Morse Theory for Periodic Solutions of Hamiltonian Systems and the Maslov Index

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This paper is dedicated to Natascha A. Brunswick.

Abstract

In this paper we prove Morse type inequalities for the contractible 1-periodic solutions of time dependent Hamiltonian differential equations on those compact symplectic manifolds M for which the symplectic form and the first Chern class of the tangent bundle vanish over $\pi_2(M)$. The proof is based on a version of infinite dimensional Morse theory which is due to Floer. The key point is an index theorem for the Fredholm operator which plays a central role in Floer homology. The index formula involves the Maslov index of nondegenerate contractible periodic solutions. This Maslov index plays the same role as the Morse index of a nondegenerate critical point does in finite dimensional Morse theory. We shall use this connection between Floer homology and Maslov index to establish the existence of infinitely many periodic solutions having integer periods provided that every 1-periodic solution has at least one Floquet multiplier which is not equal to 1.

1. Introduction

Let (M, ω) be a compact symplectic manifold of dimension $2n$. Here ω is a closed 2-form on M which is nondegenerate and therefore induces an isomorphism $T^*M \to TM$. Thus every smooth, time-dependent Hamiltonian function $H : \mathbb{R} \times M \to \mathbb{R}$ gives rise to a time-dependent Hamiltonian vector field $X_H : \mathbb{R} \times M \to TM$ defined by

$$\omega(X_H(t,x), \xi) = -d_x H(t,x)\xi$$

for $\xi \in T_x M$. Assume that H (and hence X_H) is periodic in time

$$H(t+1, x) = H(t,x)$$

and consider the time-dependent Hamiltonian differential equation

$$\dot{x}(t) = X_H(t, x(t)) \tag{1.1}$$

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on M. The solutions $x(t)$ of (1.1) determine a 1-parameter family of diffeomorphisms $\psi_t \in \mathrm{Diff}(M)$ satisfying $\psi_t(x(0)) = x(t)$. These diffeomorphisms are **symplectic**, that is they preserve the symplectic form $\omega = \psi_t^* \omega$. Every symplectic diffeomorphism ψ which can be represented as the time-1-map of such a time-dependent Hamiltonian flow is called a **Hamiltonian map**. If M is simply connected then the component of the identity in the space of symplectic diffeomorphisms consists of Hamiltonian maps. Our assumptions on the second homotopy group, however, exclude the simply connected case.

The fixed points of the time-1-map ψ are the periodic solutions of (1.1) with period 1. Such a periodic solution

$$x(t) = x(t+1)$$

is called **nondegenerate** if 1 is not a Floquet multiplier or equivalently

$$\det(I - d\psi(x(0))) \neq 0 .$$

It is called **weakly nondegenerate** if at least one Floquet multiplier is not equal to 1, that is

$$\sigma(d\psi(x(0))) \neq \{1\} .$$

Denote by $[\omega] \in H^2(M; \mathbb{R})$ the cohomology class of ω. Also note that TM, with a suitable almost complex structure, can be viewed as a $\mathbb{C}^n$-bundle and therefore has Chern classes $c_j \in H^{2j}(M; \mathbb{Z})$.

THEOREM A. *Assume that $[\omega]$ and c_1 vanish over $\pi_2(M)$. Let $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ be a smooth Hamiltonian function such that the contractible 1-periodic solutions of (1.1) are weakly nondegenerate. Then there are infinitely many contractible periodic solutions having integer periods.*

Theorem A extends an earlier result in [35]. It can be rephrased in terms of Hamiltonian mappings. If $[\omega]$ and c_1 vanish over $\pi_2(M)$ then every Hamiltonian map with weakly nondegenerate fixed points has infinitely many periodic points.

Such an existence theorem cannot be expected for general symplectic diffeomorphisms as is shown by the example of an irrational translation on the torus $M = T^2$. This is a symplectic but not a Hamiltonian map and it has no periodic points at all. Also the condition on $\pi_2(M)$ cannot be removed as is shown by the example of a rotation on $M = S^2$ with irrational frequency. This is a Hamiltonian map but has only two periodic points, namely the two nondegenerate fixed points at the north and south pole.

For the $2n$-torus Theorem A was proved in [9] under the assumption that the 1-periodic solutions have no root of unity as a Floquet multiplier or, equivalently, the fixed points of the time-1-map ψ are nondegenerate with respect to all iterates of ψ. In 1984 in [6] Conley conjectured that the nondegeneracy assumption can be removed. Even in the case $M = T^{2n}$ this remains an open question.

The global nature of Theorem A is in sharp contrast to well-known local results. For example assume that the Hamiltonian system (1.1) possesses a 1-periodic solution which is elliptic in the sense that all its Floquet multipliers lie on the unit circle. If these satisfy a finite nonresonance condition and in addition the nonlinear Birkhoff invariants of this solution are nondegenerate then the local Birkhoff-Lewis theorem guarantees infinitely many periodic solutions with large integer period nearby; see [31]. If, moreover, the system is sufficiently smooth, then it follows from K.A.M. theory that the closure of the set of these periodic solutions nearby is of large Lebesgue measure; see [9]. Similar arguments still work if there is at least one Floquet multiplier on the unit circle. It is not known, however, whether such a nonhyperbolic periodic orbit always exists.

Note that the assumptions of Theorem A are formulated in terms of the 1-periodic solutions. The existence of these is related to a conjecture by V. I. Arnold which states that every Hamiltonian map $\psi : M \to M$ possesses at least as many fixed points as a function on M has critical points; see [1] and [2]. In view of Morse theory and Ljusternik-Schnirelman theory this conjecture can be reformulated as

$$\sharp \operatorname{Fix} \psi \quad \geqq \quad \text{sum of the Betti numbers of } M$$

provided that the fixed points are all nondegenerate. Moreover, dropping the nondegeneracy assumption

$$\sharp \operatorname{Fix} \psi \quad \geqq \quad \text{cup-length of } M + 1 \,.$$

In our case in which $[\omega]$ and c_1 vanish over $\pi_2(M)$ this conjecture has recently been proved by Floer in [11] and [16] and by Hofer in [23]. The statement about nondegenerate fixed points remains valid if $[\omega]$ and c_1 agree over $\pi_2(M)$; see [16]. As for the history of this conjecture we refer to [42]. It originated in old questions of celestial mechanics and is related to the Poincaré-Birkhoff fixed point theorem which states that an area and orientation preserving homeomorphism of the annulus in the plane twisting the two boundary components in opposite directions possesses at least two fixed points. This theorem also guarantees the existence of infinitely many periodic points distinguished by rotation numbers. Theorem A can be viewed as generalizing the Poincaré-Birkhoff theorem to higher dimensions.

Our proof of Theorem A is based on a well-known variational principle on the loop space of M for which the critical points are the required periodic solutions. This variational principle has also been used by Floer in his proof of the Arnold conjecture which lead to the concept of **Floer homology**. The main point of his proof is to construct a chain complex from the 1-periodic solutions of (1.1) and to show that the homology of this chain complex agrees with the singular homology of the underlying manifold M.

Another central ingredient in the proof of Theorem A is the **Maslov index** $\mu_\tau(x; H)$ of a nondegenerate contractible periodic solution $x(t) = x(t+\tau)$ of (1.1) with integer period τ. This index can roughly be described as a mean winding number for the linearized flow along $x(t)$ or the number of times an eigenvalue crosses 1. The Maslov index is a well defined integer provided that the first Chern class c_1 of the tangent bundle vanishes over $\pi_2(M)$. Postponing the precise definition of the Maslov index we assume that every contractible τ-periodic solution of (1.1) is nondegenerate and denote by

$$p_k(H, \tau)$$

the number of contractible τ-periodic solutions of (1.1) with Maslov index k. We shall prove that these numbers are related to the Betti numbers

$$b_k = \operatorname{rank} H_k(M; \mathbb{Z}/2\mathbb{Z})$$

via the following Morse type inequalities.

THEOREM B. *Assume that* $[\omega]$ *and* c_1 *vanish over* $\pi_2(M)$. *Let* $H : \mathbb{R}/\tau\mathbb{Z} \times M \to \mathbb{R}$ *be a smooth Hamiltonian function such that the contractible* τ*-periodic solutions of* (1.1) *are nondegenerate. Then*

$$p_k(H,\tau) - p_{k-1}(H,\tau) \pm \cdots \geqq b_{n+k} - b_{n+k-1} + \cdots + (-1)^{n+k} b_0 \tag{1.2}$$

for every integer k.

In particular Theorem B shows that

$$p_k(H, \tau) \geqq b_{n+k}$$

and this refines the estimate conjectured by V. I. Arnold.

As a special case consider a Hamiltonian

$$H(t, x) = H(x)$$

which is time independent and is a Morse function on M. Then every critical point of H is a stationary solution of (1.1). If, moreover, the second derivatives of H are sufficiently small then every nonconstant periodic solution of (1.1) is of period larger than τ. Also the Maslov index $\mu_\tau(x; H)$ of a constant periodic orbit $x(t) \equiv x$ is related to the Morse index $\operatorname{ind}_H(x)$ when regarded as a critical point of H via the formula

$$\mu_\tau(x; H) = \operatorname{ind}_H(x) - n\ . \tag{1.3}$$

So in this case Theorem B reduces to the classical Morse inequalities.

Theorem B also implies the Lefschetz fixed point formula. To see this we observe that the Maslov index of a contractible τ-periodic solution $x(t) = x(t+\tau)$ of (1.1) satisfies

$$\operatorname{sign}\det(I - d\psi_\tau(x(0))) = (-1)^{\mu_\tau(x;H)-n} . \tag{1.4}$$

So the Lefschetz fixed point formula can be written in the form

$$\sum_{k\in\mathbb{Z}} (-1)^k p_k(H,\tau) = (-1)^n \chi(M)$$

and this identity follows from (1.2).

In contrast to finite dimensional Morse theory one cannot expect the numbers $p_k(H,\tau)$ to be zero for $|k| > n$ even though the associated (Floer) homology groups are zero in these dimensions. We do, however, know that in the nondegenerate case there are only finitely many contractible τ-periodic solutions and hence the sum on the left-hand side of (1.2) is finite.

We sketch the proof of Theorem A. The key observation is that under the assumptions of Theorem B there must be contractible τ-periodic solutions having Maslov indices $+n$ and $-n$. We shall use an iterated index formula to show that if there are only finitely many 1-periodic solutions of (1.1) and these are all weakly nondegenerate then the iterated 1-periodic solutions have Maslov index $|\mu_\tau(x;H)| \neq n$ for every sufficiently large prime $\tau \in \mathbb{Z}$. This shows that there must be a periodic solution of minimal period τ provided that no root of unity occurs as a Floquet multiplier of the 1-periodic solutions. The general case requires a perturbation argument.

In the following we shall outline the main ideas of Floer homology (Section 2) and give an exposition of the Maslov index for paths of symplectic matrices (Section 3). We prove an index formula for a suitable Fredholm operator involving the Maslov index (Section 4). This leads to a natural grading of the Floer homology groups which is invariant under continuation (Sections 5 and 6). The relation between the Maslov index and the Morse index gives rise to a proof of Theorem B (Sections 7 and 8). We shall then carry out the perturbation argument and the proof of Theorem A (Section 9).

2. The Variational Approach and Floer Homology

Fix $\tau > 0$ and assume that $H(t,x) = H(t+\tau,x)$. Let $L = L_\tau M$ denote the space of contractible loops in M which are represented by smooth curves $\gamma : \mathbb{R} \to M$ satisfying $\gamma(t+\tau) = \gamma(t)$. Then the contractible τ-periodic solutions of (1.1) can be characterized as the critical points of the functional $f = f_\tau : L \to \mathbb{R}$ defined by

$$f_\tau(\gamma) = -\int_D u^*\omega + \int_0^\tau H(t,\gamma(t))\,dt .$$

Here $D \subset \mathbb{C}$ denotes the closed unit disc and $u : D \to M$ is a smooth function which on the boundary agrees with γ, that is $u(\exp\{2\pi i\theta\}) = \gamma(\tau\theta)$. Such a function u exists whenever γ is a contractible loop. Since $[\omega]$ vanishes over every sphere $f(\gamma)$ is independent of the choice of u. To compute the differential of f note that the tangent space $T_\gamma L$ is the space of vector fields $\xi \in C^\infty(\gamma^* TM)$ along γ satisfying $\xi(t+\tau) = \xi(t)$. For the 1-form $df : TL \to \mathbb{R}$ one finds

$$df_\tau(\gamma)\xi = \int_0^\tau (\omega(\dot\gamma, \xi) + dH(t,\gamma)\xi)\, dt$$

and therefore the critical points of f are contractible loops in L which in addition satisfy the Hamiltonian equation (1.1). Thus the critical points are precisely the required τ-periodic solution of (1.1). We denote the set of contractible τ-periodic solutions by

$$\mathscr{P}_\tau = \mathscr{P}_\tau(H) = \{x : \mathbb{R} \to M;\ (1.1),\ x(t+\tau) = x(t),\ x \sim c\}.$$

If all τ-periodic solutions of (1.1) are nondegenerate then they are isolated and it follows that $\mathscr{P}_\tau$ is a finite set.

To describe the gradient of f we choose an **almost complex structure** on M which is compatible with ω. This is an endomorphism $J \in C^\infty(\mathrm{End}(TM))$ satisfying $J^2 = -I$ such that

$$g(\xi, \eta) = \omega(\xi, J(x)\eta), \qquad \xi, \eta \in T_x M, \tag{2.1}$$

defines a Riemannian metric on M. The Hamiltonian vector field is then represented by $X_H(t,x) = J(x)\nabla H(t,x)$ where ∇ denotes the gradient with respect to the x-variable using the metric (2.1). Moreover, the gradient of f with respect to the induced metric on L is given by

$$\operatorname{grad} f(\gamma) = J(\gamma)\dot\gamma + \nabla H(t,\gamma), \qquad \gamma \in L.$$

In studying the critical points of f one is confronted with the well-known difficulty that the variational principle is neither bounded from below nor from above. Moreover, at every possible critical point the Hessian of f has an infinite dimensional positive and an infinite dimensional negative subspace so that standard Morse theory is not applicable. In addition, the gradient vector field on the loop space L

$$\frac{d}{ds}\gamma = -\operatorname{grad} f(\gamma)$$

does not define a well posed Cauchy problem.

In the special case of the torus $M = T^{2n}$ these obstacles have first been overcome in [7]. The idea is to reduce the gradient flow to a finite dimensional submanifold of the loop space L. Then one can study the space $\mathscr{M}$ of **bounded solutions** consisting of the critical points together with their connecting orbits.

This space turns out to be a compact isolated invariant set and hence possesses a Conley index $h(\mathscr{M})$. The index homology agrees with that of the underlying manifold T^{2n} up to a shift in dimension and this proves the Arnold conjecture for the torus.

For general symplectic manifolds M such a finite dimensional reduction has not been found. Recently, however, Floer found a beautiful way to analyse the space $\mathscr{M}$ of bounded solutions directly; see [11], [12], and [13]. Combining the variational approach of [7] with Gromov's elliptic techniques in [22] he defined a **relative Morse index** for a pair of critical points and then used the structure of the space $\mathscr{M}$ to extract an invariant which is now called **Floer homology**. This represents a new approach to infinite dimensional Morse theory which does not require a finite dimensional reduction.

A beautiful exposition of Floer homology in the context of global symplectic geometry and elliptic regularity can be found in [27]. We describe the main points of this approach. A gradient flow line of f is, by definition, a smooth solution $u : \mathbb{R}^2 \to M$ of the partial differential equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t,u) = 0 \tag{2.2}$$

which satisfies $u(s, t + \tau) = u(s,t)$. The key point is to think of (2.2) not as a flow on the loop space but as an elliptic boundary value problem. As a matter of fact, in the case $H = 0$ the solutions of (2.2) are precisely Gromov's pseudoholomorphic curves and these have been studied extensively in [22], [32], and [41]. It turns out that the fundamental regularity and compactness results remain valid in the presence of a nonzero Hamiltonian term; see [34].

We denote by $\mathscr{M}_\tau = \mathscr{M}_\tau(H,J)$ the space of **bounded solutions** of (2.2). That is the space of smooth functions $u : \mathbb{C}/i\tau\mathbb{Z} \to M$ which are contractible, solve the partial differential equation (2.2), and have finite flow energy

$$\Phi_\tau(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^\tau \left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} - X_H(t,u)\right|^2\right) dt\,ds \quad < \quad \infty\,.$$

Since M is compact and $[\omega]$ vanishes over $\pi_2(M)$ the space $\mathscr{M}_\tau$ is compact in the topology of uniform convergence with all derivatives on compact sets; see [13], [16], and [34]. Moreover, for every $u \in \mathscr{M}_\tau$ there exists a pair $x, y \in \mathscr{P}_\tau$ such that u is a connecting orbit from y to x meaning that

$$\lim_{s\to-\infty} u(s,t) = y(t)\,, \qquad \lim_{s\to+\infty} u(s,t) = x(t)\,. \tag{2.3}$$

Here convergence is to be understood uniformly in t and in addition with $\partial u/\partial s$ converging to zero, again uniformly in t as $|s|$ tends to ∞. As a matter of fact, every $u \in \mathscr{M}_\tau$ converges exponentially with all derivatives as s tends to $\pm\infty$ provided that all τ-periodic solutions of (1.1) are nondegenerate.

Given two periodic solutions $x, y \in \mathscr{P}_\tau(H)$ we denote by

$$\mathscr{M}_\tau(y,x) = \mathscr{M}_\tau(y,x;H,J)$$

the space of all $u \in \mathscr{M}_\tau$ which satisfy (2.3). This is the space of absolute minima of the energy functional Φ_τ subject to the asymptotic boundary condition (2.3) and we have

$$\Phi_\tau(u) = f_\tau(y) - f_\tau(x)\ , \qquad u \in \mathscr{M}_\tau(y,x)\ .$$

For a generic choice of the Hamiltonian H the space $\mathscr{M}_\tau(y,x)$ is a finite dimensional manifold. To see this linearize equation (2.2) in the direction of a vector field $\xi \in C^\infty(u^*TM)$ along u. This leads to the linear first-order differential operator

$$F_\tau(u)\xi = \nabla_s\xi + J(u)\nabla_t\xi + \nabla_\xi J(u)\frac{\partial u}{\partial t} + \nabla_\xi\nabla H(t,u) \tag{2.4}$$

where ∇_s, ∇_t and ∇_ξ denote the covariant derivative with respect to the metric (2.1). If u satisfies (2.3) and $x, y \in \mathscr{P}_\tau$ are nondegenerate then $F_\tau(u)$ is a Fredholm operator between appropriate Sobolev spaces. The pair (H, J) with J satisfying (2.1) is called **regular** if every contractible τ-periodic solution of (1.1) is nondegenerate and $F_\tau(u)$ is onto for $u \in \mathscr{M}_\tau$. In Section 8 we prove that the set

$$(\mathscr{H} \times \mathscr{J})_{\text{reg}} \subset \mathscr{H} \times \mathscr{J}$$

of regular pairs is dense with respect to the C^∞-topology. For regular pairs it follows from an implicit function theorem that $\mathscr{M}_\tau(y,x)$ is indeed a finite dimensional manifold whose local dimension near u is the Fredholm index of $F_\tau(u)$.

