and ϕ as in Theorem 2. Rather remarkably, Gromov does not use the variational formulation. Instead, he applies an indirect argument which involves manifolds of "pseudo-holomorphic" discs in a way reminiscent of the use of Yang-Mills moduli spaces in four dimensional topology.

In some sense, our method in proving Theorem 1 interpolates between these two approaches. To outline the proof of Theorem 1, note that the Morse inequality (1.5) with $Q \in \mathbf{N}[t]$ is equivalent to the fact that the intersections serve as a model for the $\mathbf{Z}_2$-cohomology of L. To be more precise, let C^p denote the free $\mathbf{Z}_2$-module over the set of $x \in I(L,\phi)$ with $\mu(x) = p$. Then (1.5) holds if and only if there exists a $\mathbf{Z}_2$-module-homomorphism $\delta : C^p \to C^{p+1}$ so that $\delta\delta = 0$ and so that

$$\ker \delta / \operatorname{im} \delta = H^*(L, \mathbf{Z}_2). \tag{1.6}$$

Now define the space

$$\begin{aligned}\Omega := \Omega(L,\phi) = \{z \in C^\infty([0,1],P) \mid z(0) \in L \text{ and } z(1) \in \phi_1(L) \text{ and}\\ [\phi_t(z(t))] = 0 \text{ in } \pi_1(P,L)\}.\end{aligned} \tag{1.7}$$

The tangent space $T_z\Omega$ consists of tangent fields ξ of P along z which are tangent to L at 0 and to $L' := \phi_1(L)$ at 1. Then ω induces a "1-form"

$$\alpha(\xi) = \int_0^1 \omega(\dot{z}(t)\xi(t))\, dt$$

on Ω. Since ω is closed and L and L' are Lagrangian, this form is closed in the sense that it can be integrated locally to a real valued function $\mathscr{a}$ on Ω, so that

$$\alpha(z) = d\mathscr{a}(z). \tag{1.8}$$

Moreover, under the hypothesis of Theorem 1, $\mathscr{a}$ can be defined globally on a certain component of Ω (see Proposition 2.3 below). Since ω is nondegenerate, α vanishes exactly at the constant loops, so that critical points of $\mathscr{a}$ correspond to intersections of L_0 and L_1. Moreover, a critical point of $\mathscr{a}$ is nondegenerate if and only if the corresponding intersection is transversal.

There are two main obstructions against applying standard methods of Morse theory to $\mathscr{a}$. First, in order for $\mathscr{a}$ to satisfy the Palais-Smale condition (see [17]), we would have to extend it to the Sobolev space $H^{1/2}(([0,1],0,1),(P,L,L'))$, which is ill defined unless L and P are linear spaces. Second, one easily verifies that the second derivative of $\mathscr{a}$ at each critical point is a quadratic form with infinite dimensional positive and negative definite subspaces. Hence the critical points cannot be expected to be related to the topology of Ω itself. Because of the first problem, we will not try to define a gradientlike flow for $\mathscr{a}$ with respect to some Hilbert structure on Ω, but proceed as follows: Let $J \in \operatorname{End}(P)$ be an almost complex structure on P, i.e. $J^2 = -\operatorname{id}$, so that the bilinear form $g = \omega(\cdot, J\cdot)$ is a metric on P, i.e. g is positive and symmetric. The triple (ω, J, g) defines an almost Kähler structure on P, i.e. it satisfies all relations of a Kähler structure except that J is not integrable. Then the "L^2-gradient" of $\mathscr{a}$ with respect to g is $\mathscr{g}(z) = J\dot{z}$, since for all $\xi \in T_z\Omega$:

$$\langle J\dot{z}, \xi\rangle := \int_0^1 g(J\dot{z}, \xi) = \int_0^1 \omega(\dot{z}, \xi) = d\mathscr{a}(z)\xi. \tag{1.9}$$

Then define a trajectory

$$u : \mathbf{R} \to \Omega, \quad u(\tau)(t) = u(\tau, t)$$

of the L^2-gradient flow of $\mathscr{a}$ as the solution of the Cauchy-Riemann equation

$$\bar{\partial} u := \frac{\partial u}{\partial \tau} + J\frac{\partial u}{\partial t} = 0. \tag{1.10}$$

In other words, trajectories are holomorphic maps u of the complex manifold

$$\Theta := \mathbf{R} \times [0,1] \cong \{z \in \mathbf{C} \mid 0 \le \operatorname{Im} z \le 1\}$$

into the almost complex manifold (P, J) so that $u(\{0\} \times \mathbf{R}) \subset L$ and $u(\{1\} \times \mathbf{R}) \subset L'$ (compare [12]). This does not define a flow on all of Ω, since the "Cauchy problem" for the (elliptic) Cauchy-Riemann equation is ill posed, i.e. a smooth map $\{0\} \times [0,1] \to P$ representing an element of Ω does not in general extend to a holomorphic map on any neighborhood of $\{0\} \times [0,1]$. On the other hand, however, the ellipticity of $\overline{\partial}$ greatly simplifies the analysis of "bounded" trajectories connecting two critical points. In fact, if all critical points of α are nondegenerate, i.e., if L' meets L transversally, then we can define a Banach manifold $\mathscr{P}(x,y)$ of paths $u : \mathbf{R} \to \Omega$ so that in a precisely defined sense $\lim_{\tau\to\infty} u(\tau) = y$ and $\lim_{\tau\to-\infty} u(\tau) = x$ and so that $u \to \overline{\partial} u$ is a smooth section of a Banach space bundle over $\mathscr{P}(x,y)$. Then the ellipticity of $\overline{\partial}$ ensures that it is a Fredholm section, i.e. the linearization of $\overline{\partial}$ at any trajectory has closed range and a finite dimensional kernel and cokernel. Moreover, an arbitrarily small perturbation of the almost Kähler structure J makes all these linearizations surjective, so that the space of trajectories connecting x with y is a finite dimensional manifold. This procedure was motivated partly by Taubes analysis of instantons on (asymptotically) cylindrical manifolds and uses very similar analytic techniques (see [22]). It was carried out in [7] and the results are summarized in §2. We call this manifold a cell in the Morse complex of bounded trajectories. The dimension of this cell, i.e., the index of $\overline{\partial}$ on $\mathscr{P}(x,y)$, can be computed as the difference $\mu(x) - \mu(y)$ of a suitably defined integer valued "relative Morse index" μ of x and y (see [8]). This is analogous to the gradient flow of a smooth function f on a finite dimensional manifold M: for an open and dense set of metrics on M, the set of trajectories between two critical points is a finite dimensional manifold whose dimension equals the difference of the Morse index of these two critical points (see e.g. [16]).

Since equation (1.10) is invariant under time translations, the above discussion implies that the space of trajectories between critical points of index difference 1 consists of isolated trajectories. The crucial step is to show that this set is always finite. This follows from the compactness considerations in [6] (see also [12] for similar results for holomorphic closed curves and holomorphic discs). We can now define the operator δ of (1.6) as follows: For each $y \in I(L, \phi)$ with $\mu(y) = p$, consider all trajectories ending at p and starting at any critical point of Morse index $p+1$. The above discussion allows us to define $\delta(y)$ as the formal sum of the points x over all such trajectories. However, to obtain an invariant result, we would have to attach a sign to each trajectory. In the finite dimensional theory, this sign is well defined as the "normal" intersection number of the stable manifold of y at the trajectory (see [16]). Since this definition of the sign is difficult to generalize to our infinite dimensional problem, we restrict ourselves in this paper to the mod-2 reduction of δ_p.

It turns out (see Proposition 3.2 below) that $\{\delta_p\}$ satisfies the coboundary property $\delta_p\delta_{p-1} = 0$. We can therefore define the index cohomology

$$I^p(L,\phi;J) = \ker\delta_p/\operatorname{Im}\delta_{p-1}.$$

This is motivated by the following construction: Let S be a compact invariant set of a continuous flow on a finite dimensional manifold. Then its Conley index $I(S) = [X/A, \{A\}]$ is defined in [4] as the pointed homotopy type of the topological quotient of a neighborhood X of S by a suitably defined "exit set" A. It is proved in [9] that its cohomology can be computed by a construction analogous to the definition of I^*. This observation, which represents an alternative proof of the Morse inequalities, has been partly used in [16]. With the field $\mathbf{Z}_2$ replaced by $\mathbf{R}$, it has been studied by Witten in his recent paper [27], which partly prompted our approach to the Morse theory of the symplectic action.

The crucial fact is that $I(S)$ and hence its cohomology is invariant under continuous deformations of the flow as long as the invariant set S remains "isolated" in a certain sense. We will use a similar property of I^*: In §3, we show that the groups $I^p(L,\phi;J)$ are not only independent of J, but also invariant under a change of either L or L' by an exact deformation ϕ_t of P. This even holds if in the course of such a deformation, L and $\phi_t(L')$ have a nontransversal intersection.

In order to complete the proof of Theorem 1, it now suffices to calculate $I^*(L,\phi,J)$ in the case where ϕ is a small exact deformation and J is chosen conveniently. This problem is solved in [9]: If ϕ is induced by a time independent function H on L which is extended to P in a certain well-defined way, then bounded trajectories in $\Omega(L,\phi)$ with respect to J correspond to trajectories of the gradient flow of H on L with respect to $g = \omega(J\cdot,\cdot)$. In particular, the spaces of trajectories in $\Omega(L,\phi)$ are regular if and only if the gradient flow of H on L is of Morse-Smale type. The index $I^*(L,\phi)$ can then be calculated by counting trajectories of the gradient flow on L (see Theorem 3 of [9]). It then follows from Theorem 1 of [9] that it is isomorphic to $H^*(L;\mathbf{Z}_2)$. This completes the proof of Theorem 1.

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2. The Morse complex

In this section, we essentially recall some results from [7] and [8]. For any two Lagrangian submanifolds L and L' of P and for $k > 2/p$, consider the space of L^p_k-paths

$$\mathscr{P}^p_{k;\mathrm{loc}}(L,\phi) = (u \in L^p_{k;\mathrm{loc}}(\Theta, P) \mid u(\mathbf{R}\times\{0\}) \subset L \text{ and } u(\mathbf{R}\times\{1\}) \subset L'\}.$$

An almost Kähler structure is a smooth section of the bundle with fiber

$$S_x = \{J \in \mathrm{End}(T_xP) \mid J^2 = -\,\mathrm{id} \text{ and } \omega(J\cdot,\cdot) \text{ is a metric}\}. \tag{2.1}$$

For technical reasons, it is often convenient to allow J to vary smoothly with $t \in [0,1]$, i.e., we denote by J a smooth section of the trivial extension of S_ω over $[0,1]\times P$. We then replace (1.10) by

$$\overline{\partial}_J u(\tau,t) = \frac{\partial u(\tau,t)}{\partial \tau} + J_t \frac{\partial u(\tau,t)}{\partial t}, \tag{2.2}$$

which is still translationally invariant in τ. We now define the space of bounded trajectories with respect to J as

$$\mathscr{M}_J = \mathscr{M}_J[L,L'] = \{u : \mathbf{R} \to \Omega \mid \overline{\partial} u = 0 \text{ and } \|\nabla u\|_2 < \infty\}.$$

Here, $\nabla u = (\partial u/\partial\tau, \partial u/\partial t)$ and $\|\ \|_2$ is the L^2-norm with respect to the metric g. If no confusion can arise, we will omit the subscript J. It follows from elliptic regularity theory (see Lemma 2.1 of [7]) that each $u \in \mathscr{M}_J$ is smooth. Other results of [7] are summarized in

Proposition 2.1. *For $x_+, x_- \in L \cap L'$, define*

$$\mathscr{M}(x_+,x_-) = \left\{u \in \mathscr{M} \mid \lim_{\tau\to\infty} u(\tau,t) = x_\pm\right\}.$$

Then we have

$$\mathscr{M} = \bigcup_{x,y\in L\cap L'} \mathscr{M}(x,y).$$

Moreover, if L intersects L' transversally, then for each $x,y \in L\cap L'$ there exist smooth Banach manifolds

$$\mathscr{P}(x,y) = \mathscr{P}^p_k(x,y) \subset \mathscr{P}^p_{k;\mathrm{loc}}$$

so that (2.2) defines a smooth section $\overline{\partial}$ of a smooth Banach space bundle over $\mathscr{P}(x,y)$ with fibers $\mathscr{L}_u = L^p(u^*TP)$, and so that $\mathscr{M}(x,y)$ is the zero set of $\overline{\partial}$. The tangent space $T_u\mathscr{P} = T_u\mathscr{P}(x,y)$ consists of all elements ξ of $L^p_1(u^*T)$ so that $\xi(\tau,0) \in TL$ and $\xi(\tau,1) \in TL'$ for all $\tau \in \mathbf{R}$. The linearizations

$$E_u := D\overline{\partial}(u) = T_u\mathscr{P} \to \mathscr{L}_u \tag{2.3}$$

are Fredholm operators for $u \in \mathcal{M}(x,y)$. *There is a dense set* $\mathcal{J}_{\text{reg}}(L,L')$ *of* C^∞*-almost Kähler structures on* P *so that if* $J \in \mathcal{J}_{\text{reg}}(L,L')$, *then* E_u *is surjective for all* $u \in \mathcal{M}(x,y)$.