In the regular case the dimension of the manifold $\mathscr{M}_\tau(y,x)$ was used by Floer to define a relative Morse index for a pair of critical points $x, y \in \mathscr{P}_\tau$. More precisely, if the first Chern class $c_1 \in H^2(M;\mathbb{Z})$ of the tangent bundle vanishes over $\pi_2(M)$ then the Fredholm index of $F_\tau(u)$ depends only on the boundary conditions (2.3) and we denote this index by $m_\tau(y,x)$. These numbers are additive in the sense that

$$m_\tau(z,y) + m_\tau(y,x) = m_\tau(z,x)$$

for periodic solutions $x, y, z \in \mathscr{P}_\tau$. Thus there exists a function $m_\tau : \mathscr{P}_\tau \to \mathbb{Z}$, defined only up to an additive integer, such that for every smooth function $u : \mathbb{C}/i\tau\mathbb{Z} \to M$ which satisfies (2.3) the index of the operator (2.4) is given by

$$\operatorname{index} F_\tau(u) = m_\tau(y) - m_\tau(x)\ .$$

Denote by $C = C(M; H, \tau)$ the vector space over $\mathbb{Z}/2\mathbb{Z}$ generated by the finitely many elements of $\mathscr{P}_\tau$. This vector space is graded by m_τ so that

$$C = \bigoplus_k C_k\,, \qquad C_k(M; H, \tau) = \mathrm{span}_{\mathbb{Z}/2\mathbb{Z}}\,\{x \in \mathscr{P}_\tau;\ m_\tau(x) = k\}\,.$$

It follows from Gromov's compactness combined with the manifold structure that $\mathscr{M}_\tau(y, x)$ consists of finitely many orbits (modulo time shift) whenever $m_\tau(y) - m_\tau(x) = 1$ (see, for example, [34]). This observation can be used to construct a boundary operator $\partial_k = \partial_k(M; H, J, \tau) : C_{k+1} \to C_k$ via the formula

$$\partial y = \sum_{m_\tau(x)=k} \langle \partial y, x\rangle x$$

for $y \in \mathscr{P}_\tau$ with $m_\tau(y) = k + 1$. The matrix element $\langle \partial y, x\rangle$ is defined to be the number of components of $\mathscr{M}_\tau(y, x)$ counted modulo 2. In [16] Floer proves that this operator satisfies $\partial \circ \partial = 0$ so that (C, ∂) defines a chain complex. Its homology

$$HF_*(M; H, J, \tau) = \frac{\ker \partial}{\mathrm{im}\,\partial}$$

is called the **Floer homology** of the pair (H, J).

This chain complex is determined by the one-dimensional components of the space $\mathscr{M}_\tau$ of bounded solutions of (2.2). It is constructed in analogy with with the Morse complex for Morse-Smale gradient flows on finite dimensional manifolds; see [15], [28], [34], and [40]. The transversality condition corresponds to the assumption on the Fredholm operator $F_\tau(u)$ to be onto which as in finite dimensional situation is satisfied generically.

Now, a priori, the Floer homology groups might be trivial and so far we have not even shown that the space $\mathscr{M}_\tau$ of bounded solutions is nonempty. As a matter of fact, it is precisely the point of the above construction to estimate the minimal number of τ-periodic solutions (from below) by proving the nontriviality of the Floer homology groups. For this we recall that a crucial property of the Conley index for finite dimensional flows lies in the fact that it is invariant under continuation. An analogous homotopy invariance property for the Floer homology groups can be used to show that they are independent of H and J and, indeed, nontrivial. More precisely, in [16] Floer proved the following theorem.

THEOREM 2.1.

(i) *If (H, J) and (H', J') are regular pairs with respect to the periods τ and τ', respectively, then there exists a natural chain homomorphism which induces an isomorphism of Floer homology*

$$HF_*(M; H, J, \tau) \cong HF_*(M; H', J', \tau')\,.$$

(ii) *There is a natural isomorphism between the Floer homology of the loop space and the singular homology of M*

$$HF_*(M; H, J, \tau) \cong H_*(M; \mathbb{Z}/2\mathbb{Z}) .$$

Theorem 2.1 shows that the periodic solutions of the Hamiltonian system (1.1) and the bounded solutions of (2.2) can be used to construct a model for the homology of the underlying manifold M. This proves the Arnold conjecture under the assumption that $[\omega]$ and c_1 vanish over $\pi_2(M)$. Note, however, that the grading of the Floer homology groups is so far only well defined up to an additive constant and Theorem 2.1 states, more precisely, that with a suitable choice of this grading $HF_k(M; H, J, \tau)$ is isomorphic to $H_k(M; \mathbb{Z}/2\mathbb{Z})$. This ambiguity of the grading will be removed by a Maslov type index

$$\mu_\tau : \mathscr{P}_\tau \to \mathbb{Z}$$

which associates an integer to every nondegenerate contractible periodic solution $x \in \mathscr{P}_\tau$ of (1.1). This integer is a symplectic invariant obtained from the linearized Hamiltonian flow along $x(t)$. We shall prove that for $u \in \mathscr{M}_\tau(y, x)$ the Fredholm index of the operator $F_\tau(u)$ can be characterized in terms of the Maslov indices of x and y

$$\operatorname{index} F_\tau(u) = \mu_\tau(y) - \mu_\tau(x) .$$

It follows that the Maslov index defines a natural grading of the Floer homology groups. With this grading the proof of Theorem 2.1 will yield an isomorphism

$$HF_k(M; H, J, \tau) \simeq H_{k+n}(M; \mathbb{Z}/2\mathbb{Z}) , \qquad -n \leqq k \leqq n .$$

This implies the Morse inequalities of Theorem B. It also confirms the interpretation of HF_* as the 'middle dimensional' homology groups of the loop space of M; see [4].

Remarks.

(i) In [16] Floer proved the Arnold conjecture under the more general assumption that the cohomology classes $[\omega]$ and c_1 are proportional over $\pi_2(M)$ with a positive factor. If $[\omega]$ does not vanish over $\pi_2(M)$ then f_τ takes values in S^1. If c_1 does not vanish over $\pi_2(M)$ then the Maslov index μ_τ is only defined modulo an integer N. So in this case the Floer homology groups are graded modulo N.

(ii) It is an open question whether there is any compact symplectic manifold M such that $[\omega]$ vanishes over $\pi_2(M)$ but $\pi_2(M) \neq \{1\}$. There are, however, examples of symplectic manifolds with nontrivial second homotopy group and $c_1 = 0$.

(iii) The Floer homology groups HF_* can be defined with integer coefficients and hence coefficients in any abelian group G; see [16] and [34]. For this one has to assign an integer $+1$ or -1 to every connecting orbit $u \in \mathcal{M}_\tau(y,x)$ whenever $\mu_\tau(y) - \mu_\tau(x) = 1$. The determination of this sign is a subtle story and requires a consistent orientation of the moduli spaces $\mathcal{M}_\tau(y,x)$. The details have been carried out by Floer and Hofer in [18]. This refinement is not needed for the proof of Theorem A. It does, however, lead to a sharper estimate in Theorem B with coefficients in any principal ideal domain.

(iv) In [14] Floer developed a similar theory as the one described above for the Chern-Simons functional on the space of $SU(2)$-connections on a homology-3-sphere Y. In that context the connecting orbits can be interpreted as self-dual Yang-Mills connections on the 4-manifold $Y \times \mathbb{R}$ with finite Yang-Mills action. The associated Floer homology groups define new invariants of the underlying homology-3-sphere, refining the Casson invariant. This is beautifully explained in [4].

3. The Maslov Index for Symplectic Arcs

The standard Maslov index associates an integer to every loop in $\mathrm{Sp}(2n;\mathbb{R})$. In contrast, we shall need a modified version of the Maslov index which is not defined for loops but for every path in

$$\mathcal{SP}^*(\tau) = \{\Psi : [0,\tau] \to \mathrm{Sp}(2n;\mathbb{R});\ \Psi(0) = I,\ \det(I - \Psi(1)) \neq 0\}.$$

This modified index was introduced in [8]. An alternative description can be found in [26] and we present a third exposition in this section.

First note that a matrix $A \in \mathbb{R}^{2n\times 2n}$ is both symplectic and orthogonal if and only if

$$A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \tag{3.1}$$

where $X^TY - Y^TX = 0$ and $X^TX + Y^TY = I$ or equivalently

$$X + iY \in U(n).$$

Now the determinant mapping

$$\det : U(n) \to S^1$$

induces an isomorphism of fundamental groups as can be seen from the homotopy exact sequence associated to the fibration $SU(n) \hookrightarrow U(n) \to S^1$. Since the quotient $\mathrm{Sp}(2n;\mathbb{R}) \,/\, U(n)$ is contractible it follows that $\pi_1(\mathrm{Sp}(2n;\mathbb{R})) \simeq \mathbb{Z}$. This isomorphism can be represented by a natural continuous map

$$\rho : \mathrm{Sp}(2n;\mathbb{R}) \to S^1$$

which restricts to the determinant map on $\mathrm{Sp}(2n;\mathbb{R}) \cap O(2n) \simeq U(n)$. This map ρ was introduced in [35]. It is no longer a homomorphism.

More generally, let V be a **symplectic vector space** that is a finite dimensional real vector space with a nondegenerate skew-symmetric bilinear form $\omega = \omega_V : V \times V \to \mathbb{R}$. Let $\mathrm{Sp}(V) = \mathrm{Sp}(V, \omega)$ denote the group of automorphisms of V that is linear transformations $A : V \to V$ such that $A^*\omega = \omega$. In particular,

$$\mathrm{Sp}(2n;\mathbb{R}) = \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$$

where

$$\omega_0(\zeta, \zeta') = (J_0\zeta)^T \zeta', \qquad J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

THEOREM 3.1. *There is a unique collection of continuous mappings*

$$\rho : \mathrm{Sp}(V, \omega) \to S^1$$

(one for every symplectic vector space) satisfying the following conditions:

Naturality: *If $T : (V_1, \omega_1) \to (V_2, \omega_2)$ is a symplectic isomorphism, that is $T^*\omega_2 = \omega_1$, then*

$$\rho(TAT^{-1}) = \rho(A)$$

for $A \in \mathrm{Sp}(V_1, \omega_1)$.

Product: *If $(V, \omega) = (V_1 \times V_2, \omega_1 \times \omega_2)$, then*

$$\rho(A) = \rho(A_1)\rho(A_2)$$

for $A \in \mathrm{Sp}(V, \omega)$ of the form $A(z_1, z_2) = (A_1z_1, A_2z_2)$ where $A_i \in \mathrm{Sp}(V_i, \omega_i)$.

Determinant: *If $A \in \mathrm{Sp}(2n;\mathbb{R}) \cap O(2n)$ is of the form* (3.1), *then*

$$\rho(A) = \det(X + iY)\ .$$

Normalization: *If A has no eigenvalue on the unit circle then*

$$\rho(A) = \pm 1\ .$$

It follows from the determinant property that ρ induces an isomorphism of fundamental groups $\pi_1(\mathrm{Sp}(2n;\mathbb{R})) \to \pi_1(S^1)$. Also note that

$$\rho(A^T) = \rho(A^{-1}) = \overline{\rho(A)}$$

for $A \in \mathrm{Sp}(2n;\mathbb{R})$.

Proof of Theorem 3.1: We construct the map $\rho : \mathrm{Sp}(2n;\mathbb{R}) \to S^1$ explicitly. For this we first recall that the eigenvalues of a real symplectic matrix A occur in quadruples of the form $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$ unless λ is real or on the unit circle. Now every pair of simple eigenvalues on the unit circle $\lambda, \bar{\lambda} \in S^1$ can be ordered in a unique way due to the symplectic structure. To see this observe that $0 \neq \omega_0(\bar{\zeta}, \zeta) \in i\mathbb{R}$ whenever ζ is an eigenvector corresponding to a simple eigenvalue $\lambda \in S^1$. So the sign of $\mathrm{Im}\ \omega_0(\bar{\zeta}, \zeta)$ is independent of the choice of the eigenvector and we call $\lambda \in S^1$ an **eigenvalue of the first kind** if

$$\mathrm{Im}\ \omega_0(\bar{\zeta}, \zeta) > 0 .$$

Note that either λ or $\bar{\lambda}$ is an eigenvalue of the first kind whenever $\lambda \neq \pm 1$ is a simple eigenvalue on the unit circle. If $+1$ or -1 is an eigenvalue of a symplectic matrix then it occurs with even multiplicity. If $\lambda \in \sigma(A)$ is not on the unit circle we call it an eigenvalue of the first kind if $|\lambda| < 1$. This shows that if the matrix $A \in \mathrm{Sp}(2n;\mathbb{R})$ has distinct eigenvalues then these can be ordered in the form

$$\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1} \tag{3.2}$$

such that the eigenvalues $\lambda_1, \ldots, \lambda_n$ are of the first kind. Following Gelfand and Lidskii (see [21]), we define the number

$$\mathrm{Arg}(A) = \sum_{j=1}^{n} \arg(\lambda_j) \quad \in \quad \mathbb{R}/2\pi\mathbb{Z}$$

for symplectic matrices with distinct eigenvalues and

$$\rho(A) = \exp\{i\,\mathrm{Arg}(A)\} = \prod_{j=1}^{n} \frac{\lambda_j}{|\lambda_j|} . \tag{3.3}$$

This function is continuous. To see this note that only the eigenvalues on the unit circle and on the negative real axis contribute to $\mathrm{Arg}(A)$. Moreover, as has been observed by Gelfand and Lidskii in [21], two pairs on the unit circle can only move off the unit circle if they join *and* if the two eigenvalues of the first kind cancel each other. Thus the function (3.3) extends uniquely to $\mathrm{Sp}(2n;\mathbb{R})$.

To construct this extension explicitly let $\lambda \in \sigma(A)$ be an eigenvalue of multiplicity $m(\lambda)$ and denote the generalized eigenspace by

$$E_\lambda = E_\lambda(A) = \bigcup_{j \geq 1} \ker(\lambda I - A)^j \quad \subset \quad \mathbb{C}^{2n} .$$

This space is of real dimension $2m(\lambda)$ and a simple computation shows that

$$\omega_0(E_\lambda, E_\mu) = 0\,, \qquad \lambda\mu \neq 1\,. \tag{3.4}$$

If $\lambda \in S^1 \setminus \{\pm 1\}$ then it follows that the symmetric bilinear form

$$Q(\zeta_1, \zeta_2) = \operatorname{Im}\, \omega_0(\bar{\zeta}_1, \zeta_2) \tag{3.5}$$

is nondegenerate on E_λ. The identity $Q(i\zeta_1, i\zeta_2) = Q(\zeta_1, \zeta_2)$ shows that the positive part of Q on E_λ is of even dimension $2m^+(\lambda)$. Let m_0 denote the number of pairs λ, λ^{-1} of negative real eigenvalues and define

$$\rho(A) = (-1)^{m_0} \prod_{\lambda \in \sigma(A) \cap S^1 \setminus \{\pm 1\}} \lambda^{m^+(\lambda)}\,. \tag{3.6}$$

This number agrees with (3.3) provided that we count eigenvalues with their multiplicity. The eigenvalues of the first kind are all those with $|\lambda| < 1$ and if $\lambda \in S^1$ is a multiple eigenvalue on the unit circle then it counts with multiplicity $m^+(\lambda)$ as an eigenvalue of the first kind. By construction this function ρ satisfies the naturality, product, and normalization conditions.

We prove that ρ is continuous. First note that, by (3.4), the bilinear form Q vanishes on E_λ for every eigenvalue $\lambda \notin S^1$. It follows that Q is of signature zero on $E_\lambda \oplus E_{\bar{\lambda}^{-1}}$ for $\lambda \notin S^1$. Thus the continuity of ρ follows from the lower semicontinuity of the eigenspaces as functions of A.

We prove that ρ satisfies the determinant condition. Let $A \in \mathrm{Sp}(2n;\mathbb{R}) \cap O(2n)$ be of the form (3.1) and define $U = X + iY \in U(n)$. Then $AJ_0 = J_0A$ and since the eigenvalues of J_0 are $\pm i$ it follows that

$$E_\lambda = E_\lambda^+ \oplus E_\lambda^-\,, \qquad E_\lambda^\pm = E_\lambda \cap \ker(\mp iI - J_0)\,.$$

Now $\ker(-iI - J_0) = \{(\xi, -i\xi);\ \xi \in \mathbb{C}^n\}$ is a maximal positive subspace for Q, and therefore the eigenvalues of the first kind correspond to eigenvectors in E_λ^+. Since $\zeta = (\xi, -i\xi) \in E_\lambda(A)$ if and only if $\xi \in E_\lambda(U)$ we conclude that $\rho(A) = \det U$ as claimed.

The Maslov index of a path $\Psi \in \mathcal{SP}^*(\tau)$ is based on the next lemma which was proved in [8]. We present an alternative proof. We use the notation

$$\mathrm{Sp}(2n;\mathbb{R})^* = \{A \in G;\ \det(I - A) \neq 0\}\,.$$

LEMMA 3.2. $\mathrm{Sp}(2n;\mathbb{R})^*$ *has two connected components*

$$\mathrm{Sp}(2n;\mathbb{R})^\pm = \{A \in \mathrm{Sp}(2n;\mathbb{R});\ \pm\det(I - A) > 0\}\,.$$

Moreover, every loop in $\mathrm{Sp}(2n;\mathbb{R})^*$ *is contractible in* $\mathrm{Sp}(2n;\mathbb{R})$.

Proof: We prove that $\mathrm{Sp}(2n;\mathbb{R})^+$ and $\mathrm{Sp}(2n;\mathbb{R})^-$ are path connected with

$$W^+ = -I \quad \in \quad \mathrm{Sp}(2n;\mathbb{R})^+$$

and

$$W^- = \mathrm{diag}(2,-1,\dots,-1,1/2,-1,\dots,-1) \quad \in \quad \mathrm{Sp}(2n;\mathbb{R})^- .$$

Let $A \in \mathrm{Sp}(2n;\mathbb{R})^*$ be given and let $\lambda \in \sigma(A)$. Choose a basis $\zeta_1,\dots,\zeta_m$ of E_λ such that

$$(\lambda I - A)\zeta_j \quad \in \quad \mathrm{span}\{\zeta_1,\dots,\zeta_{j-1}\} .$$

Consider the case $\lambda \notin S^1 \cup \mathbb{R}$ and choose a basis $\eta_1,\dots,\eta_m$ of $E_{\bar\lambda^{-1}}$ such that

$$\omega_0(\zeta_j,\eta_k) = \delta_{jk} .$$

Define $B = B(s) \in \mathrm{Sp}(2n;\mathbb{R})$ so that B acts as the identity on E_μ for $\mu \neq \lambda,\bar\lambda,\lambda^{-1},\bar\lambda^{-1}$ and on $\mathrm{span}\{\zeta_j,\eta_j,\bar\zeta_j,\bar\eta_j;\ 2 \leqq j \leqq m\}$ and

$$B(s)\zeta_1 = (1+s)\zeta_1 , \qquad B(s)\eta_1 = \frac{1}{1+\bar s}\eta_1 ,$$
$$B(s)\bar\zeta_1 = (1+\bar s)\bar\zeta_1 , \qquad B(s)\bar\eta_1 = \frac{1}{1+s}\bar\eta_1 .$$

Then $\dim E_\lambda(B(s)A) = m-1$ and $(1+s)\lambda$ is a new eigenvalue. Using a similar argument in the cases $\lambda \in S^1$ and $\lambda \in \mathbb{R}$ we can connect A to a matrix with distinct eigenvalues. Now choose a suitable path $s : [0,1] \to \mathbb{C}$ such that $s(0) = 0$, $s(1) = -\lambda^{-1} - 1$, and $(1+s(t))\lambda \neq 1$ to move all eigenvalues to -1 except for the positive real pairs. A similar argument shows that an even number of positive real pairs can be removed from the real axis. This way $A \in \mathrm{Sp}(2n;\mathbb{R})^+$ can be connected to a matrix A_1 all of whose eigenvalues are -1. Now there exists a Hamiltonian matrix $J_0S = \log(-A_1)$ with $S = S^T$ such that $\exp\{J_0S\} = -A_1$. So the path $t \mapsto -\exp\{tJ_0S\}$ connects A_1 to $-I = W^+$. This shows that $\mathrm{Sp}(2n;\mathbb{R})^+$ is connected. A similar argument works for $\mathrm{Sp}(2n;\mathbb{R})^-$.

We prove that every loop $\gamma(t) \in \mathrm{Sp}(2n;\mathbb{R})^+$ with $\gamma(0) = \gamma(1)$ is contractible in $\mathrm{Sp}(2n;\mathbb{R})$. To see this we shall construct continuous functions $\alpha_\nu : \mathrm{Sp}(2n;\mathbb{R})^+ \to [0,2\pi]$ for $\nu = 1,\dots,n$ such that

$$\exp\left\{ i\sum_{\nu=1}^{n} \alpha_\nu(A) \right\} = \rho(A) .$$

Then the functions $\alpha_\nu(t) = \alpha_\nu(\gamma(t))$ are periodic and continuous. Hence the loop $\rho(\gamma(t)) = \exp\{i\sum_\nu \alpha_\nu(t)\} \in S^1$ is contractible and it follows from Lemma 3.1 that $\gamma(t)$ is contractible in $\mathrm{Sp}(2n;\mathbb{R})$.