We will also need certain compactness properties of the Morse complex. Note that the additive group $\mathbf{R}$ operators on each $\mathcal{M}_J(x,y)$ by translation:

$$(\rho * u)(\tau,t) = u(\tau-\rho,t).$$

Proposition 2.2. *Assume that* $J_i \in \mathcal{J}$, $J_i \to J$ *and* $L_i \to L'$ *in* C^∞. *Then each sequence* u_i *in* $\mathcal{M}_i = \mathcal{M}_{J_i}[L,L_i]$ *contains a subsequence converging to a family of adjacent trajectories* $v_\alpha \in \mathcal{M}(z_{\alpha-1},z_\alpha)$, $1 \le \alpha \le N$, *in the following sense: For each* $1 \le \alpha \le N$ *there exists a sequence* $\sigma_i \in \mathbf{R}$ *so that* $\sigma_i * u_i \to v_\alpha$ *locally in* $C^\infty(\Theta,P)$. *Moreover, if* $N=1$, *then* v_α *converges in the relative topology of* $\mathcal{P}(x,y)$.

It follows in particular that restricted to $\tau \ge 0$, the sequence $\rho * u$ for $u \in \mathcal{M}(x,y)$ converges to the constant map uniformly in all derivatives. In fact, if this were not the case, then by reparametrization one could construct a sequence violating Proposition 2.2. In the same way, it follows that for any $\varepsilon > 0$, all but finitely many u_i considered as paths in $C^k([0,1],P) \cap \Omega$ takes values in an ε-neighborhood $U_\varepsilon(v_1,\cdots,v_N)$ of the combined images of V_α.

Let us now consider the situation of Theorem 1. Define $\Omega(\phi) = \Omega(L,\phi)$ as in (1.7).

Proposition 2.3. *Let* $\{\phi_\lambda\}_{\lambda\in\mathbf{R}}$ *be an exact deformation of* P *with* $\phi_0 =$ id. *Then we have bijections*

$$\phi_{\lambda,\mu} : \Omega(\phi_\lambda) \to \Omega(\phi_{\lambda+\mu}) : \phi_{\lambda,\mu}(z)(t) = \phi_{\lambda+t\mu} z(t).$$

Now assume in addition that $\pi_2(P,L) = 0$. *Then* (1.8) *defines a function* $a : \Omega_0(\phi_\lambda) \to \mathbf{R}$. *If* $\|\nabla u\|$ *is defined with respect to the standard metric on* Θ *and the metric* $g = \omega(j\cdot,\cdot)$ *on* P, *then for all* $u \in \mathcal{P}(x,y)$ *we have*

$$\tfrac{1}{2}\|\nabla u\|_2^2 \ge a(x) - a(y), \tag{2.4}$$

with equality if and only if $u \in \mathcal{M}_J(x,y)$.

Proof. It suffices to show that for all smooth maps $u : S^1 \times [0,1] \to P$ with $u(\tau,0) \in L$ and $u(\tau,1) \in L^1$, we have

$$\int_{S^1\times[0,1]} u^*\omega = 0.$$

To this end, let us consider u as a map $u : S^1 \to \Omega$ and define the map

$$H : S^1 \times [0,1]^2 \to P : H(\tau,t,\lambda) = \phi_{0,\lambda}(u(\tau))(t).$$

It follows from the exactness of ϕ_λ and from Stokes' theorem that

$$\int u^*\omega = \int (\phi_1 u)^*\omega.$$

Hence it suffices to consider the case where $L' = L$. Moreover, since we restrict ourselves to the path connected component of the constant elements of $\Omega(L, L)$, we can assume that $u(0)$ is constant. Then we can redefine the map u to yield a map $\overline{u} : (D^2, S^1) \to (P, L)$. Since $\pi_2(P, L) = 0$ and ω is closed, the integral of $u^*\omega$ over D^2 vanishes. This proves that a is well defined. The last assertion follows from

$$|\nabla u|^2 = |\overline{\partial} u|^2 + 2\left\langle \frac{\partial u}{\partial \tau}, J\frac{\partial u}{\partial t} \right\rangle$$

and (1.9). q.e.d.

In [23], Viterbo defined a relative Morse index $\mu_u(x, y)$ for any pair (x, y) of transversal intersections, which in addition depends on the choice of a path u in Ω connecting x and y. In [8], this number was proved to be equal to the Fredholm index of $D\overline{\partial}(u)$. Moreover, it was shown that under the topological restrictions of Theorem 2, $\mu_u(x, y)$ does not depend on u, and that it can be written as the difference of suitably defined Morse indices of x and y. We summarize these results in the following proposition.

Proposition 2.4. *With P, L and ϕ as in Proposition 2.2, there exists a map*

$$\mu : I(L, \phi) \to \mathbf{Z}$$

which is well defined up to an additive constant, so that for $u \in \mathscr{M}(x, y)$:

$$\text{index}(D\overline{\partial}(u)) = \mu(x) - \mu(y). \tag{2.5}$$

If $J \in \mathscr{J}_{\text{reg}}$ as in Proposition 2.1, then (2.5) is the dimension of $\mathscr{M}(x, y)$.

3. The index cohomology

Let $L \subset P$ be as in Theorem 1, and let ϕ be an exact deformation so that $\phi(L)$ meets L transversally. We want to define $I^*(L, \phi; J)$ according to the outline given in the introduction. We also want to show that this group is independent of ϕ and J.

Let us denote by C^* the free $\mathbf{Z}_2$-vector space over the set $I(L, \phi)$. By virtue of the Morse index μ of Proposition 2.4, we have a grading,

$$C^* = \bigoplus_{p \in \mathbf{Z}} C^p, \tag{3.1}$$

where C^p is the free $\mathbf{Z}_2$-vector space over the intersection points of Morse index p. We will define a coboundary operator on C^* by counting trajectories between points in $\mathscr{C}$. Recall from §2 the operation of the translational groups $\mathbf{R}$. Since the action a is strictly decreasing on nonconstant trajectories, it has a slice

$$\widehat{\mathscr{M}}(x,y) \simeq \mathscr{M}(x,y)/\mathbf{R} \simeq \{u \in \mathscr{M}(x,y) \mid a(u(0)) = \tfrac{1}{2}(a(x)+a(y))\}. \tag{3.2}$$

Lemma 3.1. *If $J \in \mathscr{J}_{\mathrm{reg}}(L,\phi(L))$ as in Lemma 2.1, then $\widehat{\mathscr{M}}(x,y)$ is a manifold of dimension $\mu(x)-\mu(y)-1$. If $\mu(x)=\mu(y)+1$, then $\widehat{\mathscr{M}}(x,y)$ is a finite set.*

Proof. The first assertion follows from (3.2). The second assertion follows from Proposition 2.2, since if $\mu(x)-\mu(y)=1$ and $J \in \mathscr{J}_{\mathrm{reg}}$, no family of adjacent trajectories with $N>1$ can exist between x and y. q.e.d.

Now let $\langle\ ,\ \rangle$ denote the canonical $\mathbf{Z}_2$-valued inner product in C^*. Then Lemma 3.1 justifies the following construction.

Definition 3.1. For $\mu(x)=\mu(y)+1$, and $J \in \mathscr{J}_{\mathrm{reg}}(L,\phi_1(L))$, we define

$$\delta = \delta(\phi,J): C^p \to C^{p+1}, \qquad \delta(y) = \sum_{\mu(x)=p+1} x\langle x, \delta y\rangle, \tag{3.3}$$

where $\langle x,\delta y\rangle \in \mathbf{Z}_2$ is the mod-2 number of elements of $\widehat{\mathscr{M}}(x,y)$.

Lemma 3.2. $\delta\delta = 0$.

Proof. We have for $z \in C^{p-1}$ and $x \in C^{p+1}$:

$$\langle x, \delta\delta z\rangle = \sum_{\mu(y)=p} \langle x,\delta y\rangle\langle y,\delta z\rangle. \tag{3.4}$$

Geometrically, this is the number modulo 2 of pairs of adjacent trajectories joining x and z. The crucial observation is now that since both $\mathscr{M}(x,y)$ and $\mathscr{M}(y,z)$ are regular, each such pair of trajectories gives rise to a 1-parameter family of trajectories in $\widehat{\mathscr{M}}(x,z)$. In fact, by Proposition 4.1, pairs of adjacent trajectories between x and z are in 1-1 correspondence with the ends of $\widehat{\mathscr{M}}(x,y)$. Hence their number is even, which proves Lemma 3.2.

Definition 3.2. Let L be a Lagrangian submanifold of a compact symplectic manifold P, and let $\phi \in \mathscr{D}$ be an exact deformation so that $\phi_1(L)$ meets L transversally. Then for every $J \in \mathscr{J}_{\mathrm{reg}}(L,\phi_1(L))$ as in Proposition 2.1, we define the graded group

$$I^p(\phi,J) = \ker[\delta : C^p \to C^{p+1}]/\delta C^{p-1},$$

with $\delta=\delta(\phi,J)$ as in Definition 3.1.

The main result of this section is:

Proposition 3.1. *For any ϕ,ϕ' and J,J' as in Definition 3.2, there is an isomorphism $I^p(\phi,J) = I^p(\phi',J')$ of $\mathbf{Z}_2$-vector spaces.*

Proof. Since $\mathscr{D}$ is connected by definition, we can connect ϕ_0 with ϕ_1 by a smooth family $\{\phi_\lambda\}_{0\leq\lambda\leq 1}$ in $\mathscr{D}$. Of course, we cannot avoid nontransverse intersections along such a deformation. A typical nontransverse intersection is given by the following example: Let us identify (P, ω) locally with the linear symplectic space $(\mathbf{C}^n, \omega)$. Let L_1, L_2 be two transverse linear Lagrangian subspaces of $\mathbf{C}^{n-1}$ and let $\mathbf{R}$, $S^1 \subset \mathbf{C}$ be the real axis and the unit circle around i respectively. Then set

$$L = S \times L_1, \qquad L_\lambda = (\mathbf{R} + i\lambda) \times L_2. \tag{3.5}$$

Lemma 3.3. *Any $\phi_0, \phi_1 \in \mathscr{D}$ such that $\phi_0(L)$, $\phi_1(L) \pitchfork L$ can be connected by a smooth isotopy $\{\phi_\lambda\}_{0\leq\lambda\leq 1}$ so that $\phi_\lambda(L)$ and L have at most one nontransverse intersection in the vicinity of which the deformation has the form* (3.5).

Proof. In a neighborhood of each $y \in \phi_\lambda(L) \cap L$ there exists a Darboux-chart so that L is linear and $\phi_\lambda(L)$ is the graph of the differential df_λ of some smooth function f on L. Clearly, $\phi(L) \cap L$ is the set of critical points of f, and an intersection is transversal if and only if the corresponding critical point is nondegenerate. Now it is well known from singularity theory that any smooth family $\{f_\lambda\}_{0\leq\lambda\leq 1}$ of such functions can be deformed into one so that all critical points of f_λ are either nondegenerate or that the Hessian $D\,\mathrm{grad}_g f$ with respect to some metric on L has a 1-dimensional kernel k on which the quadratic differential $D^2\,\mathrm{grad}_g f_\lambda : k \to k$ is a nondegenerate quadratic function. In the latter case we can deform f_λ locally into a family of functions inducing the deformation (3.5) without introducing any new critical points. By another local deformation, we can change the critical parameter values in case one f_λ should have more than one degenerate critical point. q.e.d.

We will denote the set of critical parameter values by Λ_0. In order to prove the proposition, we first assume that $\Lambda_0 = \varnothing$. In this case, we have smooth families $\{x_\lambda\} = x$ in $I_\Lambda = \{(x, \lambda) \in L \times \Lambda \mid \phi_\lambda(x) \in L\} \subset P \times \Lambda$. Define the parametrized Morse cells

$$\mathscr{M}_\Lambda(x, y) = \{(u, \lambda) \mid u \in \mathscr{M}_{J_\lambda}(x_\lambda, y_\lambda)\}. \tag{3.6}$$

Moreover, let $\pi : \Omega \times [0, 1] \times P \to P$ denote the projection. Then if $J_a, J_b \in \mathscr{J}$, we define with (2.1)

$$\mathscr{J}_\Lambda(J_a, J_b) = \{J \in C^\infty(\pi^* S) \mid J_{|\{a\}\times P} = J_a \text{ and } J_{|\{b\}\times P} = J_b\}.$$

Proposition 3.2. *Let L_λ, $\lambda \in \Lambda = [a,b]$, be a smooth family of Lagrangian submanifolds of P so that $L \pitchfork L_\lambda$ for all $\lambda \in \Lambda$. Assume that $J_a \in \mathscr{J}_{\text{reg}}(L, L_a)$ and $J_b \in \mathscr{J}_{\text{reg}}(L, L_b)$. Then there exists a Baire set $J_{\Lambda,\text{reg}} \subset \mathscr{J}_\Lambda(J_a, J_b)$ so that for each $J_\Lambda \in \mathscr{J}_{\Lambda,\text{reg}}$, the parametrized trajectory space $\mathscr{M}_\Lambda(x,y)$ of* (3.6) *is a smooth manifold of dimension $\mu(x) - \mu(y) + 1$. Moreover, we can assume that for each $\lambda \in \Lambda$, there exists at most one trajectory in $\mathscr{M}_\lambda$ joining intersection points of equal Morse index.*

Proof. Except for the last statement, this is Theorem 5b of [7]. To show how the last statement follows, we briefly review the proof. Let $\mathscr{P}(x_\lambda, y_\lambda)$ and $\mathscr{L}(x_\lambda, y_\lambda)$ denote the Banach manifold and the Banach space bundle of Proposition 2.1 for the pair L, L_λ of Lagrangian subspaces. It was shown in Theorem 3a of [7] that they define a Banach manifold $\mathscr{P}_\Lambda(x,y) = \{(u,\lambda) \mid u \in \mathscr{P}(x_\lambda, y_\lambda)\}$ with a corresponding bundle $\mathscr{L}_\Lambda$, so that $\overline{\partial}(\lambda, u) = \overline{\partial} u$ defines a Fredholm section of $\mathscr{L}_\Lambda$ with Fredholm index $\mu(x) = \mu(y) + 1$. To prove the genericity result, we first define as in Lemma 5.1 of [7] a suitable Banach space $\mathscr{J}_\Lambda$ of smooth perturbations of J_Λ. Let $\pi_1 : \mathscr{P}_\Lambda \times \mathscr{J}_\Lambda \to \mathscr{P}_\Lambda$ be the projection and define the section

$$\overline{\partial} : \mathscr{P}_\Lambda \times \mathscr{J}_\Lambda \to \pi_1^* \mathscr{L}_\Lambda, \qquad \overline{\partial}(u, \lambda, J_\Lambda) \to \overline{\partial}_{J_\lambda} u.$$

Then the proof of [7, Lemma 5.2] applies to show that 0 is a regular value of $\overline{\partial}$ and that the projection

$$\pi_2 : \{(u, J_\Lambda, \lambda) \mid \overline{\partial}_{J_\lambda} u = 0\} \to J_\Lambda$$

is a smooth Fredholm map. For each $J \in \mathscr{J}_\Lambda$, $\pi_2^{-1}([J]) = \mathscr{M}_\Lambda$. Hence the first assertion of Proposition 3.2 follows from the Sard-Smale theorem [7].