We construct the α_ν as follows. Given $A \in \mathrm{Sp}(2n, \mathbb{R})^+$ choose real numbers

$$0 \leqq \alpha_1(A) \leqq \cdots \leqq \alpha_n(A) \leqq 2\pi$$

such that

$$\exp\{i\alpha_\nu(A)\} = \frac{\lambda_\nu}{|\lambda_\nu|}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the first kind. If there are no positive real eigenvalues then this determines the numbers $\alpha_\nu(A)$ uniquely. Since $A \in \mathrm{Sp}(2n; \mathbb{R})^+$ the total multiplicity of real eigenvalues larger than 1 is even and we choose the $\alpha_\nu(A)$ such that there is the same number of ν's with $\alpha_\nu(A) = 0$ and with $\alpha_\nu(A) = 2\pi$. For this choice the functions $\alpha_\nu : \mathrm{Sp}(2n, \mathbb{R})^+ \to [0, 2\pi]$ are continuous.

The same construction works for $A \in \mathrm{Sp}(2n; \mathbb{R})^-$ except that the total multiplicity of real eigenvalues larger than 1 is odd.

For any path $\gamma : [0, \tau] \to \mathrm{Sp}(2n; \mathbb{R})$ choose a function $\alpha : [0, \tau] \to \mathbb{R}$ such that $\rho(\gamma(t)) = e^{i\alpha(t)}$ and define

$$\Delta_\tau(\gamma) = \frac{\alpha(\tau) - \alpha(0)}{\pi} .$$

For $A \in \mathrm{Sp}(2n; \mathbb{R})^*$ choose a path $\gamma_A(t) \in \mathrm{Sp}(2n; \mathbb{R})^*$ such that $\gamma_A(0) = A$ and $\gamma_A(1) \in \{W^+, W^-\}$. Then it follows from Lemma 3.2 that $\Delta_1(\gamma_A)$ is independent of the choice of this path. Define

$$r(A) = \Delta_1(\gamma_A) , \qquad A \in \mathrm{Sp}(2n; \mathbb{R})^* .$$

The **Maslov index** of a path $\Psi \in \mathscr{S}P^*(\tau)$ is defined by

$$\mu_\tau(\Psi) = \Delta_\tau(\Psi) + r(\Psi(\tau)) .$$

This index can be interpreted as the mean winding number of the eigenvalues of the first kind or equivalently the number of such eigenvalues crossing 1.

THEOREM 3.3.

(i) *The Maslov index is an integer.*

(ii) *Two paths Ψ_0, Ψ_1 are homotopic in $\mathscr{S}P^*(\tau)$ if and only if they have the same Maslov index.*

(iii) *If $\Psi \in \mathscr{S}P^*(\tau)$ then*

$$\operatorname{sign}\det(I - \Psi(\tau)) = (-1)^{\mu_\tau(\Psi) - n} .$$

(iv) *Let* $\Psi(t) = \exp\{J_0 S t\}$ *where* $S = S^T \in \mathbb{R}^{2n\times 2n}$ *is a nonsingular symmetric matrix such that*

$$|S| < \frac{2\pi}{\tau} .$$

Then $\Psi \in \mathscr{S}P^*(\tau)$ *and*

$$\mu_\tau(\Psi) = \mu^-(S) - n$$

where $\mu^-(S)$ *denotes the number of negative eigenvalues of* S *counted with multiplicity.*

Proof: We extend $\Psi \in \mathscr{S}P^*(\tau)$ to a smooth path $\gamma : [0, \tau+1] \to \mathrm{Sp}(2n; \mathbb{R})$ which agrees with Ψ on $[0, \tau]$ and satisfies $\gamma(t) \in \mathrm{Sp}(2n; \mathbb{R})^*$ for $\tau \leqq t \leqq \tau + 1$ and $\gamma(\tau + 1) = W^\pm$. By Lemma 3.2 such an extension exists and $\mu_\tau(\Psi) = \Delta_{\tau+1}(\gamma) \in \mathbb{Z}$. It also follows from Lemma 3.2 that two paths Ψ_0 and Ψ_1 are homotopic in $\mathscr{S}P^*(\tau)$ if and only if any two such extensions γ_0 and γ_1 are homotopic with fixed end points. Since the map ρ induces an isomorphism of fundamental groups, this is equivalent to $\Delta_{\tau+1}(\gamma_0) = \Delta_{\tau+1}(\gamma_1)$. Thus we have proved statements (i) and (ii).

Now let $\Psi \in \mathscr{S}P^*(\tau)$ and let γ be an extension as above. If $\det(I - \Psi(\tau)) > 0$ then $\gamma(\tau + 1) = W^+ = -I$ and hence $\rho(\gamma(\tau + 1)) = (-1)^n$. So in this case $\mu_\tau(\Psi) - n$ is even. If $\det(I - \Psi(\tau)) < 0$ then $\gamma(\tau + 1) = W^-$ and hence $\rho(\gamma(\tau+1)) = (-1)^{n-1}$. So in this case $\mu_\tau(\Psi) - n$ is odd. This proves statement (iii).

To prove statement (iv) we choose a path of orthogonal matrices $P_\lambda \in O(2n)$ such that $P_0 = I$ and $S_1 = P_1{}^T S P_1$ is a diagonal matrix. Define

$$S_\lambda = P_\lambda{}^T S P_\lambda , \qquad \Psi_\lambda(t) = \exp\{J_0 S_\lambda t\} .$$

Then S_λ is a path of symmetric matrices connecting $S = S_0$ to a diagonal matrix such that $\mu^-(S_\lambda)$ is independent of λ. The condition $|S| < 2\pi/\tau$ guarantees that $1 \notin \sigma(\Psi_\lambda(\tau))$ and hence $\Psi_\lambda \in \mathscr{S}P^*(\tau)$ for every λ. This reduces statement (iv) to the case where S is a diagonal matrix. Without loss of generality we may assume that

$$S = \mathrm{diag}(\varepsilon, \ldots, \varepsilon, -\varepsilon, \ldots, -\varepsilon)$$

where $0 < \varepsilon < 2\pi/\tau$ and the number of negative terms is $k = \mu^-(S)$. Now decompose $\mathbb{R}^{2n}$ into n symplectic planes and consider the matrices

$$S_0 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \qquad S_1 = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \qquad S_2 = \begin{pmatrix} -\varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$$

in the case $n = 1$. The identity

$$\exp\{J_0 S_0 t\} = \begin{pmatrix} \cos \varepsilon t & \sin \varepsilon t \\ -\sin \varepsilon t & \cos \varepsilon t \end{pmatrix}$$

combined with the determinant property of the map ρ shows that

$$\rho(\exp\{J_0 S_0 t\}) = \exp\{-i\varepsilon t\} .$$

Since $0 < \varepsilon < 2\pi/\tau$ it follows that $\mu_\tau(\Psi) = -1$ in the case $S = S_0$. The case $S = S_2$ is similar with $\mu_\tau(\Psi) = 1$ while $\exp\{J_0 S_1 t\}$ has real eigenvalues so that $\mu_\tau(\Psi) = 0$ in the case $S = S_1$. We conclude that in all three cases

$$\mu_\tau(\Psi) = \mu^-(S) - 1$$

and this proves statement (iv).

The following iterated index formula will play a crucial role in the proof of Theorem A.

LEMMA 3.4. *Let $\Psi(t) \in \mathrm{Sp}(2n;\mathbb{R})$ be any path such that*

$$\Psi(k\tau + t) = \Psi(t)\Psi(\tau)^k$$

for $t \geqq 0$ and $k \in \mathbb{N}$. Then

$$\Delta_{k\tau}(\Psi) = k\Delta_\tau(\Psi)$$

for every $k \in \mathbb{N}$. Moreover, $|r(A)| < n$ for every $A \in \mathrm{Sp}(2n;\mathbb{R})^*$.

Proof: By construction of the map ρ, we have $\rho(A^k) = \rho(A)^k$ for every $A \in \mathrm{Sp}(2n;\mathbb{R})$. (This is obvious for matrices with distinct eigenvalues.) It follows that

$$\Delta_{k\tau}(\Psi) - k\Delta_\tau(\Psi) \quad \in \quad 2\mathbb{Z}$$

for every path $\Psi(t) \in \mathrm{Sp}(2n;\mathbb{R})$ which satisfies $\Psi(k\tau + t) = \Psi(t)\Psi(\tau)^k$ for $t \geqq 0$. Now let $\Psi_\lambda(t) \in \mathrm{Sp}(2n;\mathbb{R})$ be any homotopy of $\Psi = \Psi_0$ to a path in $\mathrm{Sp}(2n;\mathbb{R}) \cap O(2n)$ such that each path Ψ_λ satisfies the requirements of Lemma 3.4. Since the restriction of ρ to $\mathrm{Sp}(2n;\mathbb{R}) \cap O(2n)$ is a homomorphism we obtain

$$\Delta_{k\tau}(\Psi_0) - k\Delta_\tau(\Psi_0) = \Delta_{k\tau}(\Psi_1) - k\Delta_\tau(\Psi_1) = 0 .$$

The bound on $r(A)$ follows immediately from the construction of the maps

$$\alpha_\nu : \mathrm{Sp}(2n;\mathbb{R})^* \to [0, 2\pi]$$

in the proof of Lemma 3.2.

4. The Fredholm Index

Let $\tau > 0$ and consider the operator $F : W^{1,2}(\mathbb{R} \times \mathbb{R}/\tau\mathbb{Z}; \mathbb{R}^{2n}) \to L^2(\mathbb{R} \times \mathbb{R}/\tau\mathbb{Z}; \mathbb{R}^{2n})$ defined by

$$F\zeta = \frac{\partial \zeta}{\partial s} + J_0 \frac{\partial \zeta}{\partial t} + S\zeta \; . \tag{4.1}$$

Note that in the case $S = 0$ this is the Cauchy-Riemann operator. In general we assume that $S(s,t) \in \mathbb{R}^{2n \times 2n}$ is a continuous matrix valued function on $\mathbb{R}^2$ such that

$$S(s,t)^T = S(s,t) = S(s,t+\tau) \; .$$

We also assume that $S(s,t)$ converges, uniformly in t, as s tends to $\pm\infty$ and denote the limits by

$$S^{\pm}(t) = \lim_{s \to \pm\infty} S(s,t) \; . \tag{4.2}$$

Associated to S is the symplectic matrix function $\Psi(s,t) \in \mathrm{Sp}(2n;\mathbb{R})$ defined by

$$\frac{\partial \Psi(s,t)}{\partial t} = J_0 S(s,t) \Psi(s,t) \; , \qquad \Psi(s,0) = I \; .$$

Note that both Ψ and $\partial\Psi/\partial t$ are continuous. There is a one-to-one correspondence between Ψ and S since $S = -J_0(\partial\Psi/\partial t)\Psi^{-1}$ is symmetric whenever $\Psi(s,t) \in \mathrm{Sp}(2n;\mathbb{R})$. The periodicity condition for S corresponds to $\Psi(s,t+k\tau) = \Psi(s,t)\Psi(s,\tau)^k$ for $k \in \mathbb{Z}$. Moreover, both Ψ and $\partial\Psi/\partial t$ converge, uniformly in t, as s tends to $\pm\infty$ and we define

$$\Psi^{\pm}(t) = \lim_{s \to \pm\infty} \Psi(s,t) \; .$$

THEOREM 4.1. *Suppose that $S(s,t) = S(s,t)^T$ satisfies* (4.2) *with $\Psi^{\pm} \in \mathcal{SP}^*(\tau)$. Then F is a Fredholm operator and*

$$\operatorname{index} F = \mu_\tau(\Psi^-) - \mu_\tau(\Psi^+) \; .$$

Proof: We prove that F is a Fredholm operator. Even though this is a standard argument (see [11], [25], and [34]), we shall carry out the details in our special case. We abbreviate $X = \mathbb{R} \times Y$, $X_T = [-T,T] \times Y$ and $Y = \mathbb{R}/\tau\mathbb{Z}$. We shall prove the estimate

$$\|\zeta\|_{W^{1,2}(X)} \leqq c \left(\|F\zeta\|_{L^2(X)} + \|\zeta\|_{L^2(X_T)} \right) \tag{4.3}$$

for $T > 0$ sufficiently large. Since the injection $W^{1,2}(X_T) \hookrightarrow L^2(X_T)$ is compact it follows that F has a closed range and a finite dimensional kernel. A similar inequality holds for the adjoint operator

$$F^* = -\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S$$

and shows that the cokernel of F is finite dimensional as well.

By partial integration we have

$$\|\nabla\zeta\|_{L^2(X)}^2 = -\langle\zeta, \Delta\zeta\rangle = \left\|\frac{\partial\zeta}{\partial s} + J_0 \frac{\partial\zeta}{\partial t}\right\|_{L^2(X)}^2$$

for functions with compact support and hence

$$\|\zeta\|_{W^{1,2}(X)} \leqq c_1 \left(\|F\zeta\|_{L^2(X)} + \|\zeta\|_{L^2(X)}\right) \tag{4.4}$$

where $c_1 = 1 + \sup|S(s,t)|$. This proves (4.3) for functions ζ supported in X_T.

Now we specialize to the case where $S(s,t) = S(t)$ and $\Psi(s,t) = \Psi(t)$ are independent of s. We prove that if $\Psi \in \mathscr{S}P$ then F is invertible. To see this consider the operator $A : W^{1,2}(Y;\mathbb{R}^{2n}) \to L^2(Y;\mathbb{R}^{2n})$ defined by

$$(A\zeta)(t) = J_0 \frac{d\zeta(t)}{dt} + S(t)\zeta(t) .$$

This operator is invertible if and only if $1 \notin \sigma(\Psi(\tau))$ or equivalently $\Psi \in \mathscr{S}P^*(\tau)$. If this is the case then there exists a constant $c_0 > 0$ such that

$$\|\zeta\|_{W^{1,2}(Y)} \leqq c_0 \|A\zeta\|_{L^2(Y)} .$$

Since A is self adjoint there is an inequality

$$|\omega| \, \|\zeta\|_{L^2(Y)}^2 \leqq |\langle\zeta, i\omega\zeta + A\zeta\rangle| \leqq \|\zeta\|_{L^2(Y)} \|i\omega\zeta + A\zeta\|_{L^2(Y)}$$

for $\omega \in \mathbb{R}$. This implies

$$|\omega| \, \|\zeta\|_{L^2(Y)} \leqq \|i\omega\zeta + A\zeta\|_{L^2(Y)} , \qquad \|\zeta\|_{W^{1,2}(Y)} \leqq 2c_0 \|i\omega\zeta + A\zeta\|_{L^2(Y)} .$$

Now let $\zeta(s,t) \in \mathbb{R}^{2n}$ be a smooth compactly supported function and denote its Fourier transform by

$$\hat{\zeta}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-i\omega s\}\zeta(s,t)\, ds .$$

Then it follows from Plancherel's theorem that

$$\begin{aligned}\|\zeta\|^2_{W^{1,2}(X)} &= \int_{-\infty}^{\infty}\left(\left\|\hat{\zeta}\right\|^2_{W^{1,2}(Y)} + |\omega|^2\left\|\hat{\zeta}\right\|^2_{L^2(Y)}\right)d\omega \\ &\leqq \left(4c_0^2+1\right)\int_{-\infty}^{\infty}\left\|i\omega\hat{\zeta} + A\hat{\zeta}\right\|^2_{L^2(Y)}d\omega \\ &= \left(4c_0^2+1\right)\left\|\frac{\partial\zeta}{\partial s} + A\zeta\right\|^2_{L^2(X)} .\end{aligned}$$

This shows that the operator F is invertible whenever $S(s,t) = S(t)$ is independent of s and $\Psi \in \mathscr{SP}^*(\tau)$.

Returning to the general case we obtain that the limit operators $F^{\pm}$ with S replaced by $S^{\pm}$ are invertible. Thus there exist constants $T > 0$ and $c_2 > 0$ such that

$$\|\zeta\|_{W^{1,2}(X)} \leqq c_2\|F\zeta\|_{L^2(X)} \tag{4.5}$$

for every ζ which vanishes in X_{T-1}.

To prove that (4.4) and (4.5) imply (4.3) we choose a cutoff function $\beta(s)$ such that $\beta(s) = 1$ for $|s| \leqq T-1$ and $\beta(s) = 0$ for $|s| \geqq T$. Given ζ we use (4.4) to estimate the term $\beta\zeta$ and (4.5) to estimate $(1-\beta)\zeta$. This gives

$$\begin{aligned}\|\zeta\|_{W^{1,2}(X)} &\leqq c_1\left(\|F(\beta\zeta)\|_{L^2(X)} + \|\beta\zeta\|_{L^2(X)}\right) + c_2\|F((1-\beta)\zeta)\|_{L^2(X)} \\ &\leqq c_3\left(\|F\zeta\|_{L^2(X)} + \|\zeta\|_{L^2(X_T)}\right).\end{aligned}$$

Here we have used the fact that $\dot{\beta}$ vanished outside the interval $[-T, T]$. This proves the estimate (4.3) and therefore F is a Fredholm operator under the assumptions of Theorem 4.1.

To prove the index formula we consider the family of self adjoint operator

$$A(s) : W^{1,2}\left(\mathbf{R}/\tau\mathbb{Z}; \mathbb{R}^{2n}\right) \to L^2\left(\mathbf{R}/\tau\mathbb{Z}; \mathbb{R}^{2n}\right)$$

for $s \in \mathbf{R}$ defined by

$$(A(s)\zeta)(t) = J_0\frac{d\zeta(t)}{dt} + S(s,t)\zeta(t), \qquad 0 \leqq t \leqq \tau .$$

These operators have a compact resolvent and hence a discrete spectrum consisting of real eigenvalues with finite multiplicity. The eigenvalues of $A(s)$ occur in continuous families $\lambda_j(s)$. These can be chosen such that each

$\lambda \in \sigma(A(s))$ with multiplicity m occurs m times as $\lambda_j(s)$. With this understood the Fredholm index of the operator (4.1) is given by the **spectral flow** (see [5])

(4.6) $$\operatorname{index} F = \sharp\{j; \lambda_j(-\infty) < 0 < \lambda_j(+\infty)\} - \sharp\{j; \lambda_j(-\infty) > 0 > \lambda_j(+\infty)\} .$$

In particular, if $A(s)$ is independent of s and nonsingular then F is invertible as we have proved above and so (4.6) is obviously satisfied.

We prove that the spectral flow agrees with the index difference $\mu_\tau(\Psi^-) - \mu_\tau(\Psi^+)$. The argument rests on the fact that $0 \in \sigma(A(s))$ if and only if $\Psi(s, \cdot) \notin \mathcal{SP}^*(\tau)$. We first simplify the operator by choosing a continuous family of functions $\Psi_\lambda(s,t) \in Sp(2n;\mathbb{R})$ such that

$$\Psi_\lambda{}^{\pm}(\tau) \in \mathrm{Sp}(2n;\mathbb{R})^* .$$

Then both the Fredholm index of the operator (4.1) and the Maslov indices of $\Psi_\lambda{}^{\pm}$ are independent of λ.

The key point is that in the case $n \geqq 2$ each homotopy class of $\mathcal{SP}^*(\tau)$ contains a path of the form $\Psi(t) = \exp\{J_0St\}$ with $\Psi(\tau) = W^{\pm}$ (see [26]). More precisely, recall from Theorem 3.3 that each homotopy class in $\mathcal{SP}^*(\tau)$ is characterized by the Maslov index $\mu_\tau(\Psi) = k \in \mathbb{Z}$. If $k - n$ is odd we use a symplectic decomposition $\mathbb{R}^{2n} = (\mathbb{R}^2)^n$ and choose

$$S = \begin{pmatrix} 0 & \frac{\log 2}{\tau} \\ \frac{\log 2}{\tau} & 0 \end{pmatrix} \oplus \bigoplus_{j=1}^{n-1} \begin{pmatrix} \frac{m_j\pi}{\tau} & 0 \\ 0 & \frac{m_j\pi}{\tau} \end{pmatrix}$$

where the $m_1 = n - 2 - k$ and $m_j = -1$ for $j > 1$. If $k - n$ is even we choose

$$S = \begin{pmatrix} -\frac{\pi}{\tau} & 0 \\ 0 & -\frac{\pi}{\tau} \end{pmatrix} \oplus \bigoplus_{j=1}^{n-1} \begin{pmatrix} \frac{m_j\pi}{\tau} & 0 \\ 0 & \frac{m_j\pi}{\tau} \end{pmatrix}$$

where the $m_1 = n - 1 - k$ and $m_j = -1$ for $j > 1$. Then the path $\Psi(t) = \exp\{J_0St\}$ has Maslov index $\mu_\tau(\Psi) = k$ and satisfies $\Psi(\tau) = W^{\pm}$ as we shall see below. Therefore every path in $\mathcal{SP}^*(\tau)$ with Maslov index k is homotopic to Ψ. We point out that the condition $n \geqq 2$ does not pose any restriction on the argument; just consider the operator $F \oplus F$.

Now let $S_0{}^{\pm}$ be given by the above formulae with $k = k^{\pm} = \mu_\tau(\Psi^{\pm})$ and define

$$S_0(s) = \gamma(s)S_0{}^+ + (1 - \gamma(s))S_0{}^-$$

where $\gamma(s)$ is a smooth nondecreasing function satisfying $\gamma(s) = 0$ for $s \leqq -1$ and $\gamma(s) = 1$ for $s \geqq 1$. Then the associated symplectic matrix function $\Psi_0(s,t) = \exp\{J_0S_0(s)t\}$ is homotopic to the original function Ψ via a homotopy Ψ_λ with $\Psi_\lambda{}^{\pm} \in \mathcal{SP}^*(\tau)$. It is therefore enough to consider the case where the symmetric matrix $S(s)$ is independent of t and can be decomposed into 2×2 blocks. So we may assume $n = 1$.

We examine the spectral flow of the operators $A(s)$ for $n = 1$ and $S(s)$ independent of t. There are three cases. First suppose that

$$S^- = \begin{pmatrix} -\frac{k^-\pi}{\tau} & 0 \\ 0 & -\frac{k^-\pi}{\tau} \end{pmatrix}, \qquad S^+ = \begin{pmatrix} -\frac{k^+\pi}{\tau} & 0 \\ 0 & -\frac{k^+\pi}{\tau} \end{pmatrix}.$$

where k^+ and k^- are odd. Then

$$\Psi^\pm(t) = \exp\left\{J_0 S^\pm t\right\} = \begin{pmatrix} \cos k^\pm \pi t/\tau & -\sin k^\pm \pi t/\tau \\ \sin k^\pm \pi t/\tau & \cos k^\pm \pi t/\tau \end{pmatrix}.$$

and it follows from the determinant property of the map ρ that $\rho(\Psi^\pm(t)) = \exp\{ik^\pm \pi t/\tau\}$. This shows that $\mu_\tau(\Psi^\pm) = k^\pm$. Moreover,

$$S(s) = \begin{pmatrix} -\omega(s) & 0 \\ 0 & -\omega(s) \end{pmatrix}, \qquad \omega(s) = \left(\gamma(s)k^+ + (1-\gamma(s))k^-\right)\frac{\pi}{\tau},$$

and thus

$$\exp\left\{J_0(S(s) - \lambda I)t\right\} = \begin{pmatrix} \cos(\omega(s)+\lambda)t & -\sin(\omega(s)+\lambda)t \\ \sin(\omega(s)+\lambda)t & \cos(\omega(s)+\lambda)t \end{pmatrix}.$$

Now $\lambda \in \sigma(A(s))$ if and only if $1 \in \sigma(\exp\{J_0(S(s) - \lambda I)\tau\})$ or equivalently $(\omega(s) + \lambda)\tau \in 2\pi\mathbb{Z}$. So the eigenvalues of $A(s)$ are

$$\lambda_j(s) = -\omega(s) + \frac{2\pi j}{\tau}, \qquad j \in \mathbb{Z},$$

and each eigenvalue occurs with multiplicity 2. Also note that the eigenfunctions are independent of s. Since $\omega(s)$ varies monotonically from $k^-\pi/\tau$ to $k^+\pi/\tau$ with $k^\pm$ odd it follows that there are precisely $N = |k^- - k^+|/2$ values of s where $0 \in \sigma(A(s))$. Also note that the λ_j increase if $k^- > k^+$ and they decrease if $k^- < k^+$. This shows that the spectral flow of the operator family $A(s)$ is $k^- - k^+$ as claimed.

Secondly consider the case where

$$S^- = \begin{pmatrix} 0 & \frac{\log 2}{\tau} \\ \frac{\log 2}{\tau} & 0 \end{pmatrix}, \qquad S^+ = \begin{pmatrix} -\frac{\pi}{\tau} & 0 \\ 0 & -\frac{\pi}{\tau} \end{pmatrix}.$$

Then $\mu_\tau(\Psi^-) = 0$ and $\mu_\tau(\Psi^+) = 1$. Moreover,

$$S(s) = \begin{pmatrix} -\beta(s) & \alpha(s) \\ \alpha(s) & -\beta(s) \end{pmatrix}, \qquad \alpha(s) = (1-\gamma(s))\frac{\log 2}{\tau}, \qquad \beta(s) = \gamma(s)\frac{\pi}{\tau},$$

and a simple calculation shows that the eigenvalues of $A(s)$ are given by

$$\lambda(s) = -\beta(s) \pm \sqrt{\alpha(s)^2 + \frac{4\pi^2 j^2}{\tau^2}}, \qquad j \in \mathbb{Z}.$$

All eigenvalues are of multiplicity 2 except where $j = 0$ and $\alpha(s) \neq 0$. Since $\beta(s)$ increases from 0 to π/τ the only eigenvalue crossing zero is $\lambda(s) = \alpha(s) - \beta(s)$ with the constant eigenfunction $\zeta(t) \equiv (1,1)$. So the spectral flow of $A(s)$ in this case is -1. The third case is where the roles of S^+ and S^- are interchanged. So it follows from the second case by reversing time.

In all three cases it turns out that the eigenfunctions of $A(s)$ which correspond to those eigenvalues which cross zero are independent of s. We can therefore decompose the Hilbert space $H = L^2(\mathbb{R}/\tau\mathbb{Z}; \mathbb{R}^{2n})$ into orthogonal invariant subspaces $H = H_0 \oplus H_1$ such that H_0 is finite dimensional and the eigenvalues of $A_1(s)$ are uniformly bounded away from zero. In the associated decomposition of the Fredholm operator $F = F_0 \oplus F_1$ the operator F_1 has Fredholm index zero and F_0 is given by

$$(F_0\zeta)(s) = \frac{d\zeta(s)}{ds} + A_0(s)\zeta(s)$$

where $\zeta(s) \in \mathbb{R}^N$ and $A_0(s)$ is a diagonal $N \times N$-matrix. So the index of F_0 is obviously given by the spectral flow of $A_0(s)$ with

$$\begin{aligned} \dim\ker F_0 &= \sharp\{j; \lambda_j(-1) < 0 < \lambda_j(1)\}, \\ \dim\operatorname{coker} F_0 &= \sharp\{j; \lambda_j(-1) > 0 > \lambda_j(1)\}. \end{aligned}$$

(Note that in the finite dimensional case the Fredholm index of F_0 is the difference of the Morse indices corresponding to $s = -\infty$ and $s = +\infty$; see [34].) This proves Theorem 4.1.

We shall now consider the more general operator

$$F\zeta = \frac{\partial\zeta}{\partial s} + J_0\frac{\partial\zeta}{\partial t} + (S + A)\zeta . \tag{4.7}$$

where $S = S^T$ is symmetric as before and $A(s,t) \in \mathbb{R}^{2n\times 2n}$ is a continuous matrix valued function on $\mathbb{R}^2$ such that

$$A(s,t) = A(s,t+\tau) = -A(s,t)^T.$$

We also assume that $A(s,t)$ converges to zero, uniformly in t, as s tends to infinity

$$\lim_{s\to\pm\infty} A(s,t) = 0 . \tag{4.8}$$

Under this assumption the operator (4.7) is a compact perturbation of (4.1) and is therefore a Fredholm operator of the same index.

If both S and A are independent of t and the limit matrices $S^{\pm}$ are nonsingular then the operator $F_0 : W^{1,2}(\mathbb{R};\mathbb{R}^{2n}) \to L^2(\mathbb{R};\mathbb{R}^{2n})$ defined by

$$F_0\zeta = \frac{d\zeta}{ds} + (S + A)\zeta \tag{4.9}$$

is Fredholm and in [34] it is shown that

$$\operatorname{index} F_0 = \mu^-(S^-) - \mu^-(S^+) . \tag{4.10}$$

If, moreover, S and A are sufficiently small then we shall prove that the kernel of (4.7) is naturally isomorphic to the kernel of (4.9).

Proposition 4.2. *Suppose that $S(s) = S(s)^T$ satisfies (4.2) and $A(s) = -A(s)^T$ satisfies (4.8) and that*

$$\|S(s)\| + \|A(s)\| \leqq c < \frac{1}{\tau}$$

for every $s \in \mathbb{R}$. Then every function $\zeta \in \ker F$ is independent of t.

Proof: Let $\zeta \in W^{1,2}(\mathbb{R} \times \mathbb{R}/\tau\mathbb{Z};\mathbb{R}^{2n})$ be in the kernel of F. Then

$$\zeta_0(s) = \frac{1}{\tau}\int_0^\tau \zeta(s,t)\,dt$$

is in the kernel of F_0. We shall show that $\zeta(s,t) = \zeta_0(s)$ for all s and t. Since $\zeta - \zeta_0 \in \ker F$ we may assume $\zeta_0 \equiv 0$. We integrate the identity

$$\zeta(s,T) - \zeta(s,t) = \int_t^T \frac{\partial\zeta}{\partial t}(s,\theta)\,d\theta$$

with respect to t

$$\begin{aligned} |\zeta(s,T)| &= \frac{1}{\tau}\left|\int_0^\tau\int_t^T \frac{\partial\zeta}{\partial t}(s,\theta)\,d\theta\,dt\right| \\ &\leqq \int_0^\tau \left|\frac{\partial\zeta}{\partial t}(s,\theta)\right| d\theta \\ &\leqq \tau^{1/2}\left(\int_0^\tau \left|\frac{\partial\zeta}{\partial t}(s,\theta)\right|^2 d\theta\right)^{1/2} . \end{aligned}$$

We integrate the square of this inequality with respect to T and s

$$\|\zeta\|_{L^2} \leqq \tau\left\|\frac{\partial\zeta}{\partial t}\right\|_{L^2} .$$

As in the proof of Theorem 4.1 we obtain

$$\begin{aligned}\|\nabla\zeta\|_{L^2} &= \left\|\frac{\partial\zeta}{\partial s} + J_0\frac{\partial\zeta}{\partial t}\right\|_{L^2} \\ &\leqq \|F\zeta\|_{L^2} + \|(S+A)\zeta\|_{L^2} \\ &\leqq \|F\zeta\|_{L^2} + c\,\|\zeta\|_{L^2} \\ &\leqq \|F\zeta\|_{L^2} + c\tau\,\|\nabla\zeta\|_{L^2}\ .\end{aligned}$$

Since $F\zeta = 0$ and $c\tau < 1$ it follows that $\zeta = 0$. This proves Proposition 4.2.

COROLLARY 4.3. *Let S and A be as in Proposition 4.2 and assume that the matrices $S^{\pm}$ are nonsingular. Then the operator F is onto if and only if F_0 is onto.*

Proof: Let $\Psi^{\pm} \in \mathscr{S}P^*(\tau)$ denote the path $\Psi^{\pm}(t) = \exp\{J_0 S^{\pm} t\}$. Then

$$\begin{aligned}\operatorname{index} F &= \mu(\Psi^-) - \mu(\Psi^+) \\ &= \mu^-(S^-) - \mu^-(S^+) \\ &= \operatorname{index} F_0\ .\end{aligned}$$

The first identity follows from Theorem 4.1, the second from Theorem 3.3, and the third is (4.10). By Proposition 4.2 the kernel of F is isomorphic to the kernel of F_0 and hence the cokernel of F is of the same dimension as the cokernel of F_0.

The proof of Proposition 4.2 actually shows that the restriction F_1 of the operator F to the subspace of those functions ζ which for every s have mean value zero has a bounded inverse. In other words, if S and A are independent of t then the operator F decomposes as

$$F = F_0 \oplus F_1\ .$$

If, moreover, S and A are sufficiently small then F_1 is an isomorphism and does not contribute to the kernel and the cokernel. This can also be shown by a Fourier series decomposition of ζ with respect to which F_0 corresponds to the constant part.

5. The Maslov Index for Periodic Orbits

The Maslov index can be defined for *nondegenerate* periodic solutions x of (1.1) which are *contractible* loops on the symplectic manifold M provided that the *first Chern class c_1 of the tangent bundle vanishes over $\pi_2(M)$*. To see this choose a fixed almost complex structure J on M and denote by $g(\xi, \eta) = \omega(\xi, J\eta)$ the induced Riemannian metric on M with the associated Levi-Civita connection ∇. Thus the tangent TM is a complex vector bundle with a Hermitian structure

$$\langle \xi, \eta \rangle = g(\xi, \eta) + i\omega(\xi, \eta) .$$

(In our convention $\langle \xi, \eta \rangle$ is complex anti-linear in ξ and complex linear in η.) It is well known that for any smooth disc $\phi : D \to M$ the vector bundle $\phi^* TM$ admits a unitary trivialization.

LEMMA 5.1. *For any smooth map $\phi : D \to M$ there exists a trivialization*

$$D \times \mathbb{R}^{2n} \to \phi^* TM : (z, \zeta) \mapsto \Phi(z)\zeta$$

such that

$$J\Phi = \Phi J_0 , \qquad \Phi^* \omega = \omega_0 , \qquad g(\Phi\zeta, \Phi\zeta') = \zeta^T \zeta' .$$

Any two such trivializations are homotopic.

Proof: To prove existence choose any complex trivialization, for example with parallel transport of a complex frame $Z_1, \ldots, Z_n$ along a curve γ such that $\nabla Z_j = J\nabla_{Z_j} J\dot{\gamma}$, and use Gram-Schmidt over $\mathbb{C}$.

If $\Phi(z)$ and $\Psi(z)$ are two unitary trivializations then $\Psi(z)^{-1}\Phi(z)$ is a unitary matrix for every $z \in D$. Every smooth map $D \to U(n)$ is smoothly homotopic to the constant map $z \mapsto 1$.

To define the Maslov index for a nondegenerate contractible τ-periodic solution x of (1.1) choose a smooth function $\phi : D \to M$ such that $\phi(\exp\{2\pi i\theta\}) = x(\tau\theta)$. By Lemma 5.1 there exists a unitary trivialization of $\phi^* TM$. This gives rise to a unitary trivialization

$$\Phi_x(t) = \Phi(\exp\{2\pi i t/\tau\}) : \mathbb{R}^{2n} \to T_{x(t)}M$$

of $x^* TM$ such that

$$\Phi_x(t + \tau) = \Phi_x(t) .$$

In general the homotopy class of Φ_x may depend on the choice of the extension $\phi : D \to M$ with $\phi(\exp\{2\pi i t/\tau\}) = x(t)$ unless the first Chern class of TM vanishes over $\pi_2(M)$.

LEMMA 5.2. *If the first Chern class $c_1(TM)$ vanishes over $\pi_2(M)$ then the homotopy class of Φ_x is independent of the choice of the extension $\phi : D \to M$.*

Proof: Let $\phi : D \to M$ and $\phi' : D \to M$ be any two smooth maps such that $\phi(\exp\{2\pi it/\tau\}) = \phi'(\exp\{2\pi it/\tau\}) = x(t)$ with associated orthogonal symplectic trivializations Φ and Φ' of the pullback tangent bundle. Assume without loss of generality that

$$\phi(z) = \phi(z/|z|)\,, \quad \Phi(z) = \Phi(z/|z|) \qquad \text{for} \qquad 1 - \varepsilon \leqq |z| \leqq 1$$

and likewise for ϕ' and Φ'. Then the map $u : S^2 = \mathbb{C} \cup \{\infty\} \to M$ defined by

$$u(z) = \phi(z) \qquad \text{for} \qquad |z| \leqq 1$$

and

$$u(z) = \phi'(1/\bar{z}) \qquad \text{for} \qquad |z| > 1$$

is smooth. If u^*c_1 vanishes then the unitary $\mathbb{C}^n$-bundle u^*TM is trivial. Hence there exists an orthogonal symplectic trivialization $\Theta : S^2 \times \mathbb{R}^{2n} \to u^*TM$. It follows from Lemma 5.1 that $\Phi_x(t)$ and $\Phi'_x(t)$ are homotopic to $\Theta(\exp\{2\pi it/\tau\})$.

Given a trivialization Φ_x of x^*TM as above we consider the linearized flow along $x(t)$ and define the path

$$\Psi_x(t) = \Phi_x(t)^{-1}\, d\psi_t(x(0))\Phi_x(0)\,, \qquad 0 \leqq t \leqq \tau\,. \tag{5.1}$$

Then $\Psi_x(t) \in \mathrm{Sp}(2n;\mathbb{R})$ for every t and $\Psi_x(\tau)$ is similar to $d\psi_\tau(x(0))$. This shows that the **mean winding number**

$$\Delta_\tau(x) = \Delta_\tau(x; H) = \Delta_\tau(\Psi_x)$$

is independent of the choice of the trivialization. Moreover, the periodic solution $x \in \mathscr{P}_\tau$ is nondegenerate if and only if $\Psi_x \in \mathscr{S}P^*(\tau)$. In this case the **Maslov index** of the periodic solution $x(t)$ of (1.1) is defined as

$$\mu_\tau(x) = \mu_\tau(x; H) = \mu_\tau(\Psi_x)\,. \tag{5.2}$$

In view of Lemma 5.3 this index is uniquely determined by the requirement that the trivialization Φ_x extends over a disc which bounds x.

Let $x^+, x^- \in \mathscr{P}_\tau$ be a pair of nondegenerate periodic solutions of (1.1) and let $u : \mathbb{R}^2 \to M$ be a smooth function which satisfies $u(s, t + \tau) = u(s, t)$ and

$$\lim_{s\to-\infty} u(s,t) = x^-(t)\,, \qquad \lim_{s\to+\infty} u(s,t) = x^+(t)\,. \tag{5.3}$$

The convergence is to be understood as uniform in t and with $\partial u/\partial t$ converging to $\dot{x}^\pm$ and $\partial u/\partial s$ converging to zero, again uniformly in t, as s tends to $\pm\infty$. For any such u we introduce the Hilbert space $L^2_\tau(u)$ as the completion of the space of smooth compactly supported vector fields $\xi(s,t) \in T_{u(s,t)}M$ satisfying $\xi(s,t+\tau) = \xi(s,t)$ with respect to the norm

$$\|\xi\|_{L^2}^2 = \int_{-\infty}^{\infty} \int_0^{\tau} |\xi(s,t)|^2 \, dt \, ds \, .$$

Also define $W_\tau^{1,2}(u) = \{\xi \in L_\tau^2(u); \nabla_s\xi, \nabla_t\xi \in L_\tau^2(u)\}$ and consider

$$F_\tau(u)\xi = \nabla_s\xi + J(u)\nabla_t\xi + \nabla_\xi J(u)\frac{\partial u}{\partial t} + \nabla_\xi \nabla H(t,u) \tag{5.4}$$

as a bounded linear operator from $W_\tau^{1,2}(u)$ to $L_\tau^2(u)$. This is a Fredholm operator and its index can be characterized in terms of the Maslov index of the periodic solutions $x^\pm$ of (1.1).

THEOREM 5.3. *If $u : \mathbb{R} \times \mathbb{R}/\tau\mathbb{Z} \to M$ is a smooth function satisfying* (5.3) *for two nondegenerate periodic solutions $x^\pm \in \mathscr{P}_\tau$ then $F_\tau(u)$ is a Fredholm operator and*

$$\operatorname{index} F_\tau(u) = \mu_\tau(x^-) - \mu_\tau(x^+) \, .$$

Proof: Altering u if necessary (without changing the Fredholm index) we may assume that $u(s,t) = x^-(t)$ for $s \leqq -1$ and $u(s,t) = x^+(t)$ for $s \geqq +1$. Under this assumption Lemma 5.1 shows that there exists an orthogonal symplectic trivialization

$$\Phi(s,t) = \Phi(s,t+\tau) : \mathbb{R}^{2n} \to T_{u(s,t)}M \, , \qquad \Phi(s,t)\zeta = \sum_{j=1}^{2n} Z_j(s,t)\zeta_j \, ,$$

of u^*TM which is independent of s for $|s| \geqq 1$. In the new coordinates $\zeta = \Phi^{-1}\xi$ the operator $F_\tau(u)$ is of the form

$$F\zeta = \frac{\partial \zeta}{\partial s} + J_0\frac{\partial \zeta}{\partial t} + (S+A)\zeta \tag{5.5}$$

where the $2n \times 2n$-matrices $S(s,t)$ and $A(s,t)$ are the symmetric and skew-symmetric part of the matrix $\langle Z_j, F_\tau(u)Z_k\rangle$.