In order to prove the second assertion, we consider for any $x, x' \in C^p$ and $y, y \in C^q$ the set

$$X_1 = \{(\lambda, u, v, J) \mid u \in \widehat{\mathscr{M}_\lambda}(x, x'),\ v \in \widehat{\mathscr{M}_\lambda}(y, y'),\ J \in \mathscr{J}_\Lambda\}.$$

Again, this is a Banach manifold. The projection $X_1 \to J_\Lambda$ now has index -1. Hence for every regular value of the projection map, the counterimage is empty. This completes the proof of Proposition 3.2.

It follows from the proof of Lemma 3.1 that if $\Lambda_0 = \emptyset$, then the homomorphism $\delta_\lambda = \delta(\phi_\lambda, J_\lambda)$ and its cohomology I_λ^* is well defined on the complement of the set

$$\Lambda_1 = \{\lambda \mid \exists x, y \text{ so that } \mu(x) = \mu(y) \text{ and } \mathscr{M}_\lambda(x,y) \neq \emptyset\}.$$

In the same way as in the proof of Lemma 3.1, it also follows that Λ_1 is discrete, so that by restricting Λ to a neighborhood of 0, we can assume that $\Lambda_1 = \{0\}$. Then we have

Lemma 3.4. *If $\Lambda_0 = \Lambda_1 = \emptyset$, then $\langle x_0, \delta_0 y_0 \rangle = \langle x_1, \delta_1 y_1 \rangle$.*

Proof. As in the proof of Lemma 3.1, one shows that the set $\widehat{\mathcal{M}}(x,y)$ for $\mu(x)=\mu(y)+1$ is compact for all $\lambda\in\Lambda-\Lambda_1$. Now Lemma 3.4 follows from the fact that the number of ends of the 1-dimensional manifold $\widehat{\mathcal{M}_\Lambda}(x,y)$ is even. q.e.d.

Let us now assume that Λ is a neighborhood of 0 in $\mathbf{R}$ and that $\{0\}=\Lambda_1$, i.e. that there exists exactly one trajectory u connecting two points x_+ and x_- of equal Morse index p (see Proposition 3.2). We can define $\delta=\delta(J_\lambda,\phi_\lambda)$ for $\lambda>0$ and $\gamma=\delta(J_\lambda,\phi_\lambda)$ for $\lambda<0$. The difference between γ and δ is due to the fact that for trajectories ending at x_- or originating at x_+, the proof of Lemma 3.1 breaks down. For example, a sequence of trajectories in $\mathcal{M}(x,x_-)$ may "split" into a pair $(u,\overline{u})$ with $u\in\mathcal{M}(x,x_+)$. However, this lack of compactness can be measured algebraically:

Lemma 3.5. *For $y\in C^{p-1}$;*

$$\gamma y=\delta y+x_+\langle x_-,\delta y\rangle. \tag{3.7}$$

Moreover, we have

$$\gamma x_-=\delta x_-+\delta x_+. \tag{3.8}$$

For all other generators of C^*, we have $\gamma=\delta$.

Proof. Whenever $x\neq x_+$ and $y\neq x_-$, Proposition 2.2 yields compactness even for $\mathcal{M}_0(x,y)$, so that we conclude as in Lemma 3.4 that $\langle x,\delta y\rangle=\langle x,\gamma y\rangle$. Now consider $x\in C^{p+1}$. By Proposition 4.2, there exists for ρ large enough a local diffeomorphism

$$\#\overline{u}:\widehat{\mathcal{M}}(x,x_+)\times(\rho,\infty)\to\widehat{\mathcal{M}_\Lambda}(x,x_-),\qquad(u,\rho)\to u\#_\rho\overline{u}.$$

On the other hand, the compactness result of Proposition 2.2 together with the uniqueness statement of Proposition 4.2 implies that the complement of the image of $\#\overline{u}$ in $\widehat{\mathcal{M}_\Lambda}(x,x_-)$ is compact. It follows that the number of trajectories in $\mathcal{M}_\varepsilon(x,x_-)$ and $\mathcal{M}_{-\varepsilon}(x,x_-)$ differs modulo 2 by the number of trajectories in $\mathcal{M}_0(x,x_+)$. This proves (3.7). In the same way, we prove (3.8) by considering for $y\in C^{p-1}$ the map

$$\widehat{\mathcal{M}_0}(x_-,y)\times(0,\infty)\to\widehat{\mathcal{M}_\Lambda}(x_+,y),\quad(v,0)\to v\#\overline{\mu}.$$

Now the invariance of I^*_λ follows by purely algebraic means:

Lemma 3.6. *The map $\phi:C^*\to C^*$ defined by $\phi(x)=x+x_+\langle x_-,x\rangle$ satisfies $\phi^2=\mathrm{id}$ and*

$$\delta\phi=\phi\gamma. \tag{3.9}$$

Proof. The first assertion is obvious. To show that ϕ is a chain map, we calculate for $y\in C^{p-1}$:

$$\delta\phi(y)=\delta y=\gamma y+x_+\langle x_-,\gamma y\rangle=\phi(\gamma y).$$

Moreover, we have

$$\delta(x_-) = \delta(x_- + x_+) = \gamma(x_- + x_+) - \gamma x_+ = \gamma x_- = \phi(\gamma x_-).$$

For all other generators of C^*, equation (3.9) is obvious. q.e.d.

We have so far proved the continuation result under the hypotheses that all intersections remain transverse, i.e., that the intersection set does not change essentially. Since we want to apply the index cohomology to estimate the number of intersections, the crucial step is to show that it remains invariant even when intersections vanish through nontransverse intersections as in (3.5). As above, we set the critical parameter value to zero. Let us denote by (C^*, γ) and (D^*, δ) the chain complexes for small negative and small positive values of λ, respectively.

It is here that we make use of the variational structure of the problem. Because of relation (2.4) with a globally well-defined function a on Ω, $||\nabla u||_2$ for trajectories $u \in \mathscr{M}(y_+, y_-)$ can be estimated by $a(y_+) - a(y_-)$, which converges to zero for $\lambda \to 0$. It follows from Proposition 2.2 that for λ small enough, all trajectories in $\mathscr{M}(y_+, y_-)$ are contained in a small neighborhood of the nontransverse intersection y. Now it follows from [9, Theorem 1], that for small negative λ,

$$\langle y_+, \delta y_- \rangle = 1. \tag{3.10}$$

Let $\pi : C^* \to D^*$ denote the projection, i.e.

$$\pi x = x - y_- \langle y_-, x\rangle - y_+\langle y_+, x\rangle.$$

Let $i : D^* \to D^*$ denote the homomorphism induced by the inclusion. Unfortunately, these two homomorphisms are not chain homomorphisms in general. This is due to the fact that trajectories from x to y_- and from y_+ to z may merge to trajectories from x to z. As in the case above, we have to measure this phenomenon in algebraic terms. This will be carried out in §5. More precisely, the following formula follows from Proposition 5.1:

Lemma 3.7. *For $\mu(x) = p$, we have $\delta x = \pi\gamma(x + y_-\langle y_+, \gamma x\rangle)$. For $\mu(x) \neq p$, we have $\delta x = \pi\gamma x$.*

Now define $\phi : C^* \to D^*$ by $\phi x = \pi(x + \gamma y_- \langle y_+ x\rangle)$. In a less formal notation, we have $\delta x_+ = \pi\gamma x_-$, $\delta x_- = 0$, and $\delta x = x$ otherwise. Moreover, define $\psi : D^* \to C^*$ by $\psi x = ix + y_-\langle y_+, \gamma i x\rangle$.

Lemma 3.8. *ψ and ϕ are chain maps.*

Proof. For $\mu(x) = p$, we have $\phi x = x$ and

$$\begin{aligned}\delta\phi x = \delta x &= \pi(\gamma x + \gamma y_- \langle y_+, \gamma x\rangle) \\ &= \pi\gamma(x + y_-\langle y_+, \gamma x\rangle) = \phi\gamma x.\end{aligned}$$

For $x \in C^{p-1}$, we have

$$\phi x = \pi\gamma(\phi x) = \pi(\gamma x) = \phi(\gamma x).$$

For $x \in C^{p+1}$, we have

$$\delta\phi x = \delta(\pi(x + \gamma y_- \langle y_+, x\rangle)) = \delta\pi x + \delta\pi\gamma x + \langle y_+, x\rangle.$$

If $x \neq y$, this yields

$$\delta\phi x = \delta x = \gamma x = \phi\gamma x.$$

If $x = y_+$, we obtain

$$\begin{aligned}\delta\phi x &= \delta\pi\gamma y_- = \gamma\pi\gamma y_- = \gamma(\gamma y_- + y_+\langle y_+\gamma_-\rangle) \\ &= \gamma(y_+)\langle y_+\gamma y_-\rangle = \gamma y_+\end{aligned}$$

by (3.10). Since $\gamma y_+ = \phi\gamma y_+$, this completes the proof of the chain property of ϕ for $\mu(x) = p+1$. For $\mu(x) > p+1$ or $\mu(x) < p-1$, the chain property of ϕ is obvious.

To prove the chain property for ψ, consider $x \in D^{p-1}$. We have

$$\begin{aligned}\psi(\delta x) &= \psi(i\pi\gamma x) = i\pi\gamma x + y_-\langle y_+\gamma i\pi\gamma x\rangle \\ &= i\pi\gamma x + y_-\langle y_+, \gamma(\gamma x - y_-\langle y_-\gamma x\rangle)\rangle \\ &= i\pi\gamma x + y_-\langle y_+, \gamma y_-\langle y_-\gamma x\rangle\rangle \\ &= i\pi\gamma x + y_-\langle y_+, \gamma y_-\rangle\langle y_-, \gamma x\rangle \\ &= i\pi\gamma x + y_-\langle y_-\gamma x\rangle = \gamma x = \gamma\psi x.\end{aligned}$$

For $x \in D^p$, we have

$$\begin{aligned}\psi(\delta x) &= i(\delta x) = i\pi\gamma(x + y_-\langle y_+, \gamma i x\rangle) \\ &= i\pi\gamma\psi(x) = \gamma\psi(x).\end{aligned}$$

The last equality holds because the y_+-component of $\gamma\psi(x)$ vanishes:

$$\begin{aligned}\langle y_+, \gamma_\phi(x)\rangle &= \langle y_+, \gamma(x + y_-\langle y_+, \gamma i x\rangle)\rangle \\ &= \langle y_+, \gamma x\rangle + \langle y_+, \gamma y_-\rangle\langle y_+, \gamma x\rangle \\ &= (1 + \langle y_+, \gamma x_-\rangle)\langle y_+, \gamma x\rangle = 0,\end{aligned}$$

by (3.10). The other cases are obvious. q.e.d.

We can now define the induced maps $\psi_\#$ and $\phi_\#$ on the index groups.

Lemma 3.9. *$\psi_\#$ and $\phi_\#$ are inverse to each other.*

Proof. Since $\phi(y_-) = 0$, we calculate that $\phi \circ \psi : D^* \to D^*$ is even equal to the identity:

$$\begin{aligned}\phi(\psi(x)) &= \phi(ix + y_-\langle y_+, \gamma i x\rangle) \\ &= \phi(ix) + \phi(y_-)\langle x_+, \gamma i x\rangle \\ &= \phi(ix) = ix + \gamma x_-\langle x_-, ix\rangle = x.\end{aligned}$$

On the other hand, $\psi \circ \phi$ is not the identity, but it induces the identity in cohomology since the homomorphism $\beta = y_-\langle y_+|$ is a chain homotopy (see, for example, [20]) between $\psi\circ\phi$ and the identity on C^*, i.e., $\psi\phi - \text{id} = \beta\gamma + \gamma\beta$. To see this, first calculate

$$(\beta\gamma + \gamma\beta)x = y_-\langle y_+, \gamma x\rangle + \gamma y_-\langle y_+, x\rangle.$$

Then it follows that

$$\psi(\phi y_+) = \psi(\pi\gamma y_-) = i\pi\gamma y_- = \gamma y_- + y_+.$$

For all other generators of D^*, we have

$$\psi(\phi x) = x + y_-\langle y_+, \gamma x\rangle.$$

This completes the proof of Lemma 3.9 and hence of Proposition 3.1.