A simple calculation shows that

$$A_{jk} = \langle Z_j, \nabla_s Z_k\rangle = -A_{kj}$$

and

$$S_{jk} = \left\langle Z_j, J(u)\nabla_t Z_k + \nabla_{Z_k} J(u)\frac{\partial u}{\partial t} + \nabla_{Z_k}\nabla H(t,u)\right\rangle = S_{kj} \, .$$

This implies that $A(s,t) = 0$ for $|s| \geqq 1$. Allowing for a compact perturbation we may therefore assume that $A(s,t) = 0$ for all s and t.

Now define $S^{\pm}(t) = S(\pm 1, t)$ and $\Phi^{\pm}(t) = \Phi(\pm 1, t)$ and let $\Psi^{\pm}(t) \in \mathrm{Sp}(2n;\mathbb{R})$ be given by (5.1) with $x = x^{\pm}$ that is

$$\Phi^{\pm}(t)\Psi^{\pm}(t) = d\psi_t\left(x^{\pm}(0)\right)\Phi^{\pm}(0) .$$

Differentiating this identity with respect to t we obtain

$$\Phi^{\pm}(t)\dot{\Psi}^{\pm}(t) + \left(\nabla_t\Phi^{\pm}\right)\Psi^{\pm} = \left(\nabla_{\Phi^{\pm}}X_H\right)\Psi^{\pm} .$$

Since $\Phi J_0 = J\Phi$ this implies

$$\begin{aligned}
\Phi^{\pm}(t)J_0\dot{\Psi}^{\pm}(t) &= J\Phi^{\pm}(t)\dot{\Psi}^{\pm}(t) \\
&= J\left(\nabla_{\Phi^{\pm}}X_H - \nabla_t\Phi^{\pm}\right)\Psi^{\pm} \\
&= J\left(J\nabla_{\Phi^{\pm}}\nabla H + (\nabla_{\Phi^{\pm}}J)\nabla H - \nabla_t\Phi^{\pm}\right)\Psi^{\pm} \\
&= -\left(\nabla_{\Phi^{\pm}}\nabla H + (\nabla_{\Phi^{\pm}}J)X_H + J\nabla_t\Phi^{\pm}\right)\Psi^{\pm} \\
&= -\Phi^{\pm}(t)S^{\pm}(t)\Psi^{\pm}(t) .
\end{aligned}$$

Hence

$$\dot{\Psi}^{\pm}(t) = J_0 S^{\pm}(t)\Psi^{\pm}(t)$$

and it follows from Theorem 4.1 that

$$\begin{aligned}
\operatorname{index} F_\tau(u) &= \operatorname{index} F \\
&= \mu_\tau(\Psi^-) - \mu_\tau(\Psi^+) \\
&= \mu_\tau(x^-) - \mu_\tau(x^+) .
\end{aligned}$$

This proves Theorem 5.3.

In the context of Morse theory for Lagrangian intersections Floer proved a similar result (see [12]) relating the corresponding Fredholm index to Viterbo's relative Maslov index (see [38]). In that case, however, the Maslov index does not single out a natural grading for the Floer homology groups.

6. Continuation of Floer Homology

In this section we show how the Maslov index removes the ambiguity of the grading of the Floer homology groups. This means that the isomorphism of Theorem 2.1 is of degree 0 if the grading is provided by the Maslov index. In the proof we follow closely the line of argument in [16] which has also been described in [27]. We combine this with the index theorem of the previous section. In order to simplify the notation we consider only the 1-periodic solutions and drop the subscript τ.

Recall that a pair (H, J) consisting of a Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ and an almost complex structure J on M is called *regular* if every contractible 1-periodic solution $x \in \mathscr{P}(H)$ is nondegenerate and if the Fredholm operator $F(u)$ defined by (2.4) is onto for every $u \in \mathscr{M}(H, J)$. The space

$$(\mathscr{H} \times \mathscr{J})_{\text{reg}}$$

of regular pairs is dense in $\mathscr{H} \times \mathscr{J}$ with respect to the C^∞-topology (see [17] and Section 8). For any such regular pair we denote by

$$C_k = C_k(M; H) = \text{span}_{\mathbb{Z}/2\mathbb{Z}} \{x \in \mathscr{P}(H);\ \mu(x; H) = k\}$$

the chain complex of Section 2 graded by the Maslov index of Sections 3 and 5. Let $\partial = \partial(M; H, J)$ denote the associated boundary operator so that the Floer homology groups of the pair (H, J) are given by

$$HF_k(M; H, J) = \frac{\ker(\partial : C_k \to C_{k-1})}{\operatorname{im}(\partial : C_{k+1} \to C_k)} .$$

We shall prove that these homology groups are independent of the Hamiltonian function H and the almost complex structure J used to construct them.

THEOREM 6.1. *For any two regular pairs (H^α, J^α) and (H^β, J^β) there exists a natural isomorphism*

$$HF_k^{\beta\alpha} : HF_k(M; H^\alpha, J^\alpha) \to HF_k\left(M; H^\beta, J^\beta\right) .$$

If (H^γ, J^γ) is another regular pair then

$$HF_*^{\gamma\beta} \circ HF_*^{\beta\alpha} = HF_*^{\gamma\alpha} , \qquad HF_*^{\alpha\alpha} = \text{id} . \tag{6.1}$$

Theorem 6.1 shows that there is a category whose objects are the Floer homology groups associated to regular pairs (H^α, J^α) and whose morphisms are the natural isomorphisms $HF_*^{\beta\alpha}$ which are induced by regular homotopies as we shall see below. In the terminology of Conley such a category with unique morphisms is called a **connected simple system.**

The proof of Theorem 6.1 occupies the remainder of this section. To construct the homomorphisms $HF_*^{\beta\alpha}$ we consider a smooth **homotopy from** (H^α, J^α) **to** (H^β, J^β). By this we mean a smooth homotopy of Hamiltonians $H^{\alpha\beta} : \mathbb{R} \times S^1 \times M \to \mathbb{R}$ and a smooth homotopy of almost complex structures

$J^{\alpha\beta} : \mathbb{R}\times M \to \mathrm{End}(TM)$ such that $H^{\alpha\beta}(s,t,x)$ and $J^{\alpha\beta}(s,x)$ are independent of s for $|s|$ sufficiently large and satisfy

$$\lim_{s\to-\infty} H^{\alpha\beta}(s,t,x) = H^{\alpha}(t,x) , \qquad \lim_{s\to+\infty} H^{\alpha\beta}(s,t,x) = H^{\beta}(t,x)$$

and

$$\lim_{s\to-\infty} J^{\alpha\beta}(s,x) = J^{\alpha}(x) , \qquad \lim_{s\to+\infty} J^{\alpha\beta}(s,x) = J^{\beta}(x) .$$

(It follows from the contractibility of the space of almost complex structures at $x \in M$ which are compatible with ω_x that any two almost complex structures on M which are compatible with ω can be connected by a smooth homotopy; see [22].) Given such a pair of homotopies $(H,J) = (H^{\alpha\beta}, J^{\alpha\beta})$ we consider the solutions $u : \mathbb{R}^2 \to M$ of the elliptic partial differential equation

$$\frac{\partial u}{\partial s} + J(s,u)\frac{\partial u}{\partial t} + \nabla H(s,t,u) = 0 \tag{6.2}$$

which satisfy $u(s,t+1) = u(s,t)$ and the boundary condition

$$\lim_{s\to-\infty} u(s,t) = x^{\alpha}(t) , \qquad \lim_{s\to+\infty} u(s,t) = x^{\beta}(t) \tag{6.3}$$

where $x^{\alpha} \in \mathscr{P}(H^{\alpha})$ and $x^{\beta} \in \mathscr{P}(H^{\beta})$. This equation can be thought of as the time dependent version of (2.2) or the gradient flow of the time dependent action functional $f_s = f_{H_s}$ on the loop space with respect to the time dependent metric determined by the almost complex structure J_s. As before every solution of (6.2) and (6.3) has bounded energy. Conversely, if a solution of (6.2) has bounded energy then the limits (6.3) exist. Denote by

$$\mathscr{M}(x^{\alpha}, x^{\beta}; H^{\alpha\beta}, J^{\alpha\beta})$$

the space of (smooth) solutions $u : \mathbb{C}/i\mathbb{Z} \to M$ of (6.2) and (6.3). As in Section 2, this space can be analysed locally by linearizing equation (6.2) in the direction of a vector field $\xi \in C^{\infty}(u^*TM)$ along u. This leads to the differential operator $F(u) : W^{1,2}(u) \to L^2(u)$ given by

$$F(u)\xi = \nabla_s\xi + J(s,u)\nabla_t\xi + \nabla_{\xi}J(s,u)\frac{\partial u}{\partial t} + \nabla_{\xi}H(s,t,u) . \tag{6.4}$$

All the results of Section 5 remain valid in the time dependent case. In particular (6.4) defines a Fredholm operator whenever u satisfies (6.3) and the periodic solutions $x^{\alpha} \in \mathscr{P}(H^{\alpha})$ and $x^{\beta} \in \mathscr{P}(H^{\beta})$ are nondegenerate. The proof of Theorem 5.3 shows that the Fredholm index is given by

$$\mathrm{index} F(u) = \mu(x^{\alpha}; H^{\alpha}) - \mu(x^{\beta}; H^{\beta}) . \tag{6.5}$$

Moreover, for a generic homotopy from (H^α, J^α) to (H^β, J^β), this operator is onto. More precisely, the pair $(H^{\alpha\beta}, J^{\alpha\beta})$ is called a **regular homotopy** if both pairs (H^α, J^α) and (H^β, J^β) are regular and if the operator (6.4) is onto for every solution u of (6.2) and (6.3) and every pair $x^\alpha \in \mathscr{P}(H^\alpha)$, $x^\beta \in \mathscr{P}(H^\beta)$. If (H^α, J^α) and (H^β, J^β) are regular pairs then the space of regular homotopies is dense in the space of all smooth homotopies from (H^α, J^α) to (H^β, J^β) with respect to the topology of uniform convergence with all derivatives on compact sets (see [17] and Section 8).

It follows from (6.5) and the surjectivity of the operator (6.4) that for every regular homotopy $(H^{\alpha\beta}, J^{\alpha\beta})$ and every pair of contractible 1-periodic solutions $x^\alpha \in \mathscr{P}(H^\alpha)$, $x^\beta \in \mathscr{P}(H^\beta)$ the space $\mathscr{M}(x^\alpha, x^\beta; H^{\alpha\beta}, J^{\alpha\beta})$ of **connecting orbits** is a manifold of dimension

$$\dim \mathscr{M}(x^\alpha, x^\beta; H^{\alpha\beta}, J^{\alpha\beta}) = \mu(x^\alpha; H^\alpha) - \mu(x^\beta; H^\beta) .$$

The proof relies on a Newton type iteration in an infinite dimensional setting combined with elliptic regularity. We shall not carry out the details here. We use these spaces to construct a chain homomorphism

$$\phi^{\beta\alpha} = \phi(H^{\alpha\beta}, J^{\alpha\beta}) : C_k(M; H^\alpha) \to C_k(M; H^\beta) .$$

More precisely, it follows from the compactness of the space of bounded solutions of (6.2) that $\mathscr{M}(x^\alpha, x^\beta; H^{\alpha\beta}, J^{\alpha\beta})$ is a finite set whenever

$$\mu(x^\alpha; H^\alpha) = \mu(x^\beta; H^\beta) .$$

In this case the matrix element $\langle \phi^{\beta\alpha} x^\alpha, x^\beta \rangle$ is defined to be the number of elements in $\mathscr{M}(x^\alpha, x^\beta; H^{\alpha\beta}, J^{\alpha\beta})$ modulo 2 and $\phi^{\beta\alpha}$ is given by

$$\phi^{\beta\alpha} x^\alpha = \sum_{\mu(x^\beta; H^\beta) = k} \langle \phi^{\beta\alpha} x^\alpha, x^\beta \rangle x^\beta ,$$

for $\mu(x^\alpha; H^\alpha) = k$.

LEMMA 6.2. *For every regular homotopy $(H^{\alpha\beta}, J^{\alpha\beta})$ from (H^α, J^α) to (H^β, J^β) the above map $\phi^{\beta\alpha} = \phi(H^{\alpha\beta}, J^{\alpha\beta})$ is a chain homomorphism.*

Proof: We sketch the main idea. We have to show that $\partial^\beta \phi^{\beta\alpha} = \phi^{\beta\alpha} \partial^\alpha$ or equivalently

$$\begin{aligned} &\sum_{\mu(x^\alpha; H^\alpha) = k} \langle \partial^\alpha y^\alpha, x^\alpha \rangle \langle \phi^{\beta\alpha} x^\alpha, x^\beta \rangle \\ &\quad = \sum_{\mu(y^\beta; H^\beta) = k+1} \langle \phi^{\beta\alpha} y^\alpha, y^\beta \rangle \langle \partial^\beta y^\beta, x^\beta \rangle \quad (\text{modulo}\, 2) \end{aligned}$$

for any pair of contractible periodic solutions $y^\alpha \in \mathscr{P}(H^\alpha)$ and $x^\beta \in \mathscr{P}(H^\beta)$ with $\mu(x^\beta; H^\beta) = k$ and $\mu(y^\alpha; H^\alpha) = k+1$. This follows from a gluing argument (see [16]). Namely, any pair of connecting orbits

$$u^\alpha \in \mathscr{M}\left(y^\alpha, x^\alpha; H^\alpha, J^\alpha\right) , \qquad u^{\alpha\beta} \in \mathscr{M}\left(x^\alpha, x^\beta; H^{\alpha\beta}, J^{\alpha\beta}\right)$$

with $\mu(x^\alpha; H^\alpha) = k$ can be *glued together* at x^α producing a 1-parameter family $\tilde{v}_R(s,t)$ of approximate solutions of (6.2) such that

$$\tilde{v}_R(s,t) = \begin{cases} u^\alpha(s+2R,t) , & s \leqq -R-1 , \\ u^{\alpha\beta}(s,t) , & s \geqq -R+1 . \end{cases}$$

For $R > 0$ sufficiently large there exists a solution $v_R \in \mathscr{M}(y^\alpha, x^\beta; H^{\alpha\beta}, J^{\alpha\beta})$ of (6.2) near $\tilde{v}_R$. This follows again from a Newton type iteration using the linearized solution operator (6.4) and quadratic estimates on the higher order terms. The key point is that since the homotopy $(H^{\alpha\beta}, J^{\alpha\beta})$ is regular the operators $F_R = F(\tilde{v}_R)$ are onto with a uniformly bounded inverse of $F_R F_R{}^*$ for R sufficiently large.

The solutions v_R converge to the pair $(u^\alpha, u^{\alpha\beta})$ as R tends to infinity. More precisely, $v_R(s-2R,t)$ converges to $u^\alpha(s,t)$ and $v_R(s,t)$ converges to $u^{\alpha\beta}(s,t)$, the convergence being, with all derivatives, uniform on compact sets. This is obviously the case for $\tilde{v}_R$ and v_R is exponentially close to $\tilde{v}_R$. Moreover, any connecting orbit $v \in \mathscr{M}(y^\alpha, x^\beta; H^{\alpha\beta}, J^{\alpha\beta})$ which is sufficiently close to the pair $(u^\alpha, u^{\alpha\beta})$ must be one of the v_R's. So up to a reparametrization the 1-parameter family v_R is uniquely determined by the pair $(u^\alpha, u^{\alpha\beta})$.

It is this analysis near the ends together with the compactness away from the ends which shows that the one-dimensional manifold $\mathscr{M}(y^\alpha,\ x^\beta;\ H^{\alpha\beta},\ J^{\alpha\beta})$ is paracompact and has finitely many components. By the classification theorem for compact 1-manifolds, each component which is not a circle must have two ends. Each end can be identified either with a pair $(u^\alpha,\ u^{\alpha\beta})$ as above or with a pair $(v^{\alpha\beta},\ u^\beta)$ where $v^{\alpha\beta} \in \mathscr{M}(y^\alpha,\ y^\beta;\ H^{\alpha\beta},\ J^{\alpha\beta})$ and $u^\beta \in \mathscr{M}(y^\beta,\ x^\beta;\ H^\beta,\ J^\beta)$ with $\mu(y^\beta; H^\beta) = k+1$.

$$\begin{tikzcd} y^\alpha \arrow[r, "v^{\beta\alpha}"] \arrow[d, "u^\alpha"'] & y^\beta \arrow[d, "u^\beta"] \\ x^\alpha \arrow[r, "u^{\beta\alpha}"'] & x^\beta \end{tikzcd}$$

Thus there is an even number of pairs of connecting orbits between y^α and x^β.

Lemma 6.2 shows that every regular homotopy $(H^{\alpha\beta}, J^{\alpha\beta})$ from (H^α, J^α) to (H^β, J^β) induces a natural homomorphism of Floer homology. This homomorphism turns out to be independent of the choice of the homotopy $(H^{\alpha\beta}, J^{\alpha\beta})$.

LEMMA 6.3. *Let $(H_0^{\alpha\beta}, J_0^{\alpha\beta})$ and $(H_1^{\alpha\beta}, J_1^{\alpha\beta})$ be two regular homotopies from (H^α, J^α) to (H^β, J^β). Then the associated chain homomorphisms $\phi_0^{\beta\alpha} = \phi(H_0^{\alpha\beta}, J_0^{\alpha\beta})$ and $\phi_1^{\beta\alpha} = \phi(H_1^{\alpha\beta}, J_1^{\alpha\beta})$ are chain homotopy equivalent.*

Proof: To construct a chain homotopy from $\phi_0^{\beta\alpha}$ to $\phi_1^{\beta\alpha}$ we choose a smooth homotopy of homotopies $(H_\lambda^{\alpha\beta}, J_\lambda^{\alpha\beta})$ from $(H_0^{\alpha\beta}, J_0^{\alpha\beta})$ to $(H_1^{\alpha\beta}, J_1^{\alpha\beta})$. For any pair $x^\alpha \in \mathscr{P}(H^\alpha)$, $y^\beta \in \mathscr{P}(H^\beta)$ we introduce the space

$$\mathscr{M}\left(x^\alpha, y^\beta\right) = \left\{(\lambda, u);\ 0 \leqq \lambda \leqq 1,\ u \in \mathscr{M}\left(x^\alpha, y^\beta; H_\lambda^{\alpha\beta}, J_\lambda^{\alpha\beta}\right)\right\}.$$

As before one uses Fredholm theory to show that for a generic homotopy of homotopies this space is a finite dimensional manifold with boundary and

$$\dim \mathscr{M}(x^\alpha, y^\beta) = \mu(x^\alpha; H^\alpha) - \mu(y^\beta; H^\beta) + 1\ .$$

Now suppose that

$$\mu\left(y^\beta; H^\beta\right) = \mu\left(x^\alpha; H^\alpha\right) + 1\ .$$

Then $\mathscr{M}(x^\alpha, y^\beta)$ is a finite set of pairs

$$(\lambda_1, u_1), \ldots, (\lambda_N, u_N)\ , \qquad u_j \in \mathscr{M}\left(x^\alpha, y^\beta; H_{\lambda_j}^{\alpha\beta}, J_{\lambda_j}^{\alpha\beta}\right)\ .$$

Assume otherwise that there exists an infinite sequence (λ_j, u_j) of distinct points in $\mathscr{M}(x^\alpha, y^\beta)$. By Gromov's compactness theorem for pseudoholomorphic curves (see [22] and [34]) we can extract a converging subsequence still denoted by (λ_j, u_j). Let

$$\lambda = \lim_{j\to\infty} \lambda_j\ , \qquad u = \lim_{j\to\infty} u_j\ .$$

Then u is a bounded solution of (6.2) with $(H, J) = (H_\lambda^{\alpha\beta}, J_\lambda^{\alpha\beta})$. Thus there exist $w^\alpha \in \mathscr{P}(H^\alpha)$ and $z^\beta \in \mathscr{P}(H^\beta)$ such that $u \in \mathscr{M}(w^\alpha, z^\beta; H_\lambda^{\alpha\beta}, J_\lambda^{\alpha\beta})$. If $w^\alpha \neq x^\alpha$ then $\mu(w^\alpha; H^\alpha) < \mu(x^\alpha; H^\alpha)$ and likewise, if $z^\beta \neq y^\beta$ then $\mu(z^\beta; H^\beta) > \mu(y^\beta; H^\beta)$. In either case it would follow that $\mu(w^\alpha; H^\alpha) - \mu(z^\beta; H^\beta) + 1 < 0$ and our choice of a regular homotopy of homotopies would imply that $\mathscr{M}(w^\alpha, z^\beta) = \varnothing$. This contradiction shows that $(\lambda, u) \in \mathscr{M}(x^\alpha, y^\beta)$. But this violates the manifold structure of $\mathscr{M}(x^\alpha, y^\beta)$ and hence there are only finitely many pairs as claimed.

Since $\mu(x^\alpha; H^\alpha) - \mu(y^\beta; H^\beta) = -1$ and the homotopies $(H_0^{\alpha\beta}, J_0^{\alpha\beta})$ and $(H_1^{\alpha\beta}, J_1^{\alpha\beta})$ are regular it follows that $\mathscr{M}(x^\alpha, y^\beta; H_\lambda^{\alpha\beta}, J_\lambda^{\alpha\beta}) = \varnothing$ for $\lambda = 0$ and $\lambda = 1$. This shows that $0 < \lambda_j < 1$ for every j. Intuitively speaking, connecting orbits are not allowed generically if the index difference is negative. But if the index difference is -1 then in a generic 1-parameter family connecting orbits may occur for isolated parameter values. It is these isolated connecting orbits which determine the required chain homotopy equivalence.

We define the homomorphism

$$\Phi^{\beta\alpha} : C_k(M; H^\alpha) \to C_{k+1}\left(M; H^\beta\right), \quad \Phi^{\beta\alpha} x^\alpha = \sum_{\mu(y^\beta; H^\beta) = k+1} \left\langle \Phi^{\beta\alpha} x^\alpha, y^\beta \right\rangle y^\beta,$$

where the matrix entry $\langle \Phi^{\beta\alpha} x^\alpha, y^\beta \rangle$ is given by the number of elements in $\mathcal{M}(x^\alpha, y^\beta)$ modulo 2 whenever the index difference is -1. We claim that

$$\partial^\beta \Phi^{\beta\alpha} + \Phi^{\beta\alpha} \partial^\alpha = \phi_1^{\beta\alpha} - \phi_0^{\beta\alpha} \tag{6.6}$$

or equivalently

$$\sum_{\mu(y^\beta; H^\beta) = k+1} \left\langle \Phi^{\beta\alpha} x^\alpha, y^\beta \right\rangle \left\langle \partial^\beta y^\beta, x^\beta \right\rangle + \sum_{\mu(w^\alpha; H^\alpha) = k-1} \langle \partial^\alpha x^\alpha, w^\alpha \rangle \left\langle \Phi^{\beta\alpha} w^\alpha, x^\beta \right\rangle$$
$$= \left\langle \phi_0^{\beta\alpha} x^\alpha, x^\beta \right\rangle - \left\langle \phi_1^{\beta\alpha} x^\alpha, x^\beta \right\rangle \quad (\text{modulo } 2)$$

for every pair $x^\alpha \in \mathscr{P}(H^\alpha)$, $x^\beta \in \mathscr{P}(H^\beta)$ with $\mu(x^\alpha; H^\alpha) = \mu(x^\beta; H^\beta) = k$. As in the proof of Lemma 6.2 this will follow from a gluing argument involving the ends of the paracompact 1-manifold $\mathcal{M}(x^\alpha, x^\beta)$.

There are four cases. If $(0, u) \in \mathcal{M}(x^\alpha, x^\beta)$ then u is one of the isolated points in $\mathcal{M}(x^\alpha, x^\beta; H_0^{\alpha\beta}, J_0^{\alpha\beta})$ contributing to the matrix entry $\langle \phi_0^{\beta\alpha} x^\alpha, x^\beta \rangle$. Similarly a boundary point of the form $(1, u)$ corresponds to $\langle \phi_1^{\beta\alpha} x^\alpha, x^\beta \rangle$.

Now the manifold $\mathcal{M}(x^\alpha, x^\beta)$ will in general not be compact. There may be a sequence $(\lambda_\nu, u_\nu) \in \mathcal{M}(x^\alpha, x^\beta)$ converging to a pair $(\lambda, u) \notin \mathcal{M}(x^\alpha, x^\beta)$. In this case the pair $(H_\lambda^{\alpha\beta}, J_\lambda^{\alpha\beta})$ cannot be a regular homotopy since otherwise it would follow as before that the limit $u = \lim u_\nu$ would be in $\mathcal{M}(x^\alpha, x^\beta; H_\lambda^{\alpha\beta}, J_\lambda^{\alpha\beta})$.

So in particular $0 < \lambda < 1$ and u is a bounded solution of (6.2) with $(H, J) = (H_\lambda^{\alpha\beta}, J_\lambda^{\alpha\beta})$. Since $\mu(x^\alpha; H^\alpha) = \mu(x^\beta; H^\beta) = k$ there are only two possibilities. Either

$$u \in \mathcal{M}\left(x^\alpha, y^\beta; H_\lambda^{\alpha\beta}, J_\lambda^{\alpha\beta}\right)$$

for some $y^\beta \in \mathscr{P}(H^\beta)$ with Maslov index $\mu(y^\beta; H^\beta) = k + 1$ or

$$u \in \mathcal{M}\left(w^\alpha, x^\beta; H_\lambda^{\alpha\beta}, J_\lambda^{\alpha\beta}\right)$$

for some $w^\alpha \in \mathscr{P}(H^\alpha)$ with Maslov index $\mu(w^\alpha; H^\alpha) = k - 1$. In the first case there exists a sequence s_ν tending to $+\infty$ such that $u_\nu(s + s_\nu, t)$ converges to a solution $u^\beta \in \mathcal{M}(y^\beta, x^\beta; H_\lambda^\beta, J_\lambda^\beta)$ and the pair (u, u^β) contributes to the first term in (6.6). In the second case there exists a sequence s_ν tending to $-\infty$ such that $u_\nu(s+s_\nu,t)$ converges to a solution $u^\alpha \in \mathcal{M}(x^\alpha, w^\alpha; H^\alpha_\lambda, J^\alpha_\lambda)$ and the pair (u^α, u) contributes to the second term in (6.6).

It follows from a gluing argument as outlined in the proof of Lemma 6.2 that each such pair (u^α, u) or (u, u^β) occurs as a unique end of the 1-manifold $\mathcal{M}(x^\alpha, x^\beta)$. Since the total number of ends and boundary points is even we have proved (6.6).

The previous result shows that for any two regular pairs (H^α, J^α) and (H^β, J^β) there is a unique homomorphism of Floer homology which we denote by

$$HF_*^{\beta\alpha} : HF_*(M; H^\alpha, J^\alpha) \to HF_*\left(M; H^\beta, J^\beta\right) .$$

These homomorphisms satisfy (6.1).

LEMMA 6.4. *Let (H^α, J^α), (H^β, J^β), and (H^γ, J^γ) be regular pairs. Then*

$$HF_*^{\gamma\beta} \circ HF_*^{\beta\alpha} = HF_*^{\gamma\alpha} , \qquad HF_*^{\alpha\alpha} = \mathrm{id} .$$

Proof: That $HF_*^{\alpha\alpha} = \mathrm{id}$ follows by choosing the constant homotopy. Now suppose that $(H^{\alpha\beta}, J^{\alpha\beta})$ is a regular homotopy from (H^α, J^α) to (H^β, J^β) and $(H^{\beta\gamma}, J^{\beta\gamma})$ is a regular homotopy from (H^β, J^β) to (H^γ, J^γ). Given $R > 0$ sufficiently large we define

$$H_R^{\alpha\gamma}(s,t,x) = \begin{cases} H^{\alpha\beta}(s+R,t,x), & s \leqq 0, \\ H^{\beta\gamma}(s-R,t,x), & s \geqq 0, \end{cases} \quad J_R^{\alpha\gamma}(s,x) = \begin{cases} J^{\alpha\beta}(s+R,x), & s \leqq 0, \\ J^{\beta\gamma}(s-R,x), & s \geqq 0. \end{cases}$$

Then $(H_R^{\alpha\gamma}, J_R^{\alpha\gamma})$ is a regular homotopy from (H^α, J^α) to (H^γ, J^γ). Let $\phi^{\beta\alpha}$, $\phi^{\gamma\beta}$, and $\phi_R^{\gamma\alpha}$ denote the associated chain homomorphisms. We must prove that $\phi^{\gamma\beta} \circ \phi^{\beta\alpha} = \phi_R^{\gamma\alpha}$ for R sufficiently large.

It follows from a gluing argument as in Lemma 6.2 that every pair

$$u^{\alpha\beta} \in \mathcal{M}\left(x^\alpha, x^\beta; H^{\alpha\beta}, J^{\alpha\beta}\right) , \qquad u^{\beta\gamma} \in \mathcal{M}\left(x^\beta, x^\gamma; H^{\beta\gamma}, J^{\beta\gamma}\right)$$

with $\mu(x^\alpha; H^\alpha) = \mu(x^\beta; H^\beta) = \mu(x^\gamma; H^\gamma)$ gives rise to a unique connecting orbit

$$u_R^{\alpha\gamma} \in \mathcal{M}\left(x^\alpha, x^\gamma; H_R^{\alpha\gamma}, J_R^{\alpha\gamma}\right) .$$

Conversely, it follows from Gromov's compactness that for R sufficiently large there is no other connecting orbit in $\mathcal{M}(x^\alpha, x^\gamma; H_R^{\alpha\gamma}, J_R^{\alpha\gamma})$. This implies

$$\langle \phi_R^{\gamma\alpha} x^\alpha, x^\gamma \rangle = \sum_{\mu(x^\beta; H^\beta)=k} \left\langle \phi^{\beta\alpha} x^\alpha, x^\beta \right\rangle \left\langle \phi^{\gamma\beta} x^\beta, x^\gamma \right\rangle$$

for $\mu(x^\alpha; H^\alpha) = \mu(x^\gamma; H^\gamma) = k$.

It follows from Lemma 6.4 with $\gamma = \alpha$ that the morphisms of Floer homology are isomorphisms. This proves Theorem 6.1.

7. The Maslov Index and the Morse Index

In this section we shall prove that the Floer homology groups of a regular pair $(H, J) \in (\mathscr{H} \times \mathscr{J})_{\mathrm{reg}}$ are isomorphic to the singular homology of M.

THEOREM 7.1. *For every regular pair (H^α, J^α) there exists a natural isomorphism*

$$HF_k^\alpha : HF_k(M; H^\alpha, J^\alpha) \to H_{k+n}(M; \mathbb{Z}/2\mathbb{Z}) , \qquad k \in \mathbb{Z} ,$$

between the Floer homology of the pair (H^α, J^α) and the singular homology of M. If (H^β, J^β) is another regular pair then

$$HF_*^\alpha = HF_*^\beta \circ HF_*^{\beta\alpha} .$$

In particular, the Floer homology groups vanish for $|k| > n$.

To prove Theorem 7.1 we consider the case where

$$H(t, x) = H(x)$$

is a Morse function on M which is independent of t. Then there is a constant $\varepsilon > 0$ such that every nonconstant periodic solution of H is of period greater than ε and hence

$$\mathscr{P}_\tau(H) = \{x(t) \equiv x \in M;\, dH(x) = 0\}$$

for $\tau < \varepsilon$. In this situation it turns out that the Maslov index of a constant periodic orbit $x(t) \equiv x$ of (1.1) is related to its Morse index $\mathrm{ind}_H(x)$ when regarded as a critical point of H. To be more precise we fix a Riemannian metric on M associated to some almost complex structure which is compatible with ω.

LEMMA 7.2. *There exists a constant $\varepsilon > 0$ such that if $H : M \to \mathbb{R}$ is a Morse function with $\|H\|_{C^2} < \varepsilon/\tau$ then*

$$\mu_\tau(x; H) = \mathrm{ind}_H(x) - n$$

for every critical point x.

Proof: Let $v_1, \ldots, v_{2n}$ be a symplectic orthonormal basis of T_xM with respect to which $J(x)$ is represented by the matrix $J_0 \in \mathbb{R}^{2n\times 2n}$. Also let $S = S^T \in \mathbb{R}^{2n\times 2n}$ represent the Hessian of H in the same basis. Then

$$\mu(x;H) = \mu(\Psi_x)\ , \qquad \Psi_x(t) = \exp\{J_0 St\}\ , \quad 0 \leqq t \leqq 1\ .$$

Moreover, S is nonsingular and $\mathrm{ind}_H(x)$ is the number of negative eigenvalues of S. Hence Lemma 7.2 follows from statement (iv) of Theorem 3.3.

In the context of closed geodesics an analogous relationship between the Maslov index and the Morse index was described by Duistermaat in [10]. We also point out that Lemma 7.2 was independently proved by Viterbo in [39].

The proof of Theorem 7.1 relies on the observation that if $H(t,x) = H(x)$ is independent of t then every solution $u(s,t)$ of (2.2) which is also independent of t satisfies

$$\frac{du(s)}{ds} = -\nabla H(u(s)) \tag{7.1}$$

and is therefore a gradient flow line of H. This brings us into the realm of finite dimensional Morse theory. The gradient flow (7.1) is called a **Morse-Smale flow** if for any two critical points x and y of H the stable and unstable manifold $W^s(x)$ and $W^u(y)$ intersect transversally. In the next section we shall prove that this condition is satisfied for an open and dense set of almost complex structures J. If (7.1) is a Morse-Smale flow then there is a finite number $n(y,x)$ of solutions $u(s)$ of (7.1) (modulo time shift) such that

$$\lim_{s\to-\infty} u(s) = y\ , \qquad \lim_{s\to+\infty} u(s) = x \tag{7.2}$$

provided that

$$\mathrm{ind}_H(y) - \mathrm{ind}_H(x) = 1\ .$$

These solutions can be used to build a chain complex in which $C_k(H)$ is the $\mathbb{Z}/2\mathbb{Z}$ vector space generated by the critical points x of H with $\mathrm{ind}_H(x) = k$ and the (y,x)-entry of the boundary operator

$$\partial_k = \partial_k(H,J) : C_{k+1} \to C_k$$

is the above number $n(y,x)$ modulo 2. This chain complex is called the **Morse complex** of the pair (H,J) and its homology groups are denoted by

$$HM_k(M;H,J) = \frac{\ker \partial_{k-1}}{\operatorname{im} \partial_k}\ .$$

This is a special case of the Conley-Franzosa connection matrix in [19] and [20]. In [15], [28], [34], [36], and [40] it was proved that the homology groups of the Morse complex agree with the singular homology of M:

$$HM_k(M;H,J) \simeq H_k(M;\mathbb{Z}/2\mathbb{Z})\ .$$

To relate the Floer homology groups to the homology of the above Morse complex we must find for a given Morse-function $H : M \to \mathbb{R}$ an almost complex structure J such that both (7.1) is a Morse-Smale flow and the pair (H, J) is regular in the sense of Section 2. This is possible for a sufficiently small period τ.

THEOREM 7.3 0.1. *Let $H : M \to \mathbb{R}$ be a Morse function and let J be an almost complex structure compatible with ω such that (7.1) is a Morse-Smale flow. Then the following holds for $\tau > 0$ sufficiently small.*

(1) *The operator $F_\tau(u)$ defined by (2.4) is onto for every solution $u : \mathbb{R} \to M$ of (7.1).*
(2) *Every bounded solution $u(s,t) = u(s, t+\tau)$ of (2.2) is independent of t.*

Proof: We prove statement (1). Let $Z_j(s) \in T_{u(s)}M$ be a symplectic orthonormal basis as in the proof of Lemma 5.1 and let the matrices $A = -A^T$ and $S = S^T$ be defined as in the proof of Theorem 5.4. Then the operator $F_\tau(u)$ is in local coordinates given by

$$F\zeta = \frac{\partial \zeta}{\partial s} + J_0 \frac{\partial \zeta}{\partial t} + (S + A)\zeta \ .$$

Since $u(s,t)$ is independent of t so are the matrices A and S. Hence it follows from Corollary 4.3 that this operator is onto if and only if the operator

$$F_0\zeta = \frac{d\zeta}{ds} + (S + A)\zeta$$

is onto provided that τ is sufficiently small. But F_0 is the local coordinate representation of the operator $\xi \mapsto \nabla\xi + \nabla_\xi \nabla H(u)$. So F_0 is onto for every gradient flow line $u(s)$ if and only if (7.1) is a Morse-Smale flow; see [34].

We prove statement (2). Assume by contradiction that there is a sequence of periods $\tau_\nu > 0$ converging to zero and a sequence of bounded solutions u_ν of (2.2) such that

$$u_\nu(s, t + \tau_\nu) = u_\nu(s,t)$$

and $u_\nu(s,t)$ is not independent of t. Without loss of generality we may assume that the functions u_ν satisfy (2.3) for two critical points x, y of H. Then the flow energy of these solutions is uniformly bounded

$$\frac{1}{\tau_\nu} \int_{-\infty}^{\infty} \int_0^{\tau_\nu} \left\| \frac{\partial u_\nu}{\partial s} \right\|^2 dt\, ds = H(y) - H(x) \ .$$

By Gromov's compactness there exists a subsequence (still denoted by u_ν) and sequences of times $s_\nu^j \in \mathbb{R}$, $j = 1, \ldots, m$, such that $u_\nu(s + s_\nu^j, t)$ converges with its derivatives uniformly on compact sets to a gradient flow line $u^j(s)$ of (7.1) running from x^{j-1} to x^j with $x^0 = y$ and $x^m = x$ (for details see [34]). Moreover, the sequence $u_\nu(s,t)$ lies entirely in a neighborhood of this collection of gradient flow lines in the sense that

$$\lim_{\nu\to\infty} \sup_{s,t} \inf_{s^i,j} d(u_\nu(s,t), u^j(s^i)) = 0 .$$

We prove that $u_\nu(s,t)$ must be independent of t for ν sufficiently large. For this we restrict ourselves to the relevant case for Floer homology where the index difference is 1

$$\mathrm{ind}_H(y) - \mathrm{ind}_H(x) = 1 .$$

Then $m = 1$ and with a suitable choice of time shift the sequence $u_\nu(s,t)$ converges to a gradient flow line $u(s)$ of (7.1) running from y to x. Assume first that $\tau_\nu = \tau_0/k_\nu$ for some sequence of integers k_ν where $\tau_0 > 0$ is within the range of validity of statement (1). Then the solutions u_ν of (2.2) are all of period τ_0. By statement (1) the operator $F_{\tau_0}(u)$ is onto. Hence it follows from the uniqueness part of the implicit function theorem that the orbit u is isolated in the space $\mathscr{M}_{\tau_0}(y,x)$. This means that every τ_0-periodic solution of (2.2) and (2.3) which is sufficiently close to u must agree with u up to a time shift. Hence $u_\nu(s,t) = u(s+s_\nu)$ for large ν and this contradicts our assumption that u_ν was not independent of t.

If τ_ν is an arbitrary sequence converging to zero we choose integers $k_\nu \in \mathbb{N}$ such that $k_\nu\tau_\nu$ converges to τ_0. Then the same argument works uniformly in a neighborhood of τ_0.

In the general case where the index difference is larger than 1 we shall not carry out the details of this argument. It involves a quantitative version of the uniqueness part of the gluing construction for the solutions of (2.2) and (2.3) with the relevant constants independent of the parameter τ. For details of this gluing construction see [13].

Proof of Theorem 7.1: Let $H : M \to \mathbb{R}$ be a Morse function and let $J \in \mathscr{J}_{\mathrm{reg}}(H)$. Then it follows from Theorem 7.3 that the pair (H,J) is regular in the sense of Section 2 provided that $\tau > 0$ is sufficiently small. Moreover, all solutions of (2.2) and (2.3) are independent of t. Hence it follows from Lemma 7.2 that in this case Floer's chain complex agrees with the Morse complex of the gradient flow (7.1) with the grading shifted by n and hence

$$HF_k(M;H,J,\tau) \simeq HM_{n+k}(M;H,J) \simeq H_{n+k}(M;\mathbb{Z}/2\mathbb{Z})$$

for τ sufficiently small.