4. Gluing trajectories

By Proposition 2.2, the ends of $\widehat{\mathscr{M}}(x,y)$ correspond to families of adjacent trajectories connecting x and y. In this section we want to show that in the "generic" situation $J \in \mathscr{J}_{\text{reg}}(L, \phi(L))$ of Proposition 2.1, the converse is true: any pair $(u,v) \in \widehat{\mathscr{M}}(x,y) \times \widehat{\mathscr{M}}(y,z)$ gives rise to divergent families in $\widehat{\mathscr{M}}(x,z)$. The construction of these families proceeds essentially in the same way as Taubes' construction [21] of noncompact instanton families on 4-dimensional manifolds. We first define "approximate" trajectories using "cutoffs". Note that for $u \in \widehat{\mathscr{M}}(x,y)$ and τ large enough, there exists $\xi(\tau,t)$ such that

$$u(\tau,t) = (\exp_y \xi)(\tau,t) = \exp_y(t, \xi(\tau,t)). \tag{4.1}$$

Here, $\exp_y : [0,1] \times T_yP \to P$ is a smooth family of charts of P such that $\exp_y(0, T_yL) \subset L$ and $\exp_y(1, T_yL') \subset L'$. If $K \subset \widehat{\mathscr{M}}(x,y)$ is a compact subset, then there exists a constant $\rho_0 = \rho_0(K)$ such that (4.1) holds for all $u \in K$ and $\tau \geq \rho_0$. Then, we also have a decreasing function ε: $[\rho_0, \infty) \to \mathbf{R}_+$ with $\lim_{\rho\to\infty} \varepsilon(\rho) = 0$ and

$$||\xi|_{[\rho,\infty)\times[0,1]}||_{1,p} \leq \varepsilon(\rho) \tag{4.2}$$

uniformly in $u \in K$. Similarly, if K' is a compact subset of $\widehat{\mathscr{M}}(y,z)$, we can choose ρ_0 and ε so that (4.1) and (4.2) hold for all $u(\tau,t) = v(-\tau,t)$, $v \in K'$. Now let $\beta : \mathbf{R} \to [0,1]$ be a smooth function with $\beta(\tau) = 0$ for $\tau \leq 0$ and $\beta(\tau) = 1$ for $\tau \geq 1$.

Definition 4.1. For compact sets $K \subset \widehat{\mathscr{M}}(x,y)$ and $K' \subset \widehat{\mathscr{M}}(y,z)$ we define the map

$$K \times [\rho_0, \infty) \times K' \to \mathscr{P}(x,z),$$

(4.3)

$$\chi = (u,\rho,v) \to w_\chi(\tau,t) = \begin{cases} u(\tau+\rho,t) \text{ for } \tau \le -1, \\ \exp_y(t, \beta(-\tau)\xi(\tau+\rho,t) + \beta(\tau)\varsigma(\tau-\rho,t)) \\ \qquad\qquad\qquad \text{for } -1 \le \tau \le 1, \\ v(\tau-\rho,t) \text{ for } \tau \ge 1. \end{cases}$$

and ξ, ς are defined so that $u = \exp_y \xi$ for $\tau \ge \rho - 1$ and $v = \exp_y \varsigma$ for $\tau \le -\rho + 1$.

It is easy to see that Definition 4.1 defines a continuous map, and that by (4.2), its image is almost holomorphic in the following sense.

Lemma 4.1. *For every compact set $K \subset \widehat{\mathscr{M}}(x,y)$ and $K' \subset \widehat{\mathscr{M}}(y,z)$ there exists a decreasing function $\varepsilon : \mathbf{R}_+ \to \mathbf{R}_+$ with $\lim_{\rho\to\infty} \varepsilon(\rho) = 0$ so that for $(u,v) \in K \times K'$ and $\rho \ge \rho_0$,*

$$||\bar{\partial} w_\chi||_p < \varepsilon(\rho(\chi)).$$

Now the main result of this section can be stated as follows: Consider on Ω the metric $\operatorname{dist}(x,y) = \max_{0\le t\le 1} \operatorname{dist}(x(t), y(t))$ for some metric on P. Then for every (u,v) as above, define

$$U_\varepsilon(u,v) = \{z \in \Omega \mid \operatorname{dist}(z, u(\tau)) \le \varepsilon \text{ or } \operatorname{dist}(z, v(\tau)) \le \varepsilon \text{ for some } \tau \in \mathbf{R}\}. \tag{4.4}$$

Proposition 4.1. *Assume that x, y, z are transverse intersections and that $K \subset \widehat{\mathscr{M}}(x,y)$ and $K' \subset \widehat{\mathscr{M}}(y,z)$ are compact sets containing only regular trajectories. Then there exist positive constants ρ_0 and C and a smooth map*

$$\# : K \times [\rho_0, \infty) \times K' \to \widehat{\mathscr{M}}(x,z), \qquad \chi \to \exp_{W_\chi}(\xi_\chi)$$

with $||\xi_\chi||_{1,p} \le C||\bar{\partial} w_\chi||$. Moreover, for u and v in the interior of K and K', there exists $\varepsilon > 0$ so that $\widehat{\mathscr{M}}(x,z) \cap U_\varepsilon(u,v)$ is contained in the image of $\#$.

A trajectory u is called regular if E_u is surjective. There also exists a parametrized version of Proposition 4.1. We restrict ourselves here to the simplest case, which is needed in the proof of Lemma 3.2 above. Assume that $\mu(x) = \mu(y)$ and $(u,\lambda) \in \mathscr{M}_\Lambda(x,y)$ (see (3.6)). If in addition $v \in \widehat{\mathscr{M}_\lambda}(y,z)$, then by Proposition 2.2 we expect to find trajectories in the ε-tube

$$U_\varepsilon(u,v;\Lambda)P = \{(z,\mu) \in C^0([0,1],P) \times \Lambda \mid |\lambda - \mu| \le \varepsilon \text{ and there exists } \varepsilon \in \mathbf{R}_+ \text{ so that } \operatorname{dist}(z, w(\tau)) < \varepsilon \text{ for } w = u \text{ or } w = v\}.$$

Proposition 4.2. *Assume that L_λ, $\lambda \in \mathbf{R}$, is a smooth family of Lagrangian embeddings and that $x, y, z \in L \cap L_0$ are transverse intersections. Moreover, assume that $\mu(x) = \mu(y)$ and that $u \in \widehat{\mathscr{M}_\lambda}(x, y)$ is a nondegenerate zero of $\overline{\partial}$ in the sense of Proposition* 3.2. *Then if $K \subset \widehat{\mathscr{M}_\lambda}(y, z)$ is compact and contains only regular trajectories, there exist positive constants ρ_0 and C and a local diffeomorphism*

$$\#_\rho : \widehat{\mathscr{M}_\lambda}(x, y) \times [\rho_0, \infty) \times K \to \mathscr{M}_\Lambda(x, z),$$
$$\chi = (u, \rho, v) \to \exp_{w_\chi} \xi,$$

where w_χ is as in Definition 4.1 *and $||\xi||_{1,p} \le C||\overline{\partial} w_\chi||_p$. Moreover, for every (u, v) in the interior of $K \times K'$ there exists an $\varepsilon > 0$ so that $\mathscr{M}_\Lambda(x, z) \cap U_\varepsilon(u, v; \Lambda)$ is contained in the image of $\#$.*

The main tool of the proofs of Propositions 4.1 and 4.2 is the following version of Newton's, or rather Picard's method:

Lemma 4.2. *Assume that a smooth map $f : E \to F$ between Banach spaces E and F has an expansion*

$$f(\xi) = f(0) + Df(0)\xi + N(\xi),$$

so that $Df(0)$ has a finite dimensional kernel and a right inverse G and so that for $\xi, \varsigma \in E$:

$$||GN(\xi) - GN(\varsigma)||_E \le C(||\xi||_E + ||\varsigma||_E)||\xi - \varsigma||_E$$

for some constant C. Set $\varepsilon = (5C)^{-1}$. Then if $||Gf(0)||_E \le \varepsilon/2$, the zero set of f in $B_\varepsilon = \{\xi \in E \mid ||\xi|| < \varepsilon\}$ is a smooth manifold of dimension equal to the dimension of $\ker df$. *In fact, if we define*

$$K_\varepsilon = \{\xi \in \ker Df(0) \mid ||\xi||_E < \varepsilon\},$$

then there exists a smooth function

$$\phi : K_\varepsilon \to K^\perp := GF \subset E,$$

with $\phi(\xi + \phi(\xi)) = 0$ so that all zeros of f in B_ε are of the form $\xi + \phi(\xi)$. Moreover, we have

$$||\phi(\xi)||_E \le 2||Gf(0)||_E.$$

The proof of Lemma 4.2 proceeds by the Banach fixed point theorem and can be found in [10].

Proof of Proposition 4.1. In Proposition 2.1, we quoted from [7] the fact that $\overline{\partial}$ is a smooth section of a smooth Banach space bundle $\mathscr{L}$ over $\mathscr{P}(x, z)$. In fact, explicit charts of $\mathscr{P}(x, z)$ and trivializations of $\mathscr{L}'$ were given in Theorem 3 of [7], which convert $\overline{\partial}$ locally at w into a (Frechet) smooth function

$$\begin{aligned} &\overline{\partial}_w : T_w\mathscr{P} \to L^p(w^*TP), \\ &\overline{\partial}_w(\xi) = \overline{\partial}(w) + E_w\xi + N_w(\xi). \end{aligned} \tag{4.5}$$

E_w is linear and for $w \in \mathscr{M}(x,z)$ coincides with the linearization of the section $\bar\partial$ at w. The nonlinear part N_w satisfies the estimate

$$(4.6) \qquad ||N_w(\xi) - N_w(\xi)||_p \le C(w)(||\xi||_{1,p} + ||\varsigma||_{1,p})||\xi - \varsigma||_{1,p},$$

with $C(w)$ depending only on $||\nabla w||_\infty$. We want to apply Lemma 4.2 to (4.5) with w replaced by w_χ for $\rho(\chi)$ large enough. Then the constant $C(w_\chi)$ is bounded for $u, v \in K \times K'$ and $\rho \le \rho_0$. Hence by Lemma 4.1, it suffices to show that there exists a family of right inverses G_{w_χ} of E_{w_χ} which are bounded independently of χ if $\rho(\chi)$ is large enough.

Lemma 4.3. *There exist constants C_G and ρ_0 so that if $\chi = (u,\rho,v) \in K \times [\rho_0, \infty) \times K'$ and $w = w_\chi$, then there exists a continuous right inverse $G : L^p(w) \to W(w)$ of L_w with*

$$||G\xi||_{1,p} \le C_G||\xi||_p.$$

Proof. For $\xi \in \ker E_u = T_u\mathscr{M}(x,y)$ and $\varsigma \in \ker E_v = T_v\mathscr{M}(y,z)$, we define

$$(\xi\widehat{\#}\varsigma)(\tau) = \begin{cases} \beta(\tau+1)\xi(\tau) & \text{for } \tau > 1, \\ 0 & \text{for } \tau \in [-1,1], \\ \beta(\tau-1)\varsigma(t) & \text{for } \tau < -1, \end{cases}$$

with β as in (4.3). Denote by $W^\perp(w)$ the L^2-orthogonal complement of these sections in $T_w\mathscr{P}$. We want to show that for all $\xi \in W^\perp(w)$,

$$(4.7) \qquad ||\xi||_{1,p} \le C_G||E_w\xi||_p.$$

Since the Fredholm index $\mu(x) - \mu(z)$ of E_w is equal to the dimension of $\ker E_u \oplus \ker E_v$, this proves Lemma 4.3. To prove (4.7), assume the contrary. Then there exists a family $(u_\alpha, v_\alpha) \in K \times K'$ and $\rho_\alpha \to \infty$ so that with $\chi_\alpha = (u_\alpha, \rho_\alpha, v_\alpha)$ and $w_\alpha = w_{\chi_\alpha}$, there exists $\xi_\alpha \in W^\perp(w_\alpha)$ satisfying

$$(4.8a) \qquad |\xi_\alpha||_{1,p} = 1,$$

$$(4.8b) \qquad ||E_\alpha\xi_\alpha||_p \to 0$$

with $E_\alpha = E_{w_\alpha}$. We will derive a contradiction to (4.8a). Therefore, consider first the vector field $\xi_{0\alpha}$ defined by

$$(4.9) \qquad D_2 \exp_y(t, \xi_{0\alpha}(\tau,t)) = \xi_\alpha(\tau,t)$$

on $[-3,3] \times [0,1]$. Here, ς is defined so that $\exp_y \varsigma = w$ on this interval as in (4.1). Our first aim is to show that

$$(4.10) \qquad \xi_{0\alpha} \to 0 \quad \text{in } L^p([-3,3] \times [0,1], T_yP).$$

Note therefore that $\xi_{0\alpha}$ is defined by (4.9) on increasing subsets $\Theta_\alpha \uparrow \Theta$. Choosing a suitable sequence of cutoff functions β_α, we find that the sequence

$\beta_\alpha \xi_{0\alpha}$ of smooth maps $\Theta \to T_yP$ is bounded in $L_1^p(\Theta, T_yP)$. Hence there exists a weakly convergent subsequence whose weak limit ξ_∞ satisfies $\overline{\partial}_0 \xi_\infty = 0$, where $\overline{\partial}_0$ is the standard Cauchy-Riemann operator. However, with the given boundary conditions, this operator has no kernel in L^p, so that $\xi_\infty = 0$. Now a compact Sobolev embedding implies (4.10).