Finally we observe that the Floer homology groups are independent of the choice of the period τ. Indeed, if $u(s,t) = u(s,t+\tau)$ is any solution of (2.2) and (2.3) with a τ-periodic Hamiltonian $H(t,x) = H(t+\tau,x)$ then $v(s,t) = u(\tau s, \tau t)$ satisfies (2.2) with H replaced by the 1-periodic Hamiltonian

$$H_1(t,x) = \tau H(\tau t, x) \, .$$

Moreover, the corresponding periodic solutions have the same Maslov index. Hence

$$HF_k(M; H, J, \tau) \simeq HF_k(M; H_1, J, 1) \, .$$

This proves Theorem 7.1.

Proof of Theorem B: Assume that the contractible τ-periodic solutions of (1.1) are nondegenerate and choose any regular pair $(K,J) \in (\mathscr{H} \times \mathscr{J})_{\mathrm{reg}}$ such that the Hamiltonian $K : \mathbb{R}/\tau\mathbb{Z} \times M \to \mathbb{R}$ is sufficiently close to H. Then the number of contractible τ-periodic solutions x for K with Maslov index $\mu_\tau(x;K) = k$ agrees with the corresponding number for H. Denote by $\partial_k = \partial_k(K,J,\tau) : C_{k+1}(M;K,\tau) \to C_k(M;K,\tau)$ the boundary homomorphism of Floer homology and define

$$d_k = \operatorname{rank} \partial_k \, .$$

Note that the dimension of $C_k(M;K,\tau)$ agrees with the number of contractible τ-periodic solutions of (1.1) with Maslov index k

$$p_k = p_k(H,\tau) = \dim C_k(M;K,\tau) \, .$$

So the isomorphism $HF_k(M;K,J,\tau) \simeq H_{n+k}(M;\mathbb{Z}/2\mathbb{Z})$ of Theorem 7.1 shows that

$$b_{n+k} = \dim \ker \partial_{k-1} - \operatorname{rank} \partial_k = p_k - d_{k-1} - d_k \, .$$

This is equivalent to the Morse inequalities (1.2).

8. Transversality

THEOREM 8.1. *Let $H : M \to \mathbb{R}$ be a Morse function. Then there exists a dense set $\mathscr{J}_{\mathrm{reg}}(H)$ of smooth almost complex structures on M taming ω such that the gradient flow* (7.1) *is of Morse-Smale type.*

Proof: Let $\mathscr{J}$ denote the space of all smooth almost complex structures on M which are compatible with ω and denote by

$$\mathscr{Z}(y,x;H)$$

the space of all pairs (u,J) where $J \in \mathscr{J}$ and $u \in \mathscr{M}(y,x;H,J)$ is a gradient flow line of (7.1) running from y to x. The key point is to prove that with a suitable topology this space is an infinite dimensional manifold. We may assume $x \neq y$.

To be more precise we observe that the tangent space $T_J\mathcal{J}$ to $\mathcal{J}$ is the vector space of smooth sections $X \in C^\infty(S_J)$ where $S_J \subset \mathrm{End}(TM)$ is the bundle over M whose fiber at $x \in M$ is the space $(S_J)_x$ of linear endomorphisms $X : T_xM \to T_xM$ which satisfy

$$JX + XJ = 0\,, \qquad \omega_x(X\xi, \eta) + \omega_x(\xi, X\eta) = 0\,.$$

Indeed, for any section $X \in C^\infty(S_J)$ the one-parameter family

$$J_t = J\exp\{-JXt\} \in C^\infty(\mathrm{End}(TM))$$

consists of almost complex structures compatible with ω for t sufficiently small.

The Sard-Smale theorem requires a Banach manifold structure on the space $\mathcal{J}$ of complex structures. Hence we shall not work with the usual C^∞-topology but instead introduce a stronger norm-topology. Following Floer (see [11]) we choose a sufficiently rapidly decreasing sequence $\varepsilon_k > 0$ and denote by $C_\varepsilon^\infty(S_J)$ the subspace of those smooth sections $X \in C^\infty(S_J)$ for which

$$\|X\|_\varepsilon = \sum_{k=0}^{\infty} \varepsilon_k \|X\|_{C^k(M)} \quad < \quad \infty\,.$$

This defines a separable Banach space of smooth sections which for a suitable choice of the sequence ε_k is dense in $L^2(S_J)$; see [11]. Now fix an almost complex structure $J_1 \in \mathcal{J}$ and a sufficiently small constant $\delta > 0$. Denote by $\mathcal{J}_1$ the space of all almost complex structures of the form $J = J_1\exp\{-J_1X\}$ where $X \in C_\varepsilon^\infty(S_{J_1})$ with $\|X\|_\varepsilon < \delta$. This space is diffeomorphic to an open set in a separable Banach space. Denote by $\mathcal{Z}_1(y,x)$ the subspace of all $(u, J) \in \mathcal{Z}(y,x)$ with $J \in \mathcal{J}_1$. We prove that $\mathcal{Z}_1(y,x)$ is a Banach manifold.

Denote by $\mathcal{U}$ the space of $W^{1,2}$-maps $u : \mathbb{R} \to M$ which satisfy the limit condition (7.2) in the $W^{1,2}$-sense; see [13]. Let $\mathcal{E}$ denote the bundle over $\mathcal{U} \times \mathcal{J}_1$ whose fibers are L^2-vector fields along u. The formula

$$\mathcal{F}(u,J) = \frac{du}{ds} + \nabla H(u)$$

defines a smooth section of this bundle

$$\mathcal{F} : \mathcal{U} \times \mathcal{J}_1 \to \mathcal{E}\,.$$

and

$$\mathcal{Z}_1(y,x) = \mathcal{F}^{-1}(0)\,.$$

So we have to show that $\mathscr{F}$ is transversal to the zero section. Equivalently, we must show that the linearized map

$$d\mathscr{F}(u,J)(\xi,X) = F(u)\xi + X(u)\left(\frac{du}{dt} - X_H(u)\right)$$

is onto where the operator $F(u) : W^{1,2} \to L^2$ is defined by

$$F(u)\xi = \nabla\xi + \nabla_\xi \nabla H(u) .$$

We know already that this is a Fredholm operator (see [34]) and therefore has a closed range. Hence it remains to prove that the range of the operator $d\mathscr{F}(u,J)$ is dense.

Suppose that $\mathscr{F}(u,J) = 0$ and that the L^2 vector field η along u is orthogonal to the range of $d\mathscr{F}(u,J)$. Then η satisfies the ordinary differential equation

$$\nabla_s \eta = \nabla_\eta \nabla H(u) \tag{8.1}$$

and

$$\int_{-\infty}^{\infty} \left\langle \eta, X(u)J(u)\frac{du}{ds}\right\rangle ds = 0 \tag{8.2}$$

for every $X \in T_J\mathscr{J}_1$. Since $x \neq y$ and $\mathscr{F}(u,J) = 0$ we have

$$\frac{du}{ds} \neq 0 .$$

We claim that $\eta(s) = 0$ for all s. Suppose otherwise that $\eta(s_0) \neq 0$ for some s_0 and let $x_0 = u(s_0)$. Consider an orthogonal symplectic trivialization of TM near x_0 as in Lemma 5.1. With respect to this trivialization the transformation J is represented by the matrix J_0 and the symplectic form is given by $\omega(\xi_1,\xi_0) = (J_0\zeta_1)^T\zeta_0$ where $\zeta_0, \zeta_1 \in \mathbb{R}^{2n}$ denote the coordinate vectors of $\xi_0, \xi_1 \in TM$. Represent X as a $2n \times 2n$-matrix X_0 with respect to these coordinates and observe that $X \in (S_J)_x$ if and only if

$$X_0{}^T = X_0 = J_0 X_0 J_0 .$$

For any two nonzero vectors ζ_0 and ζ_1 we show that there exists a matrix X_0 as above with $\langle \zeta_1, X_0\zeta_0\rangle \neq 0$. If $\langle \zeta_1, \zeta_0\rangle \neq 0$ choose

$$X_0 = \zeta_0\zeta_0{}^T + J_0\zeta_0\zeta_0{}^T J_0 .$$

If $\langle \zeta_1, J_0\zeta_0\rangle \neq 0$ choose

$$X_0 = J_0\zeta_0\zeta_0{}^T - \zeta_0\zeta_0{}^T J_0 .$$

If ζ_1 is orthogonal to both ζ_0 and $J_0\zeta_0$ choose

$$X_0 = \zeta_1\zeta_0{}^T + \zeta_0\zeta_1{}^T + J_0\left(\zeta_1\zeta_0{}^T + \zeta_0\zeta_1{}^T\right)J_0 .$$

Now consider the case

$$\xi_0 = J(x_0)\frac{du}{ds}(s_0) , \qquad \xi_1 = \eta(s_0) ,$$

and let $\zeta_i \in \mathbb{R}^{2n}$ denote the coordinates of $\xi_i \in T_{x_0}M$ for $i = 0, 1$. Choose the matrix X_0 as above. Since $C_\varepsilon^\infty(S_{J_1})$ is dense in $L^2(S_{J_1})$ there exists a smooth section $X \in T_J\mathcal{J}_1$ which at the point x_0 is represented by the matrix X_0 and which is supported in an arbitrarily small neighborhood of x_0. Then

$$\left\langle \eta(s_0), X(x_0)J(x_0)\frac{du}{ds}(s_0)\right\rangle \neq 0$$

and it follows that the integral on the left-hand side of (8.2) must be nonzero. This contradiction shows that η vanishes at s_0 and hence everywhere. Thus we have proved that the operator $d\mathcal{F}(u, J)$ is onto as claimed.

Since $d\mathcal{F}(u, J)$ is onto it follows that $\mathcal{Z}_1(y, x)$ is a Banach manifold. The tangent space of $\mathcal{Z}_1(y, x)$ at (u, J) consists of all pairs $(\xi, X) \in W^{1,2} \times T_J\mathcal{J}_1$ such that

$$F_\tau(u)\xi + X(u)\left(\frac{du}{dt} - X_H(u)\right) = 0 . \tag{8.3}$$

Define $\mathcal{J}_{\text{reg}}(y, x; H)$ as the set of regular values of the projection

$$\pi : \mathcal{Z}_1(y, x; H) \to \mathcal{J}_1 , \qquad (u, J) \mapsto J .$$

Then the differential of π is onto if and only if for every X there exists a ξ such that (8.3) is satisfied. Hence (u, J) is a regular point of π if and only if $F(u)$ is onto. We conclude that the space $\mathcal{M}(y, x; H, J)$ is a smooth manifold for every $J \in \mathcal{J}_{\text{reg}}(y, x; H)$. Since π is a smooth Fredholm mapping it follows from the Sard-Smale theorem (see [37]) that the set $\mathcal{J}_{\text{reg}}(y, x; H)$ has a nowhere dense complement in $\mathcal{J}_1$ and so has the intersection

$$\mathcal{J}_{\text{reg}}(H) = \bigcap_{x,y} \mathcal{J}_{\text{reg}}(y, x; H) .$$

This proves Theorem 8.1.

We shall use a similar argument as in the proof of Theorem 8.1 to show that the set of regular pairs (H, J) in the sense of Section 2 is dense in the set of all pairs of τ-periodic Hamiltonians H and almost complex structures J.

This requires an analysis of the local behavior of the solutions $u \in \mathcal{M}_\tau(y,x)$ of (2.2) and (2.3).

A point $(s,t) \in \mathbb{R}^2$ is called a **critical point** of u if either $\partial u/\partial s(s,t) = 0$ or $u(s,t) = x(t)$ or $u(s,t) = y(t)$. A point $(s_0,t_0) \in \mathbb{R}^2$ is called a **regular point** of u if it is not a critical point and $u(s,t_0) \neq u(s_0,t_0)$ for every $s \neq s_0$. We denote the set of critical points by $C(u)$ and the set of regular points by $R(u)$. In [17] Floer and Hofer proved the following theorem.

THEOREM 8.2. *Assume $x \neq y$ and $u \in \mathcal{M}_\tau(y,x)$. Then the set $C(u)$ is discrete and the set $R(u)$ is open and dense in $\mathbb{R}^2$.*

We also need a unique continuation theorem due to Aronszajn; see [3].

THEOREM 8.3. *Let $\Omega \subset \mathbb{R}^2$ be a connected open set. Suppose that $u \in C^\infty(\Omega, \mathbb{R}^m)$ satisfies the pointwise estimate*

$$|\Delta u(s,t)| \leqq c\left(|u(s,t)| + \left|\frac{\partial u}{\partial s}(s,t)\right| + \left|\frac{\partial u}{\partial t}(s,t)\right|\right)$$

and that all the partial derivatives of u vanish at a point $(s_0,t_0) \in \Omega$. Then $u \equiv 0$.

THEOREM 8.4. *Let $J : TM \to TM$ be an almost complex structure compatible with ω. Then there exists a dense set $\mathcal{H}_{\mathrm{reg}} = \mathcal{H}_{\mathrm{reg}}(J,\tau)$ of smooth τ-periodic Hamiltonian functions $H(x,t) = H(x,t+\tau)$ such that (H,J) is a regular pair for every $H \in \mathcal{H}_{\mathrm{reg}}$.*

Proof: Let $H_0(x,t) = H_0(x,t+\tau)$ be a Hamiltonian function such that every τ-periodic solution of (1.1) is nondegenerate. We denote by $C_\varepsilon^\infty(M;H_0)$ the set of smooth functions $h : M \times \mathbb{R}/\tau\mathbb{Z} \to \mathbb{R}$ whose support is bounded away from the τ-periodic solutions of (1.1). This means that $h(x,t) = 0$ for x in some fixed neighborhood of $x_0(t)$ for $x_0 \in \mathcal{P}(H_0)$. Here the norm $\|h\|_\varepsilon$ is a C_ε^∞-norm as in the proof of Theorem 8.1. Note that if $\|h\|_\varepsilon$ is sufficiently small then $\mathcal{P}_\tau(H_0 + h) = \mathcal{P}_\tau(H_0)$.

Now let $x, y \in \mathcal{P}_\tau(H_0)$. We denote by

$$\mathcal{Z}(y,x;J)$$

the set of all pairs (u,H) where $H - H_0 \in C_\varepsilon^\infty(M;H_0)$ and $u \in \mathcal{M}(y,x;H,J)$. We must prove that for $x \neq y$ this space is an infinite dimensional manifold. For this we proceed as in the proof of Theorem 8.1. Let $\mathcal{U}$ denote the space of $W^{1,p}$-maps $u : \mathbb{R} \times \mathbb{R}/\tau\mathbb{Z} \to M$ which satisfy the limit condition (2.3) in the $W^{1,p}$ sense where $p > 2$. Let $\mathscr{E}$ denote the bundle over $\mathscr{U} \times C_\varepsilon^\infty(M; H_0)$ whose fiber at u is the space of L^p vector fields along u. Then the formula

$$\mathscr{F}(u, H) = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t, u)$$

defines a smooth section $\mathscr{F} : \mathscr{U} \times C_\varepsilon^\infty(M; H_0) \to \mathscr{E}$. We must prove that $\mathscr{F}$ is transversal to the zero section. This means that the linearized map

$$d\mathscr{F}(u, H)(\xi, h) = F_\tau(u)\xi + \nabla h$$

is onto whenever $\mathscr{F}(u, H) = 0$.

Suppose that η is a nonzero L^q-vector field along u orthogonal to the range of $d\mathscr{F}(u, H)$ where $1/p + 1/q = 1$. Then it follows from elliptic regularity that η is smooth and

$$F_\tau(u)^*\eta = -\nabla_s\eta + J(u)\nabla_t\eta + \nabla_\eta J(u)\frac{\partial u}{\partial t} + \nabla_\eta\nabla H(t, u) = 0\,. \tag{8.4}$$

Moreover, it follows that

$$\int_{-\infty}^{\infty}\int_0^\tau \langle \eta, \nabla h(u, t)\rangle\, dt\, ds = 0 \tag{8.5}$$

for every $h \in C_\varepsilon^\infty(M; H_0)$.

We prove that $\eta(s, t) \neq 0$ for almost every pair $(s, t) \in \mathbb{R}^2$. Suppose otherwise that η vanishes on an open subset of $\mathbb{R}^2$. Let $\Phi(s, t) = \Phi(s, t + \tau) : \mathbb{R}^{2n} \to T_{u(s,t)}M$ be an orthogonal symplectic trivialization as in Lemma 5.1 and denote by $\zeta = \Phi^{-1}\eta$ the new coordinates of η. Then

$$F^*\zeta = -\frac{\partial \zeta}{\partial s} + J_0\frac{\partial \zeta}{\partial t} + (S + A)\zeta = 0$$

and hence

$$0 = \left(\frac{\partial}{\partial s} + J_0\frac{\partial}{\partial t}\right)F^*\zeta = -\Delta\zeta + \text{lower order terms}\,.$$

Hence it follows from Aronszajn's theorem that $\zeta \equiv 0$.

We prove that $\partial u/\partial s$ and η are linearly dependent for all $(s, t) \in \mathbb{R}^2$. Suppose otherwise that $\partial u/\partial s$ and η are linearly independent at some point $(s_0, t_0) \in \mathbb{R}^2$. Assume without loss of generality that $(s_0, t_0) \in R(u)$. Then there exists a small neighborhood $U_0 \subset M \times \mathbb{R}/\tau\mathbb{Z}$ of the point $(u(s_0, t_0), t_0)$ such that the open set

$$V_0 = \{(s, t) : (u(s, t), t) \in U_0\}$$

is a small neighborhood of the pair (s_0, t_0). Otherwise there would exist a sequence (s_ν, t_ν) such that $t_\nu \to t_0$, $u(s_\nu, t_\nu) \to u(s_0, t_0)$ and $|s_\nu - s_0| > \delta$. Since $x(t_0) \neq u(s_0, t_0) \neq y(t_0)$ the sequence s_ν must be bounded. Assume without loss of generality that s_ν converges to s^*. Then $|s^* - s_0| > \delta$ and $u(s^*, t_0) = u(s_0, t_0)$ and this contradicts the definition of $R(u)$. Thus we have proved that the above set V_0 is a small neighborhood of (s_0, t_0) for some U_0. Note that the map

$$V_0 \to U_0 : (s,t) \mapsto (u(s,t), t)$$

is a diffeomorphism. Shrinking V_0 if necessary we may assume that η and $\partial u/\partial s$ are linearly independent in V_0. Hence we may choose coordinates $\phi_t : M \to \mathbb{R}^{2n}$ in a neighborhood of $(u(s_0, t_0), t_0) \in U_0$ such that

$$\phi_t(u(s,t)) = (s - s_0, 0, \ldots, 0) \ , \qquad d\phi_t(u(s,t))\eta(s,t) = (0, 1, 0 \ldots, 0) \ .$$

Now define $g_t : \mathbb{R}^{2n} \to \mathbb{R}$ by

$$g_t(y) = \beta(t - t_0)\beta(y_1)\beta(y_2)y_2$$

where $\beta : \mathbb{R} \to [0,1]$ is a cutoff function such that $\beta(r) = 1$ for $|r| \leqq \delta$ and $\beta(r) = 0$ for $|r| \geqq 2\delta$. Then the function

$$h : M \times \mathbb{R}/\tau\mathbb{Z} \to \mathbb{R} : (x,t) \mapsto h_t(x) = g_t(\phi_t(x))$$

vanishes outside U_0 and satisfies

$$h_t(u(s,t)) = 0 \ , \quad dh_t(u(s,t), t)\eta(s,t) = \beta(s - s_0)\beta(t - t_0)$$

for $(s,t) \in V_0$. Hence the integral (8.5) is nonzero. It remains nonzero when we approximate h by a function in $C_\varepsilon^\infty(M; H_0)$. This contradicts our assumption on η.

Thus we have shown that the vectors $\eta(s,t)$ and $\partial u/\partial s(s,t)$ must be linearly dependent for all $(s,t) \in \mathbb{R}^2$. So there exists a smooth function $\lambda : \mathbb{R}^2 \setminus C(u) \to \mathbb{R}$ such that

$$\eta(s,t) = \lambda(s,t)\frac{\partial u}{\partial s}(s,t) \ .$$

We prove that $\partial\lambda/\partial s \equiv 0$. Assume otherwise that $\partial\lambda/\partial s(s_0, t_0) \neq 0$ for some point $(s_0, t_0) \in \mathbb{R}^2 \setminus C(u)$. Since $\eta(s,t) \neq 0$ for almost every $(s,t) \in \mathbb{R}^2$ we may assume without loss of generality that $\lambda(s_0, t_0) \neq 0$. By Lemma 8.2 we may also assume without loss of generality that $(s_0, t_0) \in R(u)$. Choose a small neighborhood V_0 of (s_0, t_0) as above such that $\lambda \neq 0$ and $\partial\lambda/\partial s \neq 0$ in V_0. Since $\partial\lambda/\partial s \neq 0$ there exists a compactly supported function $\alpha : V_0 \to \mathbb{R}$ such that

$$\int_{V_0} \lambda(s,t)\frac{\partial\alpha}{\partial s}(s,t)\, ds\, dt \neq 0 \ .$$

As above we construct a function $h_t = h_{t+\tau} : M \to \mathbb{R}$ such that

$$h_t(u(s,t)) = \alpha(s,t) \quad \text{for} \quad (s,t) \in V_0 \, .$$

Then we obtain

$$dh_t(u(s,t),t)\eta(s,t) = \lambda(s,t)\frac{\partial \alpha}{\partial s}(s,t) \quad \text{for} \quad (s,t) \in V_0$$

and the integral (8.5) is nonzero again.

Thus we have shown that

$$\frac{\partial \lambda}{\partial s}(s,t) = 0$$

for all $(s,t) \in \mathbb{R}^2 \setminus C(u)$. Since $C(u)$ is a discrete set it follows that $\lambda(s,t) = \lambda(t)$ is independent of s. In particular λ is a smooth function on all of $\mathbb{R}^2$. Now we observe that

$$\lambda(t) \neq 0$$

for all $t \in \mathbb{R}$. Otherwise there would exist a number $t_0 \in \mathbb{R}$ such that $\eta(s,t_0) = 0$ for all $s \in \mathbb{R}$. Since η is a solution of (8.4) it would follow that all derivatives of η vanish on the line $t = t_0$. By Aronszajn's theorem this would imply $\eta \equiv 0$ in contradiction to our assumption.

Assume without loss of generality that $\lambda(t) > 0$. Then we have

$$\int_0^\tau \left\langle \frac{\partial u}{\partial s}(s,t), \eta(s,t) \right\rangle dt = \int_0^\tau \lambda(t) \left| \frac{\partial u}{\partial s}(s,t) \right|^2 dt > 0$$

for every $s \in \mathbb{R}$. On the other hand, it follows from the identities

$$F_\tau(u)\frac{\partial u}{\partial s} = 0 \, , \qquad F_\tau(u)^* \eta = 0 \, ,$$

that

$$\begin{aligned} &\frac{d}{ds} \int_0^\tau \left\langle \frac{\partial u}{\partial s}(s,t), \eta(s,t) \right\rangle dt \\ &= \int_0^\tau \left(\left\langle \nabla_s \frac{\partial u}{\partial s}(s,t), \eta(s,t) \right\rangle + \left\langle \frac{\partial u}{\partial s}(s,t), \nabla_s \eta(s,t) \right\rangle \right) dt \\ &= 0 \, . \end{aligned}$$

This contradiction shows that the operator $d\mathscr{F}(u,H)$ is onto whenever $\mathscr{F}(u, H) = 0$ and therefore $\mathscr{Z}(y,x,J)$ is an infinite dimensional manifold. The rest of the proof is as in Theorem 8.1.

If we consider a fixed Hamiltonian H (for example a time-independent one) we have to allow a perturbation of J in the class of time dependent almost complex structures to make the previous transversality theorem work. This means that we leave the class of finite dimensional gradient flows required to prove that the Floer homology groups agree with the homology of the underlying manifold. In Section 7 we circumvented this difficulty by choosing a small period τ.

9. Perturbation of Isolated Periodic Orbits

We prove a perturbation theorem for periodic solutions of a time dependent Hamiltonian differential equation on a (paracompact) symplectic manifold (M, ω). More precisely, we assume that $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ is a smooth (C^∞) Hamiltonian function and $x_0(t) = x_0(t+1)$ is a 1-periodic solution of (1.1) which is isolated. Given any almost complex structure J we denote by $d = d_J$ the induced metric on M defined by (2.1). Choose $\delta > 0$ such that $d(x(t), x_0(t)) > \delta$ for every t and every 1-periodic solution of (1.1) other than $x_0(t)$. Let

$$N_0 = \{(t,x) \in \mathbb{R} \times M;\ d(x, x_0(t)) < \delta\}$$

be the tubular δ-neighborhood of $x_0(t)$.

THEOREM 9.1. *For every $\varepsilon > 0$ there exists a smooth Hamiltonian function $H' : S^1 \times M \to \mathbb{R}$ which satisfies*

$$\|H' - H\|_{C^2} \leqq \varepsilon ,$$

and agrees with H outside N_0 such that every 1-periodic solution of

$$\dot{x}(t) = X_{H'}(t, x(t))$$

which enters N_0 is nondegenerate.

To prove Theorem 9.1 we choose local Darboux coordinates on the manifold M near the periodic solution $x_0(t)$. We denote by ω_0 the standard symplectic form on $\mathbb{R}^{2n}$ given by $\omega_0(\zeta, \zeta') = (J_0\zeta)^T\zeta'$.

LEMMA 9.2. *Let (M, ω) be a paracompact $2n$-dimensional symplectic manifold and let $x(t) = x(t+1)$ be a smooth curve in M. Then there exists an open set $U \subset \mathbb{R}^{2n}$ containing 0 and a smooth map*

$$\mathbb{R} \times U \to M , \qquad (t,x) \mapsto \phi_t(x) ,$$

such that ϕ_t is a diffeomorphism from U onto an open neighborhood of $x(t)$ satisfying

$$\phi_t(0) = x(t) , \qquad \phi_{t+1} = \phi_t , \qquad \phi_t^*\omega = \omega_0 .$$

Proof: We first construct a symplectic trivialization

$$\Phi(t) : \mathbb{R}^{2n} \to T_{x(t)}M , \qquad \Phi(t+1) = \Phi(t) , \tag{9.1}$$

so that

$$\omega_{x(t)}(\Phi(t)\zeta, \Phi(t)\zeta') = \omega_0(\zeta, \zeta') . \tag{9.2}$$

If $x(t)$ is a contractible loop then such a trivialization exists by Lemma 5.1. In the general case we first construct a symplectic trivialization $\Phi(t)$ which is not necessarily periodic. More precisely, let $\Phi(t,s) : T_{x(s)}M \to T_{x(t)}M$ denote the solution operator of the differential equation

$$\nabla\xi + \nabla_\xi J(x)J(x)\dot{x} = 0$$

for vector fields $\xi(t)$ along $x(t)$ so that $\xi(t) = \Phi(t,s)\xi(s)$. Then $\Phi(t,s)$ is a symplectic transformation and $\Phi(t+1, s+1) = \Phi(t,s)$. Choose any linear symplectic transformation $\Phi(0) : \mathbb{R}^{2n} \to T_{x(0)}M$ and define

$$\Phi(t) = \Phi(t,0)\Phi(0) : \mathbb{R}^{2n} \to T_{x(t)}M .$$

This function satisfies (9.2) but will in general not be periodic in t. It therefore remains to construct a symplectic matrix $\Psi(t) \in \mathrm{Sp}(2n;\mathbb{R})$ such that $\Phi(t)\Psi(t)$ is of period 1 or equivalently

$$\Psi(t+1) = A\Psi(t) \tag{9.3}$$

where $A = \Phi(t+1)^{-1}\Phi(t) = \Phi(0)^{-1}\Phi(0,1)\Phi(0) \in \mathrm{Sp}(2n;\mathbb{R})$. It is enough to construct $\Psi(t)$ for $0 \leqq t \leqq 1$ such that $\Psi(t) = I$ for $0 \leqq t \leqq \varepsilon$ and $\Psi(t) = A$ for $1-\varepsilon \leqq t \leqq 1$. Then $\Psi(t)$ extends uniquely to a smooth function from $\mathbb{R}$ to $\mathrm{Sp}(2n;\mathbb{R})$ via (9.3). Since $\mathrm{Sp}(2n;\mathbb{R})$ is connected such a path $\Psi(t)$ exists.

Now suppose that $\Phi(t) : \mathbb{R}^{2n} \to T_{x(t)}M$ satisfies both (9.1) and (9.2) and define

$$\phi_t : \mathbb{R}^{2n} \to M , \qquad \phi_t(z) = \exp_{x(t)}\{\Phi(t)z\} .$$

Then $\phi_t = \phi_{t+1}$ is a local diffeomorphism near $z = 0$ and satisfies the condition $\phi_t^*\omega = \omega_0$ at $z = 0$. So Moser's proof of Darboux's theorem (see [30]) can be carried over word by word to prove that there exists a smooth 1-parameter family of local diffeomorphisms $\psi_t : U \to \mathbb{R}^{2n}$ such that

$$\psi_t(0) = 0 , \qquad \psi_{t+1} = \psi_t , \qquad \psi_t^*\phi_t^*\omega = \omega_0 .$$

Thus the composition $\phi_t \circ \psi_t$ satisfies the requirements of the lemma.

Proof of Theorem 9.1: In view of Lemma 9.2 it is enough to consider the case where $z(t) \equiv 0$ is an isolated 1-periodic solution of the Hamiltonian differential equation

$$\dot{z}(t) = J_0 \nabla H(t, z(t)) \tag{9.4}$$

in $\mathbb{R}^{2n}$. Assume without loss of generality that the Hamiltonian function

$$H(t,z) = H(t+1,z)$$

is defined on all of $\mathbb{R}^{2n}$ and that its gradient is uniformly bounded. Denote by

$$\psi_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

the 1-parameter family of symplectic diffeomorphisms determined by the solutions of (9.4), that is $\psi_t(z(0)) = z(t)$. Then $z = 0$ is an isolated fixed point of ψ_1.

Let $z = (x,y)$ and $\zeta = (\xi, \eta)$ where x, y, ξ, and η are n-vectors. So the canonical transformation $z = \psi_1(\zeta)$ can be written in the form

$$x = u(\xi, \eta), \qquad y = v(\xi, \eta). \tag{9.5}$$

Assume without loss of generality that

$$\det\left(\frac{\partial v}{\partial \eta}\right) \neq 0$$

near $\zeta = 0$. Then there exists a generating function $Q(\xi, y)$ such that equation (9.5) in a neighborhood of zero is equivalent to

$$x = \frac{\partial Q}{\partial y}(\xi, y), \qquad \eta = \frac{\partial Q}{\partial \xi}(\xi, y).$$

With $W(\xi, y) = \langle \xi, y\rangle - Q(\xi, y)$ this becomes

$$x - \xi = -\frac{\partial W}{\partial y}(\xi, y), \qquad y - \eta = \frac{\partial W}{\partial \xi}(\xi, y). \tag{9.6}$$

So the fixed points of ψ_1 are the critical points of W. Moreover $z = (x,y) = (\xi, y)$ is nondegenerate as a fixed point of ψ_1 if and only if z is nondegenerate as a critical point of W.

Choose $\delta > 0$ such that there is no critical point of W in $B_\delta(0) \setminus \{0\}$ and define

$$\rho = \inf_{B_\delta \setminus B_{\delta/2}} |\nabla W|.$$

Choose a smooth cutoff function β which is 1 in $B_{\delta/2}$ and vanishes outside B_δ and define $c = \sup|\nabla\beta|$. Let $(a, b) \in \mathbb{R}^{2n}$ be a regular value of ∇W such that

$$|a| + |b| < \frac{\rho}{1 + c\delta} .$$

Then the function

$$W'(\xi, y) = W(\xi, y) - \beta(\xi, y)(\langle a, \xi\rangle + \langle b, y\rangle)$$

has only nondegenerate critical points in B_δ and no critical points at all in $B_\delta \setminus B_{\delta/2}$. Moreover a and b can be chosen such that

$$\|W' - W\|_{C^2} < \varepsilon_1$$

for any given positive number ε_1. Let ψ' be the symplectic diffeomorphism generated by W' meaning that $z = \psi'(\zeta)$ if and only if

$$x - \xi = -\frac{\partial W'}{\partial y}(\xi, y) , \qquad y - \eta = \frac{\partial W'}{\partial \xi}(\xi, y) .$$

Then ψ' has only nondegenerate fixed points in B_δ and no fixed points in $B_\delta \setminus B_{\delta/2}$. Moreover ψ' is C^1-close to $\psi = \psi_1$ and agrees with ψ outside B_δ. Define

$$\phi = \psi^{-1} \circ \psi' .$$

Then ϕ is C^1-close to the identity map and agrees with the identity outside B_δ. Hence there exists a generating function $S = S(\xi, y)$ which is C^2-small and vanishes outside B_δ such that $z = \phi(\zeta)$ if and only if

$$x - \xi = -\frac{\partial S}{\partial y}(\xi, y) , \qquad y - \eta = \frac{\partial S}{\partial \xi}(\xi, y) .$$

Choose a smooth cutoff function $\alpha : [0, 1] \to [0, 1]$ such that $\alpha(t) = 0$ for $0 \leqq t \leqq \varepsilon$ and $\alpha(t) = 1$ for $1 - \varepsilon \leqq t \leqq 1$ and define the symplectic diffeomorphism $z = \phi_t(\zeta)$ by the implicit formula

$$x - \xi = -\alpha(t)\frac{\partial S}{\partial y}(\xi, y) , \qquad y - \eta = \alpha(t)\frac{\partial S}{\partial \xi}(\xi, y) .$$

Then $\phi_t = \mathrm{id}$ for $t \leqq \varepsilon$, $\phi_t = \phi$ for $t \geqq 1 - \varepsilon$ and $\phi_t(z) = z$ for all t if $z \notin B_\delta$. So the vector field

$$v_t = \frac{d\phi_t}{dt} \circ \phi_t^{-1}$$

is C^1-small and vanishes for $z \notin B_\delta$ and for $t \notin [\varepsilon, 1 - \varepsilon]$. Since $\phi_t^*\omega_0 = \omega_0$ it follows that $i_{v_t}\omega_0$ is a closed and hence exact 1-form on $\mathbb{R}^{2n}$. We conclude that there exists a C^2-small Hamiltonian function $F : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ such that $v_t(z) = J_0 \nabla F(t,z)$. So ϕ_t is the solution operator of the Hamiltonian differential equation

$$\dot{z} = J_0 \nabla F(t,z) \; .$$

This function $F(t,z)$ can be chosen to vanish for $z \notin B_\delta$ and for $t \notin [\varepsilon, 1-\varepsilon]$ and therefore extends uniquely to a smooth 1-periodic Hamiltonian $F : S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$. Let ϕ_t for $t \in \mathbb{R}$ be the corresponding extended 1-parameter family of solution operators and define

$$\psi_t' = \psi_t \circ \phi_t \; .$$

Then $\psi_1' = \psi'$ and therefore its fixed points in B_δ are nondegenerate. Moreover, ψ_t' is the solution operator of the Hamiltonian differential equation $\dot{z} = J_0 \nabla H'(t,z)$ where

$$H'(t,z) = H(t,z) + F\left(t, \psi_t^{-1}(z)\right)$$

In particular, $H'(t,z)$ is 1-periodic in t and agrees with $H(t,z)$ for $z \notin B_\delta$. This proves Theorem 9.1

The above perturbation argument has also been carried out in [43]. It shows that it is enough to assume that the unperturbed Hamiltonian function H is of class C^2. This will guarantee ψ_t to be of class C^1 and therefore H' to be of class C^2.

Proof of Theorem A: Let ψ_t denote the time-t-map of the Hamiltonian system (1.1).

Suppose that there are only finitely many contractible 1-periodic solutions of (1.1) and that these are all weakly nondegenerate. Then we shall prove that there exists a number $\tau_0 > 0$ such that for every prime $\tau > \tau_0$ there is a contractible periodic solution $x \in \mathscr{P}_\tau(H)$ of minimal period τ. By this we mean that $\psi_k(x(0)) \neq x(0)$ for every integer $1 \leqq k < \tau$. To prove this we introduce the number $\tau_0(x)$ for $x \in \mathscr{P}_1(H)$ as follows. If $\Delta_1(x;H) \neq 0$ we define $\tau_0(x) = 2n/|\Delta_1(x;H)|$. If $\Delta_1(x;H) = 0$ we make use of the fact that there is a Floquet multiplier $\lambda \neq 1$. If λ is not a root of unity we define $\tau_0(x) = 1$. If λ is a root of unity we define $\tau_0(x) = \min\{k \in \mathbb{N};\, \lambda^k = 1\}$.

Now suppose that $\tau \in \mathbb{N}$ is a prime number such that

$$\tau > \tau_0 = \max_{x \in \mathscr{P}_1(H)} \tau_0(x)$$

and that there is no contractible periodic solution of (1.1) with minimal period τ. Then every contractible τ-periodic solution of (1.1) must be of period 1. So it follows from the iterated index formula in Lemma 3.4 that every contractible τ-periodic solution of (1.1) satisfies either

$$|\Delta_\tau(x;H)| = \tau |\Delta_1(x;H)| > 2n$$

or

$$\Delta_\tau(x; H) = 0.$$

In the latter case the definition of $\tau_0(x)$ shows that $1 \neq \lambda^\tau \in \sigma(d\psi_\tau(x(0)))$ with λ as above. Choose $\rho > 0$ such that every contractible τ-periodic solution of (1.1) with $\Delta_\tau(x; H) = 0$ has at least one Floquet multiplier outside the ball $B_{3\pi\rho}(1)$. Then there exists a constant $\varepsilon > 0$ such that if $H' : S^1 \times M \to \mathbb{R}$ is any Hamiltonian function with $\|H' - H\|_{C^2} < \varepsilon$ then every contractible τ-periodic solution $x' \in \mathscr{P}_\tau(H')$ satisfies either

$$|\Delta_\tau(x'; H')| > 2n$$

or

$$|\Delta_\tau(x'; H')| \leqq \rho$$

and in the latter case there is a Floquet multiplier outside $B_{2\pi\rho}(1)$. By Theorem 9.1 there exists such a Hamiltonian function close to H whose contractible τ-periodic solutions are nondegenerate. If $|\Delta_\tau(x'; H')| > 2n$ then

$$|\mu_\tau(x'; H')| \geqq |\Delta_\tau(x'; H')| - |r(\Psi_{x'}(\tau))| > n\ .$$

If $|\Delta_\tau(x'; H')| \leqq \rho$ then $|r(\Psi_{x'}(\tau))| < n - 2\rho$ and therefore

$$|\mu_\tau(x'; H')| \leqq |\Delta_\tau(x'; H')| + |r(\Psi_{x'}(\tau))| < n\ .$$

Thus there is no $x' \in \mathscr{P}_\tau(H')$ whose Maslov index is n or $-n$. This contradicts the Morse inequalities of Theorem B.

10. Some Concluding Remarks

In the special case of a constant Hamiltonian there are of course infinitely many integer periodic solutions, however they are all of period 1. In contrast the proof of Theorem A leads to a more refined statement which guarantees periodic solutions of arbitrarily large integer periods τ which are minimal in the sense that the solution is not of *integer* period less than τ. More precisely, if there are only finitely many contractible 1-periodic solutions of (1.1) and these are all weakly nondegenerate then for every sufficiently large prime $\tau \in \mathbb{N}$ there exists a contractible periodic solution with minimal period τ.

In Section 5 we have introduced the mean winding number $\Delta_1(x)$ associated to every contractible 1-periodic solution x of (1.1). This number is invariant under symplectic diffeomorphisms but is of course not invariant under perturbation. It is an even integer whenever all Floquet multipliers are equal to 1. The weak nondegeneracy condition of Theorem A is only needed for those 1-periodic solutions for which this mean winding number is zero. So we have proved the following existence result. If the Hamiltonian system (1.1) possesses only finitely many 1-periodic solutions and all of these have a nonzero mean winding number Δ_1 then for every sufficiently large prime $\tau \in \mathbb{N}$ there exists a contractible periodic solution with minimal period τ. This phenomenon is related to a recent result due to J. Franks; see [20]. It states that if an area and orientation preserving homeomorphism of the 2-dimensional plane possesses 2 fixed points with different mean rotation numbers then there are infinitely many periodic points.

We have not addressed the question as to how much smoothness is needed for the Hamiltonian function H in order for Theorems A and B to remain valid. It should be enough to assume that H has Lipschitz continuous derivatives. The details, however, have yet to be fully worked out. Also, we have not considered the existence of periodic solutions which are not contractible.

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