Now choose a cutoff function $\beta_0 : \mathbf{R} \to [0,1]$ so that $\beta_0(\tau) = 0$ for $|\tau| > 3$ and $\beta_0(\tau) = 1$ for $|\tau| < 2$. Then we have

$$\begin{aligned}||\beta_0 \xi_\alpha||_{1,p} &\leq C_1 ||\beta_0 \xi_{0\alpha}||_{1,p} \leq C_2 ||\overline{\partial}_0 \beta_0 \xi_{0\alpha}|| \\ &\leq C_2(||\beta_0 \overline{\partial}_0 \xi_\alpha||_p + ||\beta_0' \xi_\alpha||_p) \\ &\leq C_3(||E_\alpha \xi_\alpha||_p + ||\beta_0' \xi_\alpha||_p).\end{aligned}$$

This converges to zero by (4.8b) and (4.10). Hence ξ converges to zero in the L_1^p-norm on Θ_2.

Now define $\xi_\alpha^\pm = \beta_\pm \xi_\alpha$ for $\beta_\pm(\tau) = \beta(\pm(\tau+1))$. We want to show that

$$\begin{aligned}\zeta_\alpha &= \rho_\alpha * \xi_\alpha^- \to 0 \quad \text{in } L_1^p(u_\alpha^* TP), \\ \eta_\alpha &= -\rho_\alpha * \xi_\alpha^+ \to 0 \quad \text{in } L_1^p(v_\alpha^* TP).\end{aligned} \tag{4.11}$$

This will complete the proof of Lemma 4.3, since then

$$||\xi_\alpha||_{1,p} \leq ||\beta_0 \xi_\alpha||_{1,p} + ||\xi_\alpha^+||_{1,p} + ||\xi_\alpha^-||_{1,p}$$

converges to zero, in contradiction to (4.8a). To prove (4.11), we use the fact that E_u is uniformly invertible away from its kernel for $u \in K$. Therefore,

$$\begin{aligned}||\zeta_\alpha||_{1,p} &\leq C||E_{u_\alpha} \zeta_\alpha||_p = C||E_{u_\alpha} \xi_\alpha^+||_p \\ &\leq C(||\beta_- E_{u_\alpha}||_p + ||\beta_-' \xi_\alpha||_p)\end{aligned}$$

converges to zero by (4.8b) and by (4.10). This completes the proof of Lemma 4.3.

It remains to prove that the map $\#$ is surjective onto $\widehat{\mathscr{M}_\varepsilon} := \widehat{\mathscr{M}}(x,z) \cap U_\varepsilon(u,v)$. By the uniqueness assertion in Lemma 4.2, it suffices to show that for each $\delta \to 0$ there exists an $\varepsilon > 0$ so that if $w \in U_\varepsilon(u,v)$, then $w_\chi = \exp_w \xi$ in the sense of (4.1) with $||\xi||_{1,p} < \delta$. Define a map $s : \widehat{\mathscr{M}_\varepsilon} \to \mathbf{R}$ by

$$a(w(s(w))) = a(z).$$

By Proposition 2.1, we know that for $\varepsilon(u)$ small enough and $s(w) - 1 \leq \tau \leq s(w) + 1$, we have $w = \exp_z \eta$ with

$$||\eta||_{1,p} \leq \phi_1(\varepsilon). \tag{4.12}$$

Here and in the following, we denote by $\phi_\alpha : \mathbf{R} \to \mathbf{R}$ a continuous map which is independent of u and satisfies $\phi_\alpha(0) = 0$. We can define an element of $\mathscr{P}(x,y)$ by

$$\widehat{u} = \begin{cases} \exp_y \beta(-(s+\cdot))\xi & \text{for } \tau > s-1, \\ u & \text{otherwise.} \end{cases}$$

Similarly, we define $\widehat{v} \in \mathscr{P}(y,z)$. Moreover, define $\sigma, \tau \in \mathbf{R}$ by $a(u(\sigma)) = \frac{1}{2}(a(x)+a(y))$ and $a(v(\tau)) = \frac{1}{2}(a(y)+a(x))$. Then for $\widehat{\chi} = (\sigma * \widehat{u}, \sigma+\tau, \tau * \widehat{v})$, we have $w_{\widehat{\chi}} = \exp_x \xi$ with $\|\xi\|_{1,p} \leq \phi_2(\varepsilon)$. To show that $\sigma * \widehat{u}$ is close to u, note that by Proposition 2.2, we have

$$\sigma * \widehat{u} = \exp_u \xi \quad \text{with } \|\xi\|_{1,\infty} < \phi_3(\varepsilon). \tag{4.13}$$

In order to obtain estimates in integral norms, we expand as in (4.5),

$$\overline{\partial}_u : W_1^p(u^*TP) \to L^p(u^*TP), \qquad \overline{\partial}_u(\xi) = E_u\xi + N_u(\xi).$$

The zero order term vanishes since u is holomorphic. Since u is regular and since on the finite dimensional kernel of E_u all norms are equivalent, we have

$$\begin{aligned}\|\xi\|_{1,p} &\leq C(\|E_u\xi\|_p + \|\xi\|_\infty),\\ &\leq C(\|\overline{\partial}_u(\xi) - N_u(\xi)\|_p + \|\xi\|_\infty)\\ &\leq C(\|\overline{\partial}\widehat{u}\|_p + \|N_u(\xi)\|_p) + \phi_3(\varepsilon).\end{aligned} \tag{4.14}$$

Note that $\|\xi\|_p < \infty$ since $u \in \mathscr{P}(x,y)$ and $\widehat{u}$ is constant outside a compact set. Now we have estimates $\|\overline{\partial}\widehat{u}\| \leq \phi_4(\varepsilon)$. Moreover, by Theorem 3a of [7], we have

$$\|N_u(\xi)\|_p \leq C\|\xi\|_{1,\infty}\|\xi\|_p \leq \phi_5(\varepsilon)\|\xi\|_p$$

with C depending only on u. It follows that

$$\|\xi\|_{1,p} \leq \phi_6(\varepsilon)\{1 + \|\xi\|_p\},$$

which for ε small enough proves that $\lim_{\varepsilon\to 0}\|\xi\|_{1,p} = 0$. This completes the proof of Proposition 4.1.

Proof of Proposition 4.2. We apply Lemma 4.2 to the map (see [7, Theorem 3a])

$$\overline{\partial}_w : T_w\mathscr{P}_\Lambda \to L^p(w^*TP) \tag{4.15}$$

for $w = w_\chi$. Here, $T_w\mathscr{P}_\Lambda = T_w\mathscr{P} \oplus \mathbf{R}$ is the tangent space of the parametrized path space (see also the proof of Proposition 3.4 above). The additional dimension is generated by a certain section X of w^*TP. The estimates on the constant and the nonlinear parts of (4.15) are the same as in Lemma 4.1 and (4.6) (for the latter see Theorem 3a of [7]). To prove the invertibility of the linear part, we proceed as in the proof of Lemma 4.3. We have to show that for every $(\xi, \lambda X) \in T_{w_\chi}\mathscr{P}_\Lambda$ with $\xi \perp \ker E_u \# \ker E_v$,

$$\|\xi\|_{1,p} + |\lambda| \leq \mathscr{C}\|E_{w_\chi}(\xi + \lambda X)\|_p \tag{4.16}$$

for $\rho(\chi)$ large enough. Since the index of E_{w_χ} on $T_{w_\chi}\mathscr{P}$ is equal to $\dim\ker E_u + \dim\ker E_v$, (4.16) implies that E_{w_χ} is surjective and therefore has a right inverse. Again, the proof of (4.16) proceeds indirectly. For a sequence $\widehat{\xi}_\alpha = \xi_\alpha + \lambda_\alpha X$ violating the assertion, we first show that ξ_α converges to zero near the value of τ where the gluing takes place. The crucial point is then that $\lambda_\alpha \to 0$ because E_v is invertible on the parametrized space away from v'. One then obtains a contradiction as in the proof of Lemma 4.3. The proof of the uniqueness property proceeds as in the case of Proposition 4.1.

5. Vanishing critical points

It is the aim of this section to prove the formula of Lemma 3.7 for the change of the coboundary operator at a nontransversal intersection y as in Lemma 3.3. On a neighborhood of y let us fix the Kähler structure (g, J) corresponding to the Kähler structure on $\mathbf{C}^n$. We consider an exact deformation of L' generated by a time independent Hamiltonian H on P which is supported in this neighborhood of y and which has the form

$$H(z_1, \cdots, z_n) = \operatorname{Re} z_1$$

on some smaller neighborhood U of y. Clearly, under such a deformation, there exists for $\lambda < 0$ small enough a pair of transverse intersections y_λ^+, y_λ^- in U, whereas for $\lambda > 0$, U does not contain any intersections at all. If we denote the index of y_λ^- by p, then the index of y_λ^+ is $p+1$ (see [9]). Let us extend J to an almost Kähler structure for λ on P and denote as usual the Morse cells for ϕ_λ by $\mathscr{M}_\lambda(x, z)$. Then Lemma 3.7 follows from the following existence result:

Proposition 5.1. *Let $x, z \in L \cap L'$ be the transverse intersections with $\mu(x) = \mu(y_+)$ and $\mu(y) = \mu(y_-)$. Then there exists J as above and $\varepsilon > 0$ so that we have bijections between finite sets*

$$\widehat{\mathscr{M}}_\varepsilon(x, z) \simeq \widehat{\mathscr{M}}_{-\varepsilon}(x, z) \cup (\widehat{\mathscr{M}}_{-\varepsilon}(x, y_{-\varepsilon}^-) \times \widehat{\mathscr{M}}_{-\varepsilon}(y_{-\varepsilon}^+, z)).$$

For a proof, consider the selfadjoint operator $A_y := J(y)d/dt$ on $L^2([0,1], T_yP)$ whose domain is given by the boundary conditions $\xi(0) \in T_yL$ and $\xi(1) \in T_yL'$. It can be considered as the Hessian of a at the critical point $y \in \Omega$. The special problems for nontransverse intersections arise because the Fredholm property of E_u in Proposition 2.1 depends crucially on the fact that A_x and A_y have no zero eigenvalues (see Theorem 4 of [7]). This is closely related to the rate of convergence of trajectories at x and y, which is exponential in the nondegenerate case. If L and L' have a common tangent vector at y, i.e. if A_y has a zero eigenvalue, then Theorem 4 of [7] only implies that $\overline{\partial}$ is a Fredholm section of a suitable bundle over a certain Banach manifold $\mathscr{P}_{(0,\varepsilon)}(x, y)$ consisting of paths which a-priori converge exponentially at y. The exponential rate ε has to be positive but smaller than the first positive eigenvalue of A_y. By Theorem 2 of [8], its Fredholm index can be calculated by considering the "spectral flow" along u of a certain family A_z, $z \in \Omega$, of selfadjoint operators extending the operators A_x for $x \in L \cap L'$. More precisely, if x and z are transverse intersections, then the Fredholm index of $\bar\partial$ on $\mathscr{P}(x, y)$ is equal to the number of eigenvalue families of $A_{u(\tau)}$ with $\lim_{\tau\to-\infty} a(\tau) < 0$ and $\lim_{\tau\to\infty} a(\tau) > 0$. On $\mathscr{P}_{(0,\varepsilon)}(x, y)$, it is equal to the number of those families where $\lim_{\tau\to-\infty} a(\tau) < 0$ and $\lim_{\tau\to\infty} a(\tau) > \varepsilon$. One can now verify that the index of $\bar\partial$ on $\mathscr{P}_{(0,\varepsilon)}(x, y)$ must be one less than the index of $\bar\partial$ on $\mathscr{P}(x, y_-)$, which is $\mu(x) - p$. In particular, in the situation of Proposition 5.1, it is zero. Since the genericity result of Proposition 2.1 still holds on these Banach manifolds (see Theorem 5 of [7]), this implies that for $J \in \mathscr{J}_{\text{reg}}$, $\mathscr{M}(x, y) \cap \mathscr{P}_{(0,\varepsilon)}(x, y)$ is empty, i.e., no trajectory in $\mathscr{M}(x, y)$ has exponential decay. Since for possible future applications we would like to consider the general case, we define

$$\mathring{\mathscr{M}}(x, y) = \widehat{\mathscr{M}}(x, y) - \mathscr{P}_{(0,\varepsilon)}(x, y).$$

It follows from complex function theory (see also [7, Lemma 5.1]) that for every $u \in \mathring{\mathscr{M}}(x, y)$ there exists a unique representative satisfying

$$\left| u(\tau, t) + e_1 \frac{1}{\tau + it} \right| \le C_u e^{-\tau}, \tag{5.1}$$

where e_1 is the unit vector for the first component of $\mathbf{C}^n$ and C_u depends on u. We will henceforth use this gauge. A special Fredholm theory for $\mathscr{M}(x, y)$ was developed in [7]. Let us denote the derivative in the τ-variable by a prime. Then by (5.1), the function

$$\sigma_u(\tau) = \begin{cases} \|u'(0)\|_2^{-1} & \text{for } \tau \le 0, \\ \|u'(\tau)\|_2^{-1} & \text{for } \tau \ge 0 \end{cases} \tag{5.2}$$

grows like τ^2 for positive τ. Define the norms

$$\begin{aligned} \|\xi\|_{L(u)} &= \|\sigma_u^{1/2}\xi\|_p + \|\sigma_u \xi_L\|_p, \\ \|\xi\|_{W(u)} &= \|\sigma_u^{1/2}\xi\|_{1,p} + \|\sigma_u \xi_L'\|_p, \end{aligned} \tag{5.3}$$

where ξ_L is the "longitudinal component"

$$\xi_L = \beta(\tau)\langle \sigma_u(\tau) u'(\tau), \xi(\tau)\rangle$$

and $\langle\ ,\ \rangle$ is the inner product in $L^2(u(\tau)^*TP)$. If we denote by $W(u)$ and $L(u)$ the corresponding Banach spaces of sections of u^*TP, then E_u: $W(u) \to L(u)$ is Fredholm of index $\mu(x) - \mu(x_x)$ (see Theorem 4 of [7]). Moreover, we can choose the dense set $\mathscr{J}_{\text{reg}}$ above so that not only $\mathscr{M}(x, y) \cap \mathscr{P}_{(0,\varepsilon)}(x, y)$, but also all sets $\mathring{\mathscr{M}}(x, y)$ (and similarly $\mathring{\mathscr{M}}(y, z)$) are regular. E_u is then surjective and for $\xi \in W(u)$ with $\langle \xi(0), \eta(0) \rangle = 0$ for all $\eta \leq \ker E_u \cong T_u \overset{\circ}{\mathscr{M}}(x,y)$, we have

$$\|\xi\|_{W(u)} \leq C\|E_u \xi\|_{L(u)}. \tag{5.4}$$

To prove Proposition 5.1, we construct for each $\chi := (u, v, \lambda) \in \overset{\circ}{\mathscr{M}}(x,y) \times \overset{\circ}{\mathscr{M}}(y,z) \times (0, \lambda_0)$ the following "approximate trajectory": If ϕ_λ is the exact deformation generated by H, then $u_\lambda(\tau, t) = \phi_{\lambda t}(u(\tau,t))$ describes a path in $\Omega(L, L_\lambda)$. Moreover, for each positive λ small enough, there exist unique diffeomorphisms $\gamma_\chi^\pm : \mathbf{R}_\pm \to \mathbf{R}$ so that the paths

$$w_\chi(\tau, t) = \begin{cases} \phi_{\lambda t}(0) & \text{for } \tau = 0, \\ u_\lambda(\gamma_\chi^+(\tau), t) & \text{for } \tau < 0, \\ v_\lambda(\gamma_\lambda^-(\tau), t) & \text{for } \tau > 0 \end{cases} \tag{5.5}$$

in $\mathscr{P}(x,z)$ satisfy

$$\langle w_\chi'(\tau), \overline{\partial} w_\chi(\tau) \rangle = 0 \tag{5.6}$$

for all $\tau \in \mathbf{R}$. To define $\gamma_\chi^\pm$, note that the function

$$\alpha_\chi^-(\tau) = \frac{\langle u_\lambda'(\tau), \overline{\partial} u_\lambda(\tau) \rangle}{\|u_\lambda'(\tau)\|_2^2} + 1 \tag{5.7}$$

is smooth. Whenever $u(\tau, t)$ is in the domain U where the deformation is linear, we have

$$\overline{\partial} u_\lambda(\tau, t) = \lambda e_1. \tag{5.8}$$

Hence by (5.4), α_χ^- increases like τ^2 for large τ, so that there exists a unique solution γ_χ^- of the ordinary differential equation

$$\frac{\partial}{\partial \tau} \gamma_\chi^-(\tau) = \alpha_\chi^-(\gamma_\chi^-(\tau)) \tag{5.9}$$

which maps $\mathbf{R}_-$ surjectively onto $\mathbf{R}$. Similarly, we define γ_χ^+, replacing u by v. Equation (5.6) is a direct consequence of (5.9). Finally, it follows immediately from (5.4) that w_χ has continuous first derivatives at $\tau = 0$.

As in the preceding section, we now deform the approximate trajectory w_χ into an element of $\mathscr{M}(x,z)$.

Proposition 5.2. *For compact subsets $K \subset \overset{\circ}{\mathscr{M}}(x,y)$ and $K' \subset \overset{\circ}{\mathscr{M}}(y,x)$, there exists $\lambda_0 > 0$ and a smooth map*

$$\begin{aligned} \natural : K \times (0, \lambda_1) \times K' &\to \mathscr{M}_\Lambda(x,z), \\ (u, v, \lambda) &\to u \natural_\lambda v := \exp_{w_\chi}(\xi_\chi) \end{aligned}$$

with $||\xi_\chi||_{1,p} \leq C\lambda^{1/2-1/2p}$. *Moreover, if* (u,v) *is in the interior of* $K \times K'$ *then there exists an* $\varepsilon > 0$ *so that every trajectory contained in* $U_\varepsilon(u,v)$ *is in the image of* $\natural$.

In a similar way, we define inclusions of compact subsets $K \subset \mathring{\mathscr{M}}(x,y)$ in $\widehat{\mathscr{M}}_{-\varepsilon}(x, y^-_{-\varepsilon})$ and of compact sets $K' \subset \mathring{\mathscr{M}}(y,z)$ in $\widehat{\mathscr{M}}_{-\varepsilon}(y^+_{-\varepsilon}, z)$. This will complete the proof of Proposition 5.1. Clearly, it suffices to consider the first case. Let us define for $\chi = (u, \lambda)$ with $u \in \mathscr{M}(x,y)$ and $\lambda > 0$ small enough a path w_χ in $\Omega(L, L_{-\lambda})$ approximately satisfying the trajectory equation. Note that for λ_0 small enough, a smooth function $\rho : \mathring{\mathscr{M}}(x,y) \times (-\lambda_0) \to \mathbf{R}$ is defined so that $u(\rho(\chi), 0)$ coincides with $y^-_\chi(\lambda)$ in the first component. Then if R is chosen large enough,

$$u_\lambda(\tau, t) = \phi_{\lambda t}(u(\tau,t)) + \beta(\tau - R)(y^-_\chi(\lambda) - \lambda t e_1 - u(\rho(\chi), t))$$

defines a path in $\Omega(L', L_{-\lambda})$ so that $u_\lambda(\rho(\chi), t) = y^-_\chi$. As in (5.5), there exists a unique reparametrization $\gamma_\chi : \mathbf{R} \to (-\infty, \rho(\chi))$ so that

$$w_\chi(\tau, t) = u_\lambda(\gamma_\chi(\tau), t) \tag{5.10}$$

satisfies $\langle \overline{\partial} w_\chi(\tau), w'_\chi(\tau)\rangle = 0$ for all $\tau \in \mathbf{R}$. Now we have

Proposition 5.3. *For every compact subset* $K \subset \mathring{\mathscr{M}}(x,y)$ *there exists* $\lambda_1 > 0$ *and a family of maps*

$$\begin{aligned} \natural_\lambda &: K \to \mathscr{M}_{-\lambda}(x, y^-); \\ \natural_\lambda u &= \exp_{w_\chi}(\xi_\chi(u)) \end{aligned}$$

for $\lambda \in (0, \lambda_0)$, *where* w_χ *is defined by* (5.10) *and* $||\xi_\chi||_{1,p} \leq C\lambda^{1/2-1/2p}$. *Moreover, there exists an* $\varepsilon > 0$ *so that every trajectory in* $M_\lambda(x, y^-_\chi)$ *which is contained in the* ε*-tube of* u_λ *is in the image of* $\natural_\lambda$.

Propositions 5.2 and 5.3 together imply Proposition 5.1. We therefore come to the proofs of the above results.

Proof of Proposition 5.2. Throughout the proof, we will denote by C_k, $k \in \mathbf{N}$, positive real constants which depend only on the compact sets K and K'. We apply Lemma 4.2 to the map

$$\begin{aligned} &\overline{\partial}_{w_\chi} : \widehat{W}(x_\chi \to L^p(w^*_\chi TP), \\ &\overline{\partial}_{w_\chi}(\xi) = \overline{\partial} w_\chi + E_\chi \xi + N_\chi(\xi). \end{aligned} \tag{5.11}$$

As in (4.15), $\widehat{W}(w_\chi) = T_{w_\chi}\mathscr{P} \oplus \mathbf{R}$ is the tangent space to the parametrized path space. The additional direction is now generated by the vector field $X_H(w_\chi(\tau, t))$, where X_H is the Hamiltonian vector field of H. However, as opposed to the situation of the preceding section, the operator family E_χ is not uniformly invertible with respect to the standard Sobolev norms. Define therefore the weight functions

$$\sigma_\chi = \begin{cases} \|w'_\chi(\tau)\|_2^{-1} & \text{for } \gamma_\chi^-(0) \le \tau \le \gamma_\chi^+(0), \\ \|w'_\chi(\gamma_\chi^\pm(0))\|_2^{-1} & \text{for } \pm\tau \ge \gamma_\chi^\pm(0). \end{cases}$$

For any section ξ for $w_\chi^* TP$, define the longitudinal component

$$\xi_L(\tau) = \beta(\tau - \gamma_\chi^+(0))\beta(\gamma_\chi^-(0) - \tau)\langle \sigma_\chi(\tau), \xi(\tau)\rangle.$$

Now we replace the norms on $L^p(w_\chi^* TP)$ and $T_{w_\chi}\mathscr{P}$ by

$$\begin{aligned} \|\xi\|_{L_\chi} &= \|\sigma_\chi^{1/2}\xi\|_p + \|\sigma_\chi \xi_L\|_p, \\ \|\xi\|_{W_\chi} &= \|\sigma_\chi^{1/2}\xi\|_{1,p} + \|\sigma_\chi \xi'_L\|_p. \end{aligned} \tag{5.12}$$

We define the norm in the additional direction by the unit vector

$$X_\chi = \lambda^{1/2p+1} t \cdot X_H(w_\chi(\tau, t)),$$

which is chosen so that the following estimate holds:

Lemma 5.1.

$$\begin{aligned} C_1^{-1} &\le \|X_\chi\|_{W_\chi} \le C_1, \\ C_2^{-1} &\le \|E_\chi X_\chi\|_{L_\chi} \le C_2. \end{aligned}$$

Proof. The time that w_χ spends in the neighborhood U of y where the deformation is nontrivial is estimated by

$$C^{-1}\lambda^{-1/2} \le |\gamma_\chi^\pm(\pm\lambda^{1/2})| \le |\gamma_\chi^\pm(0)| \le C\lambda^{-1/2}. \tag{5.13}$$

To prove this, we can by (5.1) restrict ourselves to the case $u(\tau, t) = -(\tau + it)^{-1}$. Then for $\tau \le \lambda^{1/2}$, we have $|(\gamma_\chi^-)'(\tau)| \le 2$, so that $\gamma_\chi^-(\lambda^{1/2}) - \gamma_\chi^-(0) \le 2\lambda^{1/2}$. For $\tau \ge \lambda^{1/2}$, it follows from (5.8) that $\|w'_\chi(\tau)\|_2 \ge \lambda$, hence $0 - \gamma(\lambda^{1/2}) \ge \lambda^{-1/2}$.

Now the estimates from above follow immediately since $\sigma_\chi(\tau) = \|w'_\chi(\tau)\|_2^{-1} \le \lambda^{-1}$. To obtain an estimate from below, note that $\sigma_\chi(\tau) \ge \frac{1}{2}\lambda^{-1}$ for $\gamma_\chi^-(\lambda^{-1/2}) \le \tau \le 0$. q.e.d.

Using the explicit formulas for the nonlinear part $N_u(\xi)$ given in Lemma 3.2 of [6], we find that for all $\xi, \varsigma \in \widehat{W}_\chi$,

$$\|N_\chi(\xi) - N_\chi(\varsigma)\|_{L_\chi} \le C_1(\|\xi\|_{\widehat{W}_\chi} + \|\varsigma\|_{\widehat{W}_\chi})\|\xi - \varsigma\|_{\widehat{W}_\chi}, \tag{5.14}$$

since the weight on the longitudinal component in L_χ is only the square of the weight of the same component in $\widehat{W}_\chi$. On the other hand, although the weight σ_χ is large in the region where u and v are glued together, we still have

Lemma 5.2.

$$\|\bar\partial w_\chi\|_{L_\chi} \le C_2 \cdot \lambda^{1/2-1/2p}. \tag{5.15}$$

Proof. By (5.13), Lemma 5.2 follows from the uniform estimate

$$\sigma_\chi(\tau)|\bar\partial w_\chi(\tau,t)| \le C_3\lambda^{1/2}. \tag{5.16}$$

To prove (5.16), note that by construction of w_χ in (5.5), we have for $-a\lambda^{-1/2} < \tau < 0$, and $a > 0$ small enough,

$$w_\chi(\tau) = u(\gamma_\chi^-(\tau)) + \lambda\cdot t\cdot ie_1.$$

Therefore, with $\gamma_\chi = \gamma_\chi^-$,

$$\begin{aligned} w_\chi'(\tau) &= u'(\gamma_\chi(\tau))\gamma_\chi'(\tau),\\ J\dot w_\chi(\tau)) &= \mathcal{J}\dot u(\gamma_\chi(\tau)) + \lambda e_1 = -u'(\gamma_\chi(\tau)) + \lambda e_1. \end{aligned}$$

The reparametrization is chosen in such a way that

$$\bar\partial w_\chi(\tau) = \pi^\perp(\gamma_\chi(\tau))\cdot\lambda e_1, \tag{5.17}$$

where $\pi^\perp(\tau)$ is the projection onto the L^2-orthogonal complement of $u'(\tau)$. Using the asymptotic estimate (5.1), we find with $|X(\tau)| \le C_4 e^{-\tau}$,

$$\begin{aligned} u'(\tau,t) &= \frac{\partial}{\partial\tau}\left(\frac{e_1}{\tau+it}\right) + X(\tau)\\ &= -e_1\frac{\tau^2}{\tau^2+t^2}\left(1-\frac{2it}{\tau}-\frac{t^2}{\tau}\right) + X(\tau). \end{aligned} \tag{5.18}$$

We conclude that

$$||\pi^\perp(\tau)e_1||_2 \le C_5\tau^{-1}. \tag{5.19}$$

In order to calculate the weight, we obtain from (5.9),

$$\begin{aligned} w_\chi'(\gamma_\chi^{-1}(\tau)) &= u'(\tau)\gamma_\chi'(\gamma_\chi^{-1}(\tau))\\ &= u'(\tau)\left(\frac{\langle u'(\tau),\lambda e_1\rangle}{\langle u'(\tau),u'(\tau)\rangle}+1\right). \end{aligned}$$

Hence

$$\begin{aligned} ||w'(\gamma_\chi^{-1}(\tau))||_2 &= \lambda\left\langle\frac{u'(\tau)}{||u'(\tau)||_2}, e_1\right\rangle + ||u'(\tau)||_2\\ &\ge C_6\left(\lambda+\frac{1}{\tau_2}\right) \end{aligned}$$

for some positive constant C_6. Together with (5.17) and (5.19), this yields the estimate

$$(\sigma_\lambda|\bar\partial w_\lambda)(\gamma_\lambda^{-1}(\tau)) \le C_7\frac{\lambda\tau^{-1}}{\lambda+\tau^{-2}}.$$

Now (5.16) follows by elementary methods. q.e.d.

It remains to invert the operator E_χ. Of course, since

$$\operatorname{Index} E_\chi = \mu(x) - \mu(z) + 1 = \dim \mathring{\mathscr{M}}(x,y) + \dim \mathring{\mathscr{M}}(x,y),$$

we have to factor out a family of finite dimensional subspaces $K_\chi \simeq \ker E_u \oplus \ker E_v$ of $\widehat{W}(w_\chi)$. Define

$$W_\chi^\perp = \{\xi + \alpha X_\chi \in \widehat{W}(w_\chi) \mid \langle \eta(0), \xi(\gamma_\lambda^-(0))\rangle = 0 \text{ for all } \eta \in \ker E_u \\ \text{and } \langle \eta(0), \xi(\gamma_\lambda^+(0))\rangle = 0 \text{ for all } \eta \in \ker E_v\}.$$

Then we have

Lemma 5.3. *There exist positive constants C and λ_0 depending only on K and K' so that if $\lambda < \lambda_0$, then for each $\chi \in K \times (0, \lambda_0) \times K'$, there exists a continuous right inverse $G : L_\chi \to \widehat{W}_\chi^\perp$ of E_χ with*

$$||G\xi||_{W_\chi} \le C||\xi||_{L_\chi}.$$

Proof. In principle, we use the same method as in the proof of Lemma 4.3. Because the Fredholm index of E_χ restricted to $\widehat{W}_\chi^\perp$ is zero, it suffices to prove that we can choose λ_0 small enough so that for all $\chi \in K \times (0, \lambda_0) \times K'$ and all $\xi \in \widehat{W}_\chi^\perp$,

$$||\xi||_{W_\chi} \le C_8||E_\chi \xi||_{L_\chi}.$$

Proceeding indirectly, we assume that there exists a family $\chi_\lambda = (u_\lambda, v_\lambda, \lambda)$ indexed by $\lambda \in (0, \varepsilon)$ accumulating at zero and a family $\xi_\lambda + \alpha_\lambda X_\lambda \in \widehat{W}_\lambda^\perp = \widehat{W}_{\chi_\lambda}^\perp$ so that

$$||\xi_\lambda||_{W\lambda} = 1, \tag{5.20}$$

and, abbreviating $E_\lambda = E_{\chi_\lambda}$,

$$||E_\lambda \xi_\lambda||_{L\lambda} \to 0 \tag{5.21}$$

for $\lambda \to 0$. Again, we derive a contradiction to (5.20). However, this point is considerably more complicated than in the proof of Lemma 4.3. First note that if a is any fixed positive number and $\tau < \tau_\lambda^- : (a\lambda)^{-1/2}$, we have for λ small enough $|\alpha_\lambda^-(\tau)| \le 1/a$ (see (5.7) and (5.8)). We now decompose Θ into $\Theta_\lambda^0 := [\gamma_\lambda^-((a\lambda)^{-1/2}),\ \gamma_\lambda^+(-(a\lambda)^{-1/2})]$ and the two components $\Theta_\lambda^\pm$ of its complement. Then for $(\tau, t) \in \Theta_\lambda^-$, the reparametrization γ_λ^- satisfies

$$\left|\frac{d}{d\tau}\gamma_\lambda^-(\tau) - 1\right| \le \frac{1}{a}.$$

Therefore, the weights σ_λ can be compared with the weights σ_{u_λ} of (5.2) through

$$(1 - 1/a)\sigma_{u_\lambda}(\tau) \le \sigma_\lambda(\gamma_\lambda^-(\tau)) \le (1 + 1/a)\sigma_{u_\lambda}^-(\tau).$$

We now construct a section ξ^- of u^*TP out of ξ by

$$\xi(\tau,t) = D\phi_{\lambda t}(u(\gamma_\lambda^-(\tau),t))\xi^-(\gamma_\lambda^-(\tau),t), \tag{5.22}$$

where ϕ_λ is the deformation by the Hamiltonian H. Then if ξ is supported in Θ_λ^-, we have

$$\begin{aligned} 1-1/a &\le \|\xi^-\|_{W(u_\lambda)}\|\xi\|_{W\lambda}^{-1} \le 1+1/a, \\ 1-1/a &\le \|\xi^-\|_{L(u_\lambda)}\|\xi\|_{L\lambda}^{-1} \le 1+1/a, \end{aligned} \tag{5.23}$$

$$\|E_{u_\lambda}\xi^-\|_{L(u_\lambda)} \le (1+1/a)\|E_\lambda\xi\|_{L\lambda} + \|\xi\|_{W\lambda}/a. \tag{5.24}$$

Define now ξ_λ^- by (5.22) for $\tau \le \tau_\lambda^-$ and set $\xi_\lambda^- = 0$ otherwise. We will apply (5.24) together with (5.4) to the section $\overline{\xi}_\lambda$ obtained from ξ_λ^- by the following two modifications. First, since it is not $E_\lambda\xi_\lambda$ but $E_\lambda(\xi_\lambda + \alpha_\lambda X_\lambda)$ whose L_λ-norm converges to zero, we solve the equation

$$(E_{u_\lambda}(f_\lambda^- u_\lambda')^-)_L = (E_{u_\lambda}(\alpha_\lambda X_\lambda)^-)_L \tag{5.25}$$

explicitly for f_λ^- by integration, given the initial condition $f_\lambda^-(0) = 0$. We then have on $(-\infty,(a\lambda)^{-1/2}] \times [0,1]$:

$$\begin{aligned} &\|E_{u_\lambda}(\xi_\lambda^- + f_\lambda^- u_\lambda')\|_{L(u_\lambda)} \\ &\le \|E_{u_\lambda}(\xi_\lambda + \alpha_\lambda X_\lambda)^-\|_{L(u_\lambda)} + \alpha_\lambda\|(E_{u_\lambda}X_\lambda^-)_T\|_{L(u_\lambda)} \\ &\le 2\|E_\lambda(\xi_\lambda + \alpha_\lambda X_\lambda)\|_{L\lambda} + \|\xi\lambda + \alpha_\lambda X_\lambda\|_{W\lambda}/a + \alpha_\lambda\|(E_\lambda X_\lambda)_T\|_{W\lambda} \\ &= 1/a + \varepsilon_\lambda, \end{aligned} \tag{5.26}$$

where $\lim_{\lambda\to 0}\varepsilon_\lambda = 0$. The term $(E_\lambda X_\lambda)_T$ converges to zero since $\|(e_1)_T(\tau)\| \le C_{11}\tau^{-1}$. The second modification is necessary since $\xi_\lambda^- + f_\lambda^- u_\lambda'$ is not an element of $W(u_\lambda)$; we have to cut it off close to $\tau = \tau_\lambda^-$. Of course, this creates additional terms when we apply E_{u_λ}. In particular, the longitudinal part of $\xi_\lambda^- + f_\lambda^- u_\lambda'$ may be large at the cutoff. We therefore use the following trick: define real constants y_λ so that the longitudinal component of $(\xi_\lambda^- + (f_\lambda - y_\lambda)u')(\tau_\lambda^-) = 0$. Let β be defined as in (4.3) and define $\alpha(x) = 0$ for $x \le 0$ and $\alpha(x) = 1$ for $x > 0$. Then

$$\overline{\xi}_\lambda(\tau,t) = \xi_\lambda^- - \beta(\tau-\tau_\lambda^-)(\xi_\lambda^-)_T(\tau,t) + (f_\lambda - y_\lambda)\alpha(\tau-\tau_\lambda^-)\cdot y_\lambda u_\lambda'(\tau,t)$$

is an element of $W(u_\lambda)$, and satisfies

$$\|E_{u_\lambda}\overline{\xi}_\lambda\|_{L(u_\lambda)} \le \|E_{u_\lambda}(\xi_\lambda^- + f_\lambda^- u')\|_{L(u_\lambda)} + \|\beta'(\cdot - \tau_\lambda^-)\xi_T^-\|_{L(u_\lambda)}.$$

The second term converges to zero if

$$\lambda^{-1/2}\max\{|(\xi_\lambda^-)_T(\tau,t)\mid |\tau-\tau_\lambda^-| \le 1\} \to 0 \tag{5.27}$$

for $\lambda \to 0$. Hence assuming that (5.27) is correct, it follows with (5.26) and (5.4) that

$$\|\overline{\xi}_\lambda\|_{W(u_\lambda)} \le C/a + \varepsilon'_\lambda. \tag{5.28}$$

Here we have used the fact that the constant C in (5.4) can be chosen independently of λ, since u_λ is contained in a compact subset of $\overset{\circ}{\mathcal{M}}(x,y)$. We can then choose $a > 4C$ so that for λ small enough,

$$\|\xi_\lambda^- + f_\lambda^- u'_\lambda\|_{W(u_\lambda)} \le \tfrac{1}{3}$$

on $(-\infty, (a\lambda)^{1/2} - 1] \times [0,1]$. Performing the same procedure at the other end, we find $\xi_\lambda^+ + f_\lambda^+ v'_\lambda$ on $[-a\lambda^{-1/2}, \infty) \times [0,1]$ whose $W(v_\lambda)$-norm is less than $\frac{1}{3}$ for small λ.

Let us now consider ξ on $\Theta_\lambda^0 = [\sigma_\lambda^-, \sigma_\lambda^+]$, where $\sigma_\lambda^\pm = \gamma_\lambda^\pm(\mp(a\lambda)^{1/2})$. First, we have to prove (5.27). Clearly, for λ small enough we have $w_\lambda(\theta) = \exp_y \varsigma_\lambda$ in the sense of (4.1) on Θ_λ^0 for some $\varsigma_\lambda : \theta_\lambda^0 \to T_yP$. We can then define $\xi_\lambda^0 \in L_1^p(\theta(\lambda), T_yP)$ as in (4.19) by

$$D_2 \exp_y(\varsigma_\lambda)\xi_\lambda^0 = \xi_\lambda$$

on Θ_λ^0. To prove (5.27), we claim that

$$\lim_{\lambda\to 0}(\lambda^{-1/2}\|\xi_\lambda\|_\infty^0) = 0. \tag{5.29}$$

In fact, if there exists a family $(\tau_\lambda, t_\lambda) \in \Theta_\lambda^0$ and a positive number ε so that $|\xi_\lambda^0(\tau_\lambda, t_\lambda)| > \varepsilon\lambda^{1/2+1/p}$ for all λ, we obtain a contradiction as follows: Define

$$\varsigma_\lambda(\tau,t) = \begin{cases} \lambda^{-1/2}\xi_\lambda^0(\tau+\tau_\lambda, t) & \text{for } |\tau| \le \lambda^{-1/2}, \\ 0 & \text{otherwise.} \end{cases}$$

The crucial observation is now that for $(\tau, t) \in \Theta_\lambda^0$,

$$\lambda^{-1} \le \sigma_\lambda(\tau) \le C\lambda^{-1}.$$

Hence by (5.20), $\|\varsigma_\lambda\|_p \le C_3\|\sigma^{1/2}\xi\|_p$ is bounded. We can therefore assume that it converges weakly to some limit $\varsigma_0 \in L^p(\Theta, T_yP)$. We want to show that $\varsigma_0 = 0$. Note therefore that for any $\rho > 0$, the restriction ς_λ to $\Theta_\rho = [-\rho, \rho] \times [0,1]$ is, for λ small enough, a bounded sequence in $L_1^p(\Theta_\rho, T_yP)$. We can therefore assume that it converges weakly in this space to ς_0. Moreover, if $E_0 = \partial/\partial\tau + J\partial/\partial t$ is the standard Cauchy-Riemann operator, it follows from (5.21) that for all ρ and for $\lambda \to 0$,

$$\|(E_0\varsigma_\lambda)_{\Theta_\rho}\|_p \to 0. \tag{5.30}$$

(Note that the contribution of $\lambda^{-1/2}X_\lambda$ vanishes locally.) By weak lower semicontinuity of (5.30) we conclude that $E_0\varsigma_0 = 0$ everywhere on Θ. Together with $\varsigma_0 \in L^p$, it follows that $\varsigma_0 = 0$. Since weak convergence in $L_1^p(\Theta_\rho, T_yP)$ implies uniform convergence, this contradicts the assumption and therefore proves (5.29).

We now define the longitudinal part of ξ_λ^0 as the integral

$$\xi_{\lambda L}^0(\tau) = \int_0^1 (\xi_\lambda^0(\tau,t))_1\, dt$$

over the e_1-component of ξ_λ. Correspondingly, we define the transversal part of ξ_λ^0 as $\xi_{\lambda T}^0 = \xi_\lambda^0 - \xi_{\lambda L}^0 \cdot e_1$. It follows from (5.21) that $\lambda^{-1/2}\|(E_0\xi_\lambda^0)_T\|_p \to 0$. Using a cutoff function and (5.29), this implies

$$\lambda^{-1/2}\|\xi_{\lambda T}^0\|_{1,p} \to 0, \tag{5.31}$$

since E_0 is invertible on the transversal component of $W_{1,p}(\Theta, T_yP)$. To obtain an estimate on the longitudinal part, we first have to solve an equation corresponding to (5.25). If we choose an initial condition so that

$$\xi_{\lambda L}^0(\sigma_\lambda^-) + \alpha_\lambda f_\lambda^0(\sigma_\lambda^-) = \xi_{\lambda L}^-(\tau_\lambda^-) + \alpha_\lambda f_\lambda^-(\tau_\lambda^-)\|u_\lambda'(\tau_\lambda^-)\|^2$$

we obtain on Θ_λ^0:

$$\lambda^{-1/2}\|\xi_{\lambda L}^0 + \alpha_\lambda f_\lambda^0\|_{1,p} \le C_{12}\lambda^{-1}\|E_0(\xi_{\lambda L}^0 + \alpha_\lambda f_\lambda^0)\|_p \to 0. \tag{5.32}$$

If we replace f_λ^0 by g_λ^0, which is defined so that

$$(\xi_{\lambda L}^+ + \alpha_\lambda g_\lambda^0)(a\lambda^{-1/2}) = (\xi_{\lambda L}^+ + \alpha_\lambda f_\lambda^+\sigma^{-1})(\tau_\lambda^+),$$

we obtain in the same way

$$\lambda^{-1/2}\|\xi_{\lambda L}^0 + \alpha_\lambda g_\lambda^0\|_{1,p} \to 0. \tag{5.33}$$

However,

$$f_\lambda^0 - g_\lambda^0 = \alpha_\lambda(f_\lambda^+(\tau_\lambda^+) + f_\lambda^-(\tau_\lambda^-)) \cdot \lambda \ge C_{13}^{-1}\alpha_\lambda\lambda^{(1/2+1/2p)}.$$

By (5.13) we have on the interval $[\sigma_\lambda^-, \sigma_\lambda^+]$,

$$C^{-1} \le \lambda^{-1/2}\|\lambda^{(1/2+1/2p)}\|_p \le C.$$

Hence (5.32) and (5.33) can both be true only if $\alpha_\lambda \to 0$. Now (5.28), (5.32), and (5.33) together imply a contradiction to (5.20). This completes the proof of Lemma 5.3.

It remains to prove the uniqueness statement of Proposition 5.2. Therefore let $w_k \in \mathcal{M}_{\lambda_k}(x,z)$ be a family of trajectories contained in the $1/k$-tube of (u,v). As in the proof of the uniqueness statement of Proposition 4.1, we have to show that if we define $\overline{w}_k = w_{\chi_k}$ for $\chi_k = (u,v,\lambda_k)$, we have $w_k = \exp_{\overline{w}_k}(\xi_k)$ with $\lim_{k\to\infty}\|\xi_k\|_{W_{\chi_k}} = 0$. By applying a translation if necessary, we can assume that

$$a(w_k(0)) = a(y).$$

Choose τ_k^+ and τ_k^- so that

$$a(w_k(\tau_k^-)) = \tfrac{1}{2}(a(x) - a(y)), \qquad a(w_k(\tau_k^+)) = \tfrac{1}{2}(a(y) - a(z)).$$

Then by Proposition 2.2 we can choose a subsequence so that $\tau_k^- * w_k$ and $\tau_k^+ * w_k$ converge locally to u and v, respectively. We now split Θ for each $k \in \mathbf{N}$ into five parts: Choose ρ large enough so that u takes values in the union of a neighborhood $U(x)$ and $U(y)$ outside Θ_ρ, where $U(y)$ can be identified with open subsets of $\mathbf{C}^n$ as described in the beginning of this section. We can also assume ρ to be large enough so that v takes values in $U(y) \cup U(z)$ outside Θ_ρ. Then define

$$I_k^+ = \{(\tau + \tau_k^+, t) \mid |\tau| \le \rho\}, \qquad I_k^- = \{(\tau + \tau_k^-, t) \mid |\tau| \le \rho\}.$$

For k large enough, we can now assume that the complement of I_k^+ and I_k^- has three components Θ_k, Θ_k^0 and Θ_k^+ so that $u(\Theta q_k^-) \subset U_x$, $u(\Theta_k^-) \subset U_y$, and $u(\Theta_k^+) \subset U_z$. Now estimates of $||\xi_k||_{W_{\chi_k}}$ on $I_k^\pm$ follow immediately from the uniform convergence of Proposition 2.2. The estimate on $\Theta_k^\pm$ uses the transversality of the intersections x and z and follows essentially the lines of the proof of the uniqueness statement of Proposition 4.2. We therefore restrict ourselves to proving the estimate on Θ_k^0, which is the most delicate part because of the weights in $|| \; ||_{W_{\chi_k}}$. The idea is to compare u to certain standard holomorphic functions. Consider

$$f_\mu(\theta) = i\tan(-\mu\theta) = i\frac{e^{i\mu\theta} - e^{i\mu\theta}}{e^{i\mu\theta} + e^{i\mu\theta}}$$

for $\theta \in \mathbf{C}$. It has poles for $\theta = \frac{1}{2}(2k+1)\cdot\pi/\mu$. Moreover, it maps the interval $(-\frac{1}{2}\pi/\mu, \frac{1}{2}\pi/\mu)$ diffeomorphically onto the real axis. The line $\mathbf{R}+1$ is mapped to a circle, which is defined by

$$f_\mu(i) = i\tan(-i\mu) = i\tanh(\mu),$$

$$f\left(i + \frac{\pi}{2}\right) = i\frac{1}{\tanh(\mu)}.$$

We want to normalize this function so that $\mathbf{R}+1$ is mapped to the circle of radius 1. We therefore divide f_μ by the radius of this circle to obtain

$$g_\mu = 2i\left(\frac{1}{\tanh(\mu)} - \tanh(\mu)\right)^{-1}\tan(-\mu\theta) \tag{5.34}$$

We now show that the first component of any sequence $(w_k, \lambda_k) \in \mathscr{M}_\Lambda(x, z)$ which converges locally to the constant trajectory $\widehat{y}$ has to be asymptotic in a very strong sense to the family of functions $g_{-\mu_k}$ of (5.34) for $2\sinh^2\mu_k = \lambda_k$. Note therefore that g_{μ_k} is invertible in the neighborhood U_y to a map

$\phi_k : U_y \to \mathbf{C}$. Now define the family of holomorphic functions

$$f_k : \Theta_k^0 \to \mathbf{C}, \qquad f_k(z) = \phi_k(w_k(z)) - z$$

with the property that

$$f_k(\mathbf{R}) \subset \mathbf{R}, \qquad f_k(\mathbf{R}+1) \subset \mathbf{R}.$$

Moreover, $f_k(\tau_k^+ - \rho)$ and $f_k(\tau_k^- + \rho)$ are bounded independently of ρ. Now the Cauchy-Riemann equation for such maps can be rewritten as the ordinary differential equation $(\partial/\partial\tau) f_k(t) = A f_k(\tau)$ on the linear space of paths in $\mathbf{C}$ originating and ending at $\mathbf{R}$. Here, A is the linear operator $Az = J\dot{z}$. Therefore, it follows from linear functional analysis that

$$f_k(\tau) \leq C \exp[-\mu \min(\tau_k^+ - \rho, -\tau_k^- - \rho)]$$

for positive constants C and μ, as long as $\frac{1}{2}(\tau_k^- + \rho) \leq \frac{1}{2}(\tau_k^+ - \rho)$. For k large enough, this proves the desired estimate in Θ_k^0.

Proof of Proposition 5.3. The proof is in many aspects parallel to the proof of Proposition 5.2. One difference is that we apply Lemma 4.2 to the family of maps

$$\overline{\partial} w_\chi : T_{w_\chi}\mathscr{P} \to L^p(w_\chi^* TP)$$

rather than to the parametrized space. To define a suitable family of norms, consider the weight function

$$\sigma_\chi(\tau) = \begin{cases} ||w_\chi'(\tau)||_2^{-1} & \text{for } 0 \leq \tau \leq \frac{1}{2}\rho(\chi), \\ ||w_\chi'(\frac{1}{2}\rho(\chi))||_2^{-1} & \text{for } \tau \leq \frac{1}{2}\rho(\chi), \\ ||w_\chi'(0)||_2^{-1} & \text{for } \tau \leq 0. \end{cases}$$

Then if we define the family of norms $||\ ||_{W_\chi}$ and $||\ ||_{L_\chi}$ by the same formula as (5.12), it follows again that

$$\lim_{\lambda\to 0} ||\overline{\partial} w_\chi||_{L_\chi} = 0,$$

$$||N_\chi(\xi) - N_\chi(\varsigma)||_{L_\chi} \leq C_{14}(||\xi||_{W_\chi} + ||\varsigma||_{W_\chi})||\xi - \varsigma||_{W_\chi}.$$

The index of the operator E_χ is equal to the dimension of $\mathscr{M}(x,y)$ by Theorem 5 of [7]. To define a subspace on which to invert it, define for each $\xi \in W(w_\chi)$ the section $\overline{\xi} \in W(u)$ by

$$\xi(\tau, t) = D\phi_{-\lambda t}\overline{\xi}(\gamma_\chi(\tau), t). \tag{5.35}$$

Then define the subset

$$W_\chi^\perp = \{\xi \in W(w_\chi) \mid \langle \overline{\xi}(0), \eta(0)\rangle = 0 \text{ for all } \eta \in \ker E_u\}.$$

Since E_χ restricted to $W_\chi^\perp$ has index zero, we obtain a uniformly bounded family of inverse operators G_χ if we can show that there does not exist a sequence $\chi_\lambda \in W_\lambda^\perp = W_{\chi_\lambda}^\perp$, so that $\lambda \to 0$ and

$$||\xi_\lambda||_{W\lambda} = 1, \tag{5.36}$$

$$||E_\lambda \xi_\lambda||_{L\lambda} \to 0. \tag{5.37}$$

The crucial point is again that we can divide Θ into the part with $\tau, \frac{1}{2}\rho_\lambda = \frac{1}{2}\rho(\chi_\lambda)$ where $\gamma'_\lambda(\tau)$ is bounded, and therefore formula (5.35) essentially defines an isometry between W_λ and $W(u_\lambda)$ as well as between L_λ and $L(u_\lambda)$, and the part with $\tau \geq \frac{1}{2}\rho_\lambda$, where the weights are essentially constant. The estimate on the first part is accomplished by the same method as in the proof of Lemma 5.3. We also obtain a uniform estimate near $\frac{1}{2}\rho_\lambda$ of type (5.27). It immediately implies an estimate on the transversal component of ξ_λ on the second domain. To obtain the estimate of the longitudinal component of ξ_λ on the second domain, note that the longitudinal part of E_λ is of the form

$$E_\lambda^L f = f' - \left(\frac{\partial}{\partial\tau}||u'(\tau)||\right)||u'(\tau)||^{-1}$$

and that the second term for $\tau \to \infty$ converges to a nonzero value $a_\lambda - 2\lambda^{-1/2}$. Now instead of conjugating E_λ^L with a multiplicative operator, we conjugate it with the contraction operator

$$c_\lambda f(\tau) = \lambda^{-1/2} f(\lambda^{-1/2}\tau).$$

Then $c_\lambda E_\lambda^L c_\lambda^{-1}$ converges for $\lambda \to 0$ to the operator $f \to f' + f$, which defines an isomorphism between $W_1^p(\mathbf{R},\mathbf{R})$ and $L^p(\mathbf{R},\mathbf{R})$. But these norms correspond precisely to the longitudinal parts of $||\ ||_{W_\lambda}$ and $||\ ||_{L\lambda}$ under c_λ. This contradicts (5.36) and therefore completes the proof of the existence assertion in Proposition 5.3. To prove the uniqueness, we proceed as in the proof of Proposition 5.2, using an appropriate comparison function.

